A Generalization of Graham's Conjecture

Huanhuan Guan*
School of Mathematic and Statistics
Guizhou University of Finance and Economics
Guiyang 550025, P.R.China
guan1110h@163.com

Pingzhi Yuan
School of Mathematics
South China Normal University
Guangzhou 510631, P.R. China
yuanpz@scnu.edu.cn

Xiangneng Zeng
Sino-French Institute of Nuclear Engineering and Technology
Sun Yat-Sen University
Guangzhou 510275, P.R. China
junevab@163.com

Submitted: Apr 15, 2015; Accepted: Sep 29, 2015; Published: Oct 16, 2015
Mathematics Subject Classifications: 11B50, 11B75

Abstract

Let G be a finite Abelian group of order $|G| = n$, and let $S = g_1 \cdot \ldots \cdot g_{n-1}$ be a sequence over G such that all nonempty zero-sum subsequences of S have the same length. In this paper, we completely determine the structure of these sequences.

Keywords: Graham's conjecture; zero-sum subsequence; support of sequence

1 Introduction

Let G be a finite Abelian group (written additively), and let $\mathcal{F}(G)$ denote the free Abelian monoid with basis G, the elements of which are called *sequences* (over G). A sequence of not necessarily distinct elements from G will be written in the form

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{i=1}^{l} g_i = \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G),$$

where $g_i \in G$ are the terms of S and $\mathsf{v}_g(S) \geqslant 0$ is called the *multiplicity* of g in S. Denote by $|S| = l$ the number of terms in S (called the *length* of S) and let $\mathrm{supp}(S) = \{g \in G : \mathsf{v}_g(S) > 0\}$ be the *support* of S.

*Supported by NSFC of China (Nos.11326053, 11301556, 11271142), the Science and Technology Foundation of Guizhou Provice (No. [2013]2087) and the Guangdong Provincial Natural Science Foundation (Nos. S2012010009942, S2013040013796).

We say that S contains some $g \in G$ if $\mathsf{v}_g(S) \geqslant 1$, and a sequence $T \in \mathcal{F}(G)$ is a *subsequence* of S if $\mathsf{v}_g(T) \leqslant \mathsf{v}_g(S)$ for every $g \in G$, denoted $T|S$. If $T|S$, then let ST^{-1} denote the sequence obtained by deleting the terms of T from S. Furthermore, we denote by $\sigma(S)$ the sum of S, i.e., $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g\in G} \mathsf{v}_g(G)g \in G$. We define

$$\sum(S) = \{\sigma(T) : T \text{ is a nonempty subsequence of } S\},$$

the set of subsums of S,

$$\sum_k(S) = \left\{ \sum_{i\in I} g_i \mid I \subseteq [1, |S|] \text{ with } |I| = k \right\},$$

the set of $k-$term subsums of S, and for all $k \in \mathbb{N}$,

$$\sum_{\leqslant k}(S) = \bigcup_{j\in[1,k]} \sum_j(S) \text{ and } \sum_{\geqslant k}(S) = \bigcup_{j\geqslant k} \sum_j(S).$$

Let S be a sequence in G. We define S a zero-sum sequence if $\sigma(S) = 0$, a zero-sum free sequence if $0 \notin \sum(S)$, and a minimal zero-sum sequence if $\sigma(S) = 0$ and $\sigma(T) \neq 0$ for any proper and nontrivial subsequence T of S.

For a sequence S over G, we define

$$h(S) = \max\{\mathsf{v}_g(S) \mid g \in G\} \in [0, |S|], \text{ the maximum of the multiplicities of } S.$$

Let C_n denote the cyclic group of order n, where $n \in \mathbb{N}$, and G be a finite Abelian group (written additively) with $|G| > 1$. By the Structure Theorem of Abelian Groups, we have that $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$, where $1 < n_1 \mid \ldots \mid n_r \in \mathbb{N}$, $r = r(G)$ is the rank of G, and $n_r = \exp(G)$ is the exponent of G. If $n_1 = \ldots = n_r$, we denote $G = C_n^r$.

Here and henceforth, n is a fixed integer greater than 1, and the cyclic group of order n is identified with the additive group $\mathbb{Z}/n\mathbb{Z}$ of the integers modulo n.

Graham [4] stated the following conjecture.

Conjecture 1. Let p be a prime and S be a sequence over $\mathbb{Z}/p\mathbb{Z}$ of length p. If all nontrivial zero-sum subsequences of S is of the same length, then the number of distinct terms in S is at most 2.

In 1976, Erdős and Szemerédi [4] verified Conjecture 1 for sufficiently large prime p. However, the proof was so complicated that the details for small primes were never worked out. Recently Gao *et al* [5] proved the following result.

Theorem 2. *([5]) Let S be a sequence over $\mathbb{Z}/n\mathbb{Z}$ of length n. If all nontrivial zero-sum subsequences of S are of the same length, then the number of distinct terms in S is at most* 2.

Our objects of study can be characterized in very simple terms. To be more specific, let us recall several standard notions, see [11].

If $S = a_1 \cdot \ldots \cdot a_k$ is a sequence over $\mathbb{Z}/n\mathbb{Z}$, let $\overline{a_i}$ be the unique integer in the set $\{0, 1, \ldots, n-1\}$ which belongs to the congruence class a_i modulo n, $i = 1, \ldots, k$. The number $\overline{a_i}$ is called the least nonnegative representative of a_i modulo n.

Let $S = a_1 \cdot \ldots \cdot a_t$ be a sequence of integers. We write $m * S = (ma_1) \cdot \ldots \cdot (ma_t)$ where m is a integer.

If g is an integer coprime to n, multiplication by g preserves the zero sums in $\mathbb{Z}/n\mathbb{Z}$ and does not introduce new ones. Hence a sequence $S = a_1 \cdot \ldots \cdot a_k$ is zero-free if and only if the sequence $g * S = (ga_1) \cdot \ldots \cdot (ga_k)$ is zero-free. This fact motivates the following definition.

Let S and T be sequences over $\mathbb{Z}/n\mathbb{Z}$. We say that S is equivalent to T and write $S \cong T$ if T can be obtained from S through multiplication by an integer coprime to n and rearrangement of terms. Clearly $\cong$ is an equivalence relation. Otherwise, we say that S is not equivalent to T and write $S \ncong T$.

More recently, Grynkiewicz [8] gave an exhaustive list detailing the precise structure of S and showed that the result holds in an arbitrary finite Abelian group. He proved the following result.

Theorem 3. *([8]) Let G be an Abelian group of order n and let $S \in \mathcal{F}(G)$. Suppose there is a unique $r \in [1, n]$ such that $0 \in \sum_r(S)$. Then $|\mathrm{supp}(S)| \leqslant 2$.*

If G is non-cyclic, then $G = \langle h \rangle \oplus \langle g \rangle \cong C_2 \oplus C_{2m}, r = \frac{n}{2} = 2m$ and

$$S = g^{n-1}g' \quad or \quad S = g^{n/2+x}(h+g)^{n/2-x} \quad or \quad S = g^{n/2+x}\left(h + \frac{n+4}{4}g\right)^{n/2-x},$$

where $g \in G, h, g' \in G \setminus \langle g \rangle, ord(g) = \frac{n}{2}, ord(h) = 2$ and $x \in [1, \frac{n}{2} - 1]$ is odd.

If $G \cong \mathbb{Z}/n\mathbb{Z}$, then S is one of the following:

(i) $S \cong 1^{n-1}g$, where $g \in \mathbb{Z}/n\mathbb{Z}$.

(ii) $S \cong 2^{n-1}g$, where n is even and $g \in \mathbb{Z}/n\mathbb{Z}$ is odd.

(iii) $S \cong 1^{n-2}(\frac{n+2}{2})^2$, where n is odd and $r = \frac{n+1}{2}$.

(iv) $S \cong 2^{n/2+x}(\frac{n+4}{2})^{n/2-x}$, where $n \equiv 2 \pmod 4$ and $x \in [0, \frac{n}{2} - 1]$ is even, and $r = \frac{n}{2}$.

(v) $S \cong 1^{n/2+x}(\frac{n+2}{2})^{n/2-x}$, where n is even and $x \in [0, \frac{n}{2} - 1]$ with $\frac{n}{2} - x$ odd, and $r = \frac{n}{2}$.

For the proof in the case where G is a general Abelian group, Grynkiewicz [8] used the result of the Devos-Goddyn-Mohar Theorem [2]. The main purpose of the present paper is to give an exhaustive list detailing the precise structure of the sequences for a slight generalization of Graham's Conjecture without using the Devos-Goddyn-Mohar Theorem. The following are our main results.

Theorem 4. *Let G be a finite Abelian group of order n with $r(G) \geqslant 2$, and let S be a sequence of length $n-1$ over G. If all nonempty zero-sum subsequences have the same length $r \in [1, n-1]$, then $G \cong C_2 \oplus C_{2m}$. Moreover, either $m = 1, r = 3$ and $S = \prod_{g \in G \setminus \{0\}} g$, or $S = 0g_1g_2$, where $r = m = 1$ and $g_1, g_2 \in G \setminus \{0\}$ are distinct, or $r = \frac{n}{2} = 2m$ and S is one of the following:*

(i) $S = g^{n-1}$,

(ii) $S = g^{n-2}g'$,

(iii) $S = g^{n-3}g'(2g - g')$,

(iv) $S = g^{\frac{n}{2}+x}(h+g)^{\frac{n}{2}-1-x}$,

where $g \in G$, $h, g' \in G \setminus \langle g \rangle$, $\mathrm{ord}(g) = \frac{n}{2}$, $\mathrm{ord}(h) = 2$ and $x \in [0, \frac{n}{2} - 4]$.

Theorem 5. *Let S be a sequence of length $n-1$ over $G = \mathbb{Z}/n\mathbb{Z}$. Suppose that all nontrivial zero-sum subsequences of S have the same length $r \in [1, n-1]$, then S is one of the following:*

- $S \cong 1^{n-2} \cdot g$, *where $g \in \mathbb{Z}/n\mathbb{Z}$ and $\bar{g} \neq 1$.*
- $S \cong 1^{n-3} \cdot 0 \cdot 2$ *and $r = 1$.*
- $S \cong 1^{n-3} \cdot 2 \cdot (n-1)$ *and $r = 2$.*
- $S \cong 1^{n-3} \cdot 2^2$ *and $r = n-2$.*
- $S \cong 1^{n-3} \cdot (\frac{n+1}{2})^2$, *where n is odd, and $r = \frac{n+1}{2}$.*
- $S \cong 2^{n-1}$, *where n is even, and $r = \frac{n}{2}$.*
- $S \cong 2^{n-2} \cdot g$, *where n is even and $\bar{g}$ is odd, and $r = \frac{n}{2}$.*
- $S \cong 2^{n-3} \cdot g \cdot (4-g)$, *where n is even and $\bar{g}$ is odd, and $r = \frac{n}{2}$.*
- $S \cong 2^u \cdot (\frac{n}{2}+2)^{n-1-u}$, *where n is even, $\frac{n}{2}$ is odd, $u \geqslant \frac{n}{2} - 1$ and $r = \frac{n}{2}$.*
- $S \cong 1^u \cdot (\frac{n}{2}+1)^v$, *where n is even, $u \geqslant \frac{n}{2} - 1$ and $r = \frac{n}{2}$.*
- $n = 6$ *and* $S \cong 1^2 \cdot 3^3$, $1 \cdot 3^3 \cdot 4$ *or* $2^2 \cdot 3^3$.
- $n = 7$ *and* $S \cong 1^3 \cdot 3^3$.
- $n = 8$ *and* $S \cong 1^4 \cdot 3^3$.

Corollary 6. *Let G be a finite Abelian group of order n and let S be a sequence of length $n-1$ over G. If there is an integer $r \geqslant 0$ such that all nonempty zero-sum subsequences of S have length r, then $|\mathrm{supp}(Sx^{-1})| \leqslant 2$ for some $x \in \mathrm{supp}(S)$.*

2 Preliminaries

Given two subsets A and B of an Abelian group G, their *sumset* is the set of all pairwise sums, denoted $A + B = \{a + b : a \in A, b \in B\}$. From the basic properties of addition, we have that $A + B = B + A$, and that the sumsets of more than two sets, denoted $\sum_{i=1}^{l} A_i = \{\sum_{i=1}^{l} a_i : a_i \in A_i\}$, is well defined. We often use the convention that $\sum_{i \in \varnothing} A_i = \{0\}$. For sumsets with a single element set, we abbreviate $\{x\} + A$ to $x + A$. Substraction of sets is defined similarly, for instance, $-A = \{-a : a \in A\}$ and $A - B = \{a - b : a \in A, b \in B\}$.

Arithmetic progressions over an arbitrary Abelian group G (with length l and difference d) are sets of the form $\{\alpha + id : i = 1, 2, \cdots, l\}$ with $\alpha, d \in G$ and $l \in \mathbb{Z}$, and are closely related to the prototypical cases that arise when studying sumsets with small cardinality.

For a zero-sum free sequence S, we say that a term $x \in \mathrm{supp}(S)$ is a *1-term* if $|\sum(S)| = |\sum(Sx^{-1})| + 1$.

Definition 7. ([7], Definition 5.1.3) A sequence $S \in \mathcal{F}(G)$ is called *smooth* if $S = (n_1 g)(n_2 g) \cdot \ldots \cdot (n_l g)$, where $|S| \in \mathbb{N}$, $g \in G$, $1 = n_1 \leqslant \cdots \leqslant n_l$, $n = n_1 + \cdots + n_l < \mathrm{ord}(g)$ and $\sum(S) = \{g, \ldots, ng\}$ (in this case we say more precisely that S is g-smooth).

Let S be a sequence in an Abelian group G. Note that $\sum(S) \cup \{0\} = \{0, g_1\} + \cdots + \{0, g_l\}$, where $S = g_1 \cdot \ldots \cdot g_l$. In particular, $\sum(ST) \cup \{0\} = (\sum(T) \cup \{0\}) + \{0, g_1\} + \cdots + \{0, g_l\}$ for any nontrivial sequence $T \in \mathcal{F}(G)$. With this observation in hand, the following lemma is easily verified.

Lemma 8. *Let $A \subseteq G \setminus \{0\}$ be a finite nonempty subset of an Abelian group G. Let $b \in G \setminus \{0\}$ and $|\{0, b\} + (\{0\} \cup A)| = |A| + 2$. Then $A \cup \{0\}$ is the union of some arithmetic progression with difference b and (possibly) some disjoint $H-$cosets, where $H = \langle b \rangle$.*

Let G be an Abelian group. If A and B are subsets of G and $g \in G$, then $r_g(A, B)$ denotes the number of different representations of g as a sum of $g = a + b$ $(a \in A, b \in B)$. The following results will be needed.

Theorem 9. (Addition theorem of Kemperman-Scherk) *Let G be an Abelian group, and A, B be nonempty subsets of G. Then for each element $g \in A + B$,*

$$|A + B| \geqslant |A| + |B| - r_g(A, B).$$

The following result characterizes all zero-sum free sequences S with $|\sum(S)| \leqslant 2|S| - 1$. Moreover, if the equality holds, then S is one of the forms of (ii), (iii), (iv) and (v).

Theorem 10. ([13], Theorem 1.1) *Let G be a finite Abelian group and let S be a zero-sum free sequence over G with $|\sum(S)| \leqslant 2|S| - 1$. Then S is one of the following:*

(i) S is a-smooth for some $a \in G$.

(ii) $S = a^k b$, where $k \in \mathbb{N}$ and $a, b \in G$ are distinct.

(iii) $S = a^k b^l$, where $k \geqslant l \geqslant 1$ and $a, b \in G$ are distinct with $2a = 2b$.

(iv) $S = a^k b^l (a - b)$, where $k \geqslant l \geqslant 1$ and $a, b \in G$ are distinct with $2a = 2b$.

(v) $S = a^k bc$, where $\mathrm{ord}(a) = k + 2$ and $b, c \in G \setminus \langle a \rangle$ are distinct with $b + c = a$ and $b - c \in \langle a \rangle$. In this case, $\sum(S) = \langle a, b, c \rangle \setminus \{0\}$.

Lemma 11. ([11], Proposition 2) *Suppose a sequence $S = Ta$ is zero-sum free over the Abelian group G and $|\sum(S)| = |\sum(T)| + 1$, let $H = \langle a \rangle$ denote the subgroup of G generated by a, then:*

(i) $\sum(T)$ *is the union of a progression* $\{a, 2a, 3a, \dots, ka\}$ *and some cosets (maybe empty) of* $H = \langle a \rangle$ *where* $1 \leqslant k < \mathrm{ord}(a) - 1$.

(ii) $\sigma(T) = ka$.

(iii) a *is the unique element of* G *with such property.*

Lemma 12. *A sequence of $n-1$ integers in the interval $[0, n-1]$, assuming two distinct values, has a nonempty subsequence with sum $\equiv 0$ (mod n).*

Proof. See [1]. □

Theorem 13 (Savchev-Chen Structure Theorem [11])**.** *Let $S \in \mathcal{F}(C_n)$ be a zero-sum free sequence of length $|S| = l > \frac{n}{2}$. Then there exists a sequence $a_1 \cdot \ldots \cdot a_l \in \mathcal{F}(\mathbb{Z})$, with $a_i \in [1, n-1]$, such that*

(i) $S = (a_1 g) \cdot \ldots \cdot (a_l g)$ *for some* $g \in \mathrm{supp}(S)$, *and*

(ii) $\sum(S) = \{g, 2g, \dots, (a_1 + \ldots + a_l)g\}$.

Lemma 14. *A sequence of $n-2$ integers in the interval $[0, n-1]$, assuming more than two distinct values, has a nonempty subsequence with sum $\equiv 0$ (mod n). Furthermore, if S is a zero-sum free sequence with length $n-2$, then $S = x^{n-2}$ or $S = x^{n-3} \cdot 2x$ with $x \not\equiv 0 \pmod{n}$ and $n \geqslant 4$.*

Proof. See [1]. □

Let G be a group and $D(G)$ be the Davenport's constant of G, i.e., the smallest integer d such that every sequence S over G with $|S| \geqslant d$ satisfies $0 \in \sum(S)$. In the following, we give some properties of $D(G)$.

Lemma 15 ([6], Proposition 5.1.4 and Theorem 5.8.3)**.**

(i) Let G be a finite Abelian group and S a zero-sum free sequence over G. If $|S| = D(G) - 1$, then $\sum(S) = G \setminus \{0\}$.

(ii) Let $G = C_{n_1} \oplus C_{n_2}$ with $1 \leqslant n_1 \leqslant n_2$. Then $D(G) = n_1 + n_2 - 1$.

Lemma 16 ([9], Lemma 2.2, Theorem 2.4 and Theorem 5.1)**.**

(i) Let G be a noncyclic finite Abelian group. Then $D(G) \leqslant \frac{|G|}{2} + 1$, and equality holds when $G \cong C_2 \oplus C_{2m}$.

(ii) Let G be a finite Abelian group of rank greater than 2. Then $D(G) \leqslant \frac{|G|}{4} + 2$.

(iii) $D((\mathbb{Z}/p\mathbb{Z})^r) = r(p-1) + 1$ *for prime p and $r \geqslant 1$.*

Next we discuss the shortest length of zero-sum subsequence of a sequence S with length $n-1$ over a finite Abelian group G with $|G|=n$, and the result will be used throughout the proof of our main results.

Before stating the main result explicitly, we first introduce the important concept of setpartitions, which will be used throughout this thesis. Let S be a sequence. A *n-setpartition* of S is a partition of sequence S into n nonempty subsequences $A_1,\ldots,A_n$, such that terms in each subsequence A_i are all distinct, allowing the A_i to be regarded as sets.

Lemma 17. *Let G be an Abelian group of order $|G|=n$ and let S be a sequence of length $n-1$. If $0\notin\sum_{\leqslant h(S)+1}(S)$, then S is one of the following:*

(i) $G\cong C_n, S=g^{n-1}$ *and* $ord(g)=n$.

(ii) $G=C_2^r, \mathrm{supp}(S)=G\backslash\{0\}$ *and* $h(S)=1$.

Proof. Obviously, $0\notin \mathrm{supp}(S)$. First we consider the case $h(S)=n-1$. Then $S=g^{n-1}$, $g\neq 0$ and $g\in G$. If $\mathrm{ord}(g)\neq n$, then $\mathrm{ord}(g)\leqslant h(S)$ and $\mathrm{ord}(g)g=0$, which is a contradiction. Hence we have $\mathrm{ord}(g)=n$, $G\cong C_n$ and $S=g^{n-1}$.

For the case $h(S)=1$, we have $\mathrm{supp}(S)=G\setminus\{0\}$. If there exists some $g\in G$ such that $2g\neq 0$, then $g\neq 0$, $(-g)g|S$ and $-g+g=0$, which is a contradiction. Therefore we have $2g=0$ for all $g\in G$, and thus $G\cong C_2^r$, $r\in\mathbb{N}$.

Suppose that $2\leqslant h(S)\leqslant n-2$. We choose $b\in\mathrm{supp}(S)$ with $\mathsf{v}_b(S)=h(S)$. Since $h(S)\geqslant 2$, we have $\mathrm{ord}(b)\geqslant 3$. By Bialostocki's result in [1], the existence of a $(h+1)-$setpartition is straightforward, where $h=h(S)$. For any $(h+1)-$setpartition $\mathcal{A}=\mathcal{A}_0\mathcal{A}_1\cdots\mathcal{A}_{h+1}$ of S, we set $\mathcal{B}_i=\mathcal{A}_i\cup\{0\}$ for $i\in[0,h]$. It is clear that 0 is a unique expression element in $\sum_{i=0}^h\mathcal{B}_i$ in view of S being zero-sum free. If there exist two distinct integers $j,k\in[0,h]$ such that $|\mathcal{B}_j+\mathcal{B}_k|\geqslant|\mathcal{B}_j|+|\mathcal{B}_k|$, then by the addition theorem of Kemperman-Scherk, it follows that

$$\begin{aligned}
|\sum(\mathcal{B}_0\cdots\mathcal{B}_h)| &= |\mathcal{B}_j+\mathcal{B}_k+\sum_{i\neq j,k}\mathcal{B}_i| \\
&\geqslant |\mathcal{B}_j+\mathcal{B}_k|+|\sum_{i\neq j,k}\mathcal{B}_i|-1 \\
&\geqslant |\mathcal{B}_j|+|\mathcal{B}_k|+\sum_{i\neq j,k}|\mathcal{B}_i|-(h-2) \\
&\geqslant \sum_{i=0}^{h}|\mathcal{B}_i|-h+1 \\
&\geqslant n+1.
\end{aligned}$$

By the assumptions, we have a contradiction. By the addition theorem of Kemperman-Scherk, it follows that $|\mathcal{B}_j+\mathcal{B}_k|<|\mathcal{B}_j|+|\mathcal{B}_k|-1$ cannot hold. Therefore we have $|\mathcal{B}_j+\mathcal{B}_k|=|\mathcal{B}_j|+|\mathcal{B}_k|-1$ for any $j,k\in[0,h], j\neq k$ and $|\sum(\mathcal{B}_0\cdots\mathcal{B}_h)|=n$.

Now we choose a special setpartition $\mathcal{A} = \mathcal{A}_0\mathcal{A}_1 \cdots \mathcal{A}_h$ such that $\mathcal{A}_0 = \{b\}$ and $b \notin \mathcal{A}_h$. By Bialostocki's result in [1], the special $(h+1)-$setpartition exists. If there exists an integer $k \in [1, h]$ such that $\langle b \rangle \subseteq \mathcal{B}_k$, then we have $0 \in b + \mathcal{A}_k \subseteq \sum_{\leqslant h(S)+1}(S)$, which is impossible. Let $k \in [1, h]$ be arbitrary. Since $|\{0, b\} + \mathcal{B}_k| = |\mathcal{B}_k| + 2 - 1 = |\mathcal{A}_k| + 2$ and $\langle b \rangle \nsubseteq B_k$, Lemma 8 implies that $\mathcal{B}_k$ is the union of $\{0, b, 2b, \ldots, u_k b\}$ and some $\langle b \rangle$-cosets, where $0 \leqslant u_k \leqslant \mathrm{ord}(b) - 2$. Moreover, since $b \notin \mathcal{B}_h$, we have $u_h = 0$.

For any $g \in \mathcal{B}_k \setminus \{0, b\}$, $k \in [1, h]$, if $g \notin \langle b \rangle$ and $g \notin \mathcal{B}_t$ for some $t \in [1, h]\setminus\{k\}$, then removing g from $\mathcal{B}_k$ and appending g to $\mathcal{B}_t$, it yields a new setpartition $\mathcal{B}' = \mathcal{B}_0'\mathcal{B}_1' \cdots \mathcal{B}_h'$. By the structure of $\mathcal{B}_k$, it is easy to see that $\mathcal{B}_k'$ is not the union of $\{0, b, 2b, \ldots, u_k b\}$ and some $\langle b \rangle$- cosets, which is impossible. If $g \in \langle b \rangle$, then $g \notin \mathcal{B}_h$. Removing g from $\mathcal{B}_k$ and appending g to $\mathcal{B}_h$, it leads to a contradiction as above ($\mathcal{B}_h \cup \{g\}$ is not of that form). Therefore we obtain that $\mathcal{B}_0 = \{0, b\}, \mathcal{B}_h$ is the union of $\{0\}$ and some $\langle b \rangle$-cosets and $\mathcal{B}_i = \mathcal{B}_h \cup \{b\}$ for $i \in [1, h-1]$.

If there exists an element $g \in \mathcal{B}_h \setminus \{0\}$ such that $g + \mathcal{B}_1 \nsubseteq \mathcal{B}_1 + \langle b \rangle$, then, since $g \in \mathcal{B}_1 \setminus \langle b \rangle$ by the description of the $\mathcal{B}_i$ given above, it follows that there exists some $b_1 \in \mathcal{B}_1 \setminus \langle b \rangle$ such that $g + b_1 \notin \mathcal{B}_1 + \langle b \rangle$. Since the description of $\mathcal{B}_1$ ensures that $b_1 + \langle b \rangle \subseteq \mathcal{B}_1$, we have that the entire coset $b_1 + \langle b \rangle$ is disjoint from $\mathcal{B}_1 + \langle b \rangle$. Thus removing g from $\mathcal{B}_h$ and appending g to $\mathcal{B}_0$, it yields a new setpartition $\mathcal{B}' = \mathcal{B}_0'\mathcal{B}_1' \cdots \mathcal{B}_h'$ such that $(\mathcal{B}_1' + \mathcal{B}_0') \setminus (\mathcal{B}_1 + \langle b \rangle)$ contains some $\langle b \rangle$-cosets. Since $\mathrm{ord}(b) \geqslant 3$ and $h \geqslant 2$, it follows that $|\mathcal{B}_1' + \mathcal{B}_0'| \geqslant |\mathcal{B}_1'| + |\langle b \rangle| \geqslant |\mathcal{B}_1'| + 3 = |\mathcal{B}_1'| + |\mathcal{B}_0'|$, which is a contradiction.

If every $g \in \mathcal{B}_h \setminus \{0\}$ satisfies $g + \mathcal{B}_1 \subseteq \mathcal{B}_1 + \langle b \rangle$, we have $\mathcal{B}_1 + \mathcal{B}_h = \mathcal{B}_1 + \langle b \rangle$, which implies $|\mathcal{B}_h| \geqslant |H| + 1$, where $H = \langle b \rangle$. Recall that $|\mathcal{B}_1 + \mathcal{B}_h| = |\mathcal{B}_1| + |\mathcal{B}_h| - 1 \geqslant |\mathcal{B}_1| + |H|$, where $H = \langle b \rangle$; the latter inequality follows in view of the description of $\mathcal{B}_h$. However, $|\mathcal{B}_1 + \mathcal{B}_h| \geqslant |\mathcal{B}_1| + |H|$ is not possible in view of $\mathcal{B}_1 + \mathcal{B}_h = \mathcal{B}_1 + \langle b \rangle = \mathcal{B}_1 + H$ and the description of $\mathcal{B}_1$ given above, which makes a contradiction. This completes the proof. $\square$

3 The proof of the main results

In order to prove our main results, we need to state a property.

Property. *Let G be a finite Abelian group of order n and S be a sequence of length $n-1$ over G. Let T be a nonempty zero-sum subsequence of S. Setting $U = ST^{-1}$. If any nonempty zero-sum subsequence of S has length $r \in [1, n-1]$, then the following statements hold:*

(i) $|T| = r \leqslant D(G)$ and $|U| = n - 1 - r \leqslant D(G) - 1$, where $D(G)$ is Davenport's constant.

(ii) T is a minimal zero-sum subsequence, and U is zero-sum free.

(iii) For every $x|T$, $x \notin \sum_{\geqslant 2}(U)$.

Next we give the proof of Theorem 4.

Proof of Theorem 4. First we observe that if the rank of G is greater than 2, then, by Lemma 16(ii), we have $D(G) \leqslant \frac{|G|}{4} + 2$. If $|G| \geqslant 9$, then we have $2D(G) \leqslant |G| - 1$, and so there exist two disjoint nonempty subsequences S_1 and S_2 of S such that $\sigma(S_1) = \sigma(S_2) = 0$, which is impossible. If $|G| \leqslant 8$, then $G \cong C_2 \oplus C_2 \oplus C_2, r = D(G) = 4$ and $S = \prod_{g \in G \setminus \{0\}} g$, which is also impossible.

Now we assume that the rank of G is 2 and let $G \cong C_k \oplus C_{km}$ with $k \geqslant 2$ and $m \geqslant 1$. Then by Lemma 15(ii), we have $D(G) = km + k - 1$. By Property(i), we obtain $2D(G) - 1 \geqslant |G| - 1$. It follows that we have $k = 2, G \cong C_2 \oplus C_{2m}$ and $D(G) = 2m + 1$, or $G \cong C_3^2$ and $D(G) = 5$.

Suppose that $G \cong C_3^2$ and $D(G) = 5$. By Property, we have $4 \leqslant r \leqslant 5$. By Lemma 17, we have $h(S) \geqslant r - 1 \geqslant 3$. Let $g|S$ with $\mathsf{v}_g(S) = h(S)$. Then $\sigma(g^3) = 0$, which contradicts that $r \geqslant 4$. Therefore we have $G \cong C_2 \oplus C_{2m}$, in which case $D(G) = 2m + 1$ by Lemma 15.

If $0 \in \text{supp}(S)$, then $S0^{-1}$ is a zero-free and $4m - 2 = |G| - 2 = |S| - 1 \leqslant D(G) - 1 = 2m$, and so $m = 1$. It follows that $G \cong C_2^2$ and $S = 0g_1g_2$, where $g_1, g_2 \in G \setminus \{0\}$ are distinct. In the following argument, we consider $0 \notin \text{supp}(S)$, and so $r \geqslant 2$. We shall show $2m - 1 \leqslant r \leqslant 2m + 1$. Let T be a zero-sum subsequence of S. If $r \geqslant 2m + 2$, then $|T| = r > D(G)$, and there exists a zero-sum subsequence T_1 of T with length $< r$, which is a contradiction. If $r \leqslant 2m - 2$, then $n - r \geqslant 2m + 1 = D(G)$, and so there exists a zero-sum subsequence of ST^{-1}, which is also a contradiction. Next we divide into three cases to discuss the result.

Case 1: $r = 2m + 1$. If $0 \notin \sum_{\leqslant h(S)+1}(S)$, then by Lemma 17, we have $r = 3, m = 1$ and $S = \prod_{g \in G \setminus \{0\}} g$. Next we consider $0 \in \sum_{\leqslant h(S)+1}(S)$, then $h(S) \geqslant r - 1 = 2m$. Take $a|S$ satisfying $\mathsf{v}_a(S) = h(S) \geqslant r - 1 = 2m$. Then $a^{\text{ord}(a)}|S$ shows that $r = \text{ord}(a) \leqslant 2m$, contrary to case hypothesis.

Case 2: $r = 2m - 1$. Let $T|S$ be a zero-sum subsequence of length r. Let $U = ST^{-1}$. Then $|U| = 2m = D(G) - 1$, and so, by Lemma 15(i), $\sum(U) = G \setminus \{0\}$ and $|\sum(U)| = 2|U| - 1$. By Property(iii), for any $a \in \text{supp}(T)$, we have $a \in \text{supp}(U)$, i.e.,

$$\text{supp}(T) \subseteq \text{supp}(U).$$

By Theorem 10, we distinguish four subcases.

(i) $U = a^x b$, where $x = 2m - 1, \text{ord}(a) = 2m$ and $b \notin \langle a \rangle$. If $a \notin \text{supp}(T)$, then $T = b^{2m-1}$, which is not possible since T is zero-sum free. If $a \in \text{supp}(T)$, then $a^{2m}|S$, and $\sigma(a^{2m}) = 0$, which contradicts $r = 2m - 1 \neq 2m$.

(ii) $U = a^x(a + g)^y$, where $x + y = 2m, \text{ord}(a) = 2m, \text{ord}(g) = 2$ and y is odd. Then $T = a^s(a + g)^t$ with $s + t = 2m - 1$. However, $\sigma(T) \neq 0$, which contradicts $\sigma(T) = 0$.

(iii) $U = a^x(a+g)^y g$, where $x + y = 2m - 1, \text{ord}(a) = 2m$, and $\text{ord}(g) = 2$. Obviously, $g \notin \text{supp}(T)$. Then $T = a^s(a+g)^t$ with $s+t = 2m-1$, and so $\sigma(T) \neq 0$, which contradicts $\sigma(T) = 0$.

(iv) $U = a^x bc$, where $x = 2m - 2, \text{ord}(a) = x + 2 = 2m$, and $b, c \in G \setminus \langle a \rangle$ are distinct with $b + c = a$ and $b - c \in \langle a \rangle$. Obviously, $\langle a \rangle \cap \text{supp}(T) = \varnothing$. Otherwise, there exists $sa \in \text{supp}(S)$, with $s \in [1, 2m - 2]$, such that either $a^{2m-s} \cdot sa$ is a zero-sum subsequence with length $2m - s + 1 < r$, where $s \geqslant 3$, or $a^{2m-s-1} \cdot sa \cdot b \cdot c$ is a zero-sum subsequence with length $2m - s + 2 > r$, where $s = 1, 2$. By Property(iii) and $\sum(U) = G \setminus \{0\}$, we have $T=b^s$ or $T=c^s$ with $s=2m-1$. If $\sigma(T)=0$, we have $(2m-1)u=4m$. That is to say, $u=4, m=1$, and $r=1$, which contradicts $\sigma(T)=0$ since $\gcd(2m-1,2m)=1$.

Case 3: $r=2m$. By Lemma 17, we have $h(S)\geqslant r-1=2m-1$. We set $g\in \mathrm{supp}(S)$ with $h(S)=\mathsf{v}_g(S)$. Let $T'=Sg^{-h(S)}$. Then $\mathrm{ord}(g)=2m$ and $\mathrm{supp}(T')\cap\langle g\rangle=\varnothing$. In particular, for any $a,b\in\mathrm{supp}(T)$, we must have $2a, 2b, a+b\in\langle g\rangle$.

If $|T'|=0$, then $S=g^{4m-1}$.

If $|T'|=1$, then $S=g^{4m-2}\cdot g'$ with $g'\notin\langle g\rangle$.

If $|T'|=2$, then $T'=h_1h_2$ with $h_1,h_2\notin\langle g\rangle$, and $h_1+h_2=lg$, where $l\in[1,2m-2]$. But then $g^{2m-l}\cdot h_1\cdot h_2$ is a zero-sum subsequence, which force $h_1+h_2=2g$.

If $|T'|\geqslant 3$, choose any subsequence $h_1h_2h_3|T'$. Then as shown in the previous case, $h_1+h_2=h_1+h_3=h_2+h_3=2g$, which implies $h_1=h_2=h_3$, and so $|\mathrm{supp}(T')|=1$. In addition, $2h_1=2g$ implies $h_1=g+h$ with $\mathrm{ord}(h)=2$. Therefore, by $h(S)=\mathsf{v}_g(S)$, we have $S=g^{2m+x}(g+h)^{2m-1-x}$, where $x\in[0,2m-4]$. This completes the proof. □

In what follows, we prove the theorem 5.

Proof of Theorem 5. First we consider $r=1$, then, by Property (ii), $0\in\mathrm{supp}(S)$, and $S0^{-1}$ is zero-sum free. By Lemma 14, we have $S=0\cdot x^{n-2}$ or $S=0\cdot x^{n-3}\cdot 2x$ with $ord(x)=n$.

If $|\mathrm{supp}(S)|=1$, then $S=a^{n-1}$, where $ord(a)\geqslant\frac{n}{2}$.

In the following argument, we assume that $n\geqslant 4, |\mathrm{supp}(S)|\geqslant 2$ and $2\leqslant r\leqslant n-1$. By Lemma 17, we obtain $h(S)\geqslant r-1$. Next we divide into six cases to prove the result.

Suppose that $r\geqslant\frac{n+1}{2}$ and choose $a|S$ satisfying $\mathsf{v}_a(S)=h(S)\geqslant r-1\geqslant\frac{n-1}{2}$, it follows that $\mathrm{ord}(a)>\mathsf{v}_a(S)=h(S)\geqslant\frac{n-1}{2}$. Therefore, $\mathrm{ord}(a)=n$, and without loss of generality, we set $a=1$. Let T be a zero-sum free subsequence of S with $|\sum(T)|\geqslant 2|T|-1$ and $|T|$ maximum. Then we have $1\leqslant|T|\leqslant\lfloor\frac{|\sum(T)|+1}{2}\rfloor\leqslant\lfloor\frac{n}{2}\rfloor\leqslant r-1$. Setting $U=ST^{-1}$, then we have the following result.

Claim 1. Suppose that $r\geqslant\frac{n+1}{2}$. Let all notation be as above, then the following statements hold:

(i) If $|T|\leqslant r-2$, then $U=1^{n-1-|T|}$ and $\mathrm{supp}(T)\subseteq\{1,n-r,n-r+1\}$.

(ii) If $|T|=r-1$ and $|\sum(T)|=n-1$, then $U=1^{n-1-|T|}$ and $\mathrm{supp}(T)\subseteq\{1,n-r,n-r+1\}$.

Proof of Claim 1: (i) Since $h(S)\geqslant r-1>|T|$, we have $1|U$. Since for any $a|U$, Ta is zero-sum free and a is a $1-term$ for Ta because of the maximality of T, it follows in view of Lemma 11 that $U=a^{n-1-|T|}$, and so $U=1^{n-1-|T|}$ because of $1\in\mathrm{supp}(U)$. But now Lemma 11 further implies that $\sum(T)=\{1,2,\cdots,k\}$ and $U=1^{n-1-|T|}$ with $k\geqslant 2|T|-1$. If $|T|=1$, then $|\mathrm{supp}(S)|=1$, contrary to what we assumed above. So $|T|\geqslant 2$. Therefore, there exists a subsequence T_1 such that $\sigma(T_1)=0$ and $T|T_1$. By Property(i), we have $|T_1|=r$ and $|ST_1^{-1}|=n-1-r$. Let $x|T_1$ with $\bar{x}\neq 1$. If $\bar{x}\geqslant n-r+2$, we can obtain a zero-sum subsequence $x\cdot 1^{n-\bar{x}}$ of length less than r. If $\bar{x}\leqslant n-r-1$, then $\bar{x}=\sigma(1^{\bar{x}})$, and so $T_1\cdot x^{-1}\cdot 1^{\bar{x}}$ is a zero-sum subsequence with length $r-1+\bar{x}$, which forces $\bar{x}=1$. Hence for any $x|T$, either $n-r\leqslant\bar{x}\leqslant n-r+1$ or $\bar{x}=1$.

(ii) For $|T| = r-1$ and $|\sum(T)| = n-1$, we have $r \leqslant \frac{n}{2}+1$ and for every $a|U$, Ta is a zero-sum subsequence. Thus $U = a^{n-1-|T|}$. If $\bar{a} \neq 1$, then $T = 1^{r-1}$ and $|\sum(T)| = n-1 = r-1$ implies $r = n$, which is a contradiction. It follows that $U = 1^{n-1-|T|}$.

For any $x|T$, by the similar argument as above, we have either $n-r \leqslant \bar{x} \leqslant n-r+1$ or $\bar{x} = 1$. This proves Claim 1.

Case 1. $r \geqslant \frac{n}{2}+1$.

Choose $a|S$ satisfying $\mathsf{v}_a(S) = h(S) \geqslant r-1 \geqslant \frac{n}{2}$. Since $r \geqslant \frac{n}{2}+1$, it follows that $ord(a) > \mathsf{v}_a(S) = h(S) \geqslant \frac{n}{2}$. Therefore, $ord(a) = n$, and without loss of generality, we set $a = 1$. Let T be a zero-sum free subsequence of S with $|\Sigma(T)| \geqslant 2|T|-1$ and $|T|$ maximum. Then we have $1 \leqslant |T| \leqslant \frac{|\sum(T)|+1}{2} \leqslant \frac{n}{2} \leqslant r-1$.

Subcase 1.1. $|T| \leqslant r-2$.

By Claim 1(i), we have $U = 1^{n-1-|T|}$ and $\mathrm{supp}(T) \subseteq \{1, n-r, n-r+1\}$. There are three options need to be considered.

(i) If $n-r = 1$, then $S = 1^{n-2} \cdot 2$.

(ii) If $r = \frac{n}{2}+1$, that is, $n-r+1 = \frac{n}{2}$, then $(n-r)+(n-r+1) = n-1$. If $(n-r)\cdot(n-r+1)|S$, then $r = 3$ and $n = 4$, whence $S = 1^2 \cdot 2$. So we can assume $\mathrm{supp}(S) \subseteq \{1, n-r\}$ or $\mathrm{supp}(S) \subseteq \{1, n-r+1\}$. In the second case, let $S = 1^s \cdot (n-r+1)^t = 1^s \cdot (\frac{n}{2})^t$. If $t \geqslant 2$, then $n = 2$, which makes a contradiction. Therefore, $S = 1^{n-2} \cdot \frac{n}{2}$. In the first case, let $S = 1^s \cdot (n-r)^t = 1^s \cdot (\frac{n}{2}-1)^t$ with $s \geqslant r-1 \geqslant \frac{n}{2}$. We can assume that $t \geqslant 2$, else $S = 1^{n-2} \cdot g$, as desired. But $\mathsf{v}_1(S) \geqslant r-1 \geqslant \frac{n}{2} \geqslant 2$, so the zero-sum subsequence $1^2 \cdot (\frac{n}{2}-1)^2$ shows that $r = 4$, in which case $n = 6$, and now we must have $S \cong 1^3 \cdot 2^2$(else 2^3 would contradict that $r = 4$).

(iii) If $3 \leqslant n-r+1 < \frac{n}{2}$, then

$$n-r+2 \leqslant 2(n-r) < (n-r)+(n-r+1) < 2(n-r+1) < n,$$

and there exists an integer $s < r-2$ such that $(n-r)+(n-r+1)+s*1 = n$. If $(n-r)\cdot(n-r+1)|S$, then $1^s \cdot (n-r) \cdot (n-r+1)$ is a zero-sum subsequence with length $s+2 < r$, which is a contradiction. So we can assume $\mathrm{supp}(S) = \{1, x\}$, where $\bar{x} = n-r$ or $\bar{x} = n-r+1$.

Suppose that $S = 1^u \cdot (n-r)^v$. It is easy to show that $u \leqslant r-1$, and so $u = r-1$ and $v = n-r \geqslant 2$. If $v = n-r = 2$, then $S = 1^{n-3} \cdot 2^2$. If $v = n-r \geqslant 3$, then $0 < n-2(n-r) < r-2$, that is, $n-2(n-r) < r-2$. Therefore there exists a positive integer $t \leqslant r-3$ such that $(n-r)+(n-r)+t*1 = n$, which is a contradiction.

Suppose $S = 1^u \cdot (n-r+1)^v$, then $u \geqslant r-1$. Since $n-r+2 < 2(n-r+1) < n$, it follows that there exists a positive integer $t \leqslant r-3$ such that $2(n-r+1)+t*1 = n$. Therefore, $v = 1$ and $S = 1^{n-2} \cdot (n-r+1)$.

Subcase 1.2 : $|T| = r-1 \geqslant \frac{n}{2}$.

Since $|\Sigma(T)| \geqslant 2|T|-1$, it follows that $|\Sigma(T)| = n-1, |T| = \frac{n}{2}$ and $r = \frac{n}{2}+1$, and so, by Claim 1(ii), we have $U = 1^{\frac{n}{2}-1}$ and $\mathrm{supp}(T) \subseteq \{1, \frac{n}{2}-1, \frac{n}{2}\}$. There are two options need to be considered.

(i) If $\frac{n}{2}|T$, then we have $\mathsf{v}_{\frac{n}{2}}(S)=1$. If $\frac{n}{2}-1|T$, then $\frac{n}{2}+(\frac{n}{2}-1)+1=n$ implies that $r=3$ and $n=4$, and so $S=1^2\cdot 2$. Therefore, $S=1^{n-2}\cdot\frac{n}{2}$ for $n\geqslant 4$.

(ii) If $\frac{n}{2}-1|T$, then we must have $\mathsf{v}_1(S)\leqslant\frac{n}{2}$, and $\frac{n}{2}\nmid S$ in view of the previous paragraph. Let $S=1^s\cdot(\frac{n}{2}-1)^t$ with $s\geqslant t$ for $\mathsf{v}_1(S)=h(S)$. Since $\frac{n}{2}\geqslant s\geqslant r-1=\frac{n}{2}$, it follows that $s=\frac{n}{2}$ and $t=\frac{n}{2}-1$. If $\frac{n}{2}-1\geqslant 4$, then $\sigma((\frac{n}{2}-1)^2\cdot 1^2)=0$ and $\sigma((\frac{n}{2}-1)^4\cdot 1^4)=0$, which is a contradiction. If $\frac{n}{2}-1=3$, then $n=8, r=5$ and $S=1^4\cdot 3^3$, but the subsequence $1^2\cdot 3^2$ shows $r=4$, which is a contradiction. If $\frac{n}{2}-1=2$, then we have $n=6$ and $S=1^3\cdot 2^2$. If $\frac{n}{2}-1=1$, then we have $n=4$ and $S=1^3$, which contradicts our assumption $|\mathrm{supp}(S)|\geqslant 2$.

Therefore, by the above description, S is one of the following in this case:

(i) $S\cong 1^{n-2}\cdot(n-r+1)$ with $n\geqslant 3$.

(ii) $S\cong 1^{n-3}\cdot(n-r)^2$ with $r=n-2$.

Case 2. $\boldsymbol{r<\frac{n}{2}-1}$.

Let $T|S$ be a nontrivial zero-sum subsequence. Setting $U=ST^{-1}$, by Property(i) and (ii), we have $|T|=r$ and U is zero-sum free. It follows that

$$|U|=n-1-r>\frac{n}{2}.$$

Applying the Savchev-Chen Structure Theorem, we can suppose that $U=1^v x_1\cdot\ldots\cdot x_t$ with $v+t=n-1-r\geqslant\frac{n+1}{2}$ and $2\leqslant\overline{x_1}\leqslant\ldots\leqslant\overline{x_t}\leqslant v+\sum_{i=1}^t\overline{x_i}\leqslant n-1$ and $\sum(U)=\{1,2,\ldots,v+\sum_{i=1}^t\overline{x_i}\}$. Consequently, since $r-2\leqslant n-1-r\leqslant v+t\leqslant v+\sum_{i=1}^t\overline{x_i}$, it follows that $\{1,2,\cdots,r-2\}\subseteq\sum_{\leqslant r-2}(U)$, which implies that $\bar{x}\leqslant n-(r-2)-1=n-r+1$ for every $x|T$. Therefore, for any $x|T$, we have either $\bar{x}=1$ or

$$n-r+1\geqslant\bar{x}\geqslant 1+v+\sum_{i=1}^{t}\overline{x_i}\geqslant 1+\sigma(\bar{U})\geqslant 1+|U|=n-r.$$

Moreover, since the zero-sum subsequence T cannot be all 1's, there exists $x|T$ with $\bar{x}\neq 1$, which means that the above estimate must hold for some $x|T$. In consequence, either $U=1^{n-1-r}$ or $U=1^{n-r-2}\cdot 2$(as otherwise the above estimate can be improved to shows no much $x|T$ exists). Moreover, $n-r\nmid T$. Otherwise, $1^r\cdot(n-r)$ gives a zero-sum subsequence of length $r+1$ in view of $r\leqslant n-r-2$. Thus $T=1^t\cdot(n-r+1)^s$.

Suppose that $U=1^{n-1-r}$. Since $r\geqslant 2$, we have $n-2r+2<n$. If $n-r\leqslant n-2r+2$, then $r=2$, and so $T=1\cdot(n-1)$ and $U=1^{n-3}$, that is, $S=1^{n-2}\cdot(n-1)$. If $n-2r+2\leqslant n-r-1$ and $s\geqslant 2$, then

$$T\cdot(n-r+1)^{-2}\cdot 1^{(n-2r+2)}$$

is a zero-sum subsequence, which forces $n-2r+2=2$ and $r=\frac{n}{2}$. It arises a contradiction. Therefore, $T=(n-r+1)\cdot 1^{r-1}$ and $S=1^{n-2}\cdot(n-r+1)$.

Suppose that $U=1^{n-r-2}\cdot 2$. If $n-r+1\leqslant n-2$, then $r\geqslant 3$. However $\sigma((n-r+1)\cdot 1^{r-1})=0$ and $\sigma((n-r+1)\cdot 2\cdot 1^{r-3})=0$, which is a contradiction. If $n-r+1>n-2$, then $r=2$, and so $S=1^{n-3}\cdot 2\cdot(n-1)$.

Therefore, in this case S is one of the following:

(i) $S \cong 1^{n-2} \cdot (n-r+1)$ with $n \geqslant 3$.

(ii) $S \cong 1^{n-3} \cdot 2 \cdot (n-1) = 1^{n-3} \cdot 2 \cdot (n-r+1)$ with $r = 2$.

Case 3. $\boldsymbol{r = \frac{n}{2} - 1}$.

Let T be a nontrivial zero-sum subsequence of S, and set $U = ST^{-1}$. By Property(i) and (ii), we have $|T| = \frac{n}{2} - 1, |U| = \frac{n}{2}$ and U is zero-sum free. Then

$$\frac{n}{2} = |U| \leqslant |\sum(U)| \leqslant n - 1 = 2|U| - 1.$$

Since $r \geqslant 2$, we have $n \geqslant 6$. By Theorem 10, we divide the proof into five subcases.

Subcase 3.1. U is a-smooth for some $a \in G$. Since $|U| = \frac{n}{2}$, without loss of generality set, we may set $a = 1$. In the following, we prove that either $S = 1^{n-2} \cdot (\frac{n}{2} + 2)$ with $n \geqslant 6$ or $S = 1^2 \cdot 3^3$ or $S = 1^3 \cdot 2 \cdot 5$ with $n = 6$.

Let $U = 1^v x_1 \cdot \ldots \cdot x_t, 2 \leqslant \overline{x_1} \leqslant \cdots \leqslant \overline{x_t}$ be 1-smooth with $v + \sum_{i=1}^{t} \overline{x_i} \leqslant n - 1$ and $v \geqslant 1$. Since

$$\begin{aligned} n-1 &\geqslant v + \overline{x_1} + \cdots + \overline{x_t} \\ &\geqslant v + 2(t-1) + \overline{x_t} \\ &\geqslant v + 2t - 2 + \overline{x_t} \\ &\geqslant 2(v+t) - v - 2 + \overline{x_t} \\ &= n - v - 2 + \overline{x_t}, \end{aligned}$$

we have $\overline{x_t} \leqslant v + 1$. Let $\mathsf{v}_{x_t}(U) = b$.

(i) If $\overline{x_t} = v + 1$, then equality holds in (1), which implies that either $b = 1$ or $v = 1$. If $v = 1$, then $U = 1 \cdot 2^{\frac{n}{2}-1}$ and $|\sum(U)| = n - 1$. Thus for any $x|T$, we have $x \in \sum(U)$, and so, by Property(iii), we have either $\bar{x} = 1$ or $\bar{x} = 2$. This contradicts that T is a minimal zero-sum subsequence with length $r = \frac{n}{2} - 1$. If $b = 1$ with $v \geqslant 2$, then we have $U = 1^v \cdot 2^{\frac{n}{2}-v-1} \cdot (v+1)$ and $|\sum(U)| = n - 1$ with $v \geqslant 2$. Similarly, for any $x|T$, we have $x \in \sum(U)$, and so $\bar{x} = 1$ or $\bar{x} = 2$ or $\bar{x} = v + 1$. Since $v \geqslant 2$, we have $2 \nmid T$, so $T = 1^s \cdot (v+1)^t$ with $s + t = r$. Because the zero-sum subsequence T cannot be all 1's, there exists $x|T$ such that $\bar{x} = v+1$. If $2|U$, then by $v+1 = (v-1)+2, 1^s \cdot 2 \cdot (v+1)^{t-1} \cdot U'$ is a zero-sum subsequence with length $> r$, where U' is a subsequence of $U \cdot 2^{-1}$ satisfying $\sigma(U') = v - 1$ and $|U'| \geqslant 1$. So we can assume $U = 1^v \cdot (v+1) = 1^{\frac{n}{2}-1} \cdot \frac{n}{2}$ and $T = 1^s \cdot (v+1)^t = 1^s \cdot (\frac{n}{2})^t$. Therefore, $r = 2$, and so $n = 6$ and $S = 1^2 \cdot 3^3$.

(ii) If $\overline{x_t} \leqslant v$, then $\sum(U) = \{1, 2, \cdots, v + \sum_{i=1}^{t} \overline{x_i}\}$ and

$$\frac{n}{2} = |U| \leqslant |\sum(U)| = v + \sum_{i=1}^{t} \overline{x_i} \leqslant n - 1.$$

For any $x \in \sum(U)$ and $\bar{x} \neq 1$, if $2 \leqslant \bar{x} \leqslant v$, then there exists a subsequence $U' = 1^{\bar{x}}$ of U such that $\sigma(U') = \bar{x}$ and $|U'| = \bar{x} \geqslant 2$. If $\bar{x} \geqslant v + 1$, then there exists a subsequence U'' of U such that $\sigma(\overline{U''}) = \bar{x}$ and $|U''| \geqslant 2$. So by Property (iii), for any $x|T$, we have either $\bar{x} = 1$ or $\bar{x} \geqslant \sigma(\bar{U}) + 1 \geqslant \frac{n}{2} + 1$. If there is an element $x|T$ such that $\bar{x} > n - (r-1) = \frac{n}{2} + 2$, then there exists a subsequence U_2 of U such that $x \cdot U_2$ is a zero-sum subsequence of length less than r. Therefore, for any $x|T$, we have either $\bar{x}=1$ or $\frac{n}{2}+1 \leqslant \sigma(\bar{U})+1 \leqslant \bar{x} \leqslant \frac{n}{2}+2$, and so we have either $\sigma(\bar{U})=\frac{n}{2}$ or $\sigma(\bar{U})=\frac{n}{2}+1$. Since $|U|=\frac{n}{2}$ and U is 1-smooth, we have either $U=1^{\frac{n}{2}}$ or $U=1^{\frac{n}{2}-1}\cdot 2$.

Since $(\frac{n}{2}+1)\cdot 1^{\frac{n}{2}-1}$ is a zero-sum subsequence of length $r+1$, it follows that $\frac{n}{2}+1 \nmid T$. Thus $T=1^s\cdot(\frac{n}{2}+2)^t$ with $s+t=\frac{n}{2}-1$, and so $s+(\frac{n}{2}+2)t \equiv 0(\mathrm{mod} n)$. It follows that we have $t=1$ and $T=1^{\frac{n}{2}-2}\cdot(\frac{n}{2}+2)$.

If $U=1^{\frac{n}{2}}$, then $S=1^{n-2}\cdot(\frac{n}{2}+2)$. If $U=1^{\frac{n}{2}-1}\cdot 2$, then we must have $\frac{n}{2}-2<2$(otherwise, $1^{\frac{n}{2}-4}\cdot 2\cdot(\frac{n}{2}+2)$ is a zero-sum subsequence of S with length $\frac{n}{2}-2<r$), and so $n=6$ and $S=1^3\cdot 2\cdot 5$.

Subcase 3.2. $U=a^{\frac{n}{2}-1}\cdot b$ is not smooth with $\mathrm{ord}(a)\geqslant \frac{n}{2}$.

(i) $\mathrm{ord}(a)=\frac{n}{2}$. Without loss of generality, we set $a=2$. Then $U=2^{\frac{n}{2}-1}\cdot b$, where b is odd. By Property(iii), for any $x|T$, we have either $\bar{x}=2$ or $x=b$. Clearly, $2\nmid T$. Hence we have $x=b$ and $T=b^{\frac{n}{2}-1}$, and so $\sigma(b^{\frac{n}{2}-1})=0$. Therefore, $\mathrm{ord}(b)=\frac{n}{2}-1$ and $\frac{n}{2}-1 \mid n$, and so $n=6$ and $S=2^3\cdot 3^2$.

(ii) $\mathrm{ord}(a)=n$. Without loss of generality, we set $a=1$. Then $U=1^{\frac{n}{2}-1}\cdot b$ with $\bar{b}\geqslant \frac{n}{2}+1$ because U is not 1-smooth. However, $0\in\sum(U)$, which is a contradiction.

Subcase 3.3. $U=a^k\cdot(a+g)^l$ is not smooth with $|U|=\frac{n}{2}, \mathrm{ord}(g)=2$ and $k\geqslant l\geqslant 1$. If l is even, then since $k\geqslant 1$ and $\mathrm{ord}(g)=2$, we must have

$$\sum(U)=\{a,\cdots,(k+l)a,a+g,\cdots,(k+l-1)a+g\}.$$

If l is odd, then, similarly, we must have

$$\sum(U)=\{a,\cdots,(k+l-1)a,a+g,\cdots,(k+l)a+g\}.$$

Thus $|\sum(U)|=(k+l-1)+(k+l)=n-1$ and $\mathrm{ord}(a)\geqslant \frac{n}{3}$.

(i) $\mathrm{ord}(a)=\frac{n}{3}$. Since $r=\frac{n}{2}-1$ and $\{a,2a,\cdots,(k+l-1)a\}\subseteq\sum(U)$ with $k+l-1=\frac{n}{2}-1$, we have $\frac{n}{3}>\frac{n}{2}-1$, which contradicting that $n\geqslant 6$.

(ii) $\mathrm{ord}(a)=\frac{n}{2}$. Without loss of generality, we set $a=2$. Then $U=2^k\cdot(\frac{n}{2}+2)^l$ with l odd. For any $x|T$, by Property(iii), we have either $\bar{x}=2$ or $\bar{x}=\frac{n}{2}+2$. If $2|T$, then $2^k\cdot(\frac{n}{2}+2)^{l-1}\cdot 2$ is a zero-sum subsequence of length $\frac{n}{2}>r$, which is a contradiction. Therefore, $T=(\frac{n}{2}+2)^{\frac{n}{2}-1}$, and so $n\leqslant 4$, which is also a contradiction.

(iii) $ord(a)=n$. Without loss of generality, we set $a=1$. Then $U=1^k\cdot(\frac{n}{2}+1)^l$ with $k\geqslant l\geqslant 1$. Similarly, for any $x|T$, we have either $\bar{x}=1$ or $\bar{x}=\frac{n}{2}+1$. Thus $T=1^s(\frac{n}{2}+1)^t$ with $s+t=\frac{n}{2}-1$. However $\sigma(T)\neq 0$.

Subcase 3.4. $U=a^k\cdot(a+g)^l\cdot g$ is not smooth with $|U|=\frac{n}{2}, \mathrm{ord}(g)=2$ and $k\geqslant l\geqslant 1$. By a similar argument of subcase 3.3, we have $|\sum(U)|=n-1$ and $\mathrm{ord}(a)\geqslant\frac{n}{3}$.

(i) $\mathrm{ord}(a)=\frac{n}{3}$. Since $r=\frac{n}{2}-1$ and $\{a,2a,\cdots,(k+l)a\}\subseteq\sum(U)$ with $k+l=\frac{n}{2}-1$, we have $\frac{n}{3}>\frac{n}{2}-1$, which contradicting that $n\geqslant 6$.

(ii) $\mathrm{ord}(a)=\frac{n}{2}$. Without loss of generality, we set $a=2$ and $g=\frac{n}{2}$. Then $U=2^k\cdot(\frac{n}{2}+2)^l\cdot\frac{n}{2}$. By Property(iii), for any $x|T$, we have $\bar{x}=2,\frac{n}{2}$ or $\frac{n}{2}+2$. If $\frac{n}{2}|T$, then $r=2, n=6$, and $S=2\cdot 3^3\cdot 5$, which is equivalent to $S=1\cdot 3^3\cdot 4$. If $2|T$, then we have either $\sigma(2^k(\frac{n}{2}+2)^l 2)=0$ or $\sigma(2^k(\frac{n}{2}+2)^l\frac{n}{2}2)=0$. However these length $\geqslant\frac{n}{2}>r$, which is a contradiction. Therefore, $T=(\frac{n}{2}+2)^{\frac{n}{2}-1}$, and so $(\frac{n}{2}+2)^{\frac{n}{2}}$ or $(\frac{n}{2}+2)^{\frac{n}{2}}\cdot\frac{n}{2}$ is a zero-sum subsequence of length at least $\frac{n}{2}>r$, which is a contradiction.

(iii) $\mathrm{ord}(a)=n$. Without loss of generality, we set $a=1$. Then $U=1^k\cdot(\frac{n}{2}+1)^l\cdot\frac{n}{2}$. Similarly, for any $x|T$, we have $\bar{x}=1,\frac{n}{2}$ or $\frac{n}{2}+1$. If $\frac{n}{2}|T$, then we have $r=2, n=6$ and $S=1\cdot 3^3\cdot 4$. If $T=1^s\cdot(\frac{n}{2}+1)^l$, then $s+(\frac{n}{2}+1)l\neq 0 \pmod{n}$, which is a contradiction.

Subcase 3.5. $U=a^x\cdot b\cdot c$ is not smooth, where $\mathrm{ord}(a)=x+2=\frac{n}{2}$ and $b,c\in G\setminus\langle a\rangle$ are distinct with $b+c=a$ and $b-c\in\langle a\rangle$. Then $\sum(U)=\langle a,b,c\rangle\setminus\{0\}=G\setminus\{0\}$.

By Property(iii), $\langle a\rangle\cap\mathrm{supp}(T)=\varnothing$. If $bc|T$, then $T(bc)^{-1}a$ is a zero-sum subsequence with length $r-1$. So by Property(iii) and $\sum(U)=G\setminus\{0\}$, we have $T=b^s$ or $T=c^s$ with $s=\frac{n}{2}-1$. If $n>6$, we have $\sigma(T)\neq 0$, which contradicts $\sigma(T)=0$. If $n=6$, without loss of generality, we set $a=2$. Then $U=2\cdot 3\cdot 5$ and $T=3^2$. Hence, $S=2\cdot 3^3\cdot 5\cong 1\cdot 3^3\cdot 4$.

Therefore, in this case S is one of the following:

(i) $S\cong 1^{n-2}\cdot(n-r+1)$ with $n\geqslant 3$.

(ii) $S\cong 1^2\cdot 3^3$ or $S\cong 2^2\cdot 3^3$ or $S\cong 1\cdot 3^3\cdot 4$ or $S\cong 1^3\cdot 2\cdot 5$ with $n=6$ and $r=2$.

Case 4. $\boldsymbol{r=\frac{n+1}{2}}$.

By Lemma 17, we have $h(S)\geqslant r-1=\frac{n-1}{2}$. Then there exists some element $a|S$ such that $\mathsf{v}_a(S)=h(S)\geqslant r-1=\frac{n-1}{2}$ and $\mathrm{ord}(a)=n$. Without loss of generality, we set $a=1$. Let T be a zero-sum free subsequence of S such that $|\sum(T)|\geqslant 2|T|-1$ with $|T|$ maximal. Set $U=ST^{-1}$. Since $n-1\geqslant|\sum(T)|\geqslant 2|T|-1$, we have $|T|\leqslant\lfloor\frac{n}{2}\rfloor=r-1$.

Subcase 4.1. $|T|\leqslant r-2$.

By Claim 1(i), we have $U=1^{n-1-|T|}$ and

$$\mathrm{supp}(T)\subseteq\{1,n-r,n-r+1\}=\{1,\frac{n-1}{2},\frac{n+1}{2}\}.$$

Since $n-1-|T|\geqslant n-r+1=\frac{n+1}{2}$, it follows that $\frac{n-1}{2}\cdot 1^{\frac{n+1}{2}}$ is a zero-sum sequence with length $r+1$. Hence $\frac{n-1}{2}\nmid T$, and so $T=1^t\cdot(\frac{n+1}{2})^s$. If $s\geqslant 3$, then we obtain a zero-sum subsequence $(\frac{n+1}{2})^3\cdot 1^{\frac{n-3}{2}}$ of length $r+1$. If $s=2$, then $S\cong 1^{n-3}(\frac{n+1}{2})^2$. If $s=1$, then we have $S\cong 1^{n-2}\cdot\frac{n+1}{2}$.

Subcase 4.2. $|T|=r-1=\frac{n-1}{2}$.

Obviously, $|\sum(T)|\geqslant n-2$. There are three options consider.

(i) If Ta is a zero-sum free for every $a|U$, then $|\sum(Ta)|=|\sum(T)|+1$. By Lemma 11, we have $U=a^{n-1-|T|}=a^{\frac{n-1}{2}}$, $\sum(T)=\{a,2a,\cdots,(\mathrm{ord}(a)-2)a\}\cup(H-cosets)$ and $\sigma(T)=(\mathrm{ord}(a)-2)a$, where $H=\langle a\rangle$. If $a\neq 1$, then $T=1^{r-1}=1^{\frac{n-1}{2}}$ because of $\mathsf{v}_1(S)=h(S)\geqslant\frac{n-1}{2}$. It follows from $|\sum(T)|\geqslant 2|T|-1$ that $|T|=1$ and $n=3$, contradicting that $n\geqslant 4$. Thus $a=1$ and $\sum(T)=\{a,2a,\cdots,(n-2)a\}$. However, $T\cdot 1^2$ is a zero-sum subsequence of length $r+1$.

(ii) If Ta is a zero-sum subsequence for every $a|U$, then $U=a^{n-1-|T|}=a^{\frac{n-1}{2}}$. Similarly, we have $U=1^{\frac{n-1}{2}}$, and so $\sigma(T)=n-1$. For any $x|T$, since $\sigma(T)=n-1$, if $2\leqslant\bar{x}\leqslant\frac{n-3}{2}$, then $\sigma(Tx^{-1})=\bar{y}\in[\frac{n+1}{2},n-3]$, and then $1^{n-\bar{y}}\cdot T\cdot x^{-1}$ is zero-sum subsequence of length distinct from r. If $\bar{x}>\frac{n+1}{2}$, then $x\cdot 1^{\bar{x}}$ is a zero-sum subsequence of length $n-\bar{x}+1<\frac{n+1}{2}=r$, a contradiction. Thus we have either $\bar{x}=1$ or $\frac{n-1}{2}\leqslant\bar{x}\leqslant\frac{n+1}{2}$. Because the zero-sum subsequence $T\cdot a$ cannot be all 1's, there exists $x|T$ such that $\bar{x}=\frac{n-1}{2}$ or $\bar{x}=\frac{n+1}{2}$. Since $\sigma(\frac{n-1}{2}\cdot\frac{n+1}{2})=0$ and T is zero-sum free, it follows that

$\frac{n-1}{2}$ and $\frac{n+1}{2}$ are not in T simultaneously. So we can assume $\mathrm{supp}(T) \subseteq \{1, \frac{n-1}{2}\}$ or $\mathrm{supp}(T) \subseteq \{1, \frac{n+1}{2}\}$.

If $\mathrm{supp}(T) \subseteq \{1, \frac{n-1}{2}\}$, then $T = 1^t \cdot (\frac{n-1}{2})^s$ with $t + s = \frac{n-1}{2}$. If $s = 1$, then $T = 1^{r-2} \cdot \frac{n-1}{2} = 1^{\frac{n-3}{2}} \cdot \frac{n-1}{2}$, and $\sigma(T) = n-2$, which is a contradiction. Thus $s \geqslant 2$. Since $\frac{n-1}{2} + \frac{n-1}{2} + 1 = n$, it follows that $r = 3, n = 5$ and $T = (\frac{n-1}{2})^2$, and so $S = 1^2 \cdot 2^2$.

If $\mathrm{supp}(T) \subseteq \{1, \frac{n+1}{2}\}$, then $T = 1^t \cdot (\frac{n+1}{2})^s$ with $t + s = \frac{n-1}{2}$. If $s \geqslant 3$, then $1^{\frac{n-3}{2}} \cdot (\frac{n+1}{2})^3$ is z zero-sum subsequence of length $> r$. If $s = 2$, then $T = 1^{r-3} \cdot (\frac{n+1}{2})^2$. However, $\sigma(1 \cdot T) \neq 0$. Therefore, $T = 1^{r-2} \cdot \frac{n+1}{2}$ and $S = 1^{n-2} \cdot \frac{n+1}{2}$.

(iii) Suppose that there exist $a, b \in \mathrm{supp}(U)$ such that Ta is zero-sum free and Tb is zero-sum. Then by Lemma 11, we have $\sum(T) = \{a, 2a, \cdots, (\mathrm{ord}(a)-2)a\} \cup (H - cosets)$ with $H = \langle a \rangle$ and $\sigma(T) = (\mathrm{ord}(a) - 2)a$, and so $b = 2a$. Since $T \cdot a^2$ and $T \cdot b$ are zero-sum, by Lemma 11(iii), we must have $U = a \cdot (2a)^{n-r-1} = a \cdot (2a)^{\frac{n-3}{2}}$. If $a \neq 1$ and $2a \neq 1 \pmod n$, then we have $T = 1^{r-1} = 1^{\frac{n-1}{2}}$ and $2a \equiv 1 \pmod n$ because of $\mathsf{v}_1(S) = h(S) \geqslant \frac{n-1}{2}$. So $U = (\frac{n+1}{2})^{\frac{n-3}{2}} \cdot a$. Since $1^{\frac{n-3}{2}} \cdot (\frac{n+1}{2})^3$ is zero-sum of length $r+1$, we have $\frac{n-3}{2} \leqslant 2$, and so $n \leqslant 7$. If $\bar{a} \geqslant \frac{n+3}{2}$, then $1^{n-\bar{a}} \cdot a$ is zero-sum of length $n - \bar{a} + 1 < r$. If $2 \leqslant \bar{a} \leqslant \frac{n-1}{2}$, then $1^{\frac{n-1}{2} - \bar{a}} \cdot \frac{n+1}{2} \cdot a$ is zero-sum of length $\frac{n-1}{2} - \bar{a} + 2 < r$. Then we have $\bar{a} = \frac{n+1}{2}$, so $2a \equiv 1 \pmod n$, which is a contradiction. Thus $a = 1$ or $2a \equiv 1 \pmod n$, so $U = 1 \cdot 2^{\frac{n-3}{2}}$, or $U = 1^{\frac{n-3}{2}} \cdot \frac{n+1}{2}$.

If $U = 1 \cdot 2^{\frac{n-3}{2}}$, then $\sum(U) = \{1, 2, \cdots, n-2\}$. Applying Property(iii) to $T \cdot (2a) = T \cdot 2$ and recalling that $|T| = r - 1$ and $\sum(T) = \{1, 2, \cdots, n-2\}$(since $a = 1$), for any $x | T$, we have $\bar{x} = 1$ or $\bar{x} = 2$ or $\bar{x} = n - 1$. If $n - 1 | T$, then the zero-sum subsequence $-1 \cdot 1$ implies that $r = 2$, in which case $n = 3$, contrary to $n \geqslant 4$. So for any $x | T$, we have $\bar{x} = 1$ or $\bar{x} = 2$. We can assume $T = 1^s \cdot 2^t$ with $s + t = \frac{n-1}{2}$ and $s + 2t = n - 2$, and so $t = \frac{n-3}{2}$ and $s = 1$. It follows that $T = 1 \cdot 2^{\frac{n-3}{2}}$ and $S = 1^2 \cdot 2^{n-3}$, which is equivalent to $S \cong 1^{n-3} \cdot (\frac{n+1}{2})^2$.

Suppose $U = 1^{\frac{n-3}{2}} \cdot \frac{n+1}{2}$. If there is a term $x | T$ satisfying $2 \leqslant \bar{x} \leqslant \frac{n-1}{2}$, then $1^{\frac{n-1}{2} - \bar{x}} \cdot \frac{n+1}{2} \cdot x$ is zero-sum of length $\frac{n+3}{2} - \bar{x} < r$. If there is a term $x | T$ satisfying $\bar{x} \geqslant \frac{n+3}{2}$, then $1^{n - \bar{x}} \cdot x$ is zero-sum of length $n - \bar{x} + 1 \leqslant \frac{n-1}{2} < r$. Thus for any $x | T$, we have $\bar{x} = 1$ or $\bar{x} = \frac{n+1}{2}$. So we can assume $T = 1^s \cdot (\frac{n+1}{2})^t$. If $t \geqslant 2$, then $1^{\frac{n-3}{2}} \cdot (\frac{n+1}{2})^3$ is zero-sum of length $\frac{n+3}{2} > r$. Thus we have $t \leqslant 1$. Since $|\sum(T)| \geqslant 2|T| - 1$, we have $T = 1^{r-2} \cdot (\frac{n+1}{2}) = 1^{\frac{n-3}{2}} \cdot \frac{n+1}{2}$, and so $S = 1^{n-3} \cdot (\frac{n+1}{2})^2$.

Therefore, in this case S is one of the following:

(i) $S \cong 1^{n-2} \cdot (n - r + 1)$ with $n \geqslant 3$.

(ii) $S \cong 1^{n-3} \cdot (\frac{n+1}{2})^2 = 1^{n-3} \cdot (n - r + 1)^2$ with $r = \frac{n+1}{2}$ and $n \geqslant 3$.

(iii) $S \cong 1^2 \cdot 2^2$ with $n = 5$ and $r = 3$.

Now we give two results. Suppose $\frac{n-1}{2} \leqslant r \leqslant \frac{n}{2}$ and $\mathsf{v}_a(S) = \mathsf{v}_1(S) = h(S) \geqslant r - 1$. Let $T = S \cdot 1^{-h(S)}$. Assuming $\sigma(\overline{T}) \geqslant n - r + 1$, define T_0 as follows. Let $T = x_1 \cdot \ldots \cdot x_l$ with $2 \leqslant \overline{x_1} \leqslant \cdots \leqslant \overline{x_l}$. Then there must be a minimal index $u \geqslant 1$ such that $\overline{x_1} + \cdots + \overline{x_u} \geqslant n - r + 1$. Set $T_0 = x_1 \cdot \ldots \cdot x_u$. Then we have the following results.

Claim 2 If $\bar{x} < r$ for every $x|T$, then either $\sigma(\overline{T}) \leqslant n - r$ or else $\sigma(\overline{T}) \geqslant n - r + 1$ and $\overline{\sigma(T_0)} = \sigma(\overline{T_0}) = n - r + |T_0|$ with $2 \leqslant |T_0| \leqslant r - 1$.

Proof of Claim 2: We assume $\sigma(\overline{T}) \geqslant n - r + 1$ and proceed to show $\sigma(\overline{T_0})$ has the desired value. Since $\overline{x_1} \leqslant r \leqslant n - r$(in view of $r \leqslant \frac{n}{2}$), we must have $|T_0| \geqslant 2$. Let $T_0^{'} = T_0 \cdot x_u^{-1}$. Then the minimality of u ensures that $\sigma(\overline{T_0^{'}}) \leqslant n - r$, which together with $\overline{x_u} \leqslant r$ gives $\sigma(\overline{T_0}) \leqslant n$.

Suppose $\sigma(\overline{T_0}) = n$. Then we must have $\overline{x_u} = r$ and T_0 is a zero-sum subsequence. Thus $|T_0| = r$ and $n = \sigma(\overline{T_0}) \geqslant r + 2(r-1)$, which implies $r \leqslant \lfloor \frac{n+2}{3} \rfloor$. In view of $r \geqslant \lceil \frac{n-1}{2} \rceil$ and $n \geqslant 4$, this is only possible if $n = 7$ or $n = 5$ or $n = 4$, with $r = \lfloor \frac{n+2}{3} \rfloor = \lceil \frac{n-1}{2} \rceil$. If $n = 4$ or $n = 5$, then $r = 2$, in which case $\mathrm{supp}(T_0) = \{2\}$. Thus, if $n = 4$, then $T_0 = 2 \cdot 2$, so that $\mathsf{v}_1(S) \leqslant |S| - 2 = 1$, contradicting that $\mathsf{v}_1(S) = h(S)$, while if $n = 5$, then $\sigma(\overline{T_0}) = n = 5$ is not possible as $\sigma(\overline{T_0})$ must be even. Finally, if $n = 7$, then $r = 3$ and $T_0 = 2 \cdot 2 \cdot 3$. But then, since $\mathsf{v}_1(S) = h(S) \geqslant r - 1 = 2$, the zero-sum $1^2 \cdot 2 \cdot 3$ contradicts that $r = 3$. So we have obtained a contradiction in all cases and can instead assume $\sigma(\overline{T_0}) \leqslant n - 1$.

Thus $n - r + 1 \leqslant \overline{\sigma(T_0)} \leqslant \sigma(\overline{T_0}) \leqslant n - 1$(with the first inequality in view of the definition of T_0). But now $T_0 \cdot 1^{n-\sigma(\overline{T_0})}$ is a zero-sum sequence of length $|T_0| + n - \sigma(\overline{T_0})$. Moreover, it is a subsequence in view of $n - \sigma(\overline{T_0}) \leqslant r - 1 \leqslant h(S) \leqslant \mathsf{v}_1(S)$, which forces $\sigma(\overline{T_0}) = n - r + |T_0|$. Since $\sigma(\overline{T_0}) \leqslant n - 1$, it follows that $|T_0| \leqslant r - 1$.

Claim 3 Suppose that $\sigma(\overline{T}) \geqslant n - r + 1$ and $\bar{x} \leqslant r$ for every $x|T$. Let $T_1 = T_0 \cdot 1^{r-|T_0|}$(which is a zero-sum subsequence of length r). Then $\mathrm{supp}(T) = \{x\}$ for some $\bar{x} \in [2, n-1]$, so that $T_1 = 1^v \cdot x^u$ and $ST_1^{-1} = 1^t \cdot x^s$ with $s \geqslant 1$. Moreover, $\bar{x} > v$ and $\bar{x} > t$.

Proof of Claim 3: First, let us show that $T_0 \neq T$. Suppose to the contrary that $T_0 = T$. Then Claim 2 implies $\sigma(\overline{T}) = n - r + |T|$ with $2 \leqslant |T| \leqslant r - 1$. Consequently, letting $T^{'} = T \cdot x_1^{-1}$, we have $\sigma(\overline{T^{'}}) = n - r + |T| - \overline{x_1}$ with $\overline{x_1} \in [2, r]$, in which case

$$3 \leqslant \overline{x_1} + 1 \leqslant n - \sigma(\overline{T^{'}}) = r - |T| + \overline{x_1} \leqslant 2r - |T| \leqslant n - |T| = \mathsf{v}_1(S) + 1.$$

If the above upper bound were tight, then we must have $\overline{x_1} = r$ and $r = \frac{n}{2}$. Hence $\mathrm{supp}(T) = \{r\} = \{\frac{n}{2}\}$ with $|T| \geqslant |T_0| \geqslant 2$, whence the zero-sum subsequence $\frac{n}{2} \cdot \frac{n}{2}$ force $\frac{n}{2} = r = 2$. But now $\mathsf{v}_2(S) \geqslant 2$ has greater multiplicity than 1, contradicting that $\mathsf{v}_1(S) = h(S)$. Therefore, we conclude that the above upper bound is not right. As a result, the zero-sum subsequence $1^{n-\sigma(\overline{T^{'}})} \cdot T^{'}$ forces

$$r = n - \sigma(\overline{T^{'}}) + |T^{'}| = n - (n - r + |T| - \overline{x_1}) + |T| - 1,$$

implying $\overline{x_1} = 1$, which contradicts that $\overline{x_1} \geqslant 2$. So we instead conclude that $T_0 \neq T$. Thus let $x \in \mathrm{supp}(T \cdot T_0^{-1})$.

Let $x_i \in \mathrm{supp}(T_0)$ and let $T_0^{'} = T_0 \cdot x_i^{-1}$. If $\sigma(\overline{T_0^{'}}) \geqslant n - r + 1$, then $1^{n-\sigma(\overline{T_0^{'}})} \cdot T_0^{'}$ will be a zero-sum subsequence (in view of $\sigma(\overline{T_0^{'}}) \leqslant \sigma(\overline{T_0}) \leqslant n - 1$) of length (in view of Claim 2)

$$n - \sigma(\overline{T_0^{'}}) + |T_0^{'}| = n - (n - r + |T_0| - \overline{x_i}) + |T_0| - 1 = r + \overline{x_i} - 1,$$

which forces $\overline{x_i}=1$, contradicting that $\overline{x_i}\geqslant 2$. Therefore we may instead assume $\sigma(\overline{T_0'})\leqslant n-r$. Thus, since $\bar{x}\leqslant r$ for every $x|T$, it follows that $\sigma(\overline{T_0'\cdot x})\leqslant n$. However, if $\sigma(\overline{T_0'\cdot x})=n$, then $T_0'\cdot x$ is a zero-sum subsequence of length $|T_0'\cdot x|=|T_0|\leqslant r-1$(in view of Claim 2), which is not possible. Therefore $\sigma(\overline{T_0'\cdot x})\leqslant n-1$. By the definitions of T_0 and x, we have $\bar{x}\geqslant\overline{x_u}\geqslant\overline{x_i}$. Thus $\sigma(\overline{T_0'\cdot x})\geqslant\sigma(\overline{T_0})\geqslant n-r+1$. Consequently, as in the proof of Claim 2, the zero-sum subsequence $1^{n-\sigma(\overline{T_0'\cdot x})}\cdot T_0'\cdot x$ forces

$$\sigma(\overline{T_0})-\overline{x_i}+\bar{x}=\sigma(\overline{T_0'}\cdot x)=n-r+|T_0'\cdot x|=n-r+|T_0|,$$

whence Claim 2 implies $x=x_i$. Since $x_i\in\mathrm{supp}(T_0)$ and $x\in\mathrm{supp}(T\cdot T_0^{-1})$ were arbitrary, we have now established that $\mathrm{supp}(T)=\{x\}$, which clearly implies that $T_1=1^v\cdot x^u$ and $S\cdot T_1^{-1}=1^t\cdot x^s$.

We have $s\geqslant 1$ since $T\neq T_0$. That T_1 is zero-sum follows by a simple calculation using Claim 2. If $\bar{x}\leqslant t$, then we could replace a single term x in the zero-sum T_1 with $\bar{x}\geqslant 2$ terms equal to 1 from $S\cdot T^{-1}$, yielding a zero-sum of length other than r. Likewise, If $\bar{x}\leqslant v$, then we could replace $v\geqslant\bar{x}\geqslant 2$ terms equal to 1 in T_1 with single term equal to x from $S\cdot T_1^{-1}$(which exists since $s\geqslant 1$) to yield a zero-sum of length other than r. Thus $\bar{x}>v$ and $\bar{x}>t$.

Case 5. $\boldsymbol{r=\frac{n}{2}}$**.**

By Lemma 17, we have $h(S)\geqslant r-1=\frac{n}{2}-1$. Then there exists $a|S$ such that $\mathsf{v}_a(S)\geqslant\frac{n}{2}-1$. There are two subcases to consider.

Subcase 5.1. $\mathrm{ord}(a)=\frac{n}{2}$. Without loss of generality, we set $a=2$. Let $T=S\cdot 2^{-h(S)}$ and $x|T$, which exists else $S=2^{n-1}$, as desired. If x is even, then $x\neq 2$ as all terms equal to 2 have already been removed, and then we obtain a zero-sum subsequence of length less than $\frac{n}{2}$, which is a contradiction, by combining x with an appropriate number of the other $\mathsf{v}_2(S)=h(S)\geqslant\frac{n}{2}-1$ terms from S equal to $x=2$. Therefore all terms of T are odd.

If S has only one odd element, then $S=2^{n-2}\cdot b$ with b odd. If S has precisely two odd elements b_1 and b_2, then b_1+b_2 is even and $S=2^{n-3}\cdot b_1\cdot b_2$. It is easy to show that $\overline{b_1+b_2}=4$. Thus $S=2^{n-3}\cdot b\cdot(4-b)$ with b odd. If S has more than two odd elements, then, letting x_1,x_2 and x_3 be three odd elements in S, we have $\overline{x_1+x_2}=\overline{x_1+x_3}=\overline{x_2+x_3}=4$, and so $\overline{x_1}=\overline{x_2}=\overline{x_3}=\frac{n}{2}+2$ with $\frac{n}{2}$ odd. Thus $S=2^v\cdot(\frac{n}{2}+2)^u$ with $v\geqslant\frac{n}{2}-1$ and $\frac{n}{2}$ odd.

Subcase 5.2. $\mathrm{ord}(a)=n$. Without loss of generality, we set $a=1$. Let $T=S\cdot 1^{-h(S)}$ and $x|T$. Then we must have $\bar{x}\leqslant\frac{n}{2}+1$, otherwise $x\cdot 1^{n-\bar{x}}$ will be zero-sum of length $<\frac{n}{2}$. So for all $x\in\mathrm{supp}(S)$, we have $\bar{x}\leqslant\frac{n}{2}+1$. First we give a Claim.

Claim 4 Let all notation be as above and $n\geqslant 6$. If there exists $x|T$ such that $\bar{x}=\frac{n}{2}+1$, then $S=1^u\cdot(\frac{n}{2}+1)^v$, where $u+v=n-1, u\geqslant\frac{n}{2}-1$ and $v\geqslant 1$.

Proof of Claim 4: For $n\geqslant 6$, we have $r\geqslant 3$. If there exists $y|T$ such that $\bar{y}<\frac{n}{2}$, then $\frac{n}{2}+1+\bar{y}>\frac{n}{2}+2$, and so $(\frac{n}{2}+1)\cdot y\cdot 1^{\frac{n}{2}-1-\bar{y}}$ is a zero-sum subsequence of length $\frac{n}{2}+1-\bar{y}<\frac{n}{2}$, which is a contradiction. Clearly, if $\frac{n}{2}|T$, then we have $\mathsf{v}_1(S)=\frac{n}{2}-1, |T|=|S|-\mathsf{v}_1(S)=\frac{n}{2}$ and $\mathsf{v}_{\frac{n}{2}}(T)=1$, as $\mathsf{v}_1(S)=\frac{n}{2}-1$ and $1^{\frac{n}{2}}\cdot\frac{n}{2}$ and $(\frac{n}{2})^2$ are zero-sum of length $\neq r$. Thus

$T = \frac{n}{2} \cdot (\frac{n}{2}+1)^{\frac{n}{2}-1}$. However, $\frac{n}{2} \cdot (\frac{n}{2}+1)^2 \cdot 1^{\frac{n}{2}-2}$ is a zero-sum subsequence of length $r+1$, which is also a contradiction.

By the above argument, we have $S = 1^u \cdot (\frac{n}{2}+1)^v$, where $u+v = n-1, u \geqslant \frac{n}{2}-1$ and $v \geqslant 1$. This proves the Claim 4.

For $n=4$, we have $r=2$ and there must exist some $x|T$ with $\bar{x} \geqslant 2 = \frac{n}{2}$. In this case, we have $S = 1 \cdot 2 \cdot 3$ or $S = 1^2 \cdot 3$ in view of $\mathsf{v}_1(S) = h(S)$. So we can assume $n \geqslant 6$. If there is some $x|T$ with $\bar{x} \geqslant \frac{n}{2}+1$, then Claim 4 completes the proof. Therefore we may assume $\bar{x} \leqslant \frac{n}{2} = r$ for all $x|T$, and now we can apply Claim 2. As a result, either $\sigma(\overline{T}) \leqslant \frac{n}{2}$ or there exists a subsequence $T_0|T$ with $\sigma(\overline{T_0}) = \frac{n}{2} + |T_0| \leqslant n-1$. Let $|T| = u \geqslant 2$.

Let $T_1 = T_0 \cdot 1^{\frac{n}{2}-u}$ be a zero-sum subsequence. By Claim 3, we have $T_1 = 1^v x^u$ and $ST_1^{-1} = 1^t x^s$ with $\bar{x} > v, \bar{x} > t$. Since $\sigma(\overline{T_1}) = n$, it follows that $\bar{x} = \frac{n-v}{u} = \frac{n-r}{u} + 1 = \frac{n}{2u} + 1$ with $\bar{x} > t$ and $\bar{x} > v$. Since $n-1 \geqslant 2\bar{x} \geqslant v+t+2 \geqslant \frac{n}{2}+1$, we have $\bar{x} \geqslant \frac{n}{4} + \frac{1}{2}$. In view of the definition of T_0 and $r = \frac{n}{2}$, we have $|T_0| \geqslant 2$, so $u=2$. Thus $T_1 = x^2 \cdot 1^{\frac{n}{2}-2}$ and $\bar{x} = \frac{n}{4}+1$ with $4|n$. For $n \geqslant 10$, $(\frac{n}{4}+1)^3 \cdot 1^{\frac{n}{4}-3}$ will be a zero-sum subsequence of incorrect length unless $\bar{x} = \frac{n}{4}+1$ has multiplicity 2, in which case $S = (\frac{n}{4}+1)^2 \cdot 1^{n-3}$ and $(\frac{n}{4}+1) \cdot 1^{\frac{3n}{4}-1}$ is this time of incorrect length, which is a contradiction. For $n=8$, then we have $x=3$, so $T_1 = 1^2 \cdot 3^2$ and $S = 1^4 \cdot 3^3$ by $\mathsf{v}_1(S) = h(S)$.

Therefore, in this case, S is one of the following:

(i) $S \cong 2^{n-1}$.

(ii) $S \cong 2^{n-2} \cdot b$, where b is odd.

(iii) $S \cong 2^{n-3} \cdot b \cdot (4-b)$, where b is odd.

(iv) $S \cong 2^u \cdot (\frac{n}{2}+2)^v$, where $\frac{n}{2}$ is odd and $u \geqslant \frac{n}{2}-1$.

(v) $S \cong 1^u \cdot (\frac{n}{2}+1)^v$ with $u \geqslant \frac{n}{2}-1$ and $n \geqslant 5$.

(v) $S \cong 1^4 \cdot 3^3$ with $n=8$.

Case 6. $r = \frac{n-1}{2}$.

By Lemma 17, we have $h(S) \geqslant r-1 = \frac{n-3}{2}$. Then there exists $a|S$ such that $\mathsf{v}_a(S) = h(S) \geqslant \frac{n-3}{2}$. Since $n \geqslant 4$, so that $\frac{n}{3} < \frac{n-1}{2}$, we may without loss of generality set $a=1$. Let $T = S \cdot 1^{-h(S)}$. For any $x|T$, it is easy to show that $\bar{x} \leqslant \frac{n+3}{2}$. We first give a Claim.

Claim 5 If there exists $x|T$ such that $\bar{x} \geqslant \frac{n+1}{2}$, then we have $S = 1^{n-2} \cdot \frac{n+3}{2}$ or $S = 1^2 \cdot 2 \cdot 4$ with $n=5$, or $S = 1^3 \cdot 5^3 \cong 1^3 \cdot 3^3$ with $n=7$.

Proof of Claim 5: Since $n \geqslant 5$, we have $r \geqslant 2$ and $h(S) \geqslant 1$. If $\frac{n+1}{2}|T$, then obviously, we have $\mathsf{v}_1(S) = \frac{n-3}{2}$. If $\frac{n-1}{2}|T$, then obviously, we have $\mathsf{v}_1(S) \leqslant \frac{n-1}{2}$. If $\mathsf{v}_{\frac{n-1}{2}}(S) \geqslant 2$, then $1 \cdot (\frac{n-1}{2})^2$ is a zero-sum subsequence of length 3, which implies $r=3$ and $n=7$. Since $\mathsf{v}_1(S) \geqslant \frac{n-3}{2} = 2$, we have $1^2 \cdot (\frac{n-1}{2})^2 = 1^2 \cdot 3^2|S$, and so 4 is not in S. If $2|S$, then $1^2 \cdot 2 \cdot 3$ is a zero-sum subsequence of length $4 > r = 3$. So we can assume $\mathrm{supp}(S) \subset \{1,3,5\}$. Because $\mathsf{v}_1(S) = h(S)$ and $1 \cdot 3 \cdot 5^2, 1^4 \cdot 3$ and $1^3 \cdot 3^2 \cdot 5$ is a zero-sum sequence of incorrect length, we must have $S = 1^3 \cdot 3^3$, which contradicts that $\bar{x} \geqslant \frac{n+1}{2}$ foe some $x|T$. So we can assume $\mathsf{v}_{\frac{n-1}{2}}(S) \leqslant 1$.

If $(\frac{n-1}{2})(\frac{n+1}{2})|T$, then we have $r=2$ and $n=5$, so $\mathsf{v}_1(S) = 1 = h(S)$ and $S = 1 \cdot 2 \cdot 3 \cdot 4$, which is a contradiction.

If $(\frac{n+3}{2})(\frac{n+1}{2})|T$, then our above work shows that $\mathsf{v}_1(S) = \frac{n-3}{2}$, so that $|T| = \frac{n+1}{2} \geqslant 3$.

If there were some $x|T$ with $\bar{x} \leqslant \frac{n-3}{2}$, then the zero-sum subsequence $1^{\frac{n-3}{2}-\bar{x}} \cdot x \cdot \frac{n+3}{2}$ must have length r, implying $\bar{x} = 1$, which contradicts the definition of T. Therefore, in view of the previous paragraph, we have $\mathrm{supp}(T) = \{\frac{n+3}{2}, \frac{n+1}{2}\}$. We must also have $\mathsf{v}_{\frac{n+1}{2}}(T) = 1$, as otherwise $1^{\frac{n-3}{2}-1} \cdot (\frac{n+1}{2})^2 \cdot \frac{n+3}{2}$ will be a zero-sum subsequence of length $r+1 \neq r$. Thus $T = \frac{n+1}{2} \cdot (\frac{n+3}{2})^{\frac{n-1}{2}}$. But now $\mathsf{v}_{\frac{n+3}{2}}(S) \geqslant \frac{n-1}{2} > \frac{n-3}{2} = \mathsf{v}_1(S)$, contradicting that $\mathsf{v}_1(S) = h(S)$.

If $(\frac{n+3}{2})(\frac{n-1}{2})|T$, then our above work shows that $\mathsf{v}_{\frac{n-1}{2}}(S) = 1, \mathsf{v}_1(S) \leqslant \frac{n-1}{2}$ and $\mathsf{v}_{\frac{n+1}{2}}(S) = 0$. In view of $\mathsf{v}_1(S) \leqslant \frac{n-1}{2}$, we have $|T| \geqslant \frac{n-1}{2}$. If there were some $x|T$ with $\bar{x} \leqslant \frac{n-3}{2}$, then the zero-sum subsequence $1^{\frac{n-3}{2}-\bar{x}} \cdot x \cdot \frac{n+3}{2}$ must have length r, implying $\bar{x} = 1$, which contradicts the definition of T. Therefore $\mathrm{supp}(T) = \{\frac{n+1}{2}, \frac{n+3}{2}\}$ with $\mathsf{v}_{\frac{n-1}{2}}(S) = 1$, which implies $T = \frac{n-1}{2} \cdot (\frac{n+3}{2})^{|T|-1}$. Thus $\mathsf{v}_{\frac{n+3}{2}}(S) \geqslant |T| - 1 \geqslant \frac{n-3}{2}$. Consequently, if $n > 5$, then the zero-sum subsequence $1^{\frac{n-5}{2}} \cdot \frac{n-1}{2} \cdot (\frac{n+3}{2})^2$ has length greater than r, which is a contradiction. On the other hand, if $n = 5$, then $S = 1^2 \cdot 2 \cdot 4$ follows in view of $\mathsf{v}_1(S) = h(S)$. In view of the above work, if $\frac{n+3}{2}|S$, then we have $\mathsf{v}_{\frac{n-1}{2}}(S) = \mathsf{v}_{\frac{n+1}{2}}(S) = 0$. Moreover, as argued in both of the above paragraphs, we also cannot have any $x|T$ with $\bar{x} \leqslant \frac{n-3}{2}$. Thus $\mathrm{supp}(T) = \{\frac{n+3}{2}\}$. If $\mathsf{v}_{\frac{n+3}{2}}(S) \geqslant 3$, then we must have $\mathsf{v}_1(S) = h(S) \geqslant 3$, which implies $n \geqslant 7$, and then the zero-sum subsequence $(\frac{n+3}{2})^3 \cdot 1^{\frac{n-3}{2}-3}$ has incorrect length for $n > 7$, while for $n = 7$, we have $S = 1^3 \cdot 5^3 \cong 1^3 \cdot 3^3$, as desired. If $\mathsf{v}_{\frac{n+3}{2}}(S) = 2$, then $S = 1^{n-3} \cdot (\frac{n+3}{2})^2$, in which case the zero-sum subsequence $1^{n-3} \cdot (\frac{n+3}{2})^2$ has length $n - 1 > r$, which is a contradiction. Finally, if $\mathsf{v}_{\frac{n+3}{2}}(S) = 1$, then $S = 1^{n-2} \cdot \frac{n+3}{2}$, as desired. So we can now assume $\frac{n+3}{2} \nmid T$.

Since $\frac{n+3}{2} \nmid T$, our hypotheses ensure that $\frac{n+1}{2}|T$. Then we must have $\mathrm{supp}(T) \subseteq \{2, \frac{n+1}{2}\}$, otherwise $1^{\frac{n-1}{2}-\bar{x}} \cdot x \cdot \frac{n+1}{2}$ would be a zero-sum subsequence of incorrect length if $3 \leqslant \bar{x} \leqslant \frac{n-1}{2}$. If $2|T$ and $\mathsf{v}_2(T) \geqslant 2$, then $1^{\frac{n-9}{2}} \cdot 2^2 \cdot \frac{n+1}{2}$ is zero-sum, and this implies $n = 7$ or $n = 5$. If $n = 5$, then $\mathsf{v}_1(S) = h(S) = \mathsf{v}_2(S) \geqslant 2$ and $\mathsf{v}_{\frac{n+1}{2}}(S) \geqslant 1$ imply $|S| \geqslant 5 \geqslant n - 1$, contrary to hypothesis. For $n = 7$, we have $1^2 \cdot 2^2 \cdot 4|S$, so $S = 1^2 \cdot 2^2 \cdot 4^2$ or $S = 1^3 \cdot 2^2 \cdot 4$. The latter has $1^3 \cdot 4$ as a length $4 > r$ zero-sum, while the former has $1^2 \cdot 2^2 \cdot 4^2$ as a length $6 > r$ zero-sum, both contradictions. So we have $\mathsf{v}_2(S) \leqslant 1$. Hence $\mathsf{v}_{\frac{n+1}{2}}(S) \geqslant \frac{n-1}{2} \geqslant 2$(as shown at the start of the proof), so $n = 5$ and $\mathsf{v}_2(S) = 1$, which implies $S = 1^2 \cdot 2 \cdot 3$ in view of $\mathsf{v}_1(S = h(S)$. This proves Claim 5.

In view of Claim 5, we may assume $\bar{x} \leqslant \frac{n-1}{2} = r$ for every $x|T$. Let $T = S \cdot 1^{-h(S)}$. By Claim 2, we have that either $\sigma(\overline{T}) \leqslant \frac{n+1}{2}$ or there exists $T_0|T$ such that $\sigma(\overline{T_0}) = \frac{n+1}{2} + |T_0| \leqslant n - 1$. If $\sigma(\overline{T}) \leqslant \frac{n+1}{2}$, then either $0 \notin \sum(S)$ or there exists a zero-sum subsequence of length $> r$, which is impossible. Therefore, there exists a subsequence $T_0|T$ such that $\sigma(\overline{T_0}) = \frac{n+1}{2} + |T_0| \leqslant n - 1$ with $2 \leqslant |T_0| \leqslant \frac{n-3}{2}$. Let $|T_0| = u \geqslant 2$.

Let $T_1 = T_0 \cdot 1^{\frac{n-1}{2}-u}$. Then $\sigma(\overline{T_1}) = n$. By Claim 3, we have $T_1 = x^u \cdot 1^v$ and $ST_1^{-1} = x^s \cdot 1^t$ with $\bar{x} > v$ and $\bar{x} > t$. Since $\sigma(\overline{T_1}) = 0$, we conclude that $\bar{x} = \frac{n+1}{2u} + 1$. Since $\mathrm{supp}(S) = \{1, x\}$ with $\mathsf{v}_1(S) = h(S)$, it follows that $v + t = \mathsf{v}_1(S) \geqslant \frac{1}{2}|S| = \frac{n-1}{2}$. Thus, since $\bar{x} > v$ and $\bar{x} > t$, we have $\bar{x} \geqslant \frac{n+3}{2}$. Hence $\frac{n+1}{2u} + 1 = \bar{x} \geqslant \frac{n+3}{4}$, which implies

$u \leqslant 3$ in view of $n \geqslant 5$.

If $u = 3$, then $\frac{n+1}{2u} + 1 = \bar{x} \geqslant \frac{n+3}{4}$ implies $n = 5, x = 2$ and $r = 2$. But $\mathrm{supp}(S) = \{1, x\}$ together with $\mathsf{v}_1(S) = h(S)$ and $|T_0| \geqslant 2$ implies $S = 1^2 \cdot 2^2$, so that the zero-sum subsequence $1 \cdot 2^2$ contradicts that $r = 2$. Therefore we may instead assume $u = 2$, in which case $\bar{x} = \frac{n+1}{4} + 1$ with $4 \mid n+1$, and $T_1 = x^2 \cdot 1^v$ with $v = r - 2 = \frac{n-5}{2}$. Since $\bar{x} > v$ and $4 \mid n+1$, we have $n = 11$ or $n = 7$. If $n = 11$, then $\mathrm{supp}(S) = \{1, 4\}, r = 5$ and $\mathsf{v}_1(S) = h(S)$ ensure that $S = 1^5 \cdot 4^5, S = 1^6 \cdot 4^4, S = 1^7 \cdot 4^3, S = 1^8 \cdot 4^2$ and $S = 1^9 \cdot 4$ are the only possibilities for S. However, the zero-sum sequences $1^2 \cdot 4^5, 1^6 \cdot 4^4$ and $1^7 \cdot 4$ yield the contradiction $r \neq 5$ in all but the final case. This completes the case $n = 11$.

If $n = 7$, then $\mathrm{supp}(S) = \{1, 3\}, r = 3$ and $\mathsf{v}_1(S) = h(S)$ ensure that $S = 1^3 \cdot 3^3, S = 1^4 \cdot 3^2$ and $S = 1^5 \cdot 3$ are the only possibilities for S. However, the zero-sum sequences $1^4 \cdot 3$ and $1^7 \cdot 4$ contradicts that $r = 3$ in the second case, leaving only the other two.

Therefore, in this case, S is one of the following:

(i) $S \cong 1^{n-2} \cdot (n - r + 1)$ with $n \geqslant 5$.

(ii) $S \cong 1^2 \cdot 2 \cdot 4$ with $n = 5$; $S \cong 1^3 \cdot 3^3$ with $n = 7$.

This completes the proof. □

References

[1] A. Bialostocki, P. Dierker and M. Lotspeich. On the developments of the Erdős-Ginzburg-Ziv Theorem II. *Acta Arith.*, 45:173-184, 2003.

[2] M. Devos, L. Goddyn and B. Mohar. A generalization of Keneser's addition theorem. *Adv. Math.*, 220:1531-1548, 2009.

[3] P. Erdős and R. L. Graham. *Old and new problems and results in combinatorial number theory.* Monographies de L'Enseignement Mathmatique, 28. Universit de Genve, L'Enseignement Mathmatique, Geneva, (1980). 128.

[4] P. Erdős and E. Szemerédi. On a problem of Graham, *Publ. Math. Debrecen.* 23:123-127, 1976.

[5] W. Gao, Y. O. Hamidoune and G. Wang. Distinct length modular zero-sum subsequences: A proof of Graham's conjecture. *J. Number Theory*, 130:1425-1431, 2010.

[6] A. Geroldinger and F. Halter-Koch. *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory.* Pure and Applied Mathematics, Vol. 278, Chapman & Hall/CRC, 2006.

[7] A. Geroldinger. *Additive group theory and non-unique factorizations.* Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2009.

[8] D. J. Grynkiewicz. Note on conjecture of Graham. *European J. Combin.*, 32:1336-1344, 2011.

[9] A. Pixton. Sequences with small subsums sets. *J. Number Theory*, 129:806-817, 2009.

[10] W. W. Smith and M. Freeze. Sumsets of zerofree sequences. *Arab. J. Sci. Eng. Sect. C Theme Issues*, 26:97-105, 2001.

[11] S. Savchev and F. Chen. Long zero-free sequences in finite cyclic groups. *Discrete Math.*, 307:2671-2679, 2007.

[12] P. Z. Yuan. On the index of minimal zero-sum sequences over finite cyclic groups. *J. Combin. Theory Ser. A* , 114:1545-1551, 2007.

[13] P. Z. Yuan. Subsequence sums of a zero-sum free sequence II. *Ars Combin.*, 116:433-444, 2014.