Infinite Gammoids: Minors and Duality

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Submitted: Mar 19, 2015; Accepted: Nov 22, 2015; Published: Dec 11, 2015
Mathematics Subject Classifications: 05B35; 05C20; 05C63

Abstract

This sequel to Afzali Borujeni et. al. (2015) considers minors and duals of infinite gammoids. We prove that the class of gammoids defined by digraphs not containing a certain type of substructure, called an outgoing comb, is minor-closed. Also, we prove that finite-rank minors of gammoids are gammoids. Furthermore, the topological gammoids of Carmesin (2014) are proved to coincide, as matroids, with the finitary gammoids. A corollary is that topological gammoids are minor-closed.

It is a well-known fact that the dual of any finite strict gammoid is a transversal matroid. The class of strict gammoids defined by digraphs not containing alternating combs, introduced in Afzali Borujeni et. al. (2015), contains examples which are not dual to any transversal matroid. However, we describe the duals of matroids in this class as a natural extension of transversal matroids. While finite gammoids are closed under duality, we construct a strict gammoid that is not dual to any gammoid.

Keywords: infinite matroids; gammoids; transversal matroids; matchings; linkages; contractions

1 Introduction

Infinite matroid theory has recently seen a surge of activity (e.g. [1], [8], [11]), after Bruhn et al [10] found axiomatizations of infinite matroids with duality, solving a long-standing problem of Rado [23]. As part of this re-launch of the subject, we initiated an investigation of infinite gammoids in [2]. In this paper we will continue to primarily use the independence axioms for matroids as stated in Definition 2.1.

A *dimaze* is a digraph equipped with a specific set of sinks, the (set of) *exits*. A dimaze *contains* another dimaze, if, in addition to digraph containment, the exits of the former

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include those of the latter. In the context of digraphs, any path or ray (i.e. one-way infinite analogue of path) is forward oriented. An *outgoing comb* is a dimaze obtained from a ray by adding infinitely many non-trivial disjoint paths, that meet the ray precisely at their initial vertices, and declaring the sinks of the resulting digraph to be the exits.

The subsets of the vertex set of (the digraph of) an infinite dimaze that are linkable to the exits by a set of disjoint paths need not be the independent sets of a matroid ([2]), but if they are, this matroid is called a *strict gammoid* and the dimaze is a *presentation* of it ([18, 21]). A *gammoid* is a matroid restriction of a strict gammoid; so, by definition, the class of gammoids is closed under matroid deletion.

A pleasant property of the class of finite gammoids is that it is also closed under matroid contractions, and hence, under taking minors. In contrast, whether the class of all gammoids, possibly infinite, is minor-closed is an open question. Our aim in the first part of this paper is to begin to address this and related questions.

A standard proof of the fact that finite gammoids are minor-closed as a class of matroids proceeds via duality. The proof of this fact can be extended to infinite dimazes whose underlying (undirected) graph does not contain any ray, but it breaks down when rays are allowed. However, analysing the construction in more details, we are able to bypass duality and relax the condition to the absence of outgoing combs (Theorem 3.11).

Theorem 1. *The class of gammoids that admit a presentation not containing any outgoing comb is minor-closed.*

If we do allow outgoing combs, combining the tools developed to prove the above theorem with a proof of Pym's linkage theorem [22], we can still establish the following (Theorem 3.13).

Theorem 2. *Any finite-rank minor of an infinite gammoid is a gammoid.*

Outgoing combs naturally appeared in [12] where Carmesin used a topological approach to extend finite gammoids to the infinite case, in response to a question raised by Diestel. Carmesin proved that the topologically linkable sets, the definition of which is given in Section 2, of a dimaze form the independent sets of a finitary[1] matroid on the vertex set of the digraph. Any matroid isomorphic to such a matroid is called a strict topological gammoid. A matroid restriction of a strict topological gammoid is called a topological gammoid. Making use of this result together with Theorem 1, we prove the following (Corollary 3.16 and Theorem 3.17).

Theorem 3. *A matroid is a topological gammoid if and only if it is a finitary gammoid. Moreover, the class of topological gammoids is minor-closed.*

In the second part of the paper, we turn to duality. Recall that a *transversal matroid* can be defined by taking a fixed vertex class of a bipartite graph as the ground set and its matchable subsets as the independent sets. Ingleton and Piff [17] proved constructively

[1] Finitary means that a set is independent if and only if all of its finite subsets are.

that finite strict gammoids and finite transversal matroids are dual to each other, a key fact to their proof that the class of finite gammoids is closed under duality. In contrast, an infinite strict gammoid need not be dual to a transversal matroid, and vice versa (Examples 4.11 and 4.15). Although these examples do not rule out the possibility that infinite gammoids are closed under duality, we will see in Section 4.3 that there is a gammoid which is not dual to any gammoid.

To better describe duals of strict gammoids, in Section 4.1, we introduce a superclass of transversal matroids called the class of *path-transversal matroids*. These matroids differ from transversal matroids in that certain matchings are forbidden in the definition of independence, thereby forming a larger class of matroids. Still, as we will see in Example 4.12, there is a strict gammoid whose dual is not a path-transversal matroid. This strict gammoid has the property that any dimaze defining it contains an *alternating comb*, a dimaze that is studied to some details in [2]. It turns out that forbidding alternating combs suffices for a strict gammoid to be dual to a path-transversal matroid (Theorem 4.9).

Theorem 4. *A strict gammoid that admits a presentation not containing any alternating comb is dual to a path-transversal matroid.*

We remark that the theorem is used in [3] to characterize cofinitary transversal matroids and cofinitary strict gammoids.

2 Preliminaries

In this paper, digraphs do not have any loops or parallel edges. We collect definitions, basic results and examples. For notions not found here, we refer to [10] and [20] for matroid theory, and [13] for graph theory.

2.1 Infinite matroids

Given a set E and a family of subsets $\mathcal{I} \subseteq 2^E$, let $\mathcal{I}^{\max}$ denote the maximal elements of $\mathcal{I}$ with respect to set inclusion. For a set $I \subseteq E$ and $x \in E$, we write $x, I + x, I - x$ for $\{x\}, I \cup \{x\}$ and $I \setminus \{x\}$ respectively.

Definition 2.1. [10] A *matroid* M is a pair $(E, \mathcal{I})$, where E is a set and $\mathcal{I} \subseteq 2^E$, which satisfies the following:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \subseteq I'$ and $I' \in \mathcal{I}$, then $I \in \mathcal{I}$.

(I3) For all $I \in \mathcal{I} \setminus \mathcal{I}^{\max}$ and $I' \in \mathcal{I}^{\max}$, there is an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}$.

(IM) Whenever $I \in \mathcal{I}$ and $I \subseteq X \subseteq E$, the set $\{I' \in \mathcal{I} : I \subseteq I' \subseteq X\}$ has a maximal element.

For a matroid $M = (E, \mathcal{I})$, a subset of the *ground set* E is *independent* if it is in $\mathcal{I}$; *dependent* otherwise. A *base* is a maximal independent set, while a *circuit* is a minimal dependent set. A circuit of size one is called a *loop*. We usually identify a matroid with its set of independent sets.

Let $X \subseteq E$. A *deletion minor* $M \setminus X$ is the pair $(E \setminus X, \mathcal{I} \cap 2^{E \setminus X})$, that is, the independent sets are those of M that are contained in $E \setminus X$. A *contraction minor* M/X is the pair $(E \setminus X, \mathcal{I}')$ where a set $I \subseteq E \setminus X$ is in $\mathcal{I}'$ if $I \cup B \in \mathcal{I}$ for some basis B of $M \setminus (E \setminus X)$. We also write $M.X$ for $M/(E \setminus X)$. In general, a *minor* of M has the form $M/X \setminus Y$ for some disjoint $X, Y \subseteq E$, and is always a matroid.

The *dual* M^* of M has as bases precisely the complements of the bases of M, and is always a matroid. As usual, we use the prefix *co* to refer to an object in the dual matroid. For example, a *coloop* of M is a loop of M^*, which is equivalent to the assertion that a coloop is contained in every base of M. Similarly, a *cocircuit* is a set minimal with the property of hitting every base of M and a set is *coindependent* if and only if it misses a base of M. We also have a useful relation between duals and minors, namely, for any $X \subseteq E$, $(M \setminus X)^* = M^*/X$. The following standard fact, a proof of which can be found in [20, Lemma 3.3.2], simplifies investigations of minors.

Lemma 2.2. *Every minor of a matroid M can be written in the form $M/S \setminus R$ where S is an independent set and R is a coindependent set of M.*

For a set system $M = (E, \mathcal{I})$ let the set $\mathcal{I}^{\text{fin}}$ consist of the sets which have all their finite subsets in $\mathcal{I}$. Then $M^{\text{fin}} := (E, \mathcal{I}^{\text{fin}})$ is called *finitarisation* of M, and M is called *finitary* if $M = M^{\text{fin}}$. Applying Zorn's Lemma one sees that finitary set systems always satisfy (IM); in particular, if M is a matroid, M^{fin} is a matroid too.

2.2 Linkability systems

Given a digraph D, let $V := V(D)$ and $B_0 \subseteq V$ be a set of sinks. Call the pair (D, B_0) a *dimaze*[2] and B_0 the (set of) *exits*. The requirement of the exits to be sinks rules out some trivial cases and will not restrict the generality of the matroids that we will consider.

Given a (directed) path or ray P, $\mathrm{Ini}(P)$ and $\mathrm{Ter}(P)$ denote the initial and the terminal vertex (if exists) of P, respectively. For a set $\mathcal{P}$ of paths and rays, let $\mathrm{Ini}(\mathcal{P}) := \{\mathrm{Ini}(P) : P \in \mathcal{P}\}$ and $\mathrm{Ter}(\mathcal{P}) := \{\mathrm{Ter}(P) : P \in \mathcal{P}\}$. A *linkage $\mathcal{P}$ from $A \subseteq V$ to $B \subseteq B_0$* is a set of vertex disjoint paths with $\mathrm{Ini}(\mathcal{P}) = A$ and $\mathrm{Ter}(\mathcal{P}) \subseteq B$. Such a set A is *linkable*, and if $\mathrm{Ter}(\mathcal{P}) = B$, it is linkable *onto* B.

Definition 2.3. Let (D, B_0) be a dimaze. The pair of $V(D)$ and the set of linkable subsets is denoted by $M_L(D, B_0)$. A *strict gammoid* is a matroid isomorphic to $M_L(D, B_0)$ for some (D, B_0). A *gammoid* is a restriction of a strict gammoid. Given a gammoid M, (D, B_0) is called a *presentation* of M if $M = M_L(D, B_0)|X$ for some $X \subseteq V(D)$.

In general, $M_L(D, B_0)$ satisfies (I1), (I2) and (I3) but not (IM); see [2].

[2]Dimaze is short for directed maze.

If D' is a subdigraph of D and $B_0' \subseteq B_0$, then (D, B_0) *contains* (D', B_0') as a *subdimaze*. A dimaze (D', B_0') is a *subdivision* of (D, B_0) if it can be obtained from (D, B_0) as follows. We first add an extra vertex b_0 and the edges $\{(b, b_0) : b \in B_0\}$ to D. Then the edges of this resulting digraph are subdivided to define a digraph D''. Set B_0' as the in-neighbourhood of b_0 in D'' and D' as $D'' - b_0$. Note that this defaults to the usual notion of subdivision if $B_0 = \emptyset$.

The following dimazes play an important role in our investigation. An undirected ray is a graph with an infinite vertex set $\{x_i : i \geqslant 1\}$ and the edge set $\{x_i x_{i+1} : i \geqslant 1\}$. We orient the edges of an undirected ray in different ways to construct three dimazes:

1. R^A by orienting (x_{i+1}, x_i) and (x_{i+1}, x_{i+2}) for each odd $i \geqslant 1$ and the set of exits is empty;

2. R^I by orienting (x_{i+1}, x_i) for each $i \geqslant 1$ and x_1 is the only exit;

3. R^O by orienting (x_i, x_{i+1}) for each $i \geqslant 1$ and the set of exits is empty.

A subdivision of R^A, R^I and R^O is called *alternating ray*, *incoming ray* and *(outgoing) ray*, respectively.

Let $Y = \{y_i : i \geqslant 1\}$ be a set disjoint from X. We extend the above types of rays to combs by adding edges (and their terminal vertices) and declaring the resulting sinks to be the exits:

1. C^A by adding no edges to R^A;

2. C^I by adding the edges (x_i, y_i) to R^I for each $i \geqslant 2$;

3. C^O by adding the edges (x_i, y_i) to R^O for each $i \geqslant 2$.

Furthermore we define the dimaze F^∞ by declaring the sinks of the digraph $(\{v, v_i : i \in \mathbb{N}\}, \{(v, v_i) : i \in \mathbb{N}\})$ to be the exits.

Any subdivision of C^A, C^I, C^O and F^∞ is called *alternating comb*, *incoming comb*, *outgoing comb* and *linking fan*, respectively. The subdivided ray in any comb is called the *spine* and the paths to the exits are the *spikes*.

A dimaze (D, B_0) is called *$\mathcal{H}$-free* for a set $\mathcal{H}$ of dimazes if it does not have a subdimaze isomorphic to a subdivision of an element in $\mathcal{H}$. We usually drop the braces when $\mathcal{H}$ is a singleton. Note that the dimaze R^A is C^A-free, since its empty set of exits cannot contain the exits of an alternating comb. A (strict) gammoid is called *$\mathcal{H}$-free* if it admits an $\mathcal{H}$-free presentation. In general, an $\mathcal{H}$-free gammoid may admit a presentation that is not $\mathcal{H}$-free (see Figure 3 for $\mathcal{H} = \{C^A\}$).

Note that a maximally linkable set is certainly linkable onto the exits and the vertices of out-degree 2 in an alternating comb is a set that can be linked onto the exits but is not maximally linkable. A main result in [2] shows that the absence of alternating combs guarantees that a set that is linkable onto the exits is maximally linkable.

Theorem 2.4. *(i) Given a dimaze, the sets linkable onto the exits are maximally linkable if and only if the dimaze is C^A-free. (ii) Any C^A-free dimaze defines a strict gammoid.*

In [2], it was shown that the matroid defined in Example 4.12 is not in the class of C^A-free gammoids, however, this class contains interesting examples including highly connected matroids which are not nearly finitary ([4]) or the duals of such ones. This class is also a fruitful source for wild matroids ([8]).

Let (D, B_0) be a dimaze and $\mathcal{Q}$ a set of disjoint paths or rays (usually a linkage). A *$\mathcal{Q}$-alternating walk* is a sequence $W = w_0 e_0 w_1 e_1 \dots$ of vertices w_i and distinct edges e_i of D not ending with an edge, such that every $e_i \in W$ is incident with w_i and w_{i+1}, and the following properties hold for each $i \geqslant 0$ (and $i < n$ in case W is finite, where w_n is the last vertex):

(W1) $e_i = (w_{i+1}, w_i)$ if and only if $e_i \in E(\mathcal{Q})$;

(W2) if $w_i = w_j$ for any $j \neq i$, then $w_i \in V(\mathcal{Q})$;

(W3) if $w_i \in V(\mathcal{Q})$, then $\{e_{i-1}, e_i\} \cap E(\mathcal{Q}) \neq \emptyset$ (with $e_{-1} := e_0$).

Let $\mathcal{P}$ be another set of disjoint paths or rays. A *$\mathcal{P}$-$\mathcal{Q}$-alternating walk* is a $\mathcal{Q}$-alternating walk whose edges are in $E(\mathcal{P})\Delta E(\mathcal{Q})$, and such that any interior vertex w_i satisfies

(W4) if $w_i \in V(\mathcal{P})$, then $\{e_{i-1}, e_i\} \cap E(\mathcal{P}) \neq \emptyset$.

Two $\mathcal{Q}$-alternating walks W_1 and W_2 are *disjoint* if they are edge disjoint, $V(W_1) \cap V(W_2) \subseteq V(\mathcal{Q})$ and $\mathrm{Ter}(W_1) \neq \mathrm{Ter}(W_2)$.

Suppose that a dimaze (D, B_0), a set $X \subseteq V$ and a linkage $\mathcal{P}$ from a subset of X to B_0 are given. An X–B_0 (vertex) separator S is a set of vertices such that every path from X to B_0 intersects S, and S is *on* $\mathcal{P}$ if it consists of exactly one vertex from each path in $\mathcal{P}$.

We recall a classical result due to Grünwald [15], which can be formulated as follows (see also [13, Lemmas 3.3.2 and 3.3.3]).

Lemma 2.5. *Let (D, B_0) be a dimaze, $\mathcal{Q}$ a linkage, and* $\mathrm{Ini}(\mathcal{Q}) \subseteq X \subseteq V$.

(i) If there is a $\mathcal{Q}$-alternating walk from $X \setminus \mathrm{Ini}(\mathcal{Q})$ to $B_0 \setminus \mathrm{Ter}(\mathcal{Q})$, then there is a linkage $\mathcal{Q}'$ with $\mathrm{Ini}(\mathcal{Q}) \subsetneq \mathrm{Ini}(\mathcal{Q}') \subseteq X$ onto $\mathrm{Ter}(\mathcal{Q}) \subsetneq \mathrm{Ter}(\mathcal{Q}') \subseteq B_0$.

(ii) If there is not any $\mathcal{Q}$-alternating walk from $X \setminus \mathrm{Ini}(\mathcal{Q})$ to $B_0 \setminus \mathrm{Ter}(\mathcal{Q})$, then there is an X–B_0 separator on $\mathcal{Q}$.

A set $X \subseteq V$ in (D, B_0) is *topologically linkable*[3] if X admits a *topological linkage*, which means that from each vertex $x \in X$, there is a *topological path* P_x, i.e. P_x is the spine of an outgoing comb, a path ending at the centre of a linking fan, or a path ending in B_0, such that P_x is disjoint from P_y for any $y \neq x$. Thus, the spikes of the outgoing combs

[3]With a suitable topology on the undirected underlying graph, this notion of linkability has a topological interpretation in which these sets of rays and paths are precisely those that link I to the topological closure of the set of the exits [12].

and the paths emanating from the centre of a linking fan may intersect other topological paths and their spikes. Roughly speaking, a topological path from a vertex v does not need to reach the exits as long as no finite vertex set avoiding that path can prevent an actual connection of v to B_0. Note that a finite topologically linkable set is linkable.

Denote by $M_{TL}(D, B_0)$ the pair of V and the set of the topologically linkable subsets. Carmesin gave the following connection between dimazes (not necessarily defining a matroid) and topological linkages.

Corollary 2.6. *[12, Corollary 5.7] Given a dimaze (D, B_0), $M_{TL}(D, B_0) = M_L(D, B_0)^{\mathrm{fin}}$. In particular, $M_{TL}(D, B_0)$ is always a finitary matroid.*

A *strict topological gammoid* is a matroid of the form $M_{TL}(D, B_0)$, and its restrictions are called *topological gammoids*. We will see in Section 3 that such matroids are precisely the finitary gammoids.

2.3 Transversal systems

Let $G = (V, W)$ be a bipartite graph and call V and W, respectively, the *left* and the *right* vertex class of G. A subset I of V is *matchable onto* $W' \subseteq W$ if there is a matching m of I such that $m \cap V = I$ and $m \cap W = W'$; where we are identifying a set of edges (and sometimes more generally a subgraph) with its vertex set. Given a set $X \subseteq V$ or $X \subseteq W$, we write $m(X)$ for the set of vertices matched to X by m and $m \restriction X$ for the subset of m incident with vertices in X. Given a matching m, an *m-alternating walk* is a walk such that the consecutive edges alternate in and out of m in G. Given another matching m', an *m-m'-alternating walk* is a walk such that consecutive edges alternate between the two matchings.

Definition 2.7. Given a bipartite graph $G = (V, W)$, the pair of V and all its matchable subsets is denoted by $M_T(G)$. A *transversal matroid* is a matroid isomorphic to $M_T(G)$ for some G. Given a transversal matroid M, G is a *presentation* of M if $M = M_T(G)$.

In general, a transversal matroid may have different presentations. The following is a well-known fact whose proof uses m-alternating walks (see [9]).

Lemma 2.8. *Let $G = (V, W)$ be a bipartite graph. Suppose there is a maximal element in $M_T(G)$, witnessed by a matching m_0. Then $M_T(G) = M_T(G - (W \setminus m_0))$, and $N(W \setminus m_0)$ is a subset of every maximal element in $M_T(G)$.*

In case $M_T(G)$ is a matroid, the second part states that $N(W \setminus m_0)$ is a set of coloops. From now on, wherever there is a maximal element in $M_T(G)$, we assume that W is covered by a matching. Thus, any presentation of a transversal matroid has the property that W is covered by a matching.

If G is finite, Edmonds and Fulkerson [14] showed that $M_T(G)$ satisfies (I3), and so is a matroid. When G is infinite, $M_T(G)$ still satisfies (I3) ([2]) but need not be a matroid.

A standard compactness proof shows that a left locally finite bipartite graph $G = (V, W)$, i.e. every vertex in V has finite degree, defines a finitary transversal matroid.

Lemma 2.9 ([19])**.** *Every left locally finite bipartite graph defines a finitary transversal matroid.*

The following corollary is a tool to show that a matroid is not transversal.

Lemma 2.10. *Any infinite circuit of a transversal matroid contains an element which does not lie in any finite circuit.*

Proof. Let C be an infinite circuit of some $M_T(G)$. Applying Lemma 2.9 on $G - (V \setminus C)$, which is a presentation of the restriction of $M_T(G)$ to C, we see that there is a vertex in C having infinite degree in G. However, such a vertex does not lie in any finite circuit. □

3 Minors

The class of gammoids is closed under deletion by definition. So, if we want to show that a minor of a gammoid is a gammoid, it suffices to show that a contraction minor M/S of a strict gammoid M is a strict gammoid; where S may be assumed to be independent by Lemma 2.2.

In [17], Ingleton and Piff gave a constructive proof that finite gammoids are closed under contraction. The main part of this construction turns a presentation of a strict gammoid into a presentation of its dual transversal matroid and vice versa. So a presentation of $M/S = (M^* \setminus S)^*$ can be obtained from one of M by going to the dual, deleting S there and dualizing again.

In case of general gammoids we can no longer appeal to duality, as we shall see in Section 4. So we will ignore the intermediate step of going to the dual and directly investigate the outcome of the construction mentioned above. We are then able to show that the class of C^O-free gammoids, i.e. gammoids that admit a C^O-free presentation, is minor-closed. In combination with the linkage theorem, we can also prove that finite rank minors of gammoids are gammoids. It remains open whether the class of gammoids is closed under taking minors.

Two other results that we will prove in this section is that finitary gammoids are precisely the topological gammoids, and that they are also closed under taking minors.

3.1 Matroid contraction and shifting along a linkage

In this section, we investigate under which conditions a contraction minor M/S is a strict gammoid.

The first case is that S is a subset of the exits.

Lemma 3.1. *Let $M = M_L(D, B_0)$ be a strict gammoid and $S \subseteq B_0$. Then a dimaze presentation of M/S is given by $M_L(D - S, B_0 \setminus S)$.*

Proof. Since $S \subseteq B_0$ is independent, $I \in \mathcal{I}(M/S) \iff I \cup S \in \mathcal{I}(M)$. Moreover,

$$\begin{aligned} I \in \mathcal{I}(M/S) &\iff I \cup S \text{ admits a linkage in } (D, B_0) \\ &\iff I \text{ admits a linkage } \mathcal{Q} \text{ with } \mathrm{Ter}(\mathcal{Q}) \cap S = \emptyset \text{ in } (D, B_0) \\ &\iff I \in \mathcal{I}(M_L(D - S, B_0 \setminus S)). \end{aligned}$$

□

Thus, it suffices to give a dimaze presentation of M such that S is a subset of the exits. For this purpose we consider the process of "shifting a dimaze along a linkage".

Throughout the section, (D, B_0) denotes a dimaze, $\mathcal{Q}$ a set of disjoint paths or rays, $S := \mathrm{Ini}(\mathcal{Q})$ and $T := \mathrm{Ter}(\mathcal{Q})$. Next, we define various maps which are dependent on $\mathcal{Q}$. We find it convenient to denote all of these maps and their inverses respectively by $\vec{\mathcal{Q}}$ and $\overleftarrow{\mathcal{Q}}$.

Define a bijection $\vec{\mathcal{Q}} : V \setminus T \to V \setminus S$ as follows: $\vec{\mathcal{Q}}(v) := v$ if $v \notin V(\mathcal{Q})$; otherwise $\vec{\mathcal{Q}}(v) := u$ where u is the unique vertex such that $(v, u) \in E(\mathcal{Q})$.

Construct the digraph $\vec{\mathcal{Q}}(D)$ from D by replacing each edge $(v, u) \in E(D) \setminus E(\mathcal{Q})$ with $(\vec{\mathcal{Q}}(v), u)$ and each edge $(v, u) \in \mathcal{Q}$ with (u, v). Equivalently, $\vec{\mathcal{Q}}(D)$ can be constructed by adding a loop (v, v) to each vertex $v \in V \setminus B_0$, then replacing each edge (v, u) with $(\vec{\mathcal{Q}}(v), u)$ (including the loops), and deleting the resulting loops. Set for the rest of this section

$$D_1 := \vec{\mathcal{Q}}(D) \text{ and } B_1 := (B_0 \setminus T) \cup S$$

and call (D_1, B_1) the $\mathcal{Q}$-*shifted* dimaze.

For the finite case, the idea behind a $\mathcal{Q}$-shifted dimaze can be found in [17], where it is called "the transformed graph". Mason [18] started the investigation of $\mathcal{Q}$-shifting in the special case when the linkage has only one edge, in order to alter the presentation of a strict gammoid, in particular the exits. This case was called "swapping" by Ardila and Ruiz in [5], where they investigated when two finite dimazes define the same strict gammoid.

In order to investigate the relation between $\mathcal{Q}$-alternating walks in a dimaze and paths or rays in the $\mathcal{Q}$-shifted dimaze, we give the following definitions: Given a $\mathcal{Q}$-alternating walk $W = w_0 e_0 w_1 e_1 w_2 \ldots$ in D, let $\vec{\mathcal{Q}}(W)$ be obtained from W by deleting all the edges e_i and each vertex $w_i \in W$ with $w_i \in V(\mathcal{Q})$ but $e_i \notin E(\mathcal{Q})$. For a path or ray $P = v_0 v_1 v_2 \ldots$ in D_1, let $\overleftarrow{\mathcal{Q}}(P)$ be obtained from P by inserting after each $v_i \in P \setminus \mathrm{Ter}(P)$ one of the following:

$\boldsymbol{(v_i, v_{i+1})}$ if $v_i \notin V(\mathcal{Q})$;

$\boldsymbol{(v_{i+1}, v_i)}$ if $v_i \in V(\mathcal{Q})$ and $(v_{i+1}, v_i) \in E(\mathcal{Q})$;

$\boldsymbol{(w, v_i) w (w, v_{i+1})}$ with $w := \overleftarrow{\mathcal{Q}}(v_i)$ if $v_i \in V(\mathcal{Q})$ but $(v_{i+1}, v_i) \notin E(\mathcal{Q})$.

We now show that $\vec{\mathcal{Q}}(W)$ is a path or ray in D_1 and $\overleftarrow{\mathcal{Q}}(P)$ is a $\mathcal{Q}$-alternating walk in D, and that disjointness is preserved under shifting. For this purpose recall the definition of disjoint $\mathcal{Q}$-alternating walks from page 6.

Lemma 3.2. *(i) A $\mathcal{Q}$-alternating walk in D that is infinite or ends at $t \in B_1$ is respectively mapped by $\vec{\mathcal{Q}}$ to a ray or a path ending at t in D_1. Disjoint such walks are mapped to disjoint paths/rays.*

(ii) A ray or a path ending at $t \in B_1$ in D_1 is respectively mapped by $\overleftarrow{\mathcal{Q}}$ to an infinite $\mathcal{Q}$-alternating walk or a finite $\mathcal{Q}$-alternating walk ending at t in D. Disjoint such paths/rays are mapped to disjoint $\mathcal{Q}$-alternating walks.

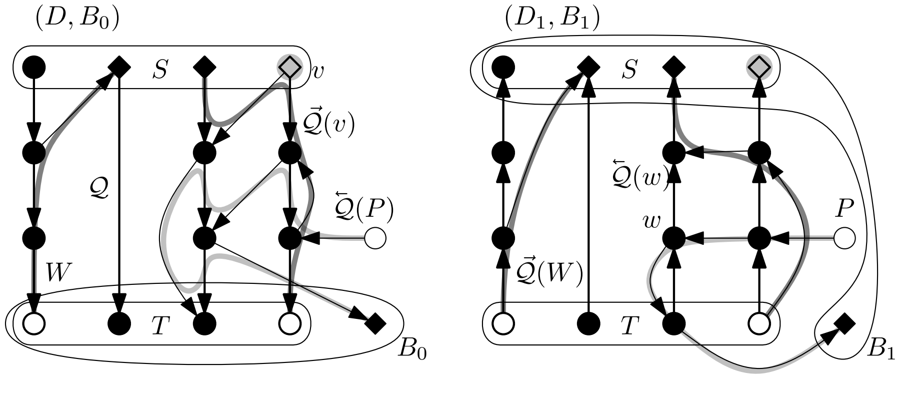

Figure 1: A $\mathcal{Q}$-shifted dimaze: $D_1 = \vec{\mathcal{Q}}(D)$, $B_1 = (B_0 \setminus T) \cup S$, where $\mathcal{Q}$ consists of the vertical downward paths. A vertex is white or diamond shaped if it is respectively the initial or terminal vertex of a (gray) $\mathcal{Q}$-alternating walk (left) or its $\vec{\mathcal{Q}}$-images (right).

Proof. We just prove (i) since (ii) can be proved with similar ideas.

Let $W = w_0 e_0 w_1 e_1 w_2 \ldots$ be a $\mathcal{Q}$-alternating walk in D. If a vertex v in W is repeated, then v occurs twice and there is i such that $v = w_i$ with $e_{i-1} = (w_i, w_{i-1}) \in E(\mathcal{Q})$ and $e_i \notin E(\mathcal{Q})$. Hence, w_i is deleted in $P := \vec{\mathcal{Q}}(W)$ and so v does not occur more than once in P, that is, P consists of distinct vertices.

By construction, the last vertex of a finite W is not deleted, hence P ends at t. In case W is infinite, by (W3), no tail of W is deleted so that P remains infinite.

Next, we show that (v_i, v_{i+1}) is an edge in D_1. Let $w_j = v_i$ be the non-deleted instance of v_i. If w_{j+1} has been deleted, then the edge (w_{j+1}, w_{j+2}) (which exists since the last vertex cannot be deleted) in D has been replaced by the edge $(\vec{\mathcal{Q}}(w_{j+1}), w_{j+2}) = (v_i, v_{i+1})$ in D_1. If both w_j and $v_{i+1} = w_{j+1}$ are in $V(\mathcal{Q})$ then the edge $(w_{j+1}, w_j) \in E(\mathcal{Q})$ has been replaced by (v_i, v_{i+1}) in D_1. In the other cases $(w_j, w_{j+1}) = (v_i, v_{i+1})$ is an edge of D and remains one in D_1.

Let W_1, W_2 be disjoint $\mathcal{Q}$-alternating walks. By construction, $\vec{\mathcal{Q}}(W_1) \cap \vec{\mathcal{Q}}(W_2) \subseteq W_1 \cap W_2 \subseteq V(\mathcal{Q})$. By disjointness, at any intersecting vertex, one of W_1 and W_2 leaves with an edge not in $E(\mathcal{Q})$. Thus, such a vertex is deleted upon application of $\vec{\mathcal{Q}}$. Hence, $\vec{\mathcal{Q}}(W_1)$ and $\vec{\mathcal{Q}}(W_2)$ are disjoint paths/rays. $\square$

Note that for a path P in D_1 and a $\mathcal{Q}$-alternating walk W in D, we have

$$\vec{\mathcal{Q}}(\overleftarrow{\mathcal{Q}}(P)) = P; \qquad \overleftarrow{\mathcal{Q}}(\vec{\mathcal{Q}}(W)) = W.$$

This correspondence of sets of disjoint $\mathcal{Q}$-alternating walks in (D, B_0) and sets of disjoint paths or rays in the $\mathcal{Q}$-shifted dimaze will be used in various situations in order to show, without the help of duality, that the independent sets associated with (D, B_0) and the $\mathcal{Q}$-shifted dimaze are the same.

Given a set $\mathcal{W}$ of $\mathcal{Q}$-alternating walks, define the graph

$$\mathcal{Q}\Delta\mathcal{W} := (V(\mathcal{Q}) \cup V(\mathcal{W}), E(\mathcal{Q})\Delta E(\mathcal{W})).$$

Lemma 3.3. *Let $J \subseteq V \setminus S$ and $\mathcal{W}$ a set of disjoint $\mathcal{Q}$-alternating walks, each of which starts from J and does not end outside of B_1. Then there is a set of disjoint rays or paths from $X := J \cup (S \setminus \mathrm{Ter}(\mathcal{W}))$ to $Y := T \cup (\mathrm{Ter}(\mathcal{W}) \cap B_0)$ in $\mathcal{Q}\Delta\mathcal{W}$.*

Proof. Every vertex in $\mathcal{Q}\Delta\mathcal{W} \setminus (X \cup Y)$ has in-degree and out-degree both 1 or both 0. Moreover, every vertex in X has in-degree 0 and out-degree 1 (or 0, if it is also in Y) and every vertex in Y has out-degree 0 and in-degree 1 (or 0, if it is also in X). Therefore every (weakly) connected component of $\mathcal{Q}\Delta\mathcal{W}$ meeting X is either a path ending in Y or a ray. $\square$

The following will be used to complete a ray to an outgoing comb in various situations.

Lemma 3.4. *Suppose $\mathcal{Q}$ is a topological linkage. Any ray R that hits infinitely many vertices of $V(\mathcal{Q})$ is the spine of an outgoing comb.*

Proof. The first step is to inductively construct an infinite linkable subset of $V(R)$. Let $\mathcal{Q}_0 := \mathcal{Q}$ and $A_0 := \emptyset$. For $i \geqslant 0$, assume that $\mathcal{Q}_i$ is a topological linkage that intersects $V(R)$ infinitely but avoids the finite set of vertices A_i. Since it is not possible to separate a vertex on a topological path from B_0 by a finite set of vertices disjoint from that topological path, there exists a path P_i from $V(R) \cap V(\mathcal{Q}_i)$ to B_0 avoiding A_i. Let $A_{i+1} := A_i \cup V(P_i)$ and $\mathcal{Q}_{i+1}$ obtained from $\mathcal{Q}_i$ by deleting from each of its elements the minimal initial segment that intersects A_{i+1}. As $\mathcal{Q}_{i+1}$ remains a topological linkage that intersects $V(R)$ infinitely, we can continue the procedure. By construction $\{P_i : i \in \mathbb{N}\}$ is an infinite set of disjoint finite paths from a subset of $V(R)$ to B_0. Let $p_i \in P_i$ be the last vertex of R on P_i, then R is the spine of the outgoing comb: $R \cup \bigcup_{i \in \mathbb{N}} p_i P_i$. $\square$

Corollary 3.5. *Any ray provided by Lemma 3.3 is in fact the spine of an outgoing comb if $\mathcal{Q}$ is a topological linkage, and the infinite forward segments of the walks in $\mathcal{W}$ are the spines of outgoing combs.*

Proof. Observe that a ray R constructed in Lemma 3.3 is obtained by alternately following the forward segments of the walks in $\mathcal{W}$ and the forward segments of elements in $\mathcal{Q}$.

Either a tail of R coincides with a tail of a walk in $\mathcal{W}$, and we are done by assumption; or R hits infinitely many vertices of $V(\mathcal{Q})$, and Lemma 3.4 applies. $\square$

With Lemma 3.3 we can transform disjoint alternating walks into disjoint paths or rays. A reverse transform is described as follows.

Lemma 3.6. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two sets of disjoint paths or rays. Let $\mathcal{W}$ be a set of maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walks starting at distinct vertices of $\mathrm{Ini}(\mathcal{P})$. Then the walks in $\mathcal{W}$ are disjoint and can only end in $(\mathrm{Ter}(\mathcal{P}) \setminus T) \cup S$.*

Proof. Let $W = w_0 e_0 w_1 \ldots$ be a maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walk. Then W is a trivial walk if and only if $w_0 \in (\mathrm{Ter}(\mathcal{P}) \setminus T) \cup S$. If W is nontrivial then $e_0 \in E(\mathcal{Q})$ if and only if $w_0 \in V(\mathcal{Q})$.

Let W_1 and $W_2 \in \mathcal{W}$. Note that for any interior vertex w_i of a $\mathcal{P}$-$\mathcal{Q}$-alternating walk, it follows from the definition that either edge in $\{e_{i-1}, e_i\}$ determines uniquely the other. So if W_1 and W_2 share an edge, then a reduction to their common initial vertex shows that they are equal by their maximality. Moreover if the two walks share a vertex $v \notin V(\mathcal{Q})$, then they are equal since they share the edge of $\mathcal{P}$ whose terminal vertex is v.

Therefore, if $W_1 \neq W_2$ and they end at the same vertex v, then $v \in V(\mathcal{P}) \cap V(\mathcal{Q})$. More precisely, we may assume that v is the initial vertex of an edge in $E(\mathcal{Q}) \cap E(W_1)$ and the terminal vertex of an edge $e \in E(\mathcal{P}) \cap E(W_2)$ (both the last edges of their alternating walk). Since v is the initial vertex of some edge, it cannot be in B_0, so the path (or ray) in $\mathcal{P}$ containing e does not end at v. Hence we can extend W_1 contradicting its maximality.

Similarly we can extend a $\mathcal{P}$-$\mathcal{Q}$-alternating walk that ends at some vertex $v \in \mathrm{Ter}(\mathcal{P}) \cap T$ by the edge in $E(\mathcal{Q})$ that has v as its terminal vertex, unless $v \in S$. So $\mathcal{W}$ is a set of disjoint $\mathcal{P}$-$\mathcal{Q}$-alternating walks that can only end in $(\mathrm{Ter}(\mathcal{P}) \setminus T) \cup S$. □

Now we investigate when a dimaze and its $\mathcal{Q}$-shifted dimaze present the same strict gammoid.

Lemma 3.7. *Suppose that $\mathcal{Q}$ is a linkage from S onto T and I a set linkable in (D_1, B_1). Then I is linkable in (D, B_0) if (i) $I \setminus S$ is finite or (ii) (D, B_0) is C^O-free.*

Proof. There is a set of disjoint finite paths from I to B_1 in (D_1, B_1), which, by Lemma 3.2, gives rise to a set of disjoint finite $\mathcal{Q}$-alternating walks from I to B_1 in (D, B_0). Let $\mathcal{W}$ be the subset of those walks starting in $J := I \setminus S$. Then Lemma 3.3 provides a set $\mathcal{P}$ of disjoint paths or rays from $J \cup (S \setminus \mathrm{Ter}(\mathcal{W})) \supseteq I$ to $Y \subseteq B_0$. It remains to argue that $\mathcal{P}$ does not contain any ray. Indeed, any such ray would meet infinitely many paths in $\mathcal{Q}$. But by Lemma 3.4, the ray is the spine of an outgoing comb, which is a contradiction. □

In fact the converse of (ii) holds.

Lemma 3.8. *Suppose that (D, B_0) is C^O-free, and $\mathcal{Q}$ is a linkage from S onto T such that there exists no linkage from S to a proper subset of T. Then a linkable set I in (D, B_0) is also linkable in (D_1, B_1), and (D_1, B_1) is C^O-free.*

Proof. For the linkability of I it suffices by Lemma 3.2 to construct a set of disjoint finite $\mathcal{Q}$-alternating walks from I to B_1. Let $\mathcal{P}$ be a linkage of I in (D, B_0).

For each vertex $v \in I$ let W_v be the maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walk starting at v. By Lemma 3.6, $\mathcal{W} := \{W_v : v \in I\}$ is a set of disjoint $\mathcal{Q}$-alternating walks that can only end in $(\mathrm{Ter}(\mathcal{P}) \setminus T) \cup S \subseteq B_1$.

If there is an infinite alternating walk $W = W_{v_0}$ in $\mathcal{W}$, then Lemma 3.3 applied on just this walk gives us a set $\mathcal{R}$ of disjoint paths or rays from $S + v_0$ to T. Since the forward segments of W are subsegments of paths in $\mathcal{P}$, by Corollary 3.5 any ray in $\mathcal{R}$ would extend to a forbidden outgoing comb. Thus, $\mathcal{R}$ is a linkage of $S + v_0$ to T. In particular, S is linked to a proper subset of T contradicting the minimality of T. Hence $\mathcal{W}$ consists of finite disjoint $\mathcal{Q}$-alternating walks, as desired.

For the second statement suppose that (D_1, B_1) contains an outgoing comb whose spine R starts at $v_0 \notin S$. Then $W := \overleftarrow{\mathcal{Q}}(R)$ is a $\mathcal{Q}$-alternating walk in (D, B_0) by

Lemma 3.2. Any infinite forward segment R' of W contains an infinite subset linkable to B_1 in (D_1, B_1). By Lemma 3.7(ii) this subset is also linkable in (D, B_0), so R' is the spine of an outgoing comb by Lemma 3.4, which is a contradiction.

On the other hand, suppose that W does not have an infinite forward tail. By investigating W as we did with W_{v_0} above, we arrive at a contradiction. Hence, there does not exist any outgoing comb in (D_1, B_1). $\square$

For later application as a refinement of Lemma 3.8, we note the following.

Lemma 3.9. *If (D, B_0) is F^∞-free, then so is (D_1, B_1).*

Proof. Suppose that (D_1, B_1) contains a subdivision of F^∞ with centre v_0. Then an infinite subset X of the out-neighbourhood of v_0, that is linkable in (D_1, B_1). By Lemma 3.2, there exists an infinite set $\mathcal{W}$ of disjoint finite $\mathcal{Q}$-alternating walks that avoid $\overleftarrow{\mathcal{Q}}(v_0)$ and start in X. Inductively construct an infinite set of of disjoint finite paths as follows. Take a walk W in $\mathcal{W}$ and follow it until it hits a vertex v of $\mathcal{Q}$ for the first time and from there follow the path $Q \in \mathcal{Q}$ containing v until B_0 to get a path in D that links the initial vertex of W to B_0. If there is no such vertex v, then W actually is a path ending in B_0 as desired. By deleting all the finitely many walks in $\mathcal{W}$ that hit Q together with W from $\mathcal{W}$, we ensure that the constructed paths will be disjoint. As there are still infinitely many walks left in $\mathcal{W}$, we can continue the construction until we get an infinite linkable subset of the out-neighbourhood of $\overleftarrow{\mathcal{Q}}(v_0)$, a forbidden linking fan in (D, B_0). $\square$

Proposition 3.10. *Suppose (D, B_0) is C^O-free and $\mathcal{Q}$ is a linkage from S onto T such that S cannot be linked to a proper subset of T. Then $M_L(D_1, B_1) = M_L(D, B_0)$.*

Proof. By Lemma 3.7(ii) and Lemma 3.8, a set $I \subseteq V$ is linkable in (D, B_0) if and only if it is linkable in (D_1, B_1). $\square$

We remark that in order to show that $M_L(D, B_0) = M_L(D_1, B_1)$, the assumption in Proposition 3.10 that (D, B_0) is C^O-free can be slightly relaxed. Only outgoing combs constructed in the proofs of Lemma 3.7(ii) and Lemma 3.8 which have the form that all the spikes are terminal segments of paths in the linkage $\mathcal{Q}$ need to be forbidden.

Theorem 3.11. *The class of C^O-free gammoids is minor-closed.*

Proof. Let $N := M_L(D, B_0)$ be a strict gammoid. It suffices to show that any minor M of N is a gammoid. By Lemma 2.2, we have $M = N/S \setminus R$ for some independent set S and coindependent set R. First extend S by elements in B_0 to a base B_1 (by (IM)). This gives us a linkage $\mathcal{Q}$ from S onto $T := B_0 \setminus B_1$ such that there exists no linkage from S to a proper subset of T.

Assume that (D, B_0) is C^O-free. Then by Lemma 3.8, (D_1, B_1) is C^O-free, and by Proposition 3.10, $M_L(D, B_0) = M_L(D_1, B_1)$. Since $S \subseteq B_1$, $M = M_L(D_1, B_1)/S \setminus R = M_L(D_1 - S, B_1 \setminus S) \setminus R$ is a C^O-free gammoid. $\square$

A partial converse of Lemma 3.7(i) can be proved by analyzing a proof of Pym's linkage theorem.

Lemma 3.12. *Let $M = M_L(D, B_0)$ be a strict gammoid, $\mathcal{Q}$ a linkage from S onto T such that $B_1 = (B_0 \setminus T) \cup S$ is a base, and $I \subseteq V \setminus S$ such that $S \cup I$ is linkable in (D, B_0). If I is finite, then it is linkable in $(D_1 - S, B_1 \setminus S)$.*

Proof. By Lemma 3.2 it suffices to construct a set of disjoint finite $\mathcal{Q}$-alternating walks from I to $B_0 \setminus T$.

Let $\mathcal{P}$ be a linkage of $S \cup I$ in (D, B_0). If we could show that the maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walks starting in I are all finite, we would be done by Lemma 3.6. Instead of $\mathcal{P}$ we will use an auxiliary linkage $\mathcal{Q}^\infty$. The linkage theorem of Pym [22] asserts the existence of such a linkage $\mathcal{Q}^\infty$ from $S \cup I$ onto some set $Y^\infty \supseteq T$. We first recall how it is constructed using the notations introduced in [2]:

For each $x \in S \cup I$, let P_x be the path in $\mathcal{P}$ containing x. Define $f_x^0 := x$ and let $\mathcal{Q}^0 := \mathcal{Q}$. For each $i > 0$ and each $x \in S \cup I$, let f_x^i be the first vertex on P_x after (and including) f_x^{i-1} that is also in $V(\mathcal{Q}^{i-1})$, or $\mathrm{Ter}(P_x)$ if no such vertex exists. Using the notation about paths in [13], f_x^i can equivalently be defined as the last vertex v on $f_x^{i-1}P_x$ such that $(f_x^{i-1}P_x\mathring{v}) \cap V(\mathcal{Q}^{i-1}) = \emptyset$. For $y \in T$, let Q_y be the path in $\mathcal{Q}$ containing y and t_y^i be the first vertex $v \in Q_y$ such that the terminal segment $\mathring{v}Q_y$ does not contain any f_x^i. Define the linkage $\mathcal{Q}^i := \mathcal{B}^i \cup \mathcal{C}^i$ with

$$\begin{aligned}
\mathcal{B}^i &:= \{P_x f_x^i Q_y : x \in S \cup I, y \in T \text{ and } f_x^i = t_y^i\},\\
\mathcal{C}^i &:= \{P_x \in \mathcal{P} : f_x^i \in B_0 \setminus T\}.
\end{aligned}$$

There exist integers $i_x, i_y \geqslant 0$ such that $f_x^{i_x} = f_x^k$, $t_y^{i_y} = t_y^l$ for all integers $k \geqslant i_x$ and $l \geqslant i_y$. Define $f_x^\infty := f_x^{i_x}, t_y^\infty := t_y^{i_y}$ and

$$\begin{aligned}
\mathcal{B}^\infty &:= \{P_x f_x^\infty Q_y : x \in S \cup I, y \in T \text{ and } f_x^\infty = t_y^\infty\},\\
\mathcal{C}^\infty &:= \{P_x \in \mathcal{P} : f_x^\infty \in B_0 \setminus T\}.
\end{aligned}$$

Then $\mathcal{Q}^\infty := \mathcal{B}^\infty \cup \mathcal{C}^\infty$ is the linkage given by the linkage theorem.

Extend the independent set $(B_0 \setminus Y^\infty) \cup (S \cup I)$ to a base B_2 using elements from the base B_1 and let $Y := Y^\infty \setminus T$. Then $B_2 \setminus B_1 = I$ and $B_1 \setminus B_2 \subseteq Y$. As $|B_2 \setminus B_1|$ is finite, by [10, Lemma 3.7], we have $|I| = |B_2 \setminus B_1| = |B_1 \setminus B_2| \leqslant |Y|$.

We next construct a family $\mathcal{W}$ of walks that will help to control the $\mathcal{Q}^\infty$-$\mathcal{Q}$-alternating walks starting in I. Let $v \in V(D)$ be a vertex with the property that $v = f_{x_{j+1}}^{j+1}$ for some integer j and a vertex $x_{j+1} \in S \cup I$ such that $f_{x_{j+1}}^j \neq f_{x_{j+1}}^{j+1}$. We backward inductively construct a walk $W(v)$ that starts from I and ends at v as follows:

Given x_{i+1} for a positive integer $i \leqslant j$, let Q_i be the path in $\mathcal{Q}$ containing $f_{x_{i+1}}^i$ (if there is no such path, then $f_{x_{i+1}}^i \in I$ and $i = 0$). Since $f_{x_{i+1}}^i \neq f_{x_{i+1}}^{i+1}$, it follows that $\mathcal{F}^i \cap \mathring{f}_{x_{i+1}}^i Q_i \neq \emptyset$, where $\mathcal{F}^i := \{f_x^i : x \in S \cup I\}$. Let x_i be such that $f_{x_i}^i$ is the first vertex of $\mathcal{F}^i$ on $\mathring{f}_{x_{i+1}}^i Q_i$. Moreover, since $f_{x_{i+1}}^i \in Q_i$, $\mathcal{F}^{i-1} \cap \mathring{f}_{x_{i+1}}^i Q_i = \emptyset$, so $f_{x_i}^{i-1} \neq f_{x_i}^i$. Hence we can complete the construction down to $i = 1$ and define:

$$W(v) := f_{x_1}^0 P_1 f_{x_1}^1 \cup \bigcup_{0<i<j} f_{x_{i+1}}^i Q_i f_{x_i}^i \cup f_{x_{i+1}}^i P_{i+1} f_{x_{i+1}}^{i+1}. \tag{1}$$

Note that $f_{x_1}^0 \neq f_{x_1}^1$ and for any $x \in S$, the definition of f_x^1 implies $f_x^0 = f_x^1$. Hence, $f_{x_1}^0$, the initial vertex of $W(v)$, is in $(S \cup I) \setminus S = I$. Now we examine the interaction between two such walks:

Claim. *Let $x, x' \in S \cup I$ be given such that $f_x^{j+1} \neq f_x^j$ and $f_{x'}^{j'+1} \neq f_{x'}^{j'}$.*

(i) If $j = j'$ and $f_x^{j+1} \neq f_{x'}^{j'+1}$, then $\mathrm{Ini}(W(f_x^{j+1})) \neq \mathrm{Ini}(W(f_{x'}^{j'+1}))$.

(ii) If $W(f_x^{j+1})$ and $W(f_{x'}^{j'+1})$ start at the same vertex in I, then one is a subwalk of the other.

Proof. For (i) we first note that f_x^{j+1} and $f_{x'}^{j'+1}$ are on distinct paths in $\mathcal{P}$ and apply induction on j. If $j = j' = 0$, then $\mathrm{Ini}(W(f_x^{j+1})) = x \neq x' = \mathrm{Ini}(W(f_{x'}^{j'+1}))$. For $j > 0$ the walk $W(f_{x'}^{j'+1})$ has the form $W(f_{x'_j}^j) \cup f_{x'_{j+1}}^j Q'_j f_{x'_j}^j \cup f_{x'_{j+1}}^j P'_{j+1} f_{x'_{j+1}}^{j+1}$ and analogue $W(f_x^{j+1})$. The vertices $f_{x'_{j+1}}^j$ and $f_{x_{j+1}}^j$ are on distinct paths in $\mathcal{P}$ and therefore distinct. Then it follows from the definition that $f_{x_j}^j \neq f_{x'_j}^j$ and we use the induction hypothesis to see that $\mathrm{Ini}(W(f_{x'_j}^j)) \neq \mathrm{Ini}(W(f_{x_j}^j))$ and hence $\mathrm{Ini}(W(f_{x'_{j+1}}^{j+1})) \neq \mathrm{Ini}(W(f_{x_{j+1}}^{j+1}))$, as desired.

For (ii) suppose that $f_x^{j+1} \neq f_{x'}^{j'+1}$, then (i) implies $j \neq j'$, say $j < j'$. If $f_{x'_{j+1}}^{j+1} \neq f_{x_{j+1}}^{j+1}$, then, by (i), $\mathrm{Ini}(W(f_x^{j+1})) \neq \mathrm{Ini}(W(f_{x'_{j+1}}^{j+1})) = \mathrm{Ini}(W(f_{x'}^{j'+1}))$. Hence $W(f_x^{j+1})$ is a subwalk of $W(f_{x'}^{j'+1})$. □

Each vertex $y \in Y \setminus I$ is on a non-trivial path in $\mathcal{Q}^\infty$, so there exists a least integer $i_y > 0$ such that $y = f_{x_{i_y}}^{i_y}$ for some $x_{i_y} \in S \cup I$. For $y \in Y \cap I$ let $W(y)$ be the trivial walk at y, so that we can define $\mathcal{W} := \{W(y) : y \in Y\}$.

Suppose y and y' are two vertices in $Y \setminus I$ such that $\mathrm{Ini}(W(y)) = \mathrm{Ini}(W(y'))$. Since there is no edge of $\mathcal{Q}$ ending at either of these vertices, (ii) implies that $W(y) = W(y')$ and therefore $y = y'$. Since the initial vertex of a non-trivial walk in $\mathcal{W}$ is not in B_0, we have $\mathrm{Ini}(W(y)) \neq \mathrm{Ini}(W(y'))$ for any two distinct vertices y, y' in Y. That means $\mathrm{Ini}(\mathcal{W}) = I$, since $|I| \leqslant |Y|$.

We are finally in the position to show that the maximal $\mathcal{Q}^\infty$-$\mathcal{Q}$-alternating walks starting in I are not only disjoint (by Lemma 3.6) but also finite. To that end, let e be an edge of such a walk. As $E(\mathcal{W})$ is finite, it suffices to show that $e \in E(W)$ for some $W \in \mathcal{W}$. By definition, $e \in E(\mathcal{Q}^\infty) \Delta E(\mathcal{Q})$. The following case analysis completes the proof.

1. $e \in E(\mathcal{Q}^\infty) \setminus E(\mathcal{Q})$: e is on some initial segment $P_x f_x^\infty$ of a path P_x in $\mathcal{P}$. More precisely, there is an integer i, such that $e \in f_x^i P_x f_x^{i+1}$. By construction $e \in W(f_x^{i+1})$ and $\mathrm{Ini}(W(f_x^{i+1})) \in I$. Let W be the walk in $\mathcal{W}$ whose initial vertex is $\mathrm{Ini}(W(f_x^{i+1}))$, then (ii) implies that e is on W.

2. $e \in E(\mathcal{Q}) \setminus E(\mathcal{Q}^\infty)$: e is on some initial segment $Q f_x^\infty$ of a path Q in $\mathcal{Q}$. More precisely, there is an integer i and $x, x' \in S \cup I$, such that $e \in f_x^i Q f_{x'}^i$. Since $f_x^i \neq f_x^{i+1}$, similar to the previous case, there is a walk in $\mathcal{W}$ containing e. □

Here is another condition under which a minor of a gammoid is a gammoid.

Theorem 3.13. *Any finite-rank minor of a gammoid is also a gammoid.*

Proof. The setting follows the first paragraph of the proof of Theorem 3.11. Suppose that $M = N/S \setminus R$ has finite rank r. Since R is coindependent, $V \setminus R$ is spanning in N. Therefore, N/S also has rank r. Let $I \in M_L(D_1 - S, B_1 \setminus S)$, then $r = |B_0 \setminus T| = |B_1 \setminus S| \geqslant |I|$ and, by Lemma 3.7(i), I is in $\mathcal{I}(N/S)$. Conversely, if $I \in \mathcal{I}(N/S)$, then I is finite. By Lemma 3.12, I is linkable in $(D_1 - S, B_1 \setminus S)$. Hence $M_L(D_1 - S, B_1 \setminus S)$ is a strict gammoid presentation of N/S and $M = M_L(D_1 - S, B_1 \setminus S) \setminus R$ is a gammoid. □

An immediate corollary is that any forbidden minor, of which there are infinitely many ([16]), for the class of finite gammoids is also a forbidden minor for infinite gammoids.

3.2 Topological gammoids

Here we show that topological gammoids coincide with the finitary gammoids. As a corollary, we see that topological gammoids are minor-closed. As the first step we investigate how a topological gammoid can be expressed as gammoid.

Lemma 3.14. *Every strict topological gammoid is a strict gammoid.*

Proof. The difference between a topological linkage and a linkage is that paths ending at the centre of a linking fan and spines of outgoing combs are allowed. Thus, it suffices to give a $\{C^O, F^\infty\}$-free dimaze presentation for a given strict topological gammoid M.

Let (D', B_0') be a dimaze such that $M = M_{TL}(D', B_0')$, and F be the set of all vertices that are the centre of a subdivision of F^∞. Let (D, B_0) be obtained from (D', B_0') by deleting all edges whose initial vertex is in F and define $B_0 := B_0' \cup F$.

We claim that $M_{TL}(D, B_0) = M_{TL}(D', B_0')$. Let $\mathcal{P}$ be a topological linkage of I in (D', B_0'). Then the collection of the initial segments of each element of $\mathcal{P}$ up to the first appearance of a vertex in F forms a topological linkage of I in (D, B_0). Conversely, let $\mathcal{P}$ be a topological linkage of I in (D, B_0). Note that any linkage in (D, B_0) is a topological linkage in (D', B_0'). In particular the spikes of an outgoing comb whose spine R is in $\mathcal{P}$ form a topological linkage. Hence, R is also the spine of an outgoing comb in (D', B_0') by Lemma 3.4. So I is topologically linkable in (D, B_0).

Let $S \cup B_0$ be a base of $M_{TL}(D, B_0)$ and $\mathcal{Q}$ a set of disjoint spines of outgoing combs starting from S. We show that a set I is topologically linkable in (D, B_0) if and only if it is linkable in the $\mathcal{Q}$-shifted dimaze (D_1, B_1).

Let $\mathcal{P}$ be a topological linkage of I in (D, B_0). By Lemma 3.6, the set $\mathcal{W}$ of maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walks starting in I is a set of disjoint $\mathcal{Q}$-alternating walks possibly ending in $\mathrm{Ter}(\mathcal{P}) \cup S \subseteq B_1$. If there were an infinite walk, then it would have to start outside S and give rise to a topologically linkable superset of $S \cup B_0$, by Lemma 3.3 and Lemma 3.4. So each walk in $\mathcal{W}$ is finite. By Lemma 3.2, I is linkable in (D_1, B_1).

Conversely let I be linkable in (D_1, B_1) and $\mathcal{W}$ a set of disjoint finite $\mathcal{Q}$-alternating walks in (D, B_0) from I to B_1 provided by Lemma 3.2. By Lemma 3.3, $\mathcal{Q}\Delta\mathcal{W}$ contains a set $\mathcal{R}$ of disjoint paths or rays in (D, B_0) from I to B_0. By Corollary 3.5, any ray in $\mathcal{R}$ is in fact the spine of an outgoing comb, so I is topologically linkable in (D, B_0). □

Now we can characterize strict topological gammoids among strict gammoids.

Theorem 3.15. *The following are equivalent:*

1. *M is a strict topological gammoid;*
2. *M is a finitary strict gammoid;*
3. *M is a strict gammoid such that any presentation is $\{C^O, F^\infty\}$-free;*
4. *M is a $\{C^O, F^\infty\}$-free strict gammoid.*

Proof. 1. $\Rightarrow$ 2. : By Corollary 2.6, M is a finitary matroid and by Lemma 3.14 it is a strict gammoid.

2. $\Rightarrow$ 3. : Let $M_L(D, B_0)$ be any presentation of M. Note that the union of any vertex $v \in V \setminus B_0$ and all the vertices in B_0 to which v is linkable forms a circuit in M (the fundamental circuit of v and B_0). Suppose (D, B_0) is not $\{C^O, F^\infty\}$-free, then there is a vertex linkable to infinitely many vertices in B_0. But then M contains an infinite circuit and is not finitary.

3. $\Rightarrow$ 4. : Trivial.

4. $\Rightarrow$ 1. : Take a $\{C^O, F^\infty\}$-free presentation of M. Then topological linkages coincide with linkages. Hence M is a topological gammoid. □

Next we also characterize topological gammoids among gammoids.

Corollary 3.16. *The following are equivalent:*

1. *M is a topological gammoid;*
2. *M is a finitary gammoid;*
3. *M is a $\{C^O, F^\infty\}$-free gammoid.*

Proof. 1. $\Rightarrow$ 3. : There exist a dimaze (D, B_0) and $X \subseteq V$ such that $M = M_{TL}(D, B_0) \setminus X$. By Theorem 3.15, there is a $\{C^O, F^\infty\}$-free dimaze (D_1, B_1) such that $M_L(D_1, B_1) = M_{TL}(D, B_0)$. Hence, M is a $\{C^O, F^\infty\}$-free gammoid.

3. $\Rightarrow$ 2. : There exists a $\{C^O, F^\infty\}$-free presentation of a strict gammoid N of which M is a restriction. By Theorem 3.15, N is finitary, thus, so is M.

2. $\Rightarrow$ 1. : There exist (D, B_0) and $X \subseteq V$ such that $M = M_L(D, B_0) \setminus X$. Since M is finitary, we have $M = M^{\text{fin}}$, which is equal to $M_L(D, B_0)^{\text{fin}} \setminus X$. By Corollary 2.6, the latter equals $M_{TL}(D, B_0) \setminus X$. Hence, M is a topological gammoid. □

Theorem 3.17. *The class of finitary gammoids (or equivalently topological gammoids) is closed under taking minors.*

Proof. Let M be a finitary gammoid. By Corollary 3.16, M is a $\{C^O, F^\infty\}$-free gammoid. Any minor of M is a C^O-free gammoid by Theorem 3.11, and also F^∞-free by Lemma 3.9. So any minor of M is a finitary gammoid by Corollary 3.16. □

4 Duality

While finite transversal matroids and finite strict gammoids are dual to each other [17], this does not hold in the infinite case: There is an infinite transversal matroid whose dual is not a strict gammoid (Example 4.15) and the dual of the strict gammoid defined by C^A is not a transversal matroid (Example 4.11). Since this latter dual happens to be a gammoid, it does not rule out the possibility that infinite gammoids are, like finite gammoids, closed under duality. However, a more badly behaved example exists: in Example 4.19, we give a strict gammoid which is not dual to any gammoid.

Trying to understand the duals of strict gammoids, we introduce a natural extension of transversal matroids and show that it captures the duals of C^A-free strict gammoids introduced in [2] (Theorem 4.9). However, as we shall see in Example 4.12, this extension does not contain the duals of all strict gammoids.

4.1 Strict gammoids and path-transversal matroids

Let us begin by introducing a dual object of a dimaze. Given a bipartite graph $G = (V, W)$ and a matching m_0 onto W, we call the pair (G, m_0) a *bimaze*[4]. We adjust two constructions of [17] (see also [20, Section 2.4]) for our purposes.

Definition 4.1. Given a dimaze (D, B_0), define a bipartite graph $D^\star_{B_0}$, with bipartition $(V, (V \setminus B_0)^\star)$, where $(V \setminus B_0)^\star := \{v^\star : v \in V \setminus B_0\}$ is disjoint from V; and $E(D^\star_{B_0}) := m_0 \cup \{vu^\star : (u, v) \in E(D)\}$, where $m_0 := \{vv^\star : v \in V \setminus B_0\}$. Call $(D, B_0)^\star := (D^\star_{B_0}, m_0)$ the *converted bimaze* of (D, B_0).

Starting from a dimaze (D, B_0), we write $(V \setminus B_0)^\star$, m_0 and $v^\star$ for the corresponding objects in Definition 4.1.

Definition 4.2. Given a bimaze (G, m_0), where $G = (V, W)$, define a digraph $G^\star_{m_0}$ such that $V(G^\star_{m_0}) := V$ and $E(G^\star_{m_0}) := \{(v, w) : wv^\star \in E(G) \setminus m_0\}$, where $v^\star$ is the vertex in W that is matched by m_0 to $v \in V$. Let $B_0 := V \setminus V(m_0)$. Call $(G, m_0)^\star := (G^\star_{m_0}, B_0)$ the *converted dimaze* of (G, m_0).

Starting from a bimaze (G, m_0), we write B_0 and $v^\star$ for the corresponding objects in Definition 4.2 and $(V \setminus B_0)^\star$ for the right vertex class of G.

Note that these constructions are inverse to each other (see Figure 2). In particular, if (G, m_0) is a bimaze, then

$$(G, m_0)^{\star\star} = (G, m_0). \tag{2}$$

Note that for any matching m, each component of $G[m_0 \cup m]$ is either a path, an even cycle, a ray or a double ray. If $G[m_0 \cup m]$ consists of only finite components, then m is called an *m_0-matching*. A set $I \subseteq V$ is *m_0-matchable* if there is an m_0-matching of I.

Definition 4.3. Given a bimaze (G, m_0), the pair of V and the set of all m_0-matchable subsets of V is denoted by $M_{PT}(G, m_0)$. If $M_{PT}(G, m_0)$ is a matroid, it is called a *path-transversal matroid*.

[4]Short for *bi*partite *maze*.

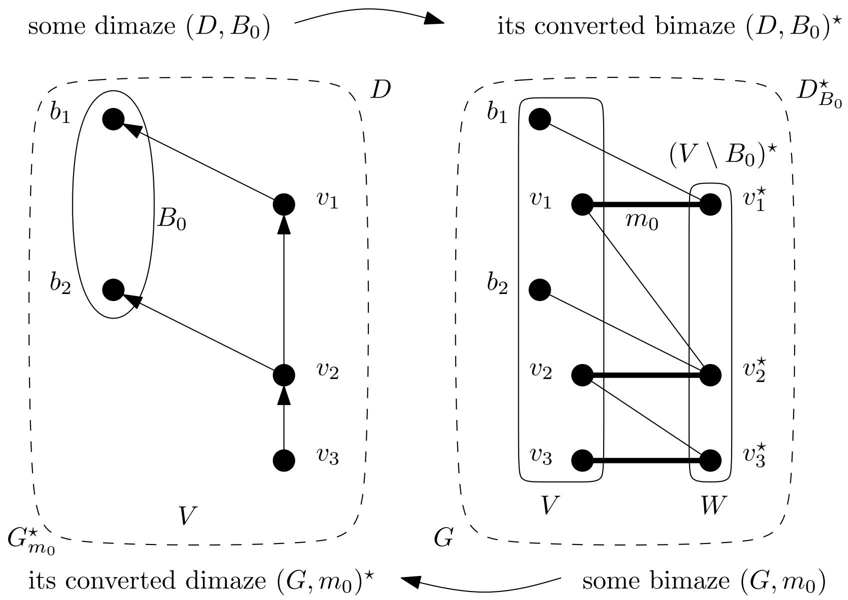

Figure 2: Converting a dimaze to a bimaze and vice versa

The correspondence between finite paths and m_0-matchings is depicted in the following lemma.

Lemma 4.4. *Let (D, B_0) be a dimaze. Then B is linkable onto B_0 in (D, B_0) if and only if $V \setminus B$ is m_0-matchable onto $(V \setminus B_0)^\star$ in $(D, B_0)^\star$.*

Proof. Suppose a linkage $\mathcal{P}$ from B onto B_0 is given. Let

$$m := \{vu^\star : (u, v) \in E(\mathcal{P})\} \cup \{ww^\star : w \notin V(\mathcal{P})\}.$$

Note that m is a matching from $V \setminus B$ onto $(V \setminus B_0)^\star$ in $D^\star_{B_0}$. Any component induced by $m_0 \cup m$ is finite, since any component which contains more than one edge corresponds to a path in $\mathcal{P}$. So m is a required m_0-matching in $(D, B_0)^\star$.

Conversely, let m be an m_0-matching from $V \setminus B$ onto $(V \setminus B_0)^\star$. Define a linkage from B onto B_0 in (D, B_0) as follows. From every vertex $v \in B$, start an m_0-m-alternating walk, which is finite because m is an m_0-matching. Moreover, the walk cannot end with an m_0-edge because m covers $(V \setminus B_0)^\star$. So the walk is either trivial or ends with an m-edge in B_0. As the m-edges on each walk correspond to a path from B to B_0, together they give us a required linkage in (D, B_0). □

Note that, for a finite dimaze, this defaults to the usual duality construction between strict gammoids and transversal matroids that was used in [17] to prove that the dual and a contraction of a finite gammoid is a gammoid. We remark that the notion of the $\mathcal{Q}$-shifted dimaze introduced in Section 3 can be expressed by the converted dimaze as follows: Given a dimaze (D, B_0) and a linkage $\mathcal{Q}$ from S onto B_0 (extended by trivial paths if needed), let m_1 be the m_0-matching of $V \setminus S$ in $(D, B_0)^\star$ whose existence is claimed by

Lemma 4.4. Then the converted dimaze of the bimaze $(D^\star_{B_0}, m_1)$ is the $\mathcal{Q}$-shifted dimaze of (D, B_0).

Let us show that path-transversal matroids are transversal matroids.

Proposition 4.5. *Let $M_T(G)$ be a transversal matroid and m_0 a matching of a base B. Then $M_T(G) = M_{PT}(G, m_0)$.*

Proof. Suppose $I \subseteq V$ admits a matching m. By the maximality of B, infinite components of $m \cup m_0$ do not intersect $V \setminus B$. To find an m_0-matching of I, in the infinite components, replace each m-edge with the m_0-edge with which it shares a vertex in V. $\square$

In fact, the class of path-transversal matroids contains the class of transversal matroids as a proper subclass; see the remark after Example 4.11.

Just as we can extend a linkage to cover the exits by trivial paths, any m_0-matching can be extended to cover W.

Lemma 4.6. *Let (G, m_0) be a bimaze. For any m_0-matchable I, there is an m_0-matching from some $B \supseteq I$ onto W.*

Proof. Let m be an m_0-matching of I. Take the union of all connected components of $m \cup m_0$ that meet $W \setminus m$. The symmetric difference of m and this union is a desired m_0-matching of a superset of I. $\square$

We find it convenient to abstract two properties of a dimaze and a bimaze. Given a dimaze (D, B_0), let (†) be

$$I \in M_L(D, B_0) \text{ is maximal } \Leftrightarrow \exists \text{ linkage from } I \text{ onto } B_0. \tag{†}$$

Analogously, given a bimaze (G, m_0), let (‡) be

$$I \in M_{PT}(G, m_0) \text{ is maximal } \Leftrightarrow \exists\ m_0\text{-matching from } I \text{ onto } (V \setminus B_0)^\star. \tag{‡}$$

In some sense (†) and (‡) are dual to each other.

Lemma 4.7. *A dimaze (D, B_0) satisfies (†) if and only if $(D, B_0)^\star$ satisfies (‡).*

Proof. Assume (D, B_0) satisfies (†). The forward direction of (‡) follows from Lemma 4.6. To prove the backward direction of (‡), suppose there is an m_0-matching from $V \setminus B$ onto $(V \setminus B_0)^\star$. By Lemma 4.4, there is a linkage from B onto B_0. Therefore, B is maximal in $M_L(D, B_0)$ by (†). By Lemma 4.6, any m_0-matchable superset of $V \setminus B$ may be extended to one, say $V \setminus I$, that is m_0-matchable onto $(V \setminus B_0)^\star$. As before, $I \subseteq B$ is maximal in $M_L(D, B_0)$, so $I = B$ and hence, $V \setminus B$ is a maximal m_0-matchable set.

Assume $(D, B_0)^\star$ satisfies (‡). The forward direction of (†) is trivial. For the backward direction, suppose there is a linkage from B onto B_0. Then there is an m_0-matching from $V \setminus B$ onto $(V \setminus B_0)^\star$ by Lemma 4.4. By (‡), $V \setminus B$ is maximal in $M_{PT}(D, B_0)^\star$. With an argument similar to the above, we can conclude that B is maximal in $M_L(D, B_0)$. $\square$

Now let us see how (†) helps to identify the dual of a strict gammoid.

Lemma 4.8. *If a dimaze (D, B_0) satisfies $(\dagger)$, then the dual of $M_L(D, B_0)$ is $M_{PT}(D, B_0)^\star$.*

Proof. By Lemma 4.7, $(D, B_0)^\star$ satisfies $(\ddagger)$. Let B be an independent set in $M_L(D, B_0)$. Then B is maximal if and only if there is a linkage from B onto B_0. By Lemma 4.4, this holds if and only if there is an m_0-matching from $V \setminus B$ onto $(V \setminus B_0)^\star$, which by $(\ddagger)$ is equivalent to $V \setminus B$ being maximal in $M_{PT}(D, B_0)^\star$.

To complete the proof, it remains to see that every m_0-matchable set can be extended to a maximal one, which follows from Lemma 4.6 and $(\ddagger)$. □

Note that while we do not need it, the twin of Lemma 4.8 is true, namely, if a bimaze (G, m_0) satisfies $(\ddagger)$, then $M_{PT}(G, m_0)$ is a matroid dual to $M_L(G, m_0)^\star$.

To summarize, the dual of strict gammoids examined in Theorem 2.4 is given as follows.

Theorem 4.9. *(i) Given a C^A-free dimaze (D, B_0), $M_L(D, B_0)$ is a matroid dual to $M_{PT}(D, B_0)^\star$.*

(ii) Given a bimaze (G, m_0), if $(G, m_0)^\star$ is C^A-free, then $M_{PT}(G, m_0)$ is a matroid dual to $M_L(G, m_0)^\star$.

Proof. (i) This is the direct consequence of Theorem 2.4 and Lemma 4.8.

(ii) Apply part (i) and (2). □

One might hope that, in the first part of the theorem, the path-transversal matroid $M_{PT}(D, B_0)^\star$ is in fact the transversal matroid $M_T(D, B_0)^\star$. However, the dimaze R^I defines a strict gammoid whose dual is not the transversal matroid defined by the converted bimaze. It turns out that R^I is the only obstruction to this hope.

Theorem 4.10. *(i) Given an $\{R^I, C^A\}$-free dimaze (D, B_0), $M_L(D, B_0)$ is a matroid dual to $M_T(D^\star_{B_0})$.*

(ii) Given a bimaze (G, m_0), if $(G, m_0)^\star$ is $\{R^I, C^A\}$-free, then $M_T(G)$ is a matroid dual to $M_L(G, m_0)^\star$.

Proof. (i) This follows from Theorem 4.9(i) and the fact that for an R^I-free dimaze (D, B_0), we have $M_T(D^\star_{B_0}) = M_{PT}(D, B_0)^\star$. The proof of the latter is similar to the one given to Proposition 4.5 and omitted.

(ii) Apply part (i) and (2). □

It appears that C^A is a natural constraint in the above theorem.

Example 4.11. The strict gammoid defined by the dimaze C^A (Figure 3a) is not co-transversal.

Proof. Since $(V \setminus B_0) + v$ is a base for every $v \in B_0$, B_0 is an infinite cocircuit. On the other hand, every vertex v of B_0 is contained in a finite cocircuit, namely v and its in-neighbours. So by Lemma 2.10, the dual is not transversal. □

We remark that C^A and C^I define the same strict gammoid via the isomorphism depicted in Figure 3. So while the dual of the strict gammoid defined by C^A is not a transversal matroid, it is the path-transversal matroid defined by the converted bimaze of C^I. It also is a gammoid by the lines above Conjecture 5.5.

Although the class of path-transversal matroids contains that of transversal matroids properly, not every strict gammoid has its dual of this type.

Example 4.12. Given a rooted tree such that each vertex has infinitely many children, let B_0 consist of the root and vertices on the alternating levels. Form a digraph $\mathcal{T}$ by directing the edges towards B_0. Then $M_L(\mathcal{T}, B_0)$ is a strict gammoid that is not dual to any path-transversal matroid.

Proof. In [2, Corollary 3.5], it was proved that $M := M_L(\mathcal{T}, B_0)$ is a matroid. Suppose that $M^* = M_{PT}(G, m')$. Let $\mathcal{Q}$ be a linkage from $B_1 := V \setminus m'$ onto B_0. Since $(\mathcal{T}, B_0)$ is C^O-free, by Proposition 3.10, we have $M = M_L(D_1, B_1)$ where (D_1, B_1) is the $\mathcal{Q}$-shifted dimaze. By construction, the underlying graph of D_1 is also a tree.

By [2, Corollary 3.6], (D_1, B_1) contains an alternating comb R. Let $\{s_i : i \geqslant 1\} := R \cap B_1$ and $U = \{u_i : i \geqslant 1\}$ be the set of vertices of out-degree 2 on R such that u_i is joined to s_i and s_{i+1} in R. Let $S_i := \{v \in \mathcal{T} : v \text{ can be linked to } s_i \text{ in } D_1 \setminus U\}$ and $U_i := \{v \in \mathcal{T} : v \text{ is separated from } R \cap B_1 \text{ by } u_i\}$, in particular U_i contains u_i. Since D_1 is a tree, $\{U_i, S_i : i \geqslant 1\}$ is a collection of pairwise disjoint sets.

Let $C := \bigcup_{i \geqslant 1} S_i$. Any linkable set in $V \setminus C$ has a linkage that misses an exit in $R \cap B_1$. Since D_1 is a tree, $(B_1 \setminus R) \cup U + c$ for any $c \in C$ is a base of M. Hence, C is a circuit in M^*. For a contradiction, we construct an m'-matching of C in (G, m').

As a first step, we note the following connection between fundamental circuits in $M_{PT}(G, m')$ and adjacency in G: If C' is the fundamental circuit of u with respect to the base matched by m', then $N(C') = m'(C' - u)$. Clearly, $N(u) \subseteq m'(C' - u)$. To see that no $v \in C' - u$ can have a neighbour outside $m'(C' - u)$, it is enough to note that there is always an m'-alternating walk from u to v. The latter follows from the fact that for any m'-matching m of $C' - v$, the m-m'-alternating walk from u ends at v.

In M^*, the fundamental circuit of s_i with respect to $V \setminus B_1$ is $S_i \cup U_{i-1} \cup U_i$ (with $U_0 := \emptyset$). Hence, we have $N(S_i \cup U_{i-1} \cup U_i) = m'(S_i \cup U_{i-1} \cup U_i - s_i)$ for $i \geqslant 1$.

We claim that for $i \geqslant 1$, in any m'-matching m of $\bigcup_{j \leqslant i} S_j$, the maximal m-m'-alternating walk from s_j ends in $m'(U_j)$ for $j \leqslant i$. Note that such a walk cannot end in $m'(S_j)$ as those vertices are incident with m-edges. Since $N(S_1) \subseteq m'(S_1 \cup U_1)$, the claim is true for $i = 1$. Assume that it is true for $i - 1$. Consider an m'-matching m_1 of $\bigcup_{j \leqslant i} S_j$. Let P_j be the maximal m_1-m'-alternating walk starting from s_j. By assumption, P_j ends in $m'(U_j)$ for each $j < i$. As P_i ends in $m'(U_{i-1} \cup U_i)$, we are done unless it ends in $m'(U_{i-1})$. In that case, the union of an m'-matching of $C \setminus \bigcup_{j \leqslant i} S_j$ with

$$(m' \restriction \bigcup_{j \leqslant i} S_j) \Delta \bigcup_{j \leqslant i} E(P_j)$$

is an m'-matching of C, a contradiction.

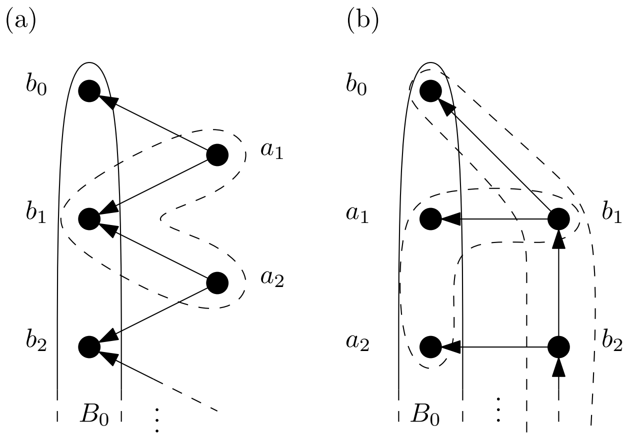

Figure 3: An alternating comb and an incoming comb which define the same gammoid

Therefore, there is a collection of pairwise disjoint m'-alternating walks $\{P'_i : i \geqslant 1\}$ where P'_i starts from s_i and ends in $m'(U_i)$. Then $m' \Delta \bigcup_{i \geqslant 1} E(P'_i)$ is an m'-matching of C, a contradiction which completes the proof. □

4.2 Finitary transversal matroids

Our aim in this section is to give a transversal matroid that is not dual to any strict gammoid. To this end, we extend some results in [6] and [7]. The following identifies edges that may be added to a presentation of a finitary transversal matroid without changing the matroid.

Lemma 4.13. *Suppose that $M_T(G)$ is finitary. Let K be a subset of $\{vw \notin E(G) : v \in V, w \in W\}$. Then the following are equivalent:*

1. *$M_T(G) \neq M_T(G + K)$;*
2. *there is $vw \in K$ and a circuit C with $v \in C$ and $w \notin N(C)$;*
3. *there is $vw \in K$ such that v is not a coloop of $M_T(G) \setminus N(w)$.*

Proof. 1. holds if and only if there is a circuit C in $M_T(G)$ which is matchable in $G + K$. This, since C is finite, in turn holds if and only if there is $v \in C$ that can be matched outside $N(C)$ in $G + K$, i.e. 2. holds.

The equivalence between 2. and 3. is clear since an element is not a coloop if and only if it lies in a circuit. □

Let $M_T(G)$ be a transversal matroid. Recall that G is a maximal presentation of $M_T(G)$ if $M_T(G + vw) \neq M_T(G)$ for any $vw \notin E(G)$ with $v \in V, w \in W$. Thus, the previous lemma implies that if $M_T(G)$ is finitary, then G is maximal if and only if $M_T(G) \setminus N(w)$ is coloop-free for any $w \in W$. Bondy [6] asserted that for any finite transversal matroid, there is a unique maximal presentation (recall that a presentation of a transversal matroid necessitates that W is covered by a matching of a base); here two presentations of a transversal matroid by bipartite graphs G and H are *isomorphic* if there is a graph isomorphism from G to H fixing the left vertex class pointwise.

Proposition 4.14. *Every finitary transversal matroid M has a unique maximal presentation.*

Proof. Let $M = M_T(G)$. Adding all vw with the property that there is not any circuit C with $v \in C$ and $w \notin N(C)$ gives a maximal presentation of M by Lemma 4.13. In particular, any coloop is always adjacent to every vertex in W. So without loss of generality, we assume that M is coloop-free.

Now let G and H be distinct maximal presentations of M.

Claim 1. For any finite subset F of V, the induced subgraphs $G[F \cup N_G(F)]$ and $H[F \cup N_H(F)]$ are isomorphic.

For every $v \in F$ pick a circuit C_v with $v \in C_v$. By Lemma 4.13, for every $vw \in \{xy \notin E(G) : x \in F, y \in N_G(F)\}$, there is a circuit C_{vw} with $v \in C_{vw}$ and $w \notin N_G(C_{vw})$. Let F_G be the union of all C_v's and C_{vw}'s. Analogously define F_H and let $F' := F_G \cup F_H$. As the circuits involved are all finite and a vertex in a finite circuit must have finite degree, the presentations $G[F' \cup N_G(F')]$ and $H[F' \cup N_H(F')]$ of $M|F'$ are finite. So we can, by Bondy's result, extend them to get maximal presentations G' and H' respectively and a graph isomorphism between them which fixes the left vertex class pointwise. As by definition of F' and Lemma 4.13, no non-edge between F and $N_G(F)$ is an edge in G' (analogously between F and $N_H(F)$ in H'), the restriction of the isomorphism to $F \cup N_G(F)$ is the claimed isomorphism of $G[F \cup N_G(F)]$ and $H[F \cup N_H(F)]$.

As G is left locally finite, for any $A \subseteq V$, $|\{w \in W(G) : N_G(w) = A\}|$ is finite. As G and H are distinct presentations, without loss of generality, there is $A \subseteq V$ such that

$$g := |\{w \in W(G) : N_G(w) = A\}| < |\{w \in W(H) : N_H(w) = A\}| =: h.$$

Note that as H is a maximal presentation, by Lemma 4.13, $M \setminus A$ is coloop-free.

As M is coloop-free, so is $M.A$. Let B_1 be a base of $M.A$ and B_2 a base of $M \setminus A$. Let m be a matching of $B_1 \cup B_2$. Since $M \setminus A$ is coloop-free, by Lemma 2.8, the neighbourhood of each vertex matched by m to a vertex in B_1 is a subset of A. Thus, $M.A$ can be presented with the subgraphs induced by $A \cup \{w \in W : N(w) \subseteq A\}$ in both graphs G and H; call these subgraphs G_1 and H_1. For any $w \in W(G_1)$, since $M \setminus N_{G_1}(w)$ is coloop-free, so is $M.A \setminus N_{G_1}(w)$. By Lemma 4.13, G_1 (analogously H_1) is a maximal presentation of $M.A$.

Claim 2. Given a family $(N_j)_{j \in J}$ of finite subsets of W, if the intersection of any finite subfamily has size at least k, then the intersection of the family has size at least k.

Let $N := \bigcap_{j \in J} N_j$. Suppose $|N| < k$. Fix some $j_0 \in J$ and for each element $y \in N_{j_0} \setminus N$ pick some N_y such that $y \notin N_y$. Then $|N_{j_0} \cap \bigcap_{y \in N_{j_0} \setminus N} N_y| = |N| < k$, which is a contradiction.

By Claim 2, there is a finite set $F \subseteq A$ such that $|\bigcap_{v \in F} N_{G_1}(v)| = g$. But Claim 1 says that F has at least $h > g$ common neighbours in H_1; this contradiction completes the proof. $\square$

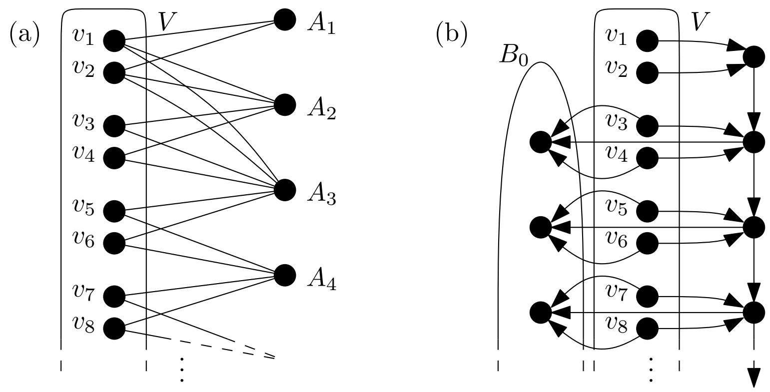

Figure 4: A transversal matroid which is not dual to a strict gammoid and a gammoid presentation of its dual

In [3], it is proved that a cofinitary strict gammoid always admits a presentation that is $\{R^I, C^A\}$-free. To show that the following finitary transversal matroid is not dual to a strict gammoid, it suffices to show that there is no bimaze presentation whose converted dimaze is C^A-free.

Example 4.15. Define a bipartite graph G as $V(G) = \{v_i, A_i : i \geqslant 1\}$ and $E(G) = \{v_1A_1, v_2A_1, v_1A_3, v_2A_3\} \cup \{v_{2i-3}A_i, v_{2i-2}A_i, v_{2i-1}A_i, v_{2i}A_i : i \geqslant 2\}$ (Figure 4(a)). Then $M := M_T(G)$ is not dual to a strict gammoid.

Proof. As G is left locally finite, M is a finitary matroid. Assume for a contradiction that $M^* = M_L(D, B_0)$. By a characterization of cofinitary strict gammoids in [3], we may assume that (D, B_0) is $\{R^I, C^A\}$-free. Then by Theorem 4.10, $M = M_T(D, B_0)^\star$.

Now it can be checked that all $M \setminus N(A_i)$ are coloop-free. By Lemma 4.13, G is the maximal presentation of M. The same lemma also implies that any minimal presentation G' is obtained by deleting edges from $\{v_1A_3, v_2A_3\}$ and at most one from $\{v_1A_2, v_2A_2\}$. In particular, all presentations of M differ from G only finitely. It is not difficult to check that with any matching m_0 of a base, $(G, m_0)^\star$ contains an alternating comb. Hence, there is no bimaze presentation of M such that the converted dimaze is C^A-free, contradicting that $(D, B_0)^\star$ is such a presentation. ☐

We remark that the above transversal matroid is dual to a gammoid, see Figure 4. However, in the next section, we give a transversal matroid that is not dual to any gammoid.

4.3 Infinite tree and gammoid duality

To show that there is a strict gammoid not dual to a gammoid, we prove the following lemmas, whose common setting is that a given dimaze (D, B_0) defines a matroid $M_L(D, B_0)$. For a linkage $\mathcal{Q}$ and any $X \subseteq \mathrm{Ini}(\mathcal{Q})$, $\mathcal{Q} \restriction X := \{Q \in \mathcal{Q} : \mathrm{Ini}(Q) \in X\}$; when $X = \{x\}$, we write simply Q_x.

Lemma 4.16. *Let b be an element in an infinite circuit C, $\mathcal{Q}$ a linkage from $C - b$. Then b can reach infinitely many vertices in C via $\mathcal{Q}$-alternating walks.*

Proof. Given any $x \in C - b$, let $\mathcal{P}$ be a linkage of $C - x$. Let W be a maximal $\mathcal{P}$-$\mathcal{Q}$-alternating walk starting from b. If W is infinite, then we are done. Otherwise, W ends in either $\mathrm{Ter}(\mathcal{P}) \setminus \mathrm{Ter}(\mathcal{Q})$ or $\mathrm{Ini}(\mathcal{Q}) \setminus \mathrm{Ini}(\mathcal{P}) = \{x\}$. The former case does not occur, since it gives rise to a linkage of C by Lemma 2.5(i), contradicting C being a circuit. As x was arbitrary, the proof is complete. □

Lemma 4.17. *For $i = 1, 2$, let C_i be a circuit of M, x_i, b_i distinct elements in $C_i \setminus C_{3-i}$. Suppose that $(C_1 \cup C_2) \setminus \{b_1, b_2\}$ admits a linkage $\mathcal{Q}$. Then any two $\mathcal{Q}$-alternating walks W_i from b_i to x_i, for $i = 1, 2$, are disjoint.*

Proof. Suppose that $W_1 = w_0^1 e_0^1 w_1^1 \dots w_n^1$ and $W_2 = w_0^2 e_0^2 w_1^2 \dots w_m^2$ are not disjoint. Then there exists a first vertex $v = w_j^1$ on W_1 such that $v = w_k^2 \in W_2$ and either $v \in V(\mathcal{Q})$ and $e_j^1 = e_k^2 \in E(\mathcal{Q})$ or $v \notin V(\mathcal{Q})$. In both cases $W_3 := W_1 v W_2$ is a $\mathcal{Q}$-alternating walk from b_1 to x_2. Let v' be the first vertex of W_3 in $V(\mathcal{Q} \upharpoonright (C_2 - b_2) \setminus C_1)$ and Q the path in $\mathcal{Q}$ containing v'. Then $W_3 v' Q$ is a $(\mathcal{Q} \upharpoonright (C_1 - b_1))$-alternating walk from b_1 to $B_0 \setminus \mathrm{Ter}(\mathcal{Q} \upharpoonright (C_1 - b_1))$, which by Lemma 2.5(i) contradicts the dependence of C_1. Hence W_1 and W_2 are disjoint. □

Lemma 4.18. *Let $\{C_i : i \in N\}$ be a set of circuits of M; x_i, b_i distinct elements in $C_i \setminus \bigcup_{j \neq i} C_j$. Suppose that $\bigcup_{i \in N} C_i \setminus \{b_i : i \in N\}$ admits a linkage $\mathcal{Q}$. Let W_i be a $\mathcal{Q}$-alternating walk from b_i to x_i. If $X \subseteq V$ is a finite set containing $C_i \cap C_j$ for any distinct i, j, then only finitely many of W_i meet $\mathcal{Q} \upharpoonright X$.*

Proof. By Lemma 4.17, the walks W_i are pairwise disjoint. Since $\mathcal{Q} \upharpoonright X$ is finite, it can be met by only finitely many W_i's. □

We are now ready to give a counterexample to classical duality in infinite gammoids.

Example 4.19. Let $(\mathcal{T}, B_0)$ be the dimaze defined in Example 4.12. The dual of the strict gammoid $M = M_L(\mathcal{T}, B_0)$ is not a gammoid.

Proof. Suppose that $M^* = M_L(D, B_1) \upharpoonright V$, where $V := V(\mathcal{T})$. Fix a linkage $\mathcal{Q}$ of $V \setminus B_0$ in (D, B_1). For $b \in B_0$, let C_b be the fundamental cocircuit of M with respect to B_0. Then for any (undirected) ray $b_0 x_0 b_1 x_1 \cdots$ in $\mathcal{T}$, $C := \bigcup_{k \in \mathbb{N}} C_{b_k} \setminus \{x_k : k \in \mathbb{N}\}$ is a cocircuit of M. We get a contradiction by building a linkage for C in (D, B_1) inductively using disjoint $\mathcal{Q}$-alternating walks.

Let b_0 be the root of $\mathcal{T}$. By Lemma 4.16, there is a $\mathcal{Q}$-alternating walk W_0 from b_0 to one of its children x_0. At step $k > 0$, from each child b of x_{k-1} in $\mathcal{T}$, by Lemma 4.16, there is a $\mathcal{Q}$-alternating walk W_b in (D, B_1) to a child x of b. Applying Lemma 4.18 on $\{C_i : i \in N^-(x_{k-1}) - b_{k-1}\}$ with $X = \{x_{k-1}\}$, we may choose $b_k := b$, $x_k := x$ such that $W_k := W_b$ avoids $Q_{x_{k-1}}$.

By Lemma 4.17, distinct W_k and $W_{k'}$ are disjoint. Moreover, as each W_k avoids $Q_{x_{k-1}}$, Lemma 2.5(i) implies that W_k can only meet $\mathcal{Q}$ at Q_x where $x \in C_{b_k} - x_{k-1}$. Then $E(\mathcal{Q}) \triangle \bigcup_{k \in \mathbb{N}} E(W_k)$ contains a linkage of C. □

By ignoring the directions of the edges of $\mathcal{T}$ and adding a leaf to each vertex in B_0, we can present $M_L(\mathcal{T}, B_0)$ as a transversal matroid [2]. Thus, not every transversal matroid is dual to a gammoid.

5 Open problems

In Section 3, we proved that the classes of C^O-free gammoids (see Section 2.2 for definitions), and the finitary gammoids are separately closed under taking minors. However, the main question remains open.

Question 5.1. *Is the class of all gammoids minor-closed?*

Theorem 4.9(i) says that if a dimaze (D, B_0) is C^A-free, then $M_L(D, B_0)$ is a matroid whose dual is a path-transversal matroid. Here is a question that is in some sense converse to the theorem.

Question 5.2. *Is every strict gammoid which is dual to a path-transversal matroid C^A-free?*

If we restrict ourselves to dimazes that are $\{C^A, R^I\}$-free, then Theorem 4.10(i) tells us that $M_L(D, B_0)$ is a matroid whose dual is a transversal matroid. Analogous to the above question is the following.

Question 5.3. *Is every cotransversal strict gammoid $\{C^A, R^I\}$-free?*

We have introduced path-transversal systems as a way to describe duals of C^A-free strict gammoids. It may be interesting to investigate path-transversal systems further. For example, while a path-transversal system need not satisfy (IM), it might be the case that (I3) always holds.

Question 5.4. *Does every path-transversal system satisfy (I3)?*

In [2], it was proved that the strict gammoid $(\mathcal{T}, B_0)$ in Example 4.19 (whose dual is not a gammoid) is not C^A-free. On the other hand, it is possible to extend Mason's construction for duality [18] to prove that any R^A-free strict gammoid is dual to a gammoid. For this purpose let (D, B_0) be an R^A-free dimaze and $M := M_L(D, B_0)$. Let $V' := \{v' : v \in V\}$ be a disjoint copy of V. Let D' be a digraph on V' obtained by reversing the directions of the edges of D. Define D'' with vertex set $V' \cup V$, edges of D', together with $\{(v, v') : v \in B_0\} \cup \{(v', v) : v \in V \setminus B_0\}$. As (D, B_0) is R^A-free, $(D'', V \setminus B_0)$ is C^A-free. So $M_L(D'', V \setminus B_0)$ defines a strict gammoid and any set linkable onto the exits is necessarily a base (see [2]). Then it can be checked that $M^* = M_L(D'', V \setminus B_0)|V$.

We conjecture that C^A-free gammoids are well-behaved under duality.

Conjecture 5.5. *The class of C^A-free gammoids is closed under duality.*

As the class of gammoids is not closed under duality, one may try to find reasonably large subclasses that are closed under duality. Another way to approach the duality problem is trying to find reasonably small superclasses of gammoids that are dual-closed.

Proposition 4.14 says that a finitary transversal matroid has a unique maximal presentation. On the other hand, minimal presentations [7] are presentations in which the removal of any edge changes the transversal system. It is easy to see that any finitary transversal matroid has a minimal presentation. Indeed, let $M = M_T(G)$ be a finitary transversal matroid where $G = (V, W)$. We construct a minimal presentation of M inductively and index each step by an ordinal. In step α, we define a bipartite graph G_α on $V \cup W$ and the set of deletable edges $E_\alpha := \{e \in E(G_\alpha) : M_T(G_\alpha) = M_T(G_\alpha - e)\}$. Set $G_0 := G$ and $G_{\alpha+1} := G_\alpha - e$ for some $e \in E_\alpha$. For a limit ordinal β, let $E(G_\beta) := \bigcap_{\alpha<\beta} E(G_\alpha)$. If $E_\alpha = \emptyset$ for some ordinal α, then it is not difficult to see that G_α is a minimal presentation of M. However, we do not know if this plan works in general.

Question 5.6. *Does every transversal matroid admit a minimal presentation?*

A finite transversal matroid is a gammoid by simply declaring the right hand side of an arbitrary presentation (V, W) as exits, directing the edges towards W and restricting the resulting strict gammoid to V. This construction fails in the infinite case due to the following example: Take a complete bipartite graph as a presentation of the free matroid on a countable ground set. Then the linkability system of the dimaze constructed as above fails to satisfy the matroid axiom (IM) because a superset of V is linkable if and only if it misses infinitely many vertices in W. As the construction works fine when we start with a minimal presentation of the free matroid, we ask:

Question 5.7. *Is every infinite transversal matroid a gammoid?*

Note that an approach to this question would be to modify the definition of gammoid in that we don't require the linkability system of the whole dimaze to define a matroid but just the one restricted to the ground set. This approach has some disadvantages, for example some proofs in Section 3 like that of Theorem 3.11 use the fact that a gammoid is a restriction of a strict gammoid.

Acknowledgements

The authors would like to thank Nathan Bowler and Johannes Carmesin for many pleasant memories and helpful discussions, in particular on the proof of Proposition 4.14, as well as a first proof that the transversal matroid in Example 4.15 is not dual to a strict gammoid.

The authors also would like to thank the referee for a thorough reading and constructive criticism.

HFL thanks the Humboldt Foundation and the Croucher Foundation for their support of this research, which was done in Hamburg over 2012–2014.

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