Bi-Cohen-Macaulay graphs

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Abstract

In this paper we consider bi-Cohen-Macaulay graphs, and give a complete classification of such graphs in the case they are bipartite or chordal. General bi-Cohen-Macaulay graphs are classified up to separation. The inseparable bi-Cohen-Macaulay graphs are determined. We establish a bijection between the set of all trees and the set of inseparable bi-Cohen-Macaulay graphs.

Keywords: Bi-Cohen–Macaulay, Bipartite and chordal graphs, Generic graphs, Inseparability

1 Introduction

A simplicial complex Δ is called *bi-Cohen-Macaulay* (bi-CM), if Δ and its Alexander dual $\Delta^\vee$ are Cohen-Macaulay. This concept was introduced by Fløystad and Vatne in [8]. In that paper the authors associated to each simplicial complex Δ in a natural way a complex of coherent sheaves and showed that this complex reduces to a coherent sheaf if and only if Δ is bi-CM.

The present paper is an attempt to classify all bi-CM graphs. Given a field K and a simple graph on the vertex set $[n] = \{1, 2, \ldots, n\}$, one associates with G the edge ideal I_G of G, whose generators are the monomials $x_i x_j$ with $\{i, j\}$ an edge of G. We say that G is bi-CM if the simplicial complex whose Stanley-Reisner ideal coincides with I_G is bi-CM. Actually, this simplicial complex is the so-called *independence complex* of G. Its faces are the independent sets of G, that is, subsets D of $[n]$ with $\{i, j\} \not\subset D$ for all edges $\{i, j\}$ of G.

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By its very definition, any bi-CM graph is also a Cohen-Macaulay graph (CM graph). A complete classification of all CM graphs is hopeless if not impossible. However, such a classification is given for bipartite graphs [10, Theorem 3.4] and for chordal graphs [11]. We refer the reader to the books [9] and [14] for a good survey on edge ideals and its algebraic and homological properties.

Based on the classification of bipartite and chordal CM graphs, we provide in Section 3 a classification of bipartite and chordal bi-CM graphs, see Theorem 3 and Theorem 4. In Section 2 we first present various characterizations of bi-CM graphs. By using the Eagon-Reiner theorem [5], one notices that the graph G is bi-CM if and only if it is CM and I_G has a linear resolution. Cohen-Macaulay ideals generated in degree 2 with linear resolution are of very special nature. They all arise as deformations of the square of the maximal ideal of a suitable polynomial ring. From this fact arise constraints on the number of edges of the graph and on the Betti numbers of I_G.

Though a complete classification of all bi-CM graphs seems to be again impossible, a classification of all bi-CM graphs up to separation can be given, and this is the subject of the remaining sections.

A *separation* of the graph G with respect to the vertex i is a graph G' whose vertex set is $V(G) \cup \{i'\}$ having the property that G is obtained from G' by identifying i with i' and such that $x_i - x_{i'}$ is a non-zerodivisor modulo $I_{G'}$. The algebraic condition on separation makes sure that the essential algebraic and homological invariants of I_G and $I_{G'}$ are the same. In particular, G is bi-CM if and only if G' is bi-CM. A graph which does not allow any separation is called *inseparable*, and an inseparable graph which is obtained by a finite number of separation steps from G is called an *inseparable model* of G. Any graph admits inseparable models and the number of inseparable models of a graph is finite. Separable and inseparable graphs from the view point of deformation theory have been studied in [1].

In Section 5 we determine all inseparable bi-CM graphs on n vertices. Indeed, in Theorem 11 it is shown that for any tree T on the vertex set $[n]$ there exists a unique inseparable bi-CM graph G_T determined by T, and any inseparable bi-CM graph is of this form. Furthermore, if G is an arbitrary bi-CM graph and T is the relation graph of the Alexander dual of I_G, then G_T is a separable model of G.

For a bi-CM graph G, the Alexander dual $J = (I_G)^\vee$ of I_G is a Cohen-Macaulay ideal of codimension 2 with linear resolution. As described in [3], one attaches to any relation matrix of J a relation tree T. Replacing the entries in this matrix by distinct variables with the same sign, one obtains the so-called *generic relation matrix* whose ideals of 2-minors J_T and its Alexander dual has been computed in [13]. This theory is described in Section 4. The Alexander dual of J_T is the edge ideal of graph, which actually is the graph G_T mentioned before and which serves as a separable model of G.

2 Preliminaries and various characterizations of Bi-CM graphs

In this section we recall some of the standard notions of graph theory which are relevant for this paper, introduce the bi-CM graphs and present various equivalent conditions of a graph to be bi-CM.

The graphs considered here will all be finite, simple graphs, that is, they will have no double edges and no loops. Furthermore we assume that G has no isolated vertices. The vertex set of G will be denoted $V(G)$ and will be the set $[n] = \{1, 2, \dots, n\}$, unless otherwise stated. The set of edges of G we denote by $E(G)$.

A subset $F \subset [n]$ is called a *clique* of G, if $\{i, j\} \in E(G)$ for all $i, j \in F$ with $i \neq j$. The set of all cliques of G is a simplicial complex, denoted $\Delta(G)$.

A subset $C \subset [n]$ is called a *vertex cover* of G if $C \cap \{i, j\} \neq \emptyset$ for all edges $\{i, j\}$ of G. The graph G is called *unmixed* if all minimal vertex covers of G have the same cardinality. This concept has an algebraic counterpart. We fix a field K and consider the ideal $I_G \subset S = K[x_1, \dots, x_n]$ which is generated by all monomials $x_i x_j$ with $\{i, j\} \in E(G)$. The ideal I_G is called the *edge ideal* of G. Let $C \subset [n]$. Then the monomial prime ideal $P_C = (\{x_i \colon i \in C\})$ is a minimal prime ideal of I_G if and only if C is a minimal vertex cover of G. Thus G is unmixed if and only if I_G is unmixed in the algebraic sense. A subset $D \subset [n]$ is called an *independent set* of G if D contains no set $\{i, j\}$ which is an edge of G. Note that D is an independent set of G if and only if $[n] \setminus D$ is a vertex cover. Thus the minimal vertex covers of G correspond to the maximal independent sets of G. The cardinality of a maximal independent set is called the *independence number* of G. It follows that the Krull dimension of S/I_G is equal to c, where c is the independence number of G.

The graph G is called *bipartite* if $V(G)$ is the disjoint union of V_1 and V_2 such that V_1 and V_2 are independent sets, and G is called *disconnected* if $V(G)$ is the disjoint union of W_1 and W_2 and there is no edge $\{i, j\}$ of G with $i \in W_1$ and $j \in W_2$. The graph G is called *connected* if it is not disconnected.

A *cycle* C (of length r) in G is a sequence of edges $\{i_k, j_k\}$ with $k = 1, 2, \dots, r$ such that $j_k = i_{k+1}$ for $k = 1, \dots, r-1$ and $j_r = i_1$. A cord of C is an edge $\{i, j\}$ of G with $i, j \in \{i_1, \dots, i_r\}$ and $\{i, j\}$ is not an edge of C. The graph G is called *chordal* if each cycle of G of length $\geqslant 4$ has a chord. A graph which has no cycle and which is connected is called a *tree*.

Now we recall the main concept we are dealing with in this paper. Let $I \subset S$ be a squarefree monomial ideal. Then $I = \bigcap_{j=1}^{m} P_j$ where each of the P_j is a monomial prime ideal of I. The ideal $I^\vee$ which is minimally generated by the monomials $u_j = \prod_{x_i \in P_j} x_i$ is called the *Alexander dual* of I. One has $(I^\vee)^\vee = I$. In the case that $I = I_G$, each P_j is generated by the variables corresponding to a minimal vertex cover of G. Therefore, $(I_G)^\vee$ is also called the *vertex cover ideal* of G.

According to [8] a squarefree monomial ideal $I \subset S$ is called *bi-Cohen-Macaulay* (or simply bi-CM) if I as well as the Alexander dual $I^\vee$ of I is a Cohen-Macaulay ideal. A graph G is called *Cohen-Macaulay* or *bi-Cohen-Macaulay (over K)* (CM or bi-CM for short), if I_G is CM or bi-CM. One important result regarding the Alexander dual that will be used frequently in this paper is the Eagon-Reiner theorem which says that I is a Cohen-Macaulay ideal if and only if $I^\vee$ has a linear resolution. Thus the Eagon-Reiner theorem implies that I is bi-CM if and only if I is a Cohen-Macaulay ideal with linear resolution. From this description it follows that a bi-CM graph is connected. Indeed, if this is not the case, then there are induced subgraphs $G_1, G_2 \subset G$ such that $V(G)$ is the disjoint union of $V(G_1)$ and $V(G_2)$. It follows that $I_G = I_{G_1} + I_{G_2}$, and the ideals I_{G_1} and I_{G_2} are ideals in a different set of variables. Therefore, the free resolution of S/I_G is obtained as the tensor product of the resolutions of S/I_{G_1} and S/I_{G_2}. This implies that I_G has relations of degree 4, so that I_G does not have a linear resolution.

From now on we will always assume that G is connected, without further mentioning it.

Proposition 1. *Let K be an infinite field and G a graph on the vertex set $[n]$ with independence number c. The following conditions are equivalent:*

(a) *G is a bi-CM graph over K;*

(b) *G is a CM graph over K, and S/I_G modulo a maximal regular sequence of linear forms is isomorphic to $T/\mathfrak{m}_T^2$ where T is the polynomial ring over K in $n-c$ variables and $\mathfrak{m}_T$ is the graded maximal ideal of T.*

Proof. We only need to show that I_G has a linear resolution if and only if condition (b) holds. Since K is infinite and since S/I_G is Cohen-Macaulay of dimension c, there exists a regular sequence $\mathbf{x}$ of linear forms on S/I_G of length c. Let $T = S/(\mathbf{x})$. Then T is isomorphic to a polynomial ring in $n-c$ variables. Let J be the image of I_G in T. Then J is generated in degree 2 and has a linear resolution if and only if I_G has linear resolution. Moreover, J is $\mathfrak{m}_T$-primary. The only $\mathfrak{m}_T$-primary ideals with linear resolution are the powers of $\mathfrak{m}_T$. Thus, I_G has a linear resolution if and only if $J = \mathfrak{m}_T^2$. ☐

Corollary 2. *Let G be a graph on the vertex set $[n]$ with independence number c. The following conditions are equivalent:*

(a) *G is a bi-CM graph over K;*

(b) *G is a CM graph over K and $|E(G)| = \binom{n-c+1}{2}$;*

(c) *G is a CM graph over K and the number of minimal vertex covers of G is equal to $n-c+1$;*

(d) *$\beta_i(I_G) = (i+1)\binom{n-c+1}{i+2}$ for $i = 0, \dots, n-c-1$.*

Proof. For the proof of the equivalent conditions we may assume that K is infinite and hence we may use Proposition 1.

(a) $\iff$ (b): With the notation of proposition 1 we have $J = \mathfrak{m}_T^2$ if and only if the number of generators of J is equal to $\binom{n-c+1}{2}$. Since I_G and J have the same number of generators and since the number of generators of I_G is equal to $|E(G)|$, the assertion follows.

(b) $\iff$ (c): Since S/I_G is Cohen-Macaulay, the multiplicity of S/I_G is equal to the length $\ell(T/J)$ of T/J. On the other hand, the multiplicity is also the number of minimal prime ideals of I_G which coincides with the number of minimal vertex covers of G. Thus the length of T/J is equal to the number of minimal vertex covers of G. Since $J = \mathfrak{m}_T^2$ if and only if $\ell(T/J) = n - c + 1$, the assertion follows.

(a) $\Rightarrow$ (d): Note that $\beta_i(I_G) = \beta_i(J)$ for all i. Since J is isomorphic to the ideal of 2-minors of the matrix

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_{n-c} & 0 \\ 0 & y_1 & \cdots & y_{n-c-1} & y_{n-c} \end{pmatrix}$$

in the variables $y_1, \ldots, y_{n-c}$, the Eagon-Northcott complex ([4], [6]) provides a free resolution of J and the desired result follows.

(d) $\Rightarrow$ (a): It follows from the description of the Betti numbers of the ideal I_G that $\operatorname{proj\,dim} S/I_G = n - c$. Thus, $\operatorname{depth} S/I_G = c$. Since $\dim S/I_G = c$, it follows that I_G is a Cohen-Macaulay ideal. Since $|E(G)| = \beta_0(I_G) = \binom{n-c+1}{2}$, condition (b) is satisfied, and hence G is bi-CM, as desired. □

Finally we note that G is a bi-CM graph over K if and only if the vertex cover ideal of G is a codimension 2 Cohen-Macaulay ideal with linear relations. Indeed, let J_G be the vertex cover ideal of G. Since $J_G = (I_G)^\vee$, it follows from the Eagon-Reiner theorem J_G is bi-CM if and only if I_G is bi-CM.

3 The classification of bipartite and chordal bi-CM graphs

In this section we give a full classification of the bipartite and chordal bi-CM graphs.

Theorem 3. *Let G be a bipartite graph on the vertex set V with bipartition $V = V_1 \cup V_2$ where $V_1 = \{v_1, \ldots, v_n\}$ and $V_2 = \{w_1, \ldots, w_m\}$. Then the following conditions are equivalent:*

(a) *G is a bi-CM graph;*

(b) *$n = m$ and $E(G) = \{\{v_i, w_j\} \colon 1 \leqslant i \leqslant j \leqslant n\}$.*

Proof. (a) $\Rightarrow$ (b): Since G is a bi-CM graph, it is in particular a CM-graph, and so $n = m$, and by [9, Theorem 9.1.13] there exists a poset $P = \{p_1, \ldots, p_n\}$ such that $G = G(P)$. Here $G(P)$ is the bipartite graph on $V = \{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ whose edges are those 2-element subset $\{v_i, w_j\}$ of V such that $p_i \leqslant p_j$. Thus $I_G = I_{G(P)} = H_P^\vee$, where

$$H_P = \bigcap_{p_i \leqslant p_j} (x_i, y_j)$$

is an ideal of $S = K[\{x_i, y_i\}_{p_i \in P}]$, the polynomial ring in $2n$ variables over K. Since G is bi-CM, it follows that H_P is Cohen–Macaulay, and hence

$$\operatorname{proj\,dim} S/H_P = 2n - \operatorname{depth} S/H_P = 2n - \dim S/H_P = \operatorname{height} H_P = 2.$$

Thus $\operatorname{proj\,dim} H_P = 1$, and hence, by [10, Corollary 2.2], the Sperner number of P, i.e., the maximum of the cardinalities of antichains of P equals 1. This implies that P is a chain, and this yields (b).

(b) $\Rightarrow$ (a): The graph G described in (b) is of the form $G = G(P)$ where P is a chain. By what is said in (a) $\Rightarrow$ (b), it follows that G is bi-CM. □

The following picture shows a bi-CM bipartite graph for $n = 4$.

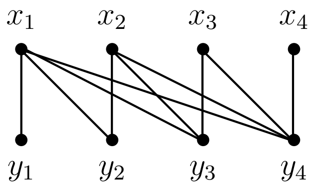

Figure 1: A bi-CM bipartite graph.

Theorem 4. *Let G be a chordal graph on the vertex set $[n]$. The following conditions are equivalent:*

(a) *G is a bi-CM graph;*

(b) *Let $F_1, \ldots, F_m$ be the facets of the clique complex of G. Then $m = 1$, or $m > 1$ and*

  (i) *$V(G) = V(F_1) \cup V(F_2) \cup \ldots \cup V(F_m)$, and this union is disjoint;*

  (ii) *each F_i has exactly one free vertex j_i;*

  (iii) *the restriction of G to $[n] \setminus \{j_1, \ldots, j_m\}$ is a clique.*

Proof. Let $I_{n,d}$ be the ideal generated by all squarefree monomials of degree d in $S = K[x_1, \ldots, x_n]$. It is known (and easy to prove) that $I_{n,d}^{\vee} = I_{n,n-d+1}$, and that all these ideals are Cohen-Macaulay, and hence all bi-CM. If $m = 1$, then $I_G = I_{n,2}$ and the result follows.

Now let $m > 1$. A bi-CM graph is a CM graph. The CM chordal graphs have been classified in [11]: they are the chordal graphs satisfying (b)(i). Thus for the proof of the theorem we may assume that (b)(i) holds and simply have to show that (b)(ii) and (b)(iii) are satisfied if and only if I_G has a linear resolution.

Let P_i be the monomial prime ideal generated by the variables x_k with $k \in V(F_i) \setminus \{j_i\}$, and let G' be subgraph of G whose edges do not belong to any F_i. It is shown in the proof of [11, Corollary 2.1] that there exists a regular sequence on S/I_G such that after reduction modulo this sequence one obtains the ideal $J \subset T$ where T is the polynomial ring on the variables x_k with $k \neq j_i$ for $i = 1, \ldots, m$ and where

$$J = (P_1^2, \ldots, P_m^2, I_{G'}). \qquad (1)$$

By Proposition 1, it follows that I_G has a linear resolution if and only if $J = \mathfrak{m}_T^2$, where $\mathfrak{m}_T$ denotes the graded maximal ideal of T.

So, now suppose first that I_G has a linear resolution, and hence $J = \mathfrak{m}_T^2$. Suppose that some F_i has more than one free vertex, say F_i has the vertex k with $k \neq j_i$. Choose any

F_t different from F_i and let $l \in F_j$ with $l \neq j_t$. Then x_k and x_l belong to T but $x_k x_l \notin J$ as can be seen from (1). This is a contradiction. Thus (b)(ii) follows.

Suppose next that the graph G'' which is the restriction of G to $[n] \setminus \{j_1, \ldots, j_m\}$ is not a clique. Then there exist $i, j \in V(G'')$ such that $\{i, j\} \notin E(G'')$. However, since all x_k with $k \in V(G'')$ belong to T and since $J = \mathfrak{m}_T^2$, it follows $x_i x_j \in J$. Thus, by (1), $x_i x_j \in P_k^2$ for some k or $x_i x_j \in I_{G'}$. Since (b)(ii) holds, this implies in both cases that $\{i, j\} \in E(G'')$, a contradiction. Thus (b)(iii) follows.

Conversely, suppose (b)(ii) and (b)(iii) hold. We want to show that $J = \mathfrak{m}_T^2$. Let $x_i, x_j \in T$. We have to show that $x_i x_j \in J$. It follows from the description of J that $x_k^2 \in J$ for all $x_k \in T$. Thus we may assume that $i \neq j$. If $\{i, j\}$ is not an edge of any F_k, then by definition it is an edge of G', and hence $x_i x_j \in I_{G'} \subset J$. On the other hand, if $\{i, j\}$ is an edge of F_k for some k, then $i, j \neq i_k$, and hence $x_i x_j \in P_k^2 \subset J$. Thus the desired conclusion follows. □

Let G be a chordal bi-CM graph as in Theorem 4(b) with $m > 1$. We call the complete graph G'' which is the restriction of G to $[n] \setminus \{j_1, \ldots, j_m\}$ the *center* of G.

The following picture shows, up to isomorphism, all bi-CM chordal graphs whose center is the complete graph K_4 on 4 vertices:

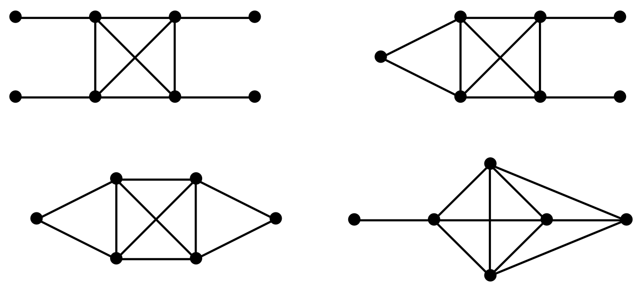

Figure 2:

4 Generic Bi-CM graphs

As we have already seen in the first section, the Alexander dual $J = I_G^\vee$ of the edge ideal of a bi-CM graph G is a Cohen–Macaulay ideal of codimension 2 with linear resolution. The ideal J may have several distinct relation matrices with respect to the unique minimal monomial set of generators of J. As shown in [3], one may attach to each of the relation matrices of J a tree as follows: let $u_1, \ldots, u_m$ be the unique minimal set of generators of J. Let A be one of the relation matrices of J. Because J has a linear resolution, the generating relations of J may be chosen all of the form $x_k u_i - x_l u_j = 0$. This implies that in each row of the $(m-1) \times m$-relation matrix A there are exactly two non-zero entries (which are variables with different signs). We call such relations, *relations of binomial type.*

Example 5. Consider the bi-CM graph G on the vertex set $[5]$ and edges $\{1,2\}$ $\{2,3\}$, $\{3,1\}$, $\{2,4\}$, $\{3,4\}$, $\{4,5\}$ as displayed in Figure 3.

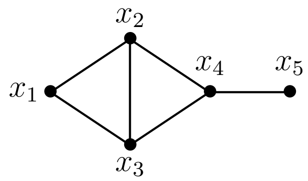

Figure 3:

The ideal $J = I_G^\vee$ is generated by $u_1 = x_2x_3x_4$, $u_2 = x_1x_3x_4$, $u_3 = x_2x_3x_5$ and $u_4 = x_1x_2x_4$. The relation matrices with respect to u_1, u_2, u_3 and u_4 are the matrices

$$A_1 = \begin{pmatrix} x_1 & -x_2 & 0 & 0 \\ x_5 & 0 & -x_4 & 0 \\ x_1 & 0 & 0 & -x_3 \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} x_1 & -x_2 & 0 & 0 \\ x_5 & 0 & -x_4 & 0 \\ 0 & x_2 & 0 & -x_3 \end{pmatrix}.$$

Coming back to the general case, one assigns to the relation matrix A the following graph Γ: the vertex set of Γ is the set $V(\Gamma) = \{1, 2, \ldots, m\}$, and $\{i, j\}$ is said to be an edge of Γ if and only if some row of A has non-zero entries for the ith- and jth-component. It is remarked in [3] and easy to see that Γ is a tree. This tree is in general not uniquely determined by G.

In our Example 5 the relation tree of A_1 is

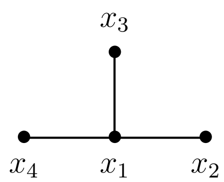

Figure 4:

while the relation tree of A_2 is

Now let J be any codimension 2 Cohen-Macaulay monomial ideal with linear resolution. Then, as observed in Section 2, $J^\vee = I_G$ where G is a bi-CM graph. Now we follow

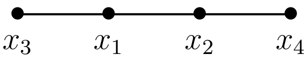

Figure 5:

Naeem [13] and define for any given tree T on the vertex set $[m] = \{1, \ldots, m\}$ with edges $e_1, \ldots, e_{m-1}$ the $(m-1) \times m$-matrix A_T whose entries a_{kl} are defined as follows: we assign to the kth edge $e_k = \{i, j\}$ of T with $i < j$ the kth row of A_T by setting

$$a_{kl} = \begin{cases} x_{ij}, & \text{if } l = i, \\ -x_{ji}, & \text{if } l = j, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

The matrix A_T is called the *generic matrix* attached to the tree T.

By the Hilbert-Burch theorem [2], the matrix A_T is the relation matrix of the ideal J_T of maximal minors of A_T, and J_T is a Cohen-Macaulay ideal of codimension 2 with linear resolution.

We let G_T be the graph such that $I_{G_T} = J^\vee$, and call G_T the *generic bi-CM graph* attached to T.

Our discussion so far yields

Proposition 6. *For any tree T, the graph G_T is bi-CM.*

In order to describe the vertices and edges of G_T, let i and j be any two vertices of the tree T. There exists a unique path $P : i = i_0, i_1, \ldots, i_r = j$ from i to j. We set $b(i, j) = i_1$ and call $b(i, j)$ the *begin* of P, and set $e(i, j) = i_{r-1}$ and call $e(i, j)$ the *end* of P.

It follows from [13, Proposition 1.4] that I_{G_T} is generated by the monomials $x_{ib(i,j)}x_{je(i,j)}$. Thus the vertex set of the graph G_T is given as

$$V(G_T) = \{(i, j), (j, i) \colon \{i, j\} \text{ is an edge of } T\}.$$

In particular, $\{(i, k), (j, l)\}$ is an edge of G_T if and only if there exists a path P from i to j such that $k = b(i, j)$ and $l = e(i, j)$.

In Example 5, let T_1 and T_2 be the relation trees of A_1 and A_2, respectively. Then the generic matrices corresponding to these trees are

$$B_1 = \begin{pmatrix} x_{12} & -x_{21} & 0 & 0 \\ x_{13} & 0 & -x_{31} & 0 \\ x_{14} & 0 & 0 & -x_{41} \end{pmatrix},$$

and

$$B_2 = \begin{pmatrix} x_{12} & -x_{21} & 0 & 0 \\ x_{13} & 0 & -x_{31} & 0 \\ 0 & x_{24} & 0 & -x_{42} \end{pmatrix}.$$

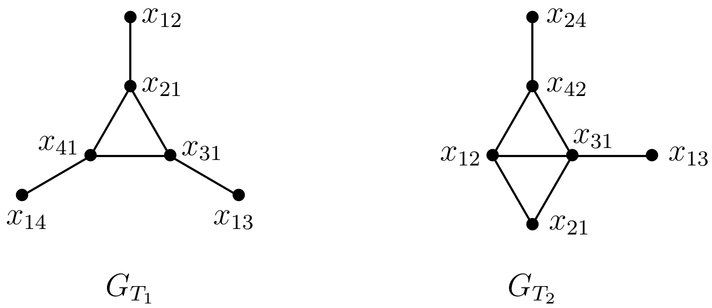

Figure 6:

The generic graphs corresponding to the trees T_1 and T_2 are displayed in Figure 6.

It follows from this description that G_T has $2(m-1)$ vertices. Since G_T is bi-CM, the number of edges of G_T is $\binom{n-c+1}{2}$, see Corollary 2. Here $n-c$ is the degree of the generators of $I_G^\vee$ which is $m-1$. Hence G_T has $\binom{m}{2}$ edges. Among the edges of G_T are in particular the $m-1$ edges $\{(i,j),(j,i)\}$ where $\{i,j\}$ is an edge of T.

Proposition 7. *Let A be the relation matrix of a codimension 2 Cohen-Macaulay monomial ideal J with linear resolution, and assume that all the variables appearing in A are pairwise distinct. Let T be the relation tree of A. Then J is isomorphic to J_T and J admits the unique relation tree, namely T.*

Proof. Since all variables appearing in A are pairwise distinct, we may rename the variables appearing in a binomial type relation and call them as in the generic matrix x_{ij} and x_{ji}. Then A becomes A_T and this shows that $J \cong J_T$.

To prove the uniqueness of the relation tree, we first notice that the shifts in the multigraded free resolution of J are uniquely determined and independent of the particular choice of the relation matrix A. A possibly different relation matrix A' can arise from A only be row operations with rows of the same multidegree. Let $r_1, \ldots, r_l$ by rows of A with the same multidegree corresponding to binomial type relations, and fix a column j. Then the non-zero jth columns of each of the r_i must be the same, up to a sign. Since we assume that the variables appearing in A are pairwise distinct, it follows that $l = 1$. In particular, there is, up to the order of the rows, only one relation matrix with rows corresponding to binomial type relations. This shows that T is uniquely determined. ☐

5 Inseparable models of Bi-CM graphs

In order to state the main result of this paper we recall the concept of inseparability introduced by Fløystad et al in [7], see also [12].

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over the field K and $I \subset S$ a squarefree monomial ideal minimally generated by the monomials $u_1, \ldots, u_m$. Let y be an indeterminate over S. A monomial ideal $J \subset S[y]$ is called a *separation* of I for the variable x_i if the following holds:

(i) the ideal I is the image of J under the K-algebra homomorphism $S[y] \to S$ with $y \mapsto x_i$ and $x_j \mapsto x_j$ for all j;

(ii) x_i as well as y divide some minimal generator of J;

(iii) $y - x_i$ is a non-zero divisor of $S[y]/J$.

The ideal I is called *separable* if it admits a separation, otherwise *inseparable*. If J is an ideal which is obtained from I by a finite number of separation steps, then we say that J *specializes* to I. If moreover, J is inseparable, then J is called an *inseparable model* of I. Each monomial ideal admits an inseparable model, but in general not only one. For example, the inseparable models of the powers of the graded maximal ideal of S have been considered by Lohne [12].

Forming the Alexander dual behaves well with respect to specialization and separation.

Proposition 8. *Let $I \subset S$ be a squarefree monomial ideal. Then the following holds:*

(a) *If J specializes to I, then $J^\vee$ specializes to $I^\vee$.*

(b) *The ideal I is separable if and only if $I^\vee$ is separable.*

Proof. (a) It follows from [7, Proposition 7.2] that if $L \subset S[y]$ is a monomial ideal such that $y - x_i$ is a regular element on $S[y]/L$ with $(S[y]/L)/(y - x_i)(S[y]/L) \cong S/I$, then $y - x_i$ is a regular element on $S[y]/L^\vee$ and $(S[y]/L^\vee)/(y - x_i)(S[y]/L^\vee) \cong S/I^\vee$. Repeated applications of this fact yields the desired result.

(b) We may assume that the ideal L as in (a) is a separation of I with respect to x_i. Since (a) holds, it remains to show that y as well as x_i divides some generator of $L^\vee$. By assumption this is the case for L. Suppose that y does not divide any generator of $L^\vee$. Then it follows from the definition of the Alexander dual that y also does not divide any generator of $(L^\vee)^\vee$. This is a contradiction, since $L = (L^\vee)^\vee$. Similarly it follows that x_i divides some generator of $L^\vee$. □

We now apply these concepts to edge ideals. Let G be a graph on the vertex set $[n]$. We call G *separable* if I_G is separable, and otherwise *inseparable*. Let J be a separation of I_G for the variable x_i. Then by the definition of separation, J is again an edge ideal, say $J = I_{G'}$ where G' is a graph with one more vertex than G. The graph G is obtained from G' by identifying this new vertex with the vertex i of G. Algebraically, this identification amounts to say that $S/I_G \cong (S'/I_{G'})/(y - x_i)(S'/I_{G'})$, where $S' = S[y]$ and $y - x_i$ is a non-zerodivisor of $S'/I_{G'}$. In particular, it follows that I_G and $I_{G'}$ have the same graded Betti-numbers. In other words, all important homological invariants of I_G and $I_{G'}$ are the same. It is therefore of interest to classify all inseparable graphs. An attempt for this classification is given in [1].

Example 9. Let G be the triangle and G' be the line graph displayed in Figure 7.

Then $I_{G'} = (x_1x_2, x_1x_3, x_2x_4)$. Since $\mathrm{Ass}(I_{G'}) = \{(x_1, x_2), (x_1, x_4), (x_2, x_3)\}$, it follows that $x_3 - x_4$ is a non-zero divisor on $S'/I_{G'}$ where $S' = K[x_1, x_2, x_3, x_4]$. Moreover,

Figure 7: A triangle and its inseparable model

$(S'/I_{G'})/(x_3 - x_4)(S'/I_{G'}) \cong S/I_G$. Therefore, the triangle in Figure 7 is obtained as a specialization from the line graph in Figure 7 by identifying the vertices x_3 and x_4.

We denote by $G^{(i)}$ the complementary graph of the restriction $G_{N(i)}$ of G to $N(i)$ where $N(i) = \{j\colon\ \{j,i\} \in E(G)\}$ is the neighborhood of i. In other words, $V(G^{(i)}) = N(i)$ and $E(G^{(i)}) = \{\{j,k\}\colon\ j,k \in N(i) \text{ and } \{j,k\} \notin E(G)\}$. Note that $G^{(i)}$ is disconnected if and only if $N(i) = A \cup B$, where $A, B \neq \emptyset$, $A \cap B = \emptyset$ and all vertices of A are adjacent to those of B.

Here we will need the following result of [1, Theorem 3.1].

Theorem 10. *The following conditions are equivalent:*

(a) *The graph G is inseparable;*

(b) *$G^{(i)}$ is connected for all i.*

Now we are ready to state our main result.

Theorem 11. (a) *Let T be a tree. Then G_T is an inseparable bi-CM graph.*

(b) *For any inseparable bi-CM graph G, there exists a unique tree T such that $G \cong G_T$.*

(c) *Let G be any bi-CM graph. Then there exists a tree T such that G_T is an inseparable model of G.*

Proof. (a) By Corollary 6, G_T is a bi-CM graph. In order to see that G_T is inseparable we apply the criterion given in Theorem 10, and thus we have to prove that for each vertex (i,j) of G_T and for each disjoint union $N((i,j)) = A \cup B$ of the neighborhood of (i,j) for which $A \neq \emptyset \neq B$, not all vertices of A are adjacent to those of B.

As follows from the discussion in Section 4,

$$N((i,j)) = \{(k,l)\colon \text{there exists a path from } i \text{ to } l, \text{ and } j = b(i,l) \text{ and } k = e(i,l)\}.$$

In particular, $(j,i) \in N((i,j))$. Let $N((i,j)) = A \cup B$, as above. We may assume that $(j,i) \in A$. Since T is a tree, then there is no path from j to any l with $(k,l) \in N((i,j))$, because otherwise we would have a loop in T. This shows that (j,i) is connected to no vertex in B, as desired.

(b) Let A be a relation matrix of $J = I_G^\vee$ and T the relation tree of A. The non-zero entries of A are variables with sign ± 1. Say the kth row of A has the non-zero entries a_{ki_k}

and a_{kj_k} with $i_k < j_k$. We may assume that the variable representing a_{ki_k} has a positive sign while that a_{kj_k} has a negative sign, and that this is so for each row. We claim that the variables appearing in the non-zero entries of A are pairwise distinct. By Proposition 7 this then implies that T is the only relation tree of J and that $G \cong G_T$.

In order to prove the claim, we consider the generic matrix A_T corresponding to T. Let S' be the polynomial ring over S in the variables x_{ij} and x_{ji} with $\{i,j\} \in E(T)$. For each k we consider the linear forms $\ell_{k1} = x_{i_k j_k} - a_{ki_k}$ and $\ell_{k2} = x_{j_k i_k} - a_{kj_k}$. For example, for the matrix A_2 in Example 5 the linear forms are $\ell_{11} = x_{12} - x_1$, $\ell_{12} = x_{21} - x_2$, $\ell_{21} = x_{13} - x_5$, $\ell_{22} = x_{31} - x_4$, $\ell_{31} = x_{24} - x_2$ and $\ell_{32} = x_{42} - x_3$.

We let ℓ be the sequence of linear form $\ell_{11}, \ell_{12}, \dots, \ell_{m-1,1}, \ell_{m-1,2}$ in S'. Then we have $(S'/J_TS')/(\ell)(S'/J_TS') \cong S/J$. Since both ideals, J as well as J_T, are Cohen-Macaulay ideals of codimension 2, it follows that ℓ is a regular sequence on S'/J_TS'. Thus, assuming the variables appearing in the non-zero entries of A are not all pairwise distinct, we see that J is separable. Indeed, suppose that the variable x_k appears at least twice in the matrix. Then we replace only one of the x_k by the corresponding generic variable x_{ij} to obtain the matrix A'. Let J' be the ideal of maximal minors of A'. It follows from the above discussions that $x_{ij} - x_k$ is a regular element of $S[x_{ij}]/J'$. In order to see that J' is a separation of J it remains to be shown that x_{ij} as well as x_k appear as factors of generators of J'. Note that J' is a specialization of J_T. The minors of A_T which are the generators of J_T are the monomials $\prod_{\substack{i=1\\ i\neq j}}^{m+1} x_{ib(i,j)}$ for $j = 1, \dots, m+1$, see [13, Proposition 1.2]. From this description of the generators of J_T it follows that all entries of A_T appear as factors of generators of J_T. Since J' is a specialization of J_T, the same holds true for J', and since x_{ij} as well as x_k are entries of A', the desired conclusion follows.

Now since we know that J is separable, Proposition 8(b) implies that G is separable as well. This is a contradiction.

(c) Let A be a relation matrix of $J = I_G^\vee$ and T the corresponding relation tree. As shown in the proof of part (b), J_T specializes to J, and hence I_{G_T} specializes to I_G, by Proposition 8(a). By part (a), the graph G_T is inseparable. Thus we conclude that G_T is an inseparable model of G, as desired. □

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