

Response to Referee:

Finite edge-transitive oriented graphs of valency four: a global approach

The authors thank the referee for the careful and helpful report. We give here a detailed response to the advice in the report, and also some words of explanation. During the period of refereeing we had some correspondence with Pablo Spiga about the paper, and in particular about Problem 1.5 – and the problem the referee had discovered with Lemma 6.2. The upshot is that Pablo has been invited, and accepted to be an author of this paper, and we have improved Theorem 1.3 by bounding the number of simple direct factors of the socle to be at most 2 in all cases. [We give more details about this below]

Comments on each point made by the referee

1. We re-worded the sentence as suggested (top paragraph of page 2). We want to comment about our use of oriented graphs rather than directed graphs:
 - a. The main reason we chose to present our results in terms of oriented graphs is that there are several families of graphs which admit rather different orientations relative to different subgroups G of automorphisms. Some of these families we have explored in depth after submitting this paper – and we have added a reference to this new work. We found it helpful to work with the underlying undirected graphs, finding normal quotients which interwove between several different families with prescribed orientations.
 - b. The other reason is that we wished to relate our work strongly with the extensive literature about half-transitive graphs, which treats the graphs as undirected graphs with valency four.
 - c. We added a comment regarding this within this paragraph.
2. We adjusted the definition of $\mathcal{G}(\Delta)$ as suggested. (page 2)
3. Table 1: we have removed the symbol for a directed cycle from this table and also other places in the paper. In the two sentences following Theorem 1.1 we explained the difference in the last two lines of the table.
4. Step (4) of the Framework on pp3-4. Thank you for the advice to be more cautious, and for the additional references. We changed the verb “determine” to “describe” in Step (4), and gave some extra discussion and references in the following paragraph.
5. In Remark 1.4(b) we point out that the graphs in [23] are half-arc-transitive, and drew attention to this also in the sentence before Construction 5.3.
6. Thank you for the reference. Reference to this is added in last sentence of the second paragraph of Section 2, and in both paragraphs of Section 2.1.
7. Sections 2.1-2.3: We have added a sentence before the first of these subsections explaining the choice of topics discussed. To Section 2.1 (which describes an alternative quotienting process) we added the reference to Sparl’s work on tightly attached graphs. We shortened Section 2.2 (explaining how many pairs in $OG(4)$ arise from regular maps).

8. We have added a reference to George Glauberman's result in Section 2.3. Thanks for pointing out this result.
9. Thank you for the reference to the still unpublished [CPS]. This is very helpful. The incorrect sentence has been changed. However, on analyzing the graph referred to by the referee we found an error in the claim in [CPS, Section 6]. Namely we found that there is indeed a unique arc-transitive valency 4 graph on 21 points. It has automorphism group of order 16×21 (not 8×21), with dihedral vertex stabilisers of order 16 (not 8). We have referred to this graph at the beginning of Section 3.1.
10. 6th line of Section 3.2: no, we believe we are correct to refer to OG (not AG) since the paper 5 lines before [36] referred to deals with directed (oriented) graphs.
11. Last half of paragraph before Proposition 3.1: thank you for picking up the lack of clarity. We have re-worded the sentence.
12. Proof of Lemma 3.5. We deleted all references to "directed cycles" in this proof. We have also tried to catch other occurrences of this
13. Cayley graphs: since some treatments in the literature have left-action admitted as automorphisms (whereas we have right action admitted), and since we need to be able to discuss the technical details in our analysis, we decided to keep the definition. We re-worded and simplified the first paragraph of Section 4.
14. We changed the symbol for semidirect product, notably in Remark 4.1, and also in many other places.
15. Comma added in line 2 of Construction 4.4.
16. Thanks for spotting this. In fact Pablo Spiga had noticed the problem during the refereeing process, and we were in discussions with him about this. Pablo is now an author of the paper and we have an improved Theorem 1.3. Consequential changes are
 - a. Problem 1.5 removed
 - b. Lemma 6.2 has new (stronger and correct) statement and proof
 - c. New Construction 5.5 and justification Lemma 5.6 for examples for Theorem 1.3(c).