A note on maxima in random walks

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Abstract

We give a combinatorial proof that a random walk attains a unique maximum with probability at least 1/2. For closed random walks with uniform step size, we recover Dwass's count of the number of length ℓ walks attaining the maximum exactly k times. We also show that the probability that there is both a unique maximum and a unique minimum is asymptotically equal to $\frac{1}{4}$ and that the probability that a Dyck word has a unique minimum is asymptotically $\frac{1}{2}$.

Keywords: Random walk, Dyck Words, Catalan Numbers

1 Introduction

A *length* ℓ *walk* is a sequence $w : \{1, \ldots, \ell\} \to \{\pm 1\}$. The *trajectory* of w is the sequence $\bar{w} : \{0, \ldots, \ell\} \to \mathbb{Z}$ defined by $\bar{w}(j) = \sum_{i=1}^{j} w(i)$. We define $\max(w) = \sup\{\bar{w}(j) : 0 \leqslant j \leqslant \ell\}$. The length ℓ walk w is *closed* if $\bar{w}(\ell) = 0$. Let $\mathcal{C}(n)$ denote the set of length $2n$ closed walks and let $\mathcal{M}(n) \subset \mathcal{C}(n)$ denote the subset consisting of those walks w for which there is a unique $i \in \{1, \ldots, 2n\}$ such that $\bar{w}(i) = \max(w)$.

The paper centers around a combinatorial proof of the following theorem which was first proven by Dwass in [5]:

Theorem 2.4. $|\mathcal{M}(n)| = \frac{1}{2} |\mathcal{C}(n)|$ *for each* $n \geqslant 1$.

In [5], this is proven by a method that computes the probabilities of events in finite random walks by relating them to events in infinite random walks for which probabilities are more readily computed. This is a general analytic method used to compute a large

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number of quantities including $\mathcal{M}(n)$ as well as the more general $\mathcal{M}(n, r)$ which we discuss in Section 3.

As is often the case, a combinatorial proof offers other intuition and insight. In this case, we will see that our method generalizes to certain cases not amenable to Dwass's method.

In Section 2 we give a first proof of Dwass's result, which uses a method we will employ fundamentally in the text. A second more transparent proof is give in Section 6. This second proof uses that the number of Dyck words is the corresponding Catalan number. In Section 3 we recover Dwass's stronger result that there are precisely $\binom{2n-r}{r-1}$ length $2n$ closed walks attaining their maximum exactly r times. In Section 4 we explain that the combinatorial proof generalizes to show that more general types of finite random walks have probability $\geqslant \frac{1}{2}$ of attaining a unique maximum. This conclusion does not assume that the walks are closed and allows an arbitrary distribution of step sizes. In Section 5 we show that the probability of having both a unique minimum and a unique maximum approaches $\frac{1}{4}$ as the length of a uniform closed walk increases. In Section 6 we show that the probability that a length n Dyck word has a unique minimum approaches $\frac{1}{2}$ as $n \to \infty$.

2 Dyck Words and Leads

A *Dyck word* of length $2n$ is a closed walk w such that $\max(w) = 0$. Let $\mathcal{D}(n)$ denote the set of length $2n$ Dyck words. The number of Dyck words of a given length is the corresponding Catalan number:

Theorem 2.1. $|\mathcal{D}(n)| = \frac{1}{n+1}\binom{2n}{n}$.

The *lead* of $w \in \mathcal{C}(k)$ is the number of values $i \in \{1, \ldots, 2k\}$ with both $w(i) > 0$ and $\bar{w}(i) > 0$. For $0 \leqslant e \leqslant k$ let $\mathcal{L}(k, e) \subset \mathcal{C}(k)$ be the set of lead e walks (so that $\mathcal{L}(k, 0) = \mathcal{D}(k)$). Thus $\mathcal{C}(k) = \sqcup_{e=0}^{k} \mathcal{L}(k, e)$.

Since $|\mathcal{C}(n)| = \binom{2n}{n}$, Theorem 2.1 follows from the Chung-Feller Theorem which states that $|\mathcal{L}(k, e)|$ is independent of e [2]. Among the many proofs of the Chung-Feller theorem is a bijective explanation given in [1, 6] the former of which traced the explanation to [4]. We now recount the bijection:

Lemma 2.2. *For each* $1 \leqslant e \leqslant k$ *there is a bijection* $\psi : \mathcal{L}(k, e-1) \to \mathcal{L}(k, e)$.

Proof. Let $w \in \mathcal{L}(k, e-1)$. Let $p > 0$ be maximal such that $w(p) = -1$ and $\bar{w}(p) = -1$. Regard w as a string in $\{\pm 1\}$, and express w as the concatenation axb where a is the initial length p subpath and x is a single symbol (which is necessarily -1). Define $\psi(w)$ to be the sequence corresponding to bxa. This lies in $\mathcal{L}(k, e)$ since a, xb, bx are all closed and the lead of bx is one greater than that of b.

The map ψ^{-1} is defined by recognizing the decomposition of $w \in \mathcal{L}(k, e)$ as a concatenation bxa by declaring b to be the length n subword where n is minimal such that $w(n+1) = -1$ and $\bar{w}(n+1) = 0$. □

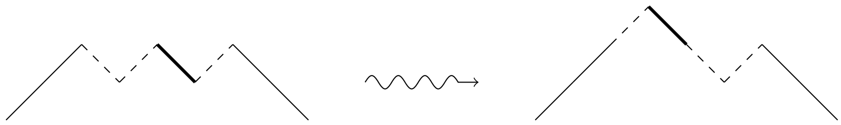

Figure 1: Performing a swap

M(1,2)	1
M(1,1) M(2,3)	1 1
M(2,2) M(3,4)	2 1
M(2,1) M(3,3) M(4,5)	3 3 1
M(3,2) M(4,4) M(5,6)	6 4 1
M(3,1) M(4,3) M(5,5) M(6,7)	10 10 5 1
M(4,2) M(5,4) M(6,6) M(7,8)	20 15 6 1
M(4,1) M(5,3) M(6,5) M(7,7) M(8,9)	35 35 21 7 1

Figure 2: The base cases (1) are the entries $1, 3, 10, 35, \ldots$ and the 1 at the top. The identity (2) states that an entry in Pascal's triangle is the sum of all the numbers in the diagonal path above it, e.g. 15=10+4+1.

Remark 2.3. We utilize $\psi : \mathcal{L}(k, 0) \to \mathcal{L}(k, 1)$ whose crucial property is that $\psi(w) \in \mathcal{M}(k)$ for $w \in \mathcal{L}(k, 0)$.

Theorem 2.4. *$|\mathcal{M}(n)| = \frac{1}{2}|\mathcal{C}(n)|$ for each $n \geqslant 1$.*

Proof. We describe a bijection $\Psi : \mathcal{C}(n) - \mathcal{M}(n) \to \mathcal{M}(n)$. Let $w \in \mathcal{C}(n) - \mathcal{M}(n)$. As $\text{rank}(w) \geqslant 2$, we let a be the nontrivial subsequence of w with domain $\{p, \ldots, q\} \subset \mathbb{N}$ where $p - 1, q$ are the minimal and maximal values of $\bar{w}^{-1}(\max(w))$. Note that $a \in \mathcal{L}(q - p + 1, 0)$ and let $\psi(a) \in \mathcal{L}(q - p + 1, 1)$ be as provided by Lemma 2.2. Define $\Psi(w)$ to be the sequence obtained from w by substituting $\psi(a)$ for a as in Figure 1. Note that $\Psi(w) \in \mathcal{M}(n)$ by Remark 2.3. $\square$

An alternate proof is given in Section 6.

3 Counting walks of arbitrary rank

The *rank* of a walk w is $|\bar{w}^{-1}(\max(w))|$. For $r \geqslant 1$ let $\mathcal{M}(n, r) \subset \mathcal{C}(n)$ denote the subset of rank r length $2n$ closed walks. As $\mathcal{M}(n, 1) = \mathcal{M}(n)$, Theorem 2.4 states that $|\mathcal{M}(n, 1)| = \frac{1}{2}\binom{2n}{n} = \binom{2n-1}{n-1}$. We now extend this result and count each $\mathcal{M}(n, r)$:

Theorem 3.1. *For $n, r \geqslant 1$ we have $|\mathcal{M}(n, r)| = \binom{2n-r}{n-1}$.*

Proof. Let $M(n, r) = \binom{2n-r}{n-1}$. We first observe that the numbers $M(n, r)$ satisfy the following recursive definition (see Figure 2).

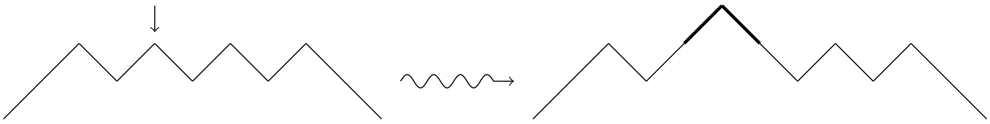

Figure 3: Inserting a peak at the third maximal peak from the right

The base cases are:

$$M(n,1) = \binom{2n-1}{n-1} \quad \text{and} \quad M(1,2) = 1 \tag{1}$$

The inductive step for $n, r \geqslant 2$ follows by iterating Pascal's identity:

$$M(n,r) = \sum_{j=r-1}^{n} M(n-1,j) \tag{2}$$

We will show that Equations (1) and (2) are satisfied with $M(a,b)$ replaced by $|\mathcal{M}(a,b)|$, from which it follows that $|\mathcal{M}(n,r)| = M(n,r)$ for all $n, r \geqslant 1$ as desired.

Equation (1) is easy as Theorem 2.4 asserts that $|\mathcal{M}(n,1)| = \binom{2n-1}{n-1}$ and obviously $|\mathcal{M}(1,2)| = 1$.

To verify Equation (2), for each $n, r \geqslant 2$, we describe a bijection χ_r between $\bigcup_{j=r-1}^{n} \mathcal{M}(n-1,j)$ and $\mathcal{M}(n,r)$. Let $w \in \bigcup_{j=r-1}^{n} \mathcal{M}(n-1,j)$. As $\operatorname{rank}(w) \geqslant r-1$, we may consider the $(r-1)$th maximal peak of w, counted from the right. As in Figure 3, we insert a peak at this point to obtain a path $\hat{w} \in \mathcal{M}(n)$. We then apply the map $\Psi^{-1} : \mathcal{M}(n) \to \mathcal{C}(n) \backslash \mathcal{M}(n)$ to $\hat{w}$ to obtain $\chi_r(w)$.

We must show that χ_r is injective and that its image is $\mathcal{M}(n,r)$. The former holds since a left-inverse to χ_r is obtained by first applying Ψ and then removing the single maximal peak. The latter holds by Lemma 3.2. □

Lemma 3.2. *Let $n, r \geqslant 2$, let $w \in \mathcal{M}(n,r)$ and let w' be obtained by removing the single maximal peak from $\Psi(w)$. Then* $\operatorname{rank}(w') \geqslant r-1$.

Proof. The general case follows from the case where $w \in \mathcal{D}(m) \cap \mathcal{M}(m,r)$ so that $w' \in \mathcal{D}(m-1)$. We express w as axb where the length i of a is maximal such that $i < m$ and $\bar{w}(i) = 0$. Then $\Psi(w) = bxa$ by definition. As $\operatorname{rank}(w) = r$, we have $\operatorname{rank}(a) = r-1$. Hence $\operatorname{rank}(w') \geqslant r-1$ as desired. □

4 Variable step lengths

In this section we provide a different generalization of Theorem 2.4: we prove that for a random closed walk of variable step size, and with a nonzero fixed number of each type of step, the probability of attaining a unique maximum is at least $\frac{1}{2}$.

Definition 4.1. Let S be a finite set. A *length ℓ S-walk* w is a sequence $\{1, \dots, \ell\} \to S$. Let $\mathcal{W}_S(n)$ denote the set of length n S-walks. Let $v : S \to \mathbb{R}$. The *v-trajectory* of w is

$\bar{w}_v(j) = \sum_{i=1}^{j} v(w(i))$. The *v-maximum* of w is $\max_v(w) = \max\{\bar{w}_v(j) : 0 \leqslant j \leqslant \ell\}$. Let $\mathcal{M}_v(n) \subset \mathcal{W}_S(n)$ denote the subset consisting of those walks w for which there is a unique $i \in \{1, \ldots, n\}$ such that $\bar{w}(i) = \max(w)$. The length ℓ S-walk w is *v-closed* if $\bar{w}_v(\ell) = 0$. Let $\mathcal{D}_v(n) \subset \mathcal{W}_S(n)$ denote the set of length N v-closed S-walks with v-maximum 0.

The proof of Lemma 2.2 can be carried over to this more general context:

Lemma 4.2. *There is an injection* $\psi_v : \mathcal{D}_v(n) \to \mathcal{M}_v(n)$

Proof. Let $w \in \mathcal{D}_v(n)$. Let $p < n$ be maximal such that $\bar{w}_v(p) = 0$. Regard w as a string in S, and express w as the concatenation axb where a is the initial length p subpath and x is a single symbol (which necessarily satisfies $v(x) < 0$). Define $\psi_v(w)$ to be the sequence corresponding to bxa. This lies in $\mathcal{M}_v(n)$ since a has v-maximum 0 and $\bar{b}_v$ obtains its unique maximum at the end.

To see that ψ_v is injective, we describe its (left-)inverse. Any walk w in the image of ψ_v has the form bxa, where $\max_v(a) = 0$, b has length q, $\bar{w}_v(q) > 0$, and $\bar{w}_v(q+1) = 0$. Moreover there can clearly be at most one such representation of w. The unique pre-image of w under ψ_v is then axb. □

Theorem 4.3. *Let S be a finite set and let $v : S \to \mathbb{R}$. Then* $|\mathcal{M}_v(n)| \geqslant \frac{1}{2} |\mathcal{W}_S(n)|$.

Proof. The proof is the same as that of Theorem 2.4; we use ψ_v in the same way to define a map $\Psi_v : \mathcal{W}_S(n) - \mathcal{M}_v(n) \to \mathcal{M}_v(n)$ and this is an injection. □

Remark 4.4. Let $X \subset \mathcal{W}_S(n)$ be any Ψ_v-invariant subset. Then by restricting Ψ_v to $X \cap (\mathcal{W}_S(n) - \mathcal{M}_v(n))$, we see that $|X \cap \mathcal{M}_v(n)| \geqslant \frac{1}{2} |X|$. For example, we could take X to be the set of v-closed walks, since Ψ_v takes v-closed walks to v-closed walks. Also, note that Ψ_v preserves the cardinality of S-fibers in the sense that $|w^{-1}(s)| = |(\Psi_v w)^{-1}(s)|$ for each $s \in S$. Hence, we could take X to be the set of walks whose S-fibers have some prescribed cardinalities.

Remark 4.5. We can also consider *weighted* random walks. That is, we let μ be a probability measure on S, and consider the induced measure μ on $\mathcal{W}(n)$ that assigns to a walk w the probability $\frac{1}{n} \sum_{s \in S} |w^{-1}(s)| \, \mu(s)$. Theorem 4.3 also generalizes to this case: with respect to this measure, the measure of $\mathcal{M}_v(n)$ is at least $\frac{1}{2}$. This works because the measure is Ψ_v-invariant, so

$$\mu_v(\mathcal{W}_S(n) - \mathcal{M}_v(n)) = \sum_{w \in \mathcal{W}_S(n) - \mathcal{M}_v(n)} \mu(w) = \sum_{w \in \mathcal{W}_S(n) - \mathcal{M}_v(n)} \mu(\Psi(w)) \leqslant \sum_{w \in \mathcal{M}_v(n)} \mu(\Psi(w)) = \mu_v(\mathcal{M}_v(n))$$

5 Estimating the probability of a unique max and a unique min

The goal of this section is to prove Theorem 5.9 which gives a $\frac{1}{4}$ asymptotic probability that a random walk with uniform step size has both a unique minimum and a unique maximum. The strategy of the proof is to show that there is a dense subset having a partition into four equal cardinality parts lying in:

$$\mathcal{M} \cap -\mathcal{M} \quad \mathcal{M}^c \cap -\mathcal{M} \quad \mathcal{M} \cap (-\mathcal{M})^c \quad \mathcal{M}^c \cap (-\mathcal{M})^c$$

Definition 5.1. Let $w \in \mathcal{C}(n)$. If $w \notin \mathcal{M}(n)$, we define the *max-interval* of w to be the largest subsequence $\{p, \ldots, q\}$ of $\{1, \ldots, 2n\}$ such that $\bar{w}(p-1) = \max(w) = \bar{w}(q)$. For $w \in \mathcal{M}(n)$, we define the *max-interval* of w to be the max-interval of $\Psi^{-1}(w)$. Note that the max-interval subtends the part of w which is modified by Ψ (or Ψ^{-1}). We define the *min-interval* of w to be the max-interval of $-w$. The *size* of the max-interval is its cardinality and likewise for the min-interval.

The max-interval and min-interval are generically small in the following sense:

Lemma 5.2. *Let $\mathcal{U}_+(n,k) \subset \mathcal{C}(n)$ be the set of walks whose max-interval has size $2k$. Similarly, $\mathcal{U}_-(n,k)$ denotes the walks with a size $2k$ min-interval. For any $\epsilon > 0$, there exists N such that for all $n \geqslant N$ we have:*

$$\frac{1}{|\mathcal{C}(n)|}\left|\bigcup_{k=N}^{n} \mathcal{U}_+(n,k)\right| = \sum_{k=N}^{n} \frac{|\mathcal{U}_+(n,k)|}{|\mathcal{C}(n)|} < \epsilon$$

and similarly for $\mathcal{U}_-$.

Proof. We will prove the claim for $\mathcal{U}_+$ as the proof for $\mathcal{U}_-$ is identical.

For $1 \leqslant k \leqslant n$, let $\mathcal{U}_+^*(n,k) = \mathcal{U}_+(n,k) \setminus \mathcal{M}(n)$. From the definitions we have $\left|\mathcal{U}_+^*(n,k)\right| = \frac{1}{2}\left|\mathcal{U}_+(n,k)\right|$ so it suffices to prove the claim with $\mathcal{U}_+$ replaced by $\mathcal{U}_+^*$.

Let $\epsilon > 0$. For any $1 \leqslant k \leqslant n$,

$$\left|\mathcal{U}_+^*(n,k)\right| \;=\; |\mathcal{D}(k)| \cdot |\mathcal{M}(n-k)| \;=\; \frac{1}{k+1}\binom{2k}{k} \cdot \frac{1}{2}\binom{2(n-k)}{n-k}$$

Indeed, each $w \in \mathcal{U}_+^*(n,k)$ corresponds to a pair (d,m) with $d \in \mathcal{D}(k)$ and $m \in \mathcal{M}(n-k)$. The correspondence arises by inserting d at the maximum of m.

We now have the following inequality which proves the claim. Its first part holds since $\bigcup_{k=1}^{n} \mathcal{U}_+^*(n,k) = \mathcal{C}(n) \setminus \mathcal{M}(n)$ and $|\mathcal{M}(n)| = \frac{1}{2}|\mathcal{C}(n)|$ by Theorem 2.4. Its second part holds by Lemma 5.3 and its last part holds for N sufficiently large by Lemma 5.4.

$$\sum_{k=N}^{n} \frac{\left|\mathcal{U}_+^*(n,k)\right|}{|\mathcal{C}(n)|} = \frac{1}{2} - \sum_{k=1}^{N-1} \frac{\left|\mathcal{U}_+^*(n,k)\right|}{|\mathcal{C}(n)|} \leqslant \frac{1}{2} - \sum_{k=1}^{N-1} \frac{1}{2}\frac{1}{k+1}\binom{2k}{k} 4^{-k} < \epsilon \qquad \square$$

Lemma 5.3. $\binom{2(n-k)}{n-k} / \binom{2n}{n} \geqslant 4^{-k}$ *for $0 \leqslant k \leqslant n$.*

Proof. For each fixed n, we prove this by induction on k.

Base case:

$$\binom{2(n-0)}{n-0} / \binom{2n}{n} = 1 \geqslant 4^0$$

Inductive step: For $0 \leqslant k < n$

$$\frac{\binom{2(n-(k+1))}{n-(k+1)} / \binom{2n}{n}}{\binom{2(n-k)}{n-k} / \binom{2n}{n}} = \frac{\binom{2(n-k)-2}{(n-k)-1}}{\binom{2(n-k)}{n-k}} = \frac{(n-k)^2}{(2(n-k)-1)(2(n-k))} > 4^{-1} \qquad \square$$

Lemma 5.4. $\sum_{k=1}^{\infty} \frac{1}{2}\frac{1}{k+1}\binom{2k}{k}4^{-k} = \frac{1}{2}$.

Proof. The well-known generating function for the Catalan numbers is

$$\sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k}x^k = \frac{1-\sqrt{1-4x}}{2x}$$

where this equality holds for $|x| < 1/4$. Setting $x = 1/4$, the left-hand side converges by the elementary estimate $\binom{2k}{k} \leqslant \frac{4^k}{\sqrt{3k+1}}$ of the central binomial coefficient. We thus obtain the following by applying $\lim_{x \nearrow \frac{1}{4}}$ to each side, and note that Abel's theorem ensures the convergence of this limit on the left.

$$\sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k}4^{-k} = 2$$

The conclusion follows since the 0-th term of this series is 1. □

Lemma 5.5. *If the max-interval and min-interval of w intersect and have size s_1 and s_2, then:*

$$\max(w) - \min(w) \leqslant \frac{1}{2}\sup(s_1, s_2)$$

Proof. Let $\{a, \ldots, b\}$ be the max-interval of w. By hypothesis, there is $c \in \{a, \ldots, b\}$ with $\bar{w}(c) = \min(w)$. Clearly, the difference between the maximum and minimum on a size s interval is at most s. Since $\max(w)$ and $\min(w)$ are both attained on $\{a, \ldots, c\}$ and on $\{c, \ldots, b\}$, and since one of these intervals is of size at most $\frac{s_1}{2}$, it follows that $\max(w) - \min(w) \leqslant \frac{s_1}{2}$. Similarly, $\max(w) - \min(w) \leqslant \frac{s_2}{2}$. □

The following is a classical fact about random walks, and we refer to [7] for an account of its history:

Lemma 5.6 (Reflection principle)**.** *For $h \geqslant 0$, the number of walks $w \in \mathcal{C}(n)$ with $\max w \geqslant h$ is equal to $\binom{2n}{n+h}$.*

Proof. For any $w \in \mathcal{C}(n)$ with $\max(w) \geqslant h$, define Rw by

$$(Rw)(i) = \begin{cases} w(i); & i \leqslant I(w) \\ -w(i) & i > I(w) \end{cases}$$

where $I(w)$ is minimal such that $\bar{w}(i) = h$. Then R is an injection onto the set of walks $w \in \mathcal{W}(2n)$ such that $\bar{w}(2n) = 2h$. The cardinality of the latter set is $\binom{2n}{n+h}$. □

Lemma 5.7 (Generically Disjoint)**.** *Let $\mathcal{J}(n) \subset \mathcal{C}(n)$ be the subset of walks whose max-interval and min-interval are disjoint. Then* $\lim_{n\to\infty} \frac{|\mathcal{J}(n)|}{|\mathcal{C}(n)|} = 1$.

Proof. Fix $\epsilon > 0$. Let N be as in Lemma 5.2 and let $n \geqslant N$. Let $\mathcal{O}(n) = \mathcal{C}(n) \setminus \mathcal{J}(n)$ consist of those walks whose max-interval and min-interval overlap. Let

$$\mathcal{K}_1(n) = \mathcal{O}(n) \cap \left(\bigcup_{k=N}^{n} \mathcal{U}_+(n,k) \cup \bigcup_{k=N}^{n} \mathcal{U}_-(n,k) \right)$$

(so that $\frac{|\mathcal{K}_1(n)|}{|\mathcal{C}(n)|} \leqslant 2\epsilon$ by Lemma 5.2) and let

$$\mathcal{K}_2(n) = \mathcal{O}(n) \cap \left(\bigcup_{k=1}^{N-1} \mathcal{U}_+(n,k) \cap \bigcup_{k=1}^{N-1} \mathcal{U}_-(n,k) \right)$$

By Lemma 5.5, for any $w \in \mathcal{K}_2(n)$ we have $\max(w) < N$. Hence, by Lemma 5.6, we have

$$\frac{|\mathcal{K}_2(n)|}{|\mathcal{C}(n)|} \leqslant \frac{\binom{2n}{n} - \binom{2n}{n+N}}{\binom{2n}{n}} = 1 - \frac{(n-N+1)\cdots(n-N+N)}{(n+1)\cdots(n+N)} \quad \xrightarrow{n\to\infty} \quad 0$$

and so

$$\lim_{n\to\infty} \frac{|\mathcal{O}(n)|}{|\mathcal{C}(n)|} = \lim_{n\to\infty} \frac{|\mathcal{K}_1(n)|}{|\mathcal{C}(n)|} + \lim_{n\to\infty} \frac{|\mathcal{K}_2(n)|}{|\mathcal{C}(n)|} < 2\epsilon + \lim_{n\to\infty} \frac{|\mathcal{K}_2(n)|}{|\mathcal{C}(n)|} = 2\epsilon.$$

Hence, $\frac{|\mathcal{J}(n)|}{|\mathcal{C}(n)|} \geqslant 1 - 2\epsilon$ for every $\epsilon > 0$, which proves the claim. □

Lemma 5.8. *$\mathcal{J}(n)$ is partitioned into 4 subsets of equal cardinality according to whether there is a unique max and/or unique min.*

Proof. Since the max-interval and min-interval of elements of $\mathcal{J}(n)$ are disjoint, it is easily seen that the restrictions of the map Ψ to $\mathcal{J}(n)$ leaves $\mathcal{J}(n) \cap -\mathcal{M}(n)$ invariant and hence provides bijections

$$(\mathcal{J}(n) \cap -\mathcal{M}(n)) \setminus \mathcal{M}(n) \to (\mathcal{J}(n) \cap -\mathcal{M}(n)) \cap \mathcal{M}(n)$$

and

$$(\mathcal{J}(n) \setminus -\mathcal{M}(n)) \setminus \mathcal{M}(n) \to (\mathcal{J}(n) \setminus -\mathcal{M}(n)) \cap \mathcal{M}(n).$$

Similarly, the map $w \mapsto -\Psi(-w)$ provides bijections

$$(\mathcal{J}(n) \setminus -\mathcal{M}(n)) \cap \mathcal{M}(n) \to (\mathcal{J}(n) \cap -\mathcal{M}(n)) \cap \mathcal{M}(n)$$

and

$$(\mathcal{J}(n) \setminus -\mathcal{M}(n)) \setminus \mathcal{M}(n) \to (\mathcal{J}(n) \cap -\mathcal{M}(n)) \setminus \mathcal{M}(n).$$

Combining these gives the desired one-to-one-to-one-to-one correspondence. □

Theorem 5.9. $\lim_{n\to\infty} \frac{|\mathcal{M}(n) \cap -\mathcal{M}(n)|}{|\mathcal{C}(n)|} = \frac{1}{4}$.

Proof. Combine Lemma 5.7 and Lemma 5.8. □

Remark 5.10. The first few terms of the sequence $|\mathcal{M}(n) \cap -\mathcal{M}(n)|$ are:

$$0, 2, 4, 18, 64, 230, 852, 3206, 12144, 46188, \ldots$$

The convergent sequence $|\mathcal{M}(n) \cap -\mathcal{M}(n)| \,/\, |\mathcal{C}(n)|$ has the following initial terms, where $d = 29099070$:

$$\frac{0}{d}, \frac{9699690}{d}, \frac{5819814}{d}, \frac{7482618}{d}, \frac{7390240}{d}, \frac{7243275}{d}, \frac{7223895}{d}, \frac{7248766}{d}, \frac{7268184}{d}, \frac{7274610}{d}, \ldots$$

It is not monotonic, but we have verified that its terms are $\geqslant 1/4$ for all $n \geqslant 1$.

6 Dyck words with a unique maximum

In this section we show that, asymptotically, one half of all Dyck words have a unique maximum. We refer to [3] for a variety of other elegant counts of frequencies of various configurations within Dyck words.

We begin by describing a second, more straightforward, proof of Theorem 2.4.

Cyclic Permutation Proof. We describe a map $\mathcal{M}(n) \to \mathcal{D}(n-1)$. We cyclically permute so that the maximum appears at the beginning and end. This yields a $(2n-1)$-to-(1) map to length $2n$ Dyck words whose trajectories are negative except at the endpoints. After removing the first and last edges, we obtain a $(2n-1)$-to-(1) map from $\mathcal{M}(n) \to \mathcal{D}(n-1)$. Since $|\mathcal{D}(n-1)| = \frac{1}{n}\binom{2n-2}{n-1}$ by Theorem 2.1, we have: $\mathcal{M}(n) = \frac{2n-1}{n}\binom{2n-2}{n-1} = \frac{1}{2}\binom{2n}{n}$. □

Theorem 6.1. $\lim_{n\to\infty} \frac{|\mathcal{D}(n) \cap -\mathcal{M}(n)|}{|\mathcal{D}(n)|} = \frac{1}{2}$.

Proof. We employ the $(2n-1)$-to-1 map $\mathcal{M}(n) \to \mathcal{D}(n-1)$ from the above proof of Theorem 2.4. Observe that an element of $\mathcal{D}(n-1)$ has a unique minimum if and only if $(2n-2)$ of its $(2n-1)$ pre-images have a unique minimum.

Thus:

$$|\mathcal{D}(n-1) \cap -\mathcal{M}(n-1)| = \frac{1}{2n-2}\,|\mathcal{M}(n) \cap -\mathcal{M}(n)|$$

and hence

$$\lim_{n\to\infty} \frac{|\mathcal{D}(n-1) \cap \mathcal{M}(n-1)|}{|\mathcal{D}(n-1)|} = \lim_{n\to\infty} \frac{\frac{1}{2n-2}\,|\mathcal{M}(n) \cap -\mathcal{M}(n)|}{\frac{1}{2n-1}\,|\mathcal{M}(n)|} = \frac{1}{2}$$

where the last equality is by Theorem 5.9. □

Remark 6.2. As in Remark 5.10, we note that $\frac{|\mathcal{D}(n) \cap -\mathcal{M}(n)|}{|\mathcal{D}(n)|} \geqslant \frac{1}{2}$ for all n.

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