

Universal and Near-Universal Cycles of Set Partitions

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Abstract

We study universal cycles of the set $\mathcal{P}(n, k)$ of k -partitions of the set $[n] := \{1, 2, \dots, n\}$ and prove that the transition digraph associated with $\mathcal{P}(n, k)$ is Eulerian. But this does not imply that universal cycles (or ucycles) exist, since vertices represent equivalence classes of partitions. We use this result to prove, however, that ucycles of $\mathcal{P}(n, k)$ exist for all $n \geq 3$ when $k = 2$. We reprove that they exist for odd n when $k = n - 1$ and that they do not exist for even n when $k = n - 1$. An infinite family of (n, k) for which ucycles do not exist is shown to be those pairs for which $\binom{n-2}{k-2}$ is odd ($3 \leq k < n - 1$). We also show that there exist universal cycles of partitions of $[n]$ into k subsets of distinct sizes when k is sufficiently smaller than n , and therefore that there exist universal packings of the partitions in $\mathcal{P}(n, k)$. An analogous result for coverings completes the investigation.

1 Introduction

A universal cycle, or ucycle, is a cyclic ordering of a set of objects \mathcal{C} , each represented as a string of length N . The ordering requires that object $b = b_0b_1\dots b_{N-1}$ follow object $a = a_0a_1\dots a_{N-1}$ only if $a_1a_2\dots a_{N-1} = b_0b_1\dots b_{N-2}$. These were originally introduced in 1992 by Chung, Diaconis, and Graham [2] as generalizations of de Bruijn cycles. As an

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example, the cyclic string 112233 encodes each of the six multisets of size 2 from the set $\{1, 2, 3\}$. Another well-quoted example, from [9], is the string

1356725 6823472 3578147 8245614 5712361 2467836 7134582 4681258,

where each block is obtained from the previous one by addition of 5 modulo 8. This string is an encoding of the $56 = \binom{8}{3}$ 3-subsets of the set $[8] := \{1, 2, 3, 4, 5, 6, 7, 8\}$. A seminal paper in the area is that of Chung, Diaconis and Graham [2] who studied ucycles of

- subsets of size k of an n -element set (as in the above example);
- set partitions (the focus of this paper); and
- permutations (with a necessarily augmented ground set and the use of order isomorphism representations, e.g., the string 124324 encodes each of the six permutations of $[3] = \{1, 2, 3\}$ in an order isomorphic fashion, which is clearly not possible to do using the ground set $[3]$).

In [2] it is shown that for $n \geq 4$, there *does exist* a ucycle of all partitions $\mathcal{P}(n)$ of the set $[n]$ into an arbitrary number of parts. For example, we have the ucycle *abcbcccdcdceec* of $\mathcal{P}(4)$, where, for example, the substring *dcd* encodes the partition $13|2|4$. Note that the alphabet used was, in this case, of size 5, though an alphabet of (minimum) size 5 is shown to suffice to encode $\mathcal{P}(5)$ as

DDDDCHHHCCDDCCCHCHCSHHS DSSDSSHSDDC
SSCHSHDHSCHSJCDC.

The above example reflects tongue-in-cheek humor, since there are 52 partitions of $[5]$ and the above ucycle has 13 cards of each suit – except that one spade has been replaced by a joker! The authors of [2] also ask how many partitions of $\mathcal{P}(n)$ using an alphabet of size $N \geq n$ exist. This question will also be of deep relevance to us, as alluded to in the later section on Open Problems.

As noted in [6], however, not much seems to be known about ucycles of the partitions $\mathcal{P}(n, k)$ ($|\mathcal{P}(n, k)| = \binom{n}{k}$) of $[n]$ into k parts. In this paper, a k -partition of $[n]$ is represented as a string $s_1, s_2 \dots s_n$ over the alphabet $[k] = \{1, 2, \dots, k\}$, where i and j are in the same block if and only if $s_i = s_j$. In general this representation is not unique and it is used in different ways in Sections 2 and 3. In [4], it was shown that for $k = n - 1$, ucycles exist if and only if n is odd. At the other end of the k -spectrum, the authors of [7] showed that for odd n , one could find a ucycle of partitions of $[n]$ into two parts, and that an “asymptotically good universal packing” could be found for $k = 3$, i.e., that there was a string of length $T(n, 3) < \binom{n}{3}$ with

$$\frac{T(n, 3)}{\binom{n}{3}} \rightarrow 1, \quad n \rightarrow \infty,$$

where each of the $T(n, 3)$ consecutive strings of length n represented a different partition of $[n]$ into 3 parts of distinct sizes. The authors of [7] also proved that ucycles of partitions of $[n]$ into 2 parts exist if $n = 4$ or 6 , as evidenced by the respective explicit examples

aabbaba

and

abaaaabbaababbbabbbaabbbbbababa.

It is this work that we build on. In Section 2, we generalize the above result on asymptotically good universal packings (upackings) to the case of all fixed k as $n \rightarrow \infty$, as well as to ucoverings, which are also shown to be “asymptotically good.” Finally, Section 3 is devoted to showing that the transition digraph associated with $\mathcal{P}(n, k)$ is Eulerian. As noted in the work of [2], this does not necessarily imply that universal cycles exist, since the digraph vertices represent equivalence classes of partitions. We use our result to prove, however, that ucycles of $\mathcal{P}(n, k)$ exist for all $n \geq 3$ when $k = 2$ and for odd n when $k = n - 1$, the latter recovering the result in [4]. We also (re)prove that ucycles do not exist for n even when $k = n - 1$. Finally, we show that for even n , ucycles do not exist when $\binom{n-2}{k-2}$ is odd ($3 \leq k < n - 1$). There are infinitely many such pairs of values (n, k) for $k \not\equiv 1 \pmod{4}$. Moreover, the technique we exhibit in Section 3 has the potential to tease out many more results along these lines.

2 Universal Packings and Coverings of Partitions of $[n]$ into k parts

One of the main results in [10] was that one could create a ucycle of all surjections from $[n]$ to $[k]$ iff $n > k$. Since there are $k!$ surjections that yield the same set partition, we need to be more careful, and proceed by showing in Theorem 1 that for sufficiently large n , it is possible to ucycle partitions of $[n]$ into k parts of *distinct* sizes. We represent such partitions as surjections from $f : [n] \rightarrow [k]$; $n > k$, with $1 \leq |f^{-1}(\{1\})| < |f^{-1}(\{2\})| < \dots < |f^{-1}(\{k\})|$. The fact that *asymptotically good upackings* exist is proved in Corollary 4.

Theorem 1. *For each fixed $k, n; n \geq \frac{(k+4)(k-1)}{2} + 1$, there exists a ucycle of all onto functions $f : [n] \rightarrow [k]$ such that the preimage cardinality function $|f^{-1}| : \{\{1\}, \{2\}, \dots, \{k\}\} \rightarrow [n]$ is strictly increasing.*

Proof. Following the usual process used to exhibit existence of deBruijn cycles (see, e.g., [6]), we create a digraph D in which the vertices are sequences of length $n - 1$, of numbers in $\{1, \dots, k\}$, for which there exists at least one number in $\{1, \dots, k\}$ which, when added as a prefix or a suffix to the vertex label creates a sequence of length n which, using the special canonical format we have adopted, represents a partition of $[n]$ into k parts of distinct sizes. For example, with $n = 10, k = 4$, 122333444 is a legal vertex, as is 123334444. On the other hand, 112233344 is not in the underlying graph. There is a directed edge from a vertex $v = v_1, \dots, v_{n-1}$ to another vertex $w = w_1 \dots w_{n-1}$ if

$v_i = w_{i-1}$ for each $i = 2, 3, \dots, n - 1$ and $v_1, \dots, v_{n-1}w_{n-1}$ represents a partition of $[n]$ into k distinct parts. The edges of the digraph, labeled by vertex label concatenation, are thus sequences representing partitions of $[n]$ into k distinct parts. It is important to mention here that the definition of edges is different in Section 3. Here edge labels *must* have, for each $1 \leq i \leq k - 1$, more of the letters $i + 1$ than the letters i , whereas in Section 3 any of the $k!$ representations may be used to label a set partition.

The problem of finding a ucycle is reduced to the problem of finding an Eulerian circuit in this digraph. We know that Eulerian circuits exist if the graph is both weakly connected and balanced, i.e., for each vertex v , the in- and out-degrees of v are equal.

If a number from 1 through k can be added as a prefix to a vertex label, then it can also be added to the end as a sequence suffix, since it is only the numbers of 1's, 2's, \dots and k 's that actually matter in determining if an edge represents a partition into distinct parts. Therefore, the digraph is balanced, though we note that the in- and out-degrees, while equal, are quite different at different vertices. We call the common value that guarantees balancedness the degree. For example for $k = 3$, the vertex 123333333 has degree 1; the vertex 122333333 has degree 2; and the vertex 122233333 has degree 3. In general, one may write down a formula for $deg(v)$ depending on the differences between the number of $i + 1$ s and the number of i s in v .

To show that the digraph is weakly connected, we will show that it is possible to reach the designated target vertex $22333 \dots (k - 1) \dots (k - 1)k \dots k$ (with j j 's for each $2 \leq j \leq k - 1$, and $n - \frac{k(k-1)}{2}$ k 's) from any other starting vertex. In fact, we will show something stronger, namely that one can traverse from any edge to the edge $122333 \dots (k - 1) \dots (k - 1)k \dots k$ from whence we may reach the target vertex in a single step. Notice that edges represent legal partitions in $\mathcal{P}(n, k)$ only if $n \geq n_0 := k(k + 1)/2$, and that this is satisfied for the target vertex. In what follows, when we say that we “switch” a number j in a given position to another number j' , we mean that after possibly rotating the word that represents the edge so as to maintain cyclic order, we replace an edge that begins with j to one that ends in j' .

Claim 2. *We are able to “switch” any $j \in \{2, 3, \dots, k - 1\}$ into a k and vice versa (possibly through several steps) provided that $n \geq n_1 := \frac{(k+4)(k-1)}{2} + 1 = n_0 + (k - 1)$, where the “extra” $(k - 1)$ digits allow us to make the switches as needed without violating monotonicity.*

Proof. Let a partition size vector (PSV) be a numerical partition of the integer n into k distinct parts. Each PSV can be used to express the number of 1's, 2's, etc., in order, of an edge. If n is too low, the values of the PSVs might not allow the above-mentioned switch from an integer j to or from k to happen; for example, if $n = 18; k = 5$, then the only possible PSV's are $(1, 2, 3, 4, 8); (1, 2, 3, 5, 7);$ and $(1, 2, 4, 5, 6)$. A 2 cannot be changed to a 5 directly. If we attempt to increase the number of 2's to three (by decreasing the number of 5's) to facilitate this change, this too causes problems with monotonicity. However, if we have $n = 19$ and $k = 5$, the PSV's are $(1, 2, 3, 4, 9), (1, 2, 3, 5, 8), (1, 2, 3, 6, 7), (1, 2, 4, 5, 7),$ and $(1, 3, 4, 5, 6)$, which will be seen to imply that no matter what our starting position,

we have enough “spaces” so that we can switch a 2, 3, or a 4 into a 5 (possibly in multiple steps) and back again eventually. Notice that for $k = 5$, $n_1 = 19$.

Given an edge of weight $n_1 := \frac{(k+4)(k-1)}{2} + 1$, the sums of the gaps between the components of the PSV may be as low as k , corresponding to the PSV $(1, 3, 4, \dots, k+1)$, or as high as $2k - 2$, corresponding to the PSV $(1, 2, \dots, k-1, 2k-1)$. To change a j into a k we cannot have the number of j s be one more than the number of $j-1$ s or one less than the number of $j+1$ s. Notice first that, by the above discussion, there must be a gap of two somewhere in the PSV, which we represent by $(\rho_1, \rho_2, \dots, \rho_k)$. Suppose first that i is the largest integer for which $\rho_i + 2 \leq \rho_{i+1}$ and that $j \geq i + 1$, where j is the integer that we wish to change into a k . Then we move the gap sequentially closer to j by successively replacing integers between $i + 1$ and $j - 1$ by k . This creates a gap of two between ρ_{j-1} and ρ_j , allowing us to switch a j with a k as desired. The process is a little different if $j \leq i$, where the leftmost gap of two in the PSV is between ρ_i and ρ_{i+1} ; in this case we first change the PSV to $(1, 2, \dots, k-1, \ell)$ where $\ell \geq 2k - 1$, and then to $(1, 2, \dots, j-1, j+1, j+2, \dots, k, \ell')$, where $\ell' \geq k + 1$. We can now replace j by k . Replacing a k by a j follows a similar and inverted process. This proves the claim. \square

To continue with the proof of Theorem 1, our algorithm to reach the target sequence will be to first, if we have more than one 1, change all extra 1’s into k ’s. We will then “underline” the single remaining 1 as something we won’t touch again. Next, we will switch the number to the right of the 1 into a k , possibly in multiple steps, and then the k back into a 2, again possibly through multiple steps. We will now underline the 1 and the 2 together, as something we won’t touch again. Next we will consider the next number to the right of this 2 and switch it to a k , and then switch back from the k into a 2 again, then underline the sequence 122, and switch all remaining 2’s in the sequence into k ’s. Next, we consider the number to right of the second 2, switch it into a k , and then switch from the k back into a 3, then underline 1223, etc., until we reach the target sequence. \square

The next theorem shows that there exists a ucovering of all partitions of $[n]$ into k parts if n is large enough; for simplicity we let the threshold n be the same as in Theorem 1. This is because any partition may be represented by a surjection satisfying the conditions of Theorem 3, though there may be multiple such representations when two or more of the part sizes are equal.

Theorem 3. *For each fixed $k, n; n \geq \frac{(k+4)(k-1)}{2} + 1$, there exists a ucycle of all onto functions $f : [n] \rightarrow [k]$ such that the preimage cardinality function $|f^{-1}| : \{\{1\}, \{2\}, \dots, \{k\}\} \rightarrow [n]$ is non-decreasing.*

Proof. Similar to that of Theorem 1; the digraph is a little different, and to show connectivity we first go from an edge to another with strictly increasing PSV – and then proceed to the same target vertex as in Theorem 1. \square

Corollary 4. *For any fixed $k \geq 3$, the upacking and ucovering given in Theorems 1 and 2 respectively are both asymptotically of size $\binom{n}{k}$, the number of partitions of $[n]$ into k parts.*

Proof. This result is a special case of asymptotic results found in [5], where a threshold of $k = n^{1/5}$ is seen to hold for the property “partitions of size k with distinct parts form a “high” fraction of all partitions of $[n]$ into k parts.” \square

3 Universal Cycles of Partitions $\mathcal{P}(n, k)$ of $[n]$ into k parts

As in the previous section, we encode a k -partition of $[n]$ as a string of length n containing k symbols where i and j are in the same subset of the partition if and only if the i th character in the string is the same symbol as the j th character. Since the cases for $k = 1$ and $k = n$ are trivial, we always assume that $2 \leq k < n$. For convenience, we use $\{1, 2, \dots, k\}$ as our alphabet. We refer to an encoding of a partition as a *representation* of that partition. Note that each k -partition of $[n]$ has $k!$ different representations, and we will refer to the representation in which symbols from $[k]$ first appear, from left to right, in their natural order as being a *restricted growth function* or RGF representation, where the value of the RGF at any index is at most one more than its maximum over all previous indices.

Following methods outlined in [2], we construct a transition digraph $G_{n,k}$ for $\mathcal{P}(n, k)$ as follows. Let the set of vertices of $G_{n,k}$ be the set of all k and $(k-1)$ -partitions of $[n-1]$. There is an edge between two vertices v and w of $G_{n,k}$ if and only if w can immediately follow v in a ustring of k -partitions of $[n]$. That is, there is an edge from v to w if and only if the last $n-2$ symbols of a representation for v match the first $n-2$ symbols of a representation for w and the string formed by overlaying these two representations at their shared $n-2$ length substring is a representation of a k -partition of $[n]$. Observe that each vertex that is a k -partition of $[n-1]$ will have indegree = outdegree = k , and each vertex that is a $(k-1)$ -partition of $[n-1]$ will have indegree = outdegree = 1. As an example, $G_{5,3}$ is shown in Figure 1, with all vertices labeled with the RGF representation.

Now, the edges of $G_{n,k}$ are precisely the k -partitions of $[n]$. Furthermore, a partition p_1 can follow another partition p_2 in a ustring for $\mathcal{P}(n, k)$ if and only if the vertex at the tail of p_1 is also at the head of p_2 . Thus, there is a bijection between the Eulerian cycles of $G_{n,k}$ and the ustrings of $\mathcal{P}(n, k)$.

Theorem 5. *Let $n, k \in \mathbf{Z}^+$ with $2 \leq k < n$, and let $G_{n,k}$ be the transition digraph for $\mathcal{P}(n, k)$. Then $G_{n,k}$ has an Eulerian cycle.*

Proof. $G_{n,k}$ is balanced as remarked above, so we must show that $G_{n,k}$ is weakly connected. To do so, we show that there exists a path from any vertex of $G_{n,k}$ to the vertex w with representation $(1, 2, \dots, k-1, k, k, \dots, k)$. Accordingly, let u be a vertex of $G_{n,k}$. We describe an algorithm for obtaining a path from u to w . We first find a path from u to a vertex v which ends in k distinct symbols. We may arrive at such a vertex in $k-1$ steps by a path $u = v_1, v_2, \dots, v_k$ where, for $i = 1, 2, \dots, k-1$, we choose v_{i+1} to be a vertex connected to v_i such that the representations of v_{i+1} end in $i+1$ distinct symbols. Note that choosing v_{i+1} this way is always possible - v_i will have representations ending in i distinct symbols and if $\text{outdegree}(v_i) = 1$ then the only possible choice for v_{i+1} has representations formed by adding the missing symbol of each representation of v_i to its

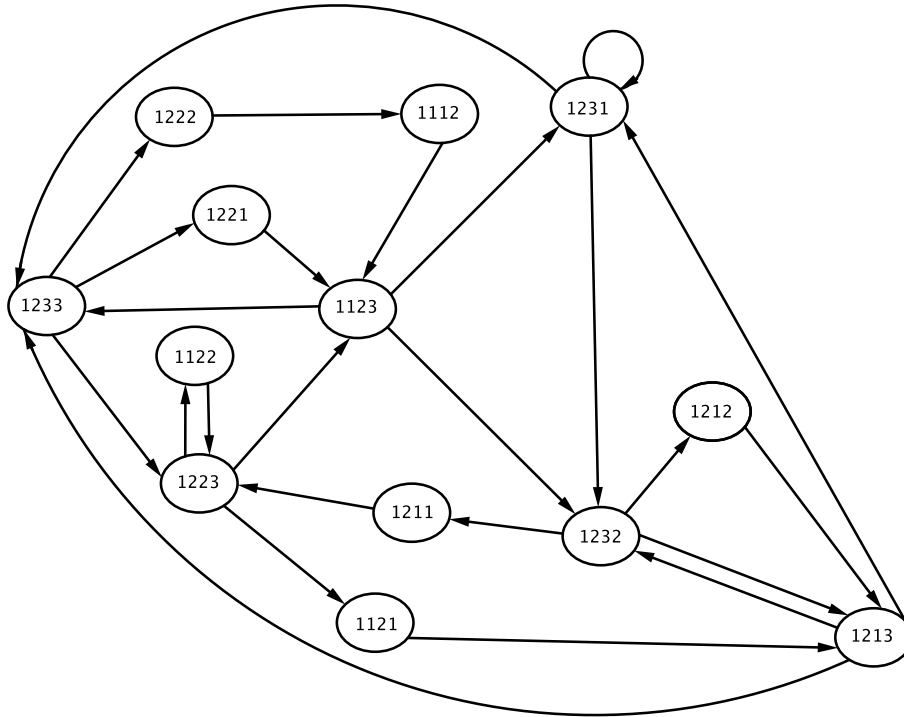


Figure 1: $G_{5,3}$

last $n - 2$ symbols (the case where $\text{outdeg}(v_i) = k$ is clear). Now, v_k has representations ending in k distinct symbols, so for any path of length $(n - 1) - k$ starting at v_k , each vertex on the path will have $\text{outdeg} = k$. Thus, there exists a path $v_k, v_{k+1}, \dots, v_{n-1}$, where v_{k+j} has representations whose last $j + 1$ symbols are all the same ($j = 0, 1, \dots, (n - 1) - k$). Then, by construction, $v_{n-1} = w$. Hence, $G_{n,k}$ is weakly connected and so it follows that $G_{n,k}$ contains an Eulerian cycle. \square

Hence, we know that Eulerian cycles exist in $G_{n,k}$, and therefore ustrings of $\mathcal{P}(n, k)$ exist as well. However, there may be ustrings which cannot be turned into ucycles, which occurs when the representations of the first and last partitions do not overlap correctly, i.e., they have their symbols permuted. This idea is illustrated in the Eulerian cycle in Figure 2:

If we start with the RGF representation 123 of the first vertex, then this Eulerian cycle represents the ustring 123312321, which cannot be turned into a ucycle. This example shows another important concept - once we choose the first representation to use, all other representations used are uniquely determined by the given Eulerian cycle. These observations motivate the following definitions.

Definition 6. Suppose v is a vertex in $G_{n,k}$ and r is a representation for v . Form a new string r_0 from r by deleting all but the first occurrence of each symbol from r and appending the missing symbol to the end if v is a $(k - 1)$ -partition. Then r_0 is a permutation of $[k]$;

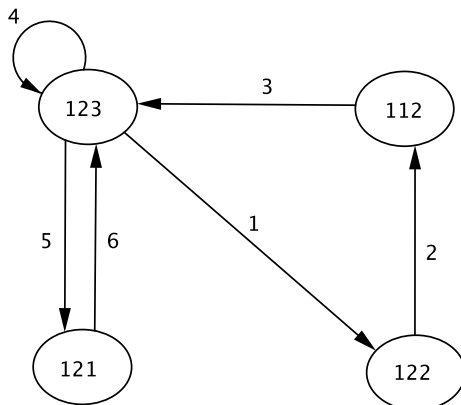


Figure 2: Eulerian cycle in $G_{4,3}$

we call r_0 the relative order of r .

Definition 7. Consider an edge vw for some vertices v and w of $G_{n,k}$. Fix a representation r_v of v and suppose it has relative order π_v . Suppose the corresponding representation of w is r_w with relative order π_w . Then $\pi_w\pi_v^{-1}$ is called the associated permutation of the edge vw .

Remark 8. We have defined the associated permutation as the $\pi \in S_k$ such that $\pi\pi_v = \pi_w$, so that this definition is independent of the choice of representation of v .

The graph $G_{5,3}$ is shown again in Figure 3 with edges labeled with their associated permutations (expressed in cycle notation with fixed points suppressed).

Definition 9. Let $E = e_1, e_2, \dots, e_{S(n,k)}$ be an Eulerian cycle in $G_{n,k}$ and let π_i be the associated permutation of e_i , $i = 1, 2, \dots, S(n, k)$. We call the product $\pi_{S(n,k)}\pi_{S(n,k)-1} \cdots \pi_2\pi_1$ the permutation product of E .

From this definition, we get the following characterization.

Theorem 10. An Eulerian cycle $E = e_1, e_2, \dots, e_{S(n,k)}$ in $G_{n,k}$ can be lifted to a ucycle of $\mathcal{P}(n, k)$ if and only if its permutation product is the identity.

Proof. Fix a representation r of the vertex at the tail of e_1 and suppose r has relative order τ . E can be lifted to a ucycle if and only if we arrive back at r at the end of the cycle, and going through E is equivalent to applying the permutation product to τ . \square

Now, we show that the associated permutation of an edge is completely determined by the vertex at its “tail”, and that only certain permutations can be associated permutations.

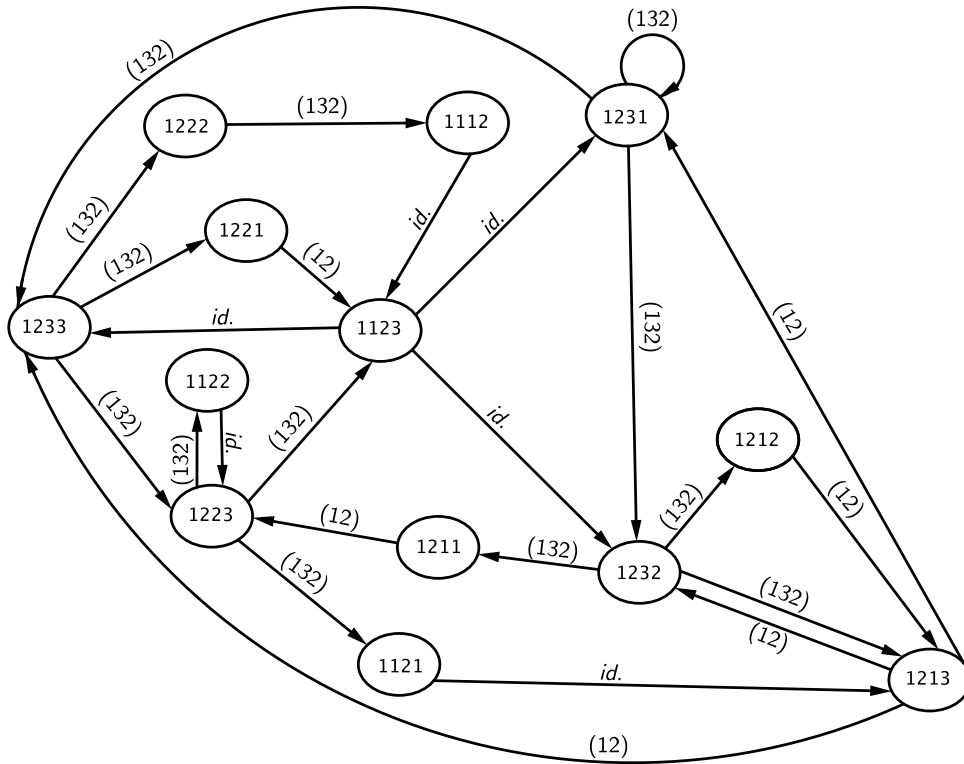


Figure 3: $G_{5,3}$ with associated permutations

Lemma 11. *Let vw_1 be an edge in $G_{n,k}$, suppose vw_1 has associated permutation π . Then π has the form $(1\ j\ j-1\ \dots\ 2)$ for some $1 \leq j \leq k$, and if vw_2 is another edge from v , then vw_2 has associated permutation π as well.*

Proof. Let $r = v_1v_2 \dots v_{n-1}$ be the representation of v with relative order $12 \dots k$. Then there are representations $r_1 = v_2 \dots v_{n-1}u_1$ and $r_2 = v_2 \dots v_{n-1}u_2$ of w_1 and w_2 corresponding to r under vw_1 and vw_2 , respectively. Suppose that $j-1$ distinct symbols appear in r after $v_1 (= 1)$ and before a second appearance of 1 (1 may only occur once in r). Since the first $n-2$ characters of both r_1 and r_2 are the same as the last $n-2$ characters of r , it follows that r_1 and r_2 both have relative order $23 \dots j\ 1\ j+1\ j+2 \dots k$. Hence, the associated permutations of vw_1 and vw_2 are both $(1\ j\ j-1 \dots 2)$. \square

Theorem 12. *For $n \geq 3$, every Eulerian cycle of $G_{n,2}$ can be lifted to a ucycle.*

Proof. Observe that the vertex with representation $11 \dots 1$ is the only vertex of outdegree 1 and that the edge coming out of this vertex has *id.* as its associated permutation. All other vertices have outdegree 2, and the two edges originating from any particular vertex both have the same associated permutation by Lemma 11. In particular, there is an even number of (12) permutations. Since S_2 is abelian, the permutation product of an Eulerian cycle will be the identity so that the result follows by Theorem 10. \square

Corollary 13. For $n \geq 3$, ucycles of $\mathcal{P}(n, 2)$ exist.

Proof. This follows directly from Theorem 5 and Theorem 12. \square

We can also use the permutation product to determine cases when ucycles do not exist. The easiest way for this to occur is if the multiset consisting of all associated permutations in $G_{n,k}$ contains an odd number of odd permutations since this ensures that there is no ordering of the associated permutations which multiplies to the identity.

Definition 14. We call the multiset consisting of all associated permutations in $G_{n,k}$ the permutation multiset of $G_{n,k}$.

Definition 15. Let \mathcal{O} be the multiset which contains all odd permutations of the permutation multiset of $G_{n,k}$. Define the parity function by

$$Par(n, k) = \begin{cases} 0 & \text{if } |\mathcal{O}| \equiv 0 \pmod{2} \\ 1 & \text{if } |\mathcal{O}| \equiv 1 \pmod{2} \end{cases}$$

Lemma 16. If $Par(n, k) = 1$, then there does not exist a ucycle of $\mathcal{P}(n, k)$.

The following formula gives a recursive formula for calculating $Par(n, k)$.

Lemma 17. The function $Par(n, k)$ satisfies the following recurrence relation:

$$Par(n, k) \equiv k \cdot Par(n-1, k) + Par(n-1, k-1) + \begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \pmod{2} \quad (1)$$

with initial conditions $Par(n, 2) = 0$ for all n , and

$$Par(n, n-1) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{4} \\ 0 & \text{otherwise} \end{cases}$$

Proof. We establish a relationship between the permutation multiset of $G_{n,k}$ and those of $G_{n-1,k}$ and $G_{n-1,k-1}$. Suppose v is a vertex in $G_{n,k}$, so v represents a k or $k-1$ -partition p of $[n-1]$. We consider the edge e_p which represents p in either $G_{n-1,k}$ or $G_{n-1,k-1}$. We know that the associated permutation of e_p is determined by the location of the second occurrence of the first symbol in a representation r_p of the vertex w_p at the tail of e_p by Lemma 11. First, suppose v represents a k -partition of $[n-1]$. If the first symbol does actually occur for a second time in r_p , then since there is a representation of v whose first $n-2$ characters are precisely r_p , it follows that e_p has the same associated permutation as all the edges coming from v . If the first symbol of r_p does not occur a second time, then e_p has associated permutation $(1 \ k \ k-1 \ \cdots \ 2)$. If w_p has outdegree 1, then the representations of v do not have a second occurrence of their first symbols, and so all edges from v have associated permutation $(1 \ k \ k-1 \ \cdots \ 2)$. If w_p has outdegree k , then the representations of v have an occurrence of all symbols before a second occurrence of the first symbol, so we get that the edges from v have associated permutation $(1 \ k \ k-1 \ \cdots \ 2)$ again. Since each

vertex of $G_{n,k}$ which represents a k -partition of $[n - 1]$ has outdegree k , we get the term $k \cdot Par(n - 1, k)$.

Now, suppose v represents a $k - 1$ -partition of $[n - 1]$.

Case 1, the first symbol of v appears a second time. Then either the first symbol of r_p appears a second time, or the first symbol of r_p is appended by following e_p (if the second appearance in v is at the last character). If the first symbol of r_p appears a second time, then by previous reasoning e_p has the same associated permutation as all edges from v . If the first symbol of r_p does not occur a second time, then e_p and the edges from v all have associated permutation $(1\ k\ k-1 \cdots 2)$.

Case 2, the first symbol of v does not appear a second time. Then the first symbol of r_p does not appear a second time, and so e_p must have associated permutation $(1\ k-1\ k-2 \cdots 2)$. However, in this case v has associated permutation $(1\ k\ k-1 \cdots 2)$. Note that since the first symbol of v does not appear a second time, the last $n - 2$ characters of v represent a $k - 2$ -partition of $[n - 2]$, so this case occurs exactly $\binom{n-2}{k-2}$ times. Thus, we have $\binom{n-2}{k-2}$ partitions that either switch from even to odd or odd to even; in either case adding $\binom{n-2}{k-2}$ affects the parity in the desired manner.

Thus, each vertex in $G_{n,k}$ which represents a $k - 1$ -partition of $[n - 1]$ has the same associated permutation as it does in the graph $G_{n-1,k-1}$ except for $\binom{n-2}{k-2}$ permutations which change sign. Since each such vertex has outdegree 1, we get the $Par(n - 1, k - 1) + \binom{n-2}{k-2}$ term.

Finally, the initial condition $Par(n, 2) = 0$ for all n follows from Theorem 12, and the initial condition

$$Par(n, n - 1) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{4} \\ 0 & \text{otherwise} \end{cases}$$

follows from the remark following Theorem 5 in [4] □

Corollary 18. For $n \geq 4$ and $2 \leq k < n$,

$$Par(n, k) \equiv \begin{cases} 0 \pmod{2} & \text{if } n \text{ is odd} \\ \binom{n-2}{k-2} \pmod{2} & \text{if } n \text{ is even} \end{cases} \quad (2)$$

Proof. We proceed by induction. For the base cases, we show that the initial conditions of the recursion in Corollary 17 satisfy Equation (2). If we define $\binom{n}{0} := 0$ for all n , then Equation (2) yields $Par(n, 2) = 0$ for all n . Also, if n is even, then Equation 2 yields

$$\begin{aligned} Par(n, n - 1) &= \binom{n-2}{n-3} = \binom{n-2}{2} = \frac{(n-2)(n-3)}{2} \\ &\equiv \begin{cases} 0 \pmod{2} & \text{if } n \equiv 2 \pmod{4} \\ 1 \pmod{2} & \text{if } n \equiv 0 \pmod{4} \end{cases} \end{aligned}$$

as desired.

Now, suppose that (2) holds for $Par(n-1, k)$ and $Par(n-1, k-1)$. If n is odd then $n-1$ is even, so by (1), the induction hypothesis, and the fundamental Stirling number recurrence,

$$\begin{aligned}
 Par(n, k) &\equiv k \cdot Par(n-1, k) + Par(n-1, k-1) + \begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} && \text{mod } 2 \\
 &\equiv k \cdot \begin{Bmatrix} n-3 \\ k-2 \end{Bmatrix} + \begin{Bmatrix} n-3 \\ k-3 \end{Bmatrix} + \begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} && \text{mod } 2 \\
 &\equiv 2 \begin{Bmatrix} n-3 \\ k-2 \end{Bmatrix} + \begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} + \begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} && \text{mod } 2 \\
 &\equiv 2 \begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} && \text{mod } 2 \\
 &\equiv 0 && \text{mod } 2
 \end{aligned}$$

If n is even, then $n-1$ is odd and so $Par(n-1, k) = Par(n-1, k-1) = 0$. Hence, $Par(n, k) \equiv \begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \text{ mod } 2$ by (1). \square

Corollary 19. *If n is even and $\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix}$ is odd, there does not exist a ucycle for k -partitions of $[n]$.*

Proof. This follows directly from Corollary 18. \square

Corollary 20. *For each $k \geq 3, k \not\equiv 1 \pmod{4}$, there are infinitely many even n 's for which ucycles of $\mathcal{P}(n, k)$ do not exist.*

Proof. To prove the corollary, we use the well-known fact [1] that

$$\begin{Bmatrix} n \\ k \end{Bmatrix} \equiv \binom{n - \lfloor k/2 \rfloor - 1}{n - k} \pmod{2}. \tag{3}$$

Thus

$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \equiv \binom{n - \lfloor k/2 \rfloor - 2}{n - k} \equiv \binom{n - \lfloor k/2 \rfloor - 2}{k - \lfloor k/2 \rfloor - 2} \pmod{2}.$$

We also know (see, e.g., Theorem 4.1.10 at <http://www.cs.columbia.edu/~cs4205/files/CM4.pdf>) that

$$\binom{n}{k} \equiv \begin{cases} 0 \pmod{2} & \text{if } n \text{ is even and } k \text{ is odd} \\ \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor} \pmod{2} & \text{otherwise} \end{cases} \tag{4}$$

If $k = 4s + 1$, then

$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \equiv \binom{n-2s-2}{2s-1} \pmod{2},$$

and no conclusion may be reached due to (4), since this is an $\binom{\text{even}}{\text{odd}}$ situation. Using the above facts, it is easy to check that for $k = 3, 4, 6$, ucycles do not exist for any even k . For

$k = 7$, $\left\{ \begin{smallmatrix} n-2 \\ 5 \end{smallmatrix} \right\} \equiv \binom{n-5}{2} = (n-5)(n-6)/2 \pmod{2}$ and thus ucycles do not exist if $n \equiv 0 \pmod{4}$. The same is true for $k = 8$. For $k = 10$, by (3) and (4), we see that ucycles do not exist when $\lfloor (n-7)/2 \rfloor$ is odd, or when $n \equiv 2 \pmod{4}$. We treat the general cases $k = 4s, k = 4s + 2, k = 4s + 3$ next, simultaneously. In these three cases respectively we have, by (3),

$$\begin{aligned} \left\{ \begin{smallmatrix} n-2 \\ k-2 \end{smallmatrix} \right\} &\equiv \binom{n-2s-2}{2s-2} \text{ or} \\ \left\{ \begin{smallmatrix} n-2 \\ k-2 \end{smallmatrix} \right\} &\equiv \binom{n-2s-3}{2s-1} \text{ or} \\ \left\{ \begin{smallmatrix} n-2 \\ k-2 \end{smallmatrix} \right\} &\equiv \binom{n-2s-3}{2s}. \end{aligned}$$

In each of these cases we cannot conclude summarily that the Stirling number $\left\{ \begin{smallmatrix} n-2 \\ k-2 \end{smallmatrix} \right\}$ is even – since we never have an $\binom{even}{odd}$ situation. Let us proceed to iterate (4) until, after roughly $\log(\frac{k}{2})$ steps, we arrive at an $\binom{M}{1} = M$ term for some $m \in \mathbb{Z}^+$. We claim that for each odd M , it is possible to choose at least one starting n for which the above iterations never contain an $\binom{even}{odd}$ term, so that we can deduce that a ucycle doesn't exist. Starting at $\binom{M}{1}$ we manipulate the backward path of equivalences to be

$$\binom{2M+1}{a_1}, \binom{4M+3}{a_2}, \binom{8M+7}{a_3},$$

etc., none of which are of the form $\binom{even}{odd}$, where the a_i are determined by the value of $k + \lfloor k/2 \rfloor - 2$ in the starting binomial coefficient. Now if r iterations of (4) are needed to get to M , then the initial binomial coefficient must equal

$$\binom{n - \lfloor k/2 \rfloor - 2}{k - \lfloor k/2 \rfloor - 2} = \binom{2^r + 2^r - 1}{k - \lfloor \frac{k}{2} \rfloor - 2},$$

so that the required solution is $n \equiv \lfloor k/2 \rfloor + 1 \pmod{2^{r+1}}$. This completes the proof, though it should be noted that we have not *entirely* characterized those n for which ucycles do not exist in terms of k , since if a_i is even there are two choices for the “top” binomial coefficient. Overall, however, the solutions get progressively sparser due to the fact [8] that “most binomial coefficients are even”. \square

In practice, for fixed k , the task of identifying suitable n 's can be done exactly, as seen by the following example.

Example 21. *Ucycles of $\mathcal{P}(n, 23)$ do not exist for $n \equiv 8, 12 \pmod{16}$.*

Proof. We have

$$\left\{ \begin{smallmatrix} n-2 \\ 23-2 \end{smallmatrix} \right\} \equiv \binom{n-13}{10} \equiv \binom{\lfloor \frac{n-13}{2} \rfloor}{5} \equiv \binom{a}{2} \equiv \binom{b}{1},$$

where a, b are respectively the “twice- and thrice-iterated half-floors” of $n - 13$. For any odd $b \geq 1$, moreover, we must have $a = 2b$ or $a = 2b + 1$. Thus

$$\left\lfloor \frac{n-13}{2} \right\rfloor = 4b, 4b+1, 4b+2, \text{ or } 4b+3,$$

whence we get $n - 13 = 8b, \dots, 8b + 7$, or $n = 13 + 8b, \dots, 20 + 8b$. Eliminating the odd n 's leads to $n = 14 + 8b, 16 + 8b, 18 + 8b, 20 + 8b$, or

$$n = \frac{16(b+1)}{2} + 6, \frac{16(b+1)}{2} + 8, \frac{16(b+1)}{2} + 10, \frac{16(b+1)}{2} + 12.$$

The solutions that are congruent to 6 or 10 (mod 16) can be checked to terminate prematurely in an $\binom{\text{even}}{\text{odd}}$ coefficient. This proves the claim. \square

4 Open Questions

We can ask the following:

(a) The smallest case remaining after our investigation is the one for $n = 5, k = 3$, for which we have found the ucycle 3112311123213233112131322. All other cases for $n = 5$ and $n = 6$ are solved. This leads to the question: What is the best result that can be proved along the following lines “For $n \geq 3$ and $3 \leq k < n$, ucycles of k -partitions of $[n]$ exist if and only if n is odd?”

(b) In general, how can one use Theorem 10 to prove results on *existence* of ucycles (rather than non-existence)?

(c) Even if ucycles of $\mathcal{P}(n, k)$ may not exist, when is it true that ucycles exist for $\mathcal{P}(n, s, t)$, the set of partitions of $[n]$ into between s and t parts; $s < t$?

(d) Throughout this paper, we have insisted on having the alphabet size equal k . How do our results change if we relax this condition?

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