Stability for intersecting families in $PGL(2, q)$

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Abstract

We consider the action of the 2-dimensional projective general linear group $PGL(2, q)$ on the projective line $PG(1, q)$. A subset S of $PGL(2, q)$ is said to be an intersecting family if for every $g_1, g_2 \in S$, there exists $\alpha \in PG(1, q)$ such that $\alpha^{g_1} = \alpha^{g_2}$. It was proved by Meagher and Spiga that the intersecting families of maximum size in $PGL(2, q)$ are precisely the cosets of point stabilizers. We prove that if an intersecting family $S \subset PGL(2, q)$ has size close to the maximum then it must be "close" in structure to a coset of a point stabilizer. This phenomenon is known as stability. We use this stability result proved here to show that if the size of S is close enough to the maximum then S must be contained in a coset of a point stabilizer.

1 Introduction

The Erdős-Ko-Rado (EKR) theorem is a classical result in extremal set theory. It states that if $k < n/2$, an intersecting family of k-subsets of $[n] = \{1, 2, \dots, n\}$ has size at most $\binom{n-1}{k-1}$; equality holds if and only if the family consists of all k-subsets containing a fixed element from $[n]$. Intersecting families of maximum size are called *extremal families.* In [11], Frankl proved that these extremal families are not only unique, but also stable: Any intersecting family of size close to the maximum is "close" in structure to an extremal family. In this paper, we focus on an analogue of these results for permutations groups, in particular, to the natural right action of $PGL(2, q)$ on the projective points of $PG(1, q)$, where q is a prime power.

Let Ω be a finite set and G a finite group acting on Ω. A subset S of G is said to be an *intersecting family* if for every $g_1, g_2 \in S$ there exists an element $\alpha \in \Omega$ such that $\alpha^{g_1} = \alpha^{g_2}$. Like in the original EKR-problem, we call intersecting families of maximum size *extremal families.* Moreover, intersecting families whose sizes are close to the maximum are called *almost extremal families.*

The following problems about intersecting families in G are considered to be the basic problems in EKR theory.

I (Upper Bound) What is the maximum size of an intersecting family?

II (Uniqueness) What is the structure of extremal families?

III (Stability) Are almost extremal families similar in structure to the extremal ones?

The above three problems were solved for the symmetric group S_n. Indeed, Deza and Frankl [10] proved that the maximum size of an intersecting family in S_n is $(n-1)!$. Moreover, they conjectured that the cosets of points stabilizers are the only extremal families. This conjecture turned out to be rather harder to prove than one might expect. It was first proved by Cameron and Ku [3], and independently by Larose and Malvenuto [15]. Finally, the stability of extremal families in S_n was settled by Ellis [6], who proved that for any $\epsilon > 0$ and $n > N(\epsilon)$, any intersecting family of size at least $(1-1/e+\epsilon)(n-1)!$ must be strictly contained in an extremal family.

In [17], Meagher and Spiga studied Problems I and II for the group $G_q := PGL(2, q)$ acting on the set of points of the projective line $PG(1, q)$. These authors proved that the maximum size of an intersecting family in G_q is $q(q-1)$. Furthermore, they also solved the uniqueness problem: Every extremal family in G_q is a coset of a point stabilizer. In this paper, we prove that extremal families in G_q are also stable, like their counterparts in the symmetric group. That is, an almost extremal family in G_q must be close in structure to a coset of a point stabilizer. We make this statement explicit in the following theorem.

Theorem 1. *There exists an absolute constant C_0 such that the following holds. Let $S \subset G_q$ be an intersecting family with $|S| = (1-\delta)q(q-1)$, where $0 \leqslant \delta \leqslant 1/2$. Then there exists a coset of a point stabilizer $T \subset G_q$ such that*

$$|S \triangle T| \leqslant C_0 \left(\delta^{1/2} + \frac{1}{q+1} \right) |S|,$$

where $\triangle$ is the symmetric difference of sets.

Using Theorem 1 and some properties of intersecting families in G_q we get the following stronger result on almost extremal families in G_q.

Theorem 2. *There exists an absolute constant $\delta_0 > 0$ such that the following holds. If $S \subset G_q$ is an intersecting family with $|S| \geqslant (1-\delta_0)q(q-1)$, then S is contained within a coset of a point stabilizer in G_q.*

Theorem 2 is a direct analogue of the Cameron-Ku conjecture proved by Ellis in [6].

The main tools in this paper are the eigenvalue method and analysis of Boolean functions on G_q. The eigenvalue method was introduced by Lóvasz [16] as a new way to prove for the EKR-theorem. Since then, it has been used several times to prove analogues of the EKR theorem [7, 13, 17, 19]. The analysis of Boolean functions on finite groups has been an active research area especially in computer science. A lot of work has been done in recent years to characterize Boolean functions whose Fourier transforms are highly concentrated on some irreducible representations. Friedgut, Kalai and Naor [12] proved that a Boolean function on $\mathbb{Z}_2^n$ whose Fourier transform is close to being concentrated on the first two levels, must be close to a dictatorship (a function determined by just one coordinate). Furthermore, similar results have been obtained for other abelian groups [1, 14]. Recently, Ellis, Filmus and Friedgut [8] showed that similar results can be obtained for the symmetric group S_n. Specifically, they proved that if the Fourier transform of a Boolean function f is highly concentrated on the first two irreducible representations of S_n and $\frac{1}{n!}\sum_{x\in S_n} f(x) = O(\frac{1}{n})$ then f must be close to a union of cosets of points stabilizers.

The proof of Theorem 1 is divided into two parts. First, we prove that the Fourier transform of the characteristic function of the almost extremal families are highly concentrated on two irreducible representations of G_q. Second, we use this Fourier characterization of almost extremal families to get structural information. In particular, we note that most of the ideas used in [8], can be used to characterize Boolean functions on G_q whose Fourier transforms are highly concentrated on the trivial and standard representations of G_q. This partially answers a question of Ellis, Filmus and Friedgut in [9]. These authors asked if there were others groups (besides S_n) for which there is an elegant characterization of Boolean functions whose Fourier support is concentrated on certain irreducible representations. Actually, in Section 4, we explain that 3-transitive groups satisfying certain extra conditions have a similar characterization.

The proof of Theorem 2 follows from Theorem 1 and some basic properties of intersecting families in G_q.

The rest of the paper is organized as follows. In Section 2 we provide some notation, definitions and basic results. In Section 3 we characterize the Fourier transforms of the characteristic functions of almost extremal families. In Section 4 we prove our main theorems. Finally, in Section 5 we conclude with some remarks and open problems.

2 Background

2.1 Fourier Analysis

Let G be a finite group. We denote by $\mathbb{C}[G]$ the vector space of all complex valued functions on G.

Definition 3. Let R be a complete set of non-isomorphic irreducible matrix representations of G. The Fourier transform of $f \in \mathbb{C}[G]$ is a matrix-valued function on irreducible representations. Its value at the irreducible representation $\rho \in R$ is

$$\widehat{f}(\rho) = \frac{1}{|G|}\sum_{s\in G} f(s)\rho(s).$$

We apply the Fourier transform to decompose the vector space $\mathbb{C}[G]$ into a direct sum of subspaces indexed by the irreducible representations of G. For every $\rho \in R$, we denote by V_ρ the subspace of $\mathbb{C}[G]$ consisting of all functions whose Fourier transform is supported only on ρ, more precisely,

$$V_\rho = \{f \in \mathbb{C}[G] : \widehat{f}(\rho') = 0, \text{ for all } \rho' \neq \rho,\ \rho' \in R\}.$$

Since the Fourier transform is an invertible linear transformation, we can write

$$\mathbb{C}[G] = \bigoplus_{\rho \in R} V_\rho.$$

By abuse of notation, we will sometimes use V_{χ_ρ} to denote V_ρ where χ_ρ is the irreducible character afforded by ρ.

Moreover, we can make $\mathbb{C}[G]$ an inner product space. For any $f, g \in \mathbb{C}[G]$ we define

$$\langle f, g\rangle = \frac{1}{|G|}\sum_{x \in G} f(x)\overline{g(x)}.$$

We denote by $\|f\|$ the euclidean norm induced by the inner product

$$\|f\| = \sqrt{\frac{1}{|G|}\sum_{g \in G} |f(g)|^2}\ .$$

Let U be any subspace of $\mathbb{C}[G]$ and $f \in \mathbb{C}[G]$. We denote by $U^\perp$ the orthogonal complement of U and by $P_U(f)$ the projection of f onto U. Thus, we can write

$$f = P_U(f) + P_{U^\perp}(f).$$

Let $\Omega = \{1, \ldots, n\}$ be a set and $\mathbb{C}[\Omega]$ be the vector space of all $\mathbb{C}$-valued functions defined on Ω. For every $i \in \Omega$, we define e_i as the function on Ω which takes the value 1 at i and 0 elsewhere. Let G be a group acting on Ω on the right. This action turns $\mathbb{C}[\Omega]$ into a representation of G of degree n. Indeed, this representation is produced by a linear extension of the (left) action defined by $g(e_i) = e_{ig^{-1}}$ for all $g \in G$ and $i \in \Omega$. The vector subspace V_{std} spanned by the vectors $\{\sum_{i=1}^n x_i e_i : \sum x_i = 0\}$ is a subrepresentation of $\mathbb{C}[\Omega]$ of degree $n-1$, known as the standard representation of G. We denote by χ_{std} the character afforded by the standard representation (we will refer to χ_{std} as the standard character of G). It follows by definition that for every $g \in G$, the value $\chi_{std}(g)$ corresponds to the number of elements in Ω fixed by g minus one.

Let X be an inverse-closed subset of G. The Cayley graph on G generated by X is the graph with vertex set G such that there is an edge between $g_1, g_2 \in G$ if and only if $g_1 g_2^{-1} \in X$. We denote this graph by $Cay(G, X)$. The following lemma says that under certain conditions on X, the subspaces V_ρ are eigenspaces of $Cay(G, X)$.

Lemma 4. *(Babai [2], Diaconis-Shahshahani [4]) Let G be a finite group, and let R be a complete set of irreducible representations of G. Let $X \subset G$ be inverse-closed and*

Table 1: Character table of $PGL(2,q)$.

	I	u	d_x	v_r
λ_1	1	1	1	1
ψ_1	q	0	1	-1
η_β	$q-1$	-1	0	$-\beta(r)-\beta(r^q)$
ν_γ	$q+1$	1	$\gamma(x)+\gamma(x^{-1})$	0
λ_{-1} (q odd)	1	1	$\delta(x)$	$\delta(r)$
ψ_{-1} (q odd)	q	0	$\delta(x)$	$-\delta(r)$

conjugation invariant, and let $Cay(G,X)$ be the Cayley graph on G with generating set X. For every $\rho \in R$, the vector subspace V_ρ is an eigenspace of $Cay(G,X)$ with eigenvalue

$$\frac{1}{\chi_\rho(1)} \sum_{x\in X} \chi_\rho(x),$$

where χ_ρ is the irreducible character of ρ. Besides, if λ is an eigenvalue of $Cay(G,X)$ corresponding to the irreducible representations $\{\rho_1,\ldots,\rho_s\} \subset R$ then the dimension of the λ-eigenspace is $\sum_{i=1}^{s} \chi_{\rho_i}(1)^2$.

2.2 $PGL(2,q)$

Let $\mathbb{F}_q$ be the finite field of size q and $\mathbb{F}_{q^2}$ its unique quadratic extension. We denote by $\mathbb{F}_q^*$ and $\mathbb{F}_{q^2}^*$ the multiplicative groups of $\mathbb{F}_q$ and $\mathbb{F}_{q^2}$, respectively. Let V be a 2-dimensional vector space over $\mathbb{F}_q$ then $GL(V)$ denotes the group of all invertible linear transformations on V. The subgroup of all invertible linear transformations on V with determinant 1 is known as the special general linear group $SL(V)$. We denote by $Z(GL(V))$ and $Z(SL(V))$ the centers of the groups $GL(V)$ and $SL(V)$, respectively.

The projective general linear group of V is defined as $PGL(V) = GL(V)/Z(GL(V))$, and the projective special linear group of V is defined as $PSL(V) = SL(V)/Z(SL(V))$.

Choosing a basis for V provides an isomorphism between $GL(V)$ and the group $GL(2,q)$ of all invertible 2×2 matrices over $\mathbb{F}_q$. Analogously, the group $SL(2,q)$ of all invertible 2×2 matrices with determinant 1 is isomorphic to $SL(V)$. The center of $GL(2,q)$, denoted by $Z(GL(2,q))$, consists of all non-zero scalar matrices while the center of $SL(2,q)$ is equal to $SL(2,q)\cap Z(GL(2,q))$. Therefore, the groups

$$PGL(2,q) = GL(2,q)/Z(GL(2,q)) \quad \text{and} \quad PSL(2,q) = SL(2,q)/(SL(2,q)\cap Z(SL(2,q)))$$

are isomorphic to $PGL(V)$ and $PSL(V)$, respectively. If q is odd then $PSL(2,q)$ is a subgroup of $PGL(2,q)$ of index 2. On the other hand, if q is even then $PGL(2,q) = PSL(2,q)$.

We denote by $PG(1,q)$ the set of 1-dimensional subspaces of V. Thus, $PG(1,q)$ is a projective line over $\mathbb{F}_q$ and its elements are called projective points. An easy computation shows that $PG(1,q)$ has cardinality $q+1$. From the above definitions, it is clear that the groups $PGL(2,q)$ and $PSL(2,q)$ define a natural right action on $PG(1,q)$. Moreover, the action of $PGL(2,q)$ on $PG(1,q)$ is sharply 3-transitive.

We briefly describe the character table of $G_q := PGL(2,q)$. We refer the reader to [18] for a complete study of the complex irreducible characters of G_q. We start by describing its conjugacy classes. By abuse of notation we will denote the elements of G_q by 2 by 2 matrices with entries from $\mathbb{F}_q$. We choose the following representatives for the conjugacy classes of G_q:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad d_x = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, \quad v_r = \begin{pmatrix} 0 & 1 \\ -r^{1+q} & r+r^q \end{pmatrix},$$

where the label x runs through all the elements of $\mathbb{F}_q^*$ of order greater than 1 up to inversion, and the label r runs through all the elements of $\mathbb{F}_{q^2}^*/\mathbb{F}_q^*$ of order greater than 1 up to inversion.

The complex irreducible characters of G_q are described in Table 1. The trivial character is denoted by λ_1. The character ψ_1 corresponds to the standard character which is an irreducible character of G_q. Thus, for every $g \in G_q$, the value of $\psi_1(g)$ is equal to the number of projective points fixed by g in $PG(1,q)$ minus 1. The label β in Table 1 runs through all homomorphism $\beta : \mathbb{F}_{q^2}^*/\mathbb{F}_q^* \to \mathbb{C}$ of order greater than 2 up to inversion. Therefore, the number of irreducible characters $\{\eta_\beta\}_\beta$ is $q/2$ if q is even and $(q-1)/2$ if q is odd. The label γ in Table 1 runs through all the homomorphism $\gamma : \mathbb{F}_q^* \to \mathbb{C}$ of order greater than 2 up to inversion. Therefore, the number of irreducible characters $\{\nu_\gamma\}_\gamma$ is $q/2-1$ if q is even and $(q-3)/2$ if q is odd.

If q is odd then G_q has two more irreducible characters denoted by λ_{-1} and ψ_{-1} in Table 1. The values of these characters depend on the function δ. We define $\delta(x) = 1$ if $d_x \in PSL(2,q)$ and $\delta(x) = -1$ otherwise. Similarly, $\delta(r) = 1$ if $v_r \in PSL(2,q)$ and $\delta(r) = -1$ otherwise.

Using the notation introduced in the above paragraphs we can write

$$\mathbb{C}[G_q] = V_{\lambda_1} \oplus V_{\psi_1} \oplus \bigoplus_\beta V_{\eta_\beta} \oplus \bigoplus_\gamma V_{\nu_\gamma}$$

when q is even, and

$$\mathbb{C}[G_q] = V_{\lambda_1} \oplus V_{\lambda_{-1}} \oplus V_{\psi_1} \oplus V_{\psi_{-1}} \oplus \bigoplus_\beta V_{\eta_\beta} \oplus \bigoplus_\gamma V_{\nu_\gamma}$$

when q is odd.

2.3 The eigenvalue method

As was remarked in the introduction, the eigenvalue method has been used several times to get upper bounds on the size of intersecting families for EKR-type problems. In this section, we apply the eigenvalue method to show that the characteristic function of every extremal family of G_q has a Fourier transform supported on just two irreducible representations of G_q. In the next section, we will show that almost extremal families have a similar Fourier characterization.

The first step of the method is to reformulate the problem in graph theory terminology. Indeed, the problem of finding the maximum size of an intersecting family in G_q is equivalent to the problem of finding the maximum size of an independent set in a certain graph. Then, we can apply Hoffman's bound to get an upper bound on the size of an independent set. The following variant of Hoffman's theorem will be enough for our purposes.

Theorem 5. *(Hoffman's bound) Let Γ be a k-regular, n-vertex graph. Let A be the adjacency matrix of Γ and let λ_{min} be the minimum eigenvalue of A. If S is an independent set in Γ, then*

$$\frac{|S|}{n} \leqslant \frac{-\lambda_{\min}}{k-\lambda_{\min}}.$$

If equality holds then the characteristic function 1_S of S satisfies:

$$1_S \in V_1 \oplus V_{\lambda_{\min}}$$

where V_1 is the vector space spanned by the all-ones vector and $V_{\lambda_{\min}}$ is the $\lambda_{\min}$-eigenspace.

Recall that an element $g \in G_q$ is a derangement if for any $\alpha \in PG(1,q)$ we have that $\alpha \neq \alpha^g$. Denote by D_q the set of derangements in G_q. We define Γ as the Cayley graph on G_q with generating set D_q. This graph is known as the derangement graph of G_q. Note that every independent set in Γ corresponds to an intersecting family in G_q. Hence, an upper bound on the size of independent sets in Γ is also an upper bound on the size of intersecting families in G_q.

To apply Hoffman's bound, we need to compute the eigenvalues of Γ. Note that the set D_q is a union of conjugacy classes and inverse-closed. Therefore, Γ satisfies the conditions of Lemma 4. Thus, to compute the eigenvalues of Γ we just need to evaluate the character sum $\frac{1}{\chi(1)}\sum_{x\in D_q}\chi(x)$ for every irreducible character χ of G_q.

Now, using the character table of G_q (Table 1) and Lemma 4, Meagher and Spiga [17] computed the eigenvalues of Γ for every q:

q even	λ_1	ψ_1	η_β	ν_γ
eigenvalues	$\frac{q^2(q-1)}{2}$	$-\frac{q(q-1)}{2}$	q	0

q odd	λ_1	λ_{-1}	ψ_1	ψ_{-1}	η_β	ν_γ
eigenvalues	$\frac{q^2(q-1)}{2}$	$-\frac{q(q-1)}{2}$	$-\frac{q(q-1)}{2}$	$\frac{q-1}{2}$	q	0

Then, applying Hoffman's bound, they proved that the maximum size of an intersecting family in G_q is $q(q-1)$. Therefore, the cosets of point stabilizers in G_q are extremal families. For every $\alpha, \beta \in PG(1,q)$, we denote by $T_{\alpha,\beta}$ the coset of a point stabilizer sending α to β.

Furthermore, Hoffman's bound also provides information about the characteristic function of an extremal family. Indeed, if S is an intersecting family of maximum size then its characteristic function 1_S is contained in $V_{\lambda_1} \oplus V_{\psi_1}$ when q is even, and in $V_{\lambda_1} \oplus V_{\psi_1} \oplus V_{\lambda_{-1}}$ when q is odd. In the next lemma, we show that it is possible to improve this result in the case when q is odd.

Lemma 6. *Let q be odd. Let $S \subset G_q$ be an intersecting family of size $q(q-1)$ and denote by 1_S its characteristic function. Then*

$$1_S \in V_{\lambda_1} \oplus V_{\psi_1}.$$

To prove Lemma 6, we will need the following result proved by Meagher and Spiga in [17].

Lemma 7. *Consider the natural right action of $PSL(2,q)$ on the projective points of $PG(1,q)$. Let X be the set of derangements of $PSL(2,q)$. An independent set of maximum size in $Cay(PSL(2,q),X)$ has size $q(q-1)/2$.*

Proof of Lemma 6. We already know that $1_S \in V_{\lambda_1} \oplus V_{\psi_1} \oplus V_{\lambda_{-1}}$. The vector space $V_{\lambda_{-1}}$ is one dimensional so $V_{\lambda_{-1}} = \text{span}_{\mathbb{C}}\{\lambda_{-1}\}$. Hence, it is enough to show that $\langle 1_S, \lambda_{-1} \rangle = 0$.

Recall that $PSL(2,q)$ is a subgroup of G_q. The irreducible character λ_{-1} is a function on G_q such that $\lambda_{-1}(g) = 1$ if $g \in PSL(2,q)$ and -1, otherwise. Therefore, $\langle 1_S, \lambda_{-1} \rangle = 0$ if and only if exactly half of the elements in S are in $PSL(2,q)$.

From Lemma 7 it follows that he maximum size of an intersecting family in $PSL(2,q)$ is $q(q-1)/2$. Therefore, at most $q(q-1)/2$ elements of S are contained in $PSL(2,q)$.

Since $PSL(2,q)$ is a subgroup of index 2, there exists $g' \in G_q$ such that $G_q = g'PSL(2,q) \cup PSL(2,q)$. Assume to the contrary, that more than $q(q-1)/2$ elements of S are contained in $g'PSL(2,q)$. If we multiply each of these elements by g' then we get an intersecting family in $PSL(2,q)$. This is a contradiction because the maximum size of an intersecting family in $PSL(2,q)$ is $q(q-1)/2$. Therefore, exactly half of the elements in S are contained in $PSL(2,q)$. □

3 Fourier characterization

Let S be an intersecting family of maximum size in G_q. It follows from Section 2 that the Fourier transform of 1_S is supported only on the irreducible representations affording the characters λ_1 and ψ_1. In this section, we prove that the characteristic functions of almost extremal families in G_q have Fourier transforms highly concentrated on the irreducible representations affording the characters λ_1 and ψ_1. To do this we apply a stability version of Hoffman's bound (this term was coined by Ellis in [5]). The next two lemmas show that if an intersecting family $S \subset G_q$ satisfies that $|S|$ is very close to $q(q-1)$ then 1_S must be close to $U := V_{\lambda_1} \oplus V_{\psi_1}$.

Lemma 8. *Let S be an intersecting family in G_q. If q is a power of 2 then,*

$$\|P_{U^\perp}(1_S)\|^2 \leqslant \left(1 - \frac{|S|}{q(q-1)}\right) \|1_S\|^2.$$

Proof. Let A be the adjacency matrix of the graph $\Gamma = Cay(G_q, D_q)$. Let $\{x_1, \ldots, x_N\} \subset \mathbb{C}[G_q]$ be an orthonormal basis of real eigenvectors for A (recall that A is symmetric). Let θ_i be the eigenvalue of A such that $Ax_i = \theta_i x_i$, for $1 \leqslant i \leqslant N$. Note that,

- $1_S = \sum_{i=1}^N \epsilon_i x_i$ where $\epsilon_i = \langle 1_S, x_i \rangle$ for every $i = 1, \ldots, N$.
- $\|1_S\|^2 = \sum_{i=1}^N \epsilon_i^2$.
- $\langle 1_S, 1 \rangle = \|1_S\|^2 = \epsilon_1$.

Let x_1 be the all $1'$s vector with eigenvalue $q^2(q-1)/2$. Since every intersecting family corresponds to an independent set in the graph Γ we get

$$0 = 1_S^T A 1_S = \sum_{i=1}^N \theta_i \epsilon_i^2 = \theta_1 \epsilon_1^2 + \sum_{i: i \neq 1, \theta_i \neq \lambda_{\min}} \theta_i \epsilon_i^2 - \frac{q(q-1)}{2} \sum_{i:\theta_i = \lambda_{\min}} \epsilon_i^2, \tag{1}$$

where $\lambda_{\min} = -q(q-1)/2$.

Recall that the second smallest eigenvalue of Γ is zero. Therefore, from equation (1) we obtain the following inequality

$$\theta_1 \|1_S\|^4 - \frac{q(q-1)}{2} \sum_{i:\theta_i = \lambda_{\min}} \epsilon_i^2 \leqslant 0. \tag{2}$$

By definition we have

$$\|P_{U^\perp}(1_S)\|^2 = \sum_{i: i \neq 1, \theta_i \neq \lambda_{\min}} \epsilon_i^2,$$

hence

$$\sum_{i:\theta_i = \lambda_{\min}} \epsilon_i^2 = \|1_S\|^2 - \|1_S\|^4 - \|P_{U^\perp}(1_S)\|^2. \tag{3}$$

Combining (2) and (3) we get

$$\|P_{U^\perp}(1_S)\|^2 \leqslant \left(1 - \frac{|S|}{q(q-1)}\right) \|1_S\|^2. \qquad \square$$

The next lemma deals with the case q odd. The proof is a little more complicated because in that case the minimum eigenvalue of Γ is afforded by two distinct irreducible characters, ψ_1 and λ_{-1}.

Lemma 9. *Let S be an intersecting family in G_q such that $|S| = (1-\delta)q(q-1)$, $\delta > 0$. If q is an odd prime power then*

$$\|P_{U^\perp}(1_S)\|^2 \leqslant \left(1 - \frac{|S|}{q(q-1)}\right) \|1_S\|^2 + \left(\frac{\delta}{q+1}\right)^2.$$

Proof. Using the notation introduced in the proof of Lemma 8 we get

$$\frac{q^2(q-1)}{2}\|1_S\|^4 - \frac{q(q-1)}{2}\sum_{i:\theta_i=\lambda_{min}} \epsilon_i^2 \leqslant 0. \tag{4}$$

Recall that the vector space $V_{\lambda_{-1}}$ is one dimensional. Hence, we denote by $x_{\lambda_{-1}}$ the only eigenvector in the set $\{x_i\}_{i=1}^N$ contained in $V_{\lambda_{-1}}$. We claim that $\epsilon_{\lambda_{-1}}^2 = \langle 1_S, x_{\lambda_{-1}}\rangle^2 \leqslant (\delta/(q+1))^2$.

Note that $x_{\lambda_{-1}}$ is the irreducible character λ_{-1}. Hence, $x_{\lambda_{-1}}$ is a function on G_q such that $x_{\lambda_{-1}}(g) = 1$ if $g \in PSL(2,q)$ and -1, otherwise. Besides, note that $S \cap PSL(2,q)$ and $S \cap (G_q \setminus PSL(2,q))$ have size at most $q(q-1)/2$ because the maximum size of an intersecting family in $PSL(2,q)$ is $q(q-1)/2$. Putting all the above remarks together

$$\epsilon_{\lambda_{-1}}^2 = \langle 1_S, x_{\lambda_{-1}}\rangle^2 = \frac{1}{|G_q|^2}(|S\cap PSL(2,q)| - |S\cap(G_q\setminus PSL(2,q))|)^2 \leqslant \left(\frac{\delta}{q+1}\right)^2. \tag{5}$$

By definition we have

$$\|P_{U^\perp}(1_S)\|^2 = \sum_{i:i\neq 1,\theta_i\neq\lambda_{min}} \epsilon_i^2 + \epsilon_{\lambda_{-1}}^2,$$

hence

$$\sum_{i:\theta_i=\lambda_{min}} \epsilon_i^2 = \|1_S\|^2 - \|1_S\|^4 - \|P_{U^\perp}(1_S)\|^2 + \epsilon_{\lambda_{-1}}^2. \tag{6}$$

Combininig (4), (5) and (6) we get

$$\|P_{U^\perp}(1_S)\|^2 \leqslant \left(1 - \frac{|S|}{q(q-1)}\right)\|1_S\|^2 + \left(\frac{\delta}{q+1}\right)^2. \qquad \square$$

4 Structural Characterization

In this section we give a characterization of the structure of Boolean functions on G_q whose Fourier transform is highly concentrated on U. The technique used to prove this result is from [8]. In that paper, Ellis, Filmus and Friedgut proved that if a Boolean function on S_n has Fourier transform that is highly concentrated on the first two irreducible representations of S_n (which correspond to the trivial and standard representation) then it must be close to a union of cosets of points stabilizers. Their proof is only based on the fact that the action of S_n on $[n]$ is 3-transitive.

Let G be a group acting 3-transitively on a set Ω. It is well-known (and easy to show) that the standard representation is irreducible for any 2-transitive group. Also, recall that V_1 and $V_{\chi_{std}}$ are the vector subspaces of complex-valued functions on G whose Fourier transform has support on the trivial and the standard representation, respectively. The following proposition is a generalization of Theorem 1 from [8][1].

[1]Actually, Proposition 10 is a generalization of a special case of Theorem 1 from [8]. To fully generalize that theorem we need to consider $S \subset G$ with $|S| = c|G|/n$, where $c = o(n)$.

Proposition 10. *There exist absolute constants $C_1, \epsilon_1 > 0$ such that the following holds. Let G be a finite group acting 3-transitively on a set Ω of size n. Let $S \subset G$ with $|S| = (1-\delta)|G|/n$, where $0 \leqslant \delta < 1/2$. Let $V = V_1 \oplus V_{std}$. If $\|P_{V^\perp}(1_S)\|^2 = \epsilon \|1_S\|^2$, where $\epsilon \leqslant \epsilon_1$, then there exists $T \subset G$ such that T is a coset of the stabilizer of an element of Ω, and*

$$|S \triangle T| \leqslant C_1 \left(\epsilon^{1/2} + \frac{1}{n} \right) |S|.$$

The proof of this proposition is exactly the same as the proof of Theorem 1 in [8]. Since the action of G_q on $PG(1,q)$ is 3-transitive, Proposition 10 can be used to characterize Boolean functions on G_q whose Fourier transform is highly concentrated on U. Recall that U is the vector subspace of all of complex-valued functions on G_q whose Fourier transform has support on the trivial and the standard representation.

Corollary 11. *There exist absolute constants $C_1, \epsilon_1 > 0$ such that the following holds. Let $S \subset G_q$ with $|S| = (1-\delta)q(q-1)$, where $0 \leqslant \delta < 1/2$. If $\|P_{U^\perp}(1_S)\|^2 = \epsilon\|1_S\|^2$ where $\epsilon \leqslant \epsilon_1$, then there exist $\alpha, \beta \in PG(1,q)$ such that $T_{\alpha,\beta}$ satisfies that*

$$|S \triangle T_{\alpha,\beta}| \leqslant C_1 \left(\epsilon^{1/2} + \frac{1}{q+1} \right) |S|.$$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. We choose $C_0 = \max(\frac{4\sqrt{2}}{\sqrt{\epsilon_1}}, \sqrt{2}C_1)$ where C_1 and ϵ_1 are the absolute constants from Corollary 11. With this choice of C_0, if $\epsilon_1/2 \leqslant \delta \leqslant 1/2$ then the statement of the theorem holds trivially with any choice of a coset of a point stabilizer T.

Now, we consider the case where $\delta < \epsilon_1/2$. By assumption we know that $|S| = (1-\delta)q(q-1)$. Thus, it follows from Lemmas 8 and 9 that $\|P_{U^\perp}(1_S)\|^2 \leqslant \delta\|1_S\|^2$ when q is even and $\|P_{U^\perp}(1_S)\|^2 \leqslant 2\delta\|1_S\|^2$ when q is odd. This implies that the characteristic function 1_S is highly concentrated on U. Hence, we can apply Corollary 11 to conclude that

$$|S \triangle T| \leqslant C_0 \left(\delta^{1/2} + \frac{1}{q+1} \right) |S|,$$

where T is some coset of a point stabilizer. □

Theorem 1 implies that almost extremal families are almost contained in a coset of a point stabilizer. Furthermore, we can refine this result to conclude that almost extremal families are fully contained in a coset of a point stabilizer.

Proof of Theorem 2. First assume that $q \leqslant 4C_0 - 1$, where C_0 is the absolute constant from Theorem 1. Note that we can choose $\delta_1 > 0$ small enough such that for all $q \leqslant 4C_0 - 1$ we have

$$(1-\delta_1)q(q-1) > q(q-1) - 1.$$

Hence, if S is an intersecting family of G_q with $|S| \geqslant (1-\delta_1)q(q-1)$ then $|S| = q(q-1)$. Therefore, by the characterization of intersecting families of maximum size in G_q given in [17], we conclude that S must be equal to a coset of the stabilizer of a point.

Now, we assume that $q > 4C_0 - 1$. It is clear that if we choose δ_2 such that $0 \leqslant \delta_2 \leqslant 1/(16C_0^2)$ then

$$C_0 \left(\delta_2^{1/2} + \frac{1}{q+1} \right) < \frac{1}{2}. \tag{7}$$

From Theorem 1 it follows that if $|S| \geqslant (1 - \delta_2)q(q-1)$ then

$$|S \triangle T| \leqslant C_0 \left(\delta_2^{1/2} + \frac{1}{q+1} \right) |S|, \tag{8}$$

where T is a coset of a point stabilizer. Combining (7) and (8), we get that $|S \triangle T| < \frac{1}{2}q(q-1)$.

Suppose without loss of generality that $T = T_{\alpha,\alpha}$ for some $\alpha \in PG(1,q)$. Assume for a contradiction that there exists $g \in S$ such that $\alpha^g = \beta$ with $\beta \in PG(1,q)$, $\beta \neq \alpha$. We use this assumption to estimate the size of $T_{\alpha,\alpha} \setminus S$.

If $h \in S \cap T_{\alpha,\alpha}$ then $g^{-1}h$ contains at least one fixed point (recall that S is an intersecting family). Hence, the elements $h \in T_{\alpha,\alpha}$ such that $g^{-1}h$ is a derangement must be contained in $T_{\alpha,\alpha} \setminus S$.

We compute the number of derangements in $g^{-1}T_{\alpha,\alpha} = T_{\beta,\alpha}$. The number of derangements in $T_{\alpha,\alpha}$ is zero. Thus, the $\frac{q^2(q-1)}{2}$ derangements in G_q are contained in $\bigcup_{\delta \neq \alpha} T_{\delta,\alpha}$. Using the 2-transitivity of the action of G_q on $PG(1,q)$, we get that the number of derangements in $T_{\delta,\alpha}$ is the same for every $\delta \neq \alpha$. Indeed, for any two distinct $\delta, \delta' \in PG(1,q)$ with $\delta, \delta' \neq \alpha$, let $m \in G_q$ such that $\alpha^m = \alpha$ and $\delta^m = \delta'$. Then the bijection $\Phi : G_q \to G_q : g \mapsto m^{-1}gm$ satisfies $\Phi(D_q) = D_q$, and $\Phi(T_{\delta,\alpha}) = T_{\delta',\alpha}$, so $|T_{\delta',\alpha} \cap D_q| = |\Phi(T_{\delta,\alpha} \cap D_q)| = |T_{\delta,\alpha} \cap D_q|$.

Therefore, the number of derangements in $T_{\beta,\alpha}$ is $q(q-1)/2$. Hence, there are at least $q(q-1)/2$ elements in $T_{\alpha,\alpha} \setminus S$ which implies

$$|S \triangle T_{\alpha,\alpha}| \geqslant \frac{q(q-1)}{2}.$$

Thus, we get a contradiction. Finally, we choose the universal constant δ_0 to be equal to $\min(\delta_1, \delta_2)$. □

5 Conclusions and Open Problems

In this paper we prove that extremal families in G_q are not only unique, but also stable: any intersecting family in G_q of size close to $q(q-1)$ must be close in structure to a coset of a point stabilizer. Actually, Theorem 2 implies that for q sufficiently large the cosets of point stablizers are the only extremal families in G_q. This result was already proven by Meagher and Spiga [17] using different methods.

It is possible to apply the ideas used in this paper to prove similar results for some 3-transitive groups. Let G be a finite group acting 3-transitively on a finite set Ω. Suppose that this action satisfies the following conditions:

1. The maximum size of an intersecting family in G is $|G|/|\Omega|$ (note that this number is equal to the size of a coset of a point stabilizer in G).

2. Let D be the set of derangements of G with respect to its action on Ω. The standard character is the unique irreducible character affording the minimum eigenvalue of the derangement graph $Cay(G, D)$ (recall that since D is inverse-closed and conjugation-invariant there is a correspondence, given by Lemma 4, between the eigenvalues of $Cay(G, D)$ and the irreducible characters of G).

Thus, applying Hoffman's bound it follows that the characteristic vector of any intersecting family of maximum size lies in the vector subspace $V = V_1 \oplus V_{\chi_{std}}$ of $\mathbb{C}[G]$. Recall that V_1 and $V_{\chi_{std}}$ are the vector subspaces of complex-valued functions on G whose Fourier transform has support on the trivial and the standard representation, respectively.

Now, let $S \subset G$ be an intersecting family. If the size of S is close to $|G|/|\Omega|$ and the size of the gap between the smallest and the second-smallest eigenvalue of $Cay(G, D)$ is big enough then we can use analogues of Lemmas 8 and 9 to conclude that the characteristic function 1_S is close to V. Moreover, as was remarked in Section 4, the result of Ellis, Filmus and Friedgut in [8], for Boolean functions on S_n, can be generalized to any 3-transitive action of a finite group on a finite set. Thus, if 1_S is close to the vector space V then it must be close in structure to some coset of a point stabilizer in G. Therefore, we can use these ideas to prove that extremal families in G are unique and stable.

Consider the action of $PSL(2, q)$ on the points of $PG(1, q)$ for q an odd prime power. This action is 2-transitive. Using the eigenvalue method it is easy to prove that the maximum size of an intersecting family in $PSL(2, q)$ is $q(q-1)/2$. Furthermore, the characteristic vector of any extremal family in $PSL(2, q)$ lies in $V = V_1 \oplus V_{\chi_{std}}$ (recall that the standard character is irreducible for 2-transitive actions). However, the argument used here cannot be applied in a straightforward way to solve the uniqueness or stability problems because the action of $PSL(2, q)$ on $PG(1, q)$ is not 3-transitive. It was conjectured by Meagher and Spiga [17] that the cosets of points stabilizers are the only extremal families in $PSL(2, q)$. Here, we extend their conjecture: the extremal families in $PSL(2, q)$ are not only unique but also stable.

Conjecture 12. Let S be an intersecting family in $PSL(2, q)$ with q an odd prime power. Then

1. Uniqueness: If $|S| = \frac{q(q-1)}{2}$ then S is a coset of a point stabilizer.

2. Stability: There exists $\delta > 0$ such that if $|S| \geqslant (1-\delta)q(q-1)/2$ then S is contained within a coset of a point stabilizer.

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