

A note on the γ -coefficients of the tree Eulerian polynomial

Rafael S. González D'León*

Department of Mathematics
University of Kentucky
Lexington, KY, U.S.A. 40506-0027
rafaeldleon@uky.edu

Submitted: Jun 26, 2015; Accepted: Jan 20, 2016; Published: Feb 5, 2016
Mathematics Subject Classifications: 05A15, 05E05

Abstract

We consider the generating polynomial of the number of rooted trees on the set $\{1, 2, \dots, n\}$ counted by the number of descending edges (a parent with a greater label than a child). This polynomial is an extension of the descent generating polynomial of the set of permutations of a totally ordered n -set, known as the Eulerian polynomial. We show how this extension shares some of the properties of the classical one. A classical product formula shows that this polynomial factors completely over the integers. From this product formula it can be concluded that this polynomial has positive coefficients in the γ -basis and we show that a formula for these coefficients can also be derived. We discuss various combinatorial interpretations of these coefficients in terms of leaf-labeled binary trees and in terms of the Stirling permutations introduced by Gessel and Stanley. These interpretations are derived from previous results of Liu, Dotsenko-Khoroshkin, Bershtein-Dotsenko-Khoroshkin, González D'León-Wachs and González D'León related to the free multibracketed Lie algebra and the poset of weighted partitions.

Keywords: Gamma positivity; Eulerian polynomial; Rooted trees

1 introduction

A *labeled rooted tree* T on the set $[n] := \{1, 2, \dots, n\}$ is a tree whose nodes or vertices are the elements of $[n]$ and such that one of its nodes has been distinguished and called the *root*. For nodes x and y in T we say that x is the *child* of y or y is the *parent* of x if y is the first node following x in the unique path from x to the root of T and we

*Supported by NSF Grant DMS 1202755.

say that $y = p(x)$. Nodes that have children are said to be *internal* otherwise we call a node without children a *leaf*. If y is the parent of x , we say that the edge $\{x, y\}$ of T is *descending* (and we call x a *descent* of T) if the label of y is greater than the label of x . We denote $\text{des}(T)$ the number of descents in T . Figure 1 shows all the rooted trees on $[3]$ grouped by the number of descents. We draw the trees with the convention that parents come higher than their children and the root is the highest node. We denote \mathcal{T}_n the set of rooted trees on $[n]$ and $\mathcal{T}_{n,i}$ the set of trees in \mathcal{T}_n with exactly i descents.

For a given $n \geq 1$ define

$$T_n(t) := \sum_{T \in \mathcal{T}_n} t^{\text{des}(T)} = \sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i, \quad (1.1)$$

the descent generating polynomial of \mathcal{T}_n . We call $T_n(t)$ the *tree Eulerian polynomial* in analogy with the classical polynomial $A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{des}(\sigma)}$, that is the descent generating polynomial of the set \mathfrak{S}_n of permutations of $[n]$. We can identify the permutations in \mathfrak{S}_n with the set of rooted trees on $[n]$ that have $n - 1$ internal nodes, each of them having a unique child (and so containing a unique leaf). It is not hard to see that our definition of a descent on this set of trees coincides with the classical definition of descent in a permutation so the polynomial $T_n(t)$ is an extension of the polynomial $A_n(t)$. The polynomials $A_n(t)$ have been extensively studied in the literature and are known with the name of *Eulerian polynomials* since Euler was one of the first in studying them (see [20]). The Eulerian polynomial $A_n(t) = \sum_{k=0}^{n-1} A_{n,i} t^i$ have degree $n - 1$ and its coefficients satisfy the relation

$$A_{n,i} = A_{n,n-1-i}. \quad (1.2)$$

For example, the Eulerian polynomial for $n = 3$ is $A_3(t) = 1 + 4t + t^2$. A polynomial that satisfies Equation 1.2 is called *symmetric* or *palindromic*.

It is a simple observation that a symmetric polynomial $f(t) = \sum_{k=0}^d f_i t^i$ of degree d with $f_i \in \mathbb{Z}$ can be written in the form

$$\sum_{i=0}^d f_i t^i = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (1+t)^{d-2i}, \quad (1.3)$$

where the coefficients $\gamma_i \in \mathbb{Z}$, i.e., the set $\{t^i(1+t)^{d-2i}\}_{i=0}^{\lfloor \frac{d}{2} \rfloor}$, where $\lfloor \cdot \rfloor$ is the integer floor function, is a basis (known as the γ -basis) for the space of symmetric polynomials of degree d with integer coefficients. If $\gamma_i \geq 0$ for all i then we say that the polynomial $f(t)$ is γ -nonnegative and if $\gamma_i > 0$ for all $i \leq \lfloor \frac{d}{2} \rfloor$ then we say that the polynomial $f(t)$ is γ -positive. It is a result of Foata and Schützenberger ([8]) that $A_n(t)$ is γ -positive. And it follows from the work of Foata and Strehl in [9] that its coefficients γ_i have a nice combinatorial interpretation (see also [18]). Indeed, let $\widehat{\mathfrak{S}}_n$ be the set of permutations in \mathfrak{S}_n that have no two adjacent descents and no descent in the last position. Then

$$\gamma_j = \{ \sigma \in \widehat{\mathfrak{S}}_n \mid \text{des}(\sigma) = j \}.$$

For example, $A_3(t) = (1 + t)^2 + 2t$ with $\gamma_0 = 1$ and $\gamma_1 = 2$. The permutations in $\widehat{\mathfrak{S}}_3$ are, 123 with no descents and; 213 and 312 with one descent. Gal [10] and Brändén [2, 3] have introduced the use of the γ -basis in different contexts. Gal conjectured that the γ -coefficients of the h -polynomial of a flag simple polytope are all nonnegative. In particular, $A_n(t)$ is the h -vector of the permutahedron that is a flag simple polytope so Gal's conjecture is confirmed in this case. Postnikov, Reiner and Williams [17] have confirmed Gal's conjecture for the family of chordal nestohedra that is a large family of flag simple polytopes. For more information about γ -nonnegativity see [4].

We will show that the properties discussed above for the Eulerian polynomial $A_n(t)$ are also shared by the polynomial $T_n(t)$ in a similar fashion. The degree of $T_n(t)$ is also $n - 1$ and it is easy to see from the definition of a descent that

$$|\mathcal{T}_{n,i}| = |\mathcal{T}_{n,n-1-i}|, \quad (1.4)$$

so $T_n(t)$ is also symmetric. Indeed there is a natural bijection $\mathcal{T}_{n,i} \simeq \mathcal{T}_{n,n-1-i}$ where the image of a labeled rooted tree $T \in \mathcal{T}_{n,i}$, is the tree in $\mathcal{T}_{n,n-1-i}$ with the same shape of T but where each label i has been replaced by $n + 1 - i$. For the example in Figure 1, $T_3(t) = 2 + 5t + 2t^2$.

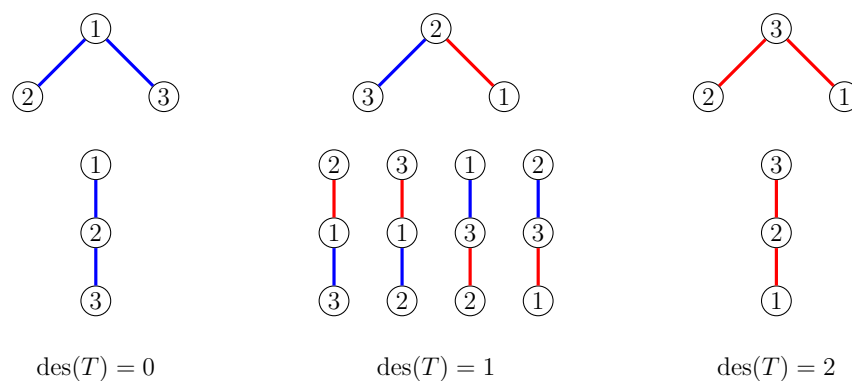


Figure 1: All labeled rooted trees on [3]

The following nice product formula for $T_n(t)$ can be obtained from the results in [7] (see also [12] and [6]).

Theorem 1. For $n \geq 1$,

$$\sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i = \prod_{i=1}^{n-1} ((n - i) + it). \quad (1.5)$$

In particular, setting $t = 1$ in (1.5) reduces to the classical formula $|\mathcal{T}_n| = n^{n-1}$. Equation (1.5) implies that all the roots of this polynomial are real and negative. It is known and not difficult to show that a real-rooted symmetric polynomial with nonnegative real coefficients is γ -nonnegative (see [4, 10]). For example $T_3(t) = 2(1 + t)^2 + t$, so $\gamma_0 = 2$ and $\gamma_1 = 1$. Although $T_n(t)$ is not in general an h -vector of a convex polytope (for example

$h_0 \neq 1$ for $n \geq 3$), it is of interest to find combinatorial formulas and interpretations of nonnegative γ -coefficients of general symmetric polynomials.

Let $\gamma_j(T_n(t))$ denote the j -th γ -coefficient of the symmetric polynomial $T_n(t)$. Equation (1.5) can be used to find a formula for the coefficients γ_j of $T_n(t)$. The values of $\gamma_j(T_n(t))$ for $n = 1, \dots, 7$ appear in Table 1.

Theorem 2. For $n \geq 1$,

$$\gamma_j(T_n(t)) = \begin{cases} \sum_{\substack{J \subset [\frac{n-1}{2}] \\ |J|=j}} \prod_{i \in J} (n-2i)^2 \prod_{s \in [\frac{n-1}{2}] \setminus J} s(n-s) & \text{if } n \text{ is odd} \\ \frac{n}{2} \sum_{\substack{J \subset [\frac{n-2}{2}] \\ |J|=j}} \prod_{i \in J} (n-2i)^2 \prod_{s \in [\frac{n-2}{2}] \setminus J} s(n-s) & \text{if } n \text{ is even} \end{cases} \quad (1.6)$$

Proof. If we multiply

$$\begin{aligned} (n-i+it)(i+(n-i)t) &= (n-i)i + [(n-i)^2 + i^2]t + (n-i)it^2 \\ &= (n-i)i(1+t^2) + [(n-i)^2 + i^2]t \\ &= (n-i)i(1+2t+t^2) + [(n-i)^2 - 2i(n-i) + i^2]t \\ &= (n-i)i(1+t)^2 + (n-2i)^2t. \end{aligned}$$

Equation 1.5 can be written as

$$T_n(t) = \begin{cases} \prod_{i=1}^{\frac{n-1}{2}} [(n-i)i(1+t)^2 + (n-2i)^2t] & \text{if } n \text{ is odd,} \\ \frac{n}{2}(1+t) \prod_{i=1}^{\frac{n-2}{2}} [(n-i)i(1+t)^2 + (n-2i)^2t] & \text{if } n \text{ is even,} \end{cases}$$

implying Formula 1.6. □

n/j	0	1	2	3
1	1			
2	1			
3	2	1		
4	6	8		
5	24	58	9	
6	120	444	192	
7	720	3708	3004	225

Table 1: Values of $\gamma_j(T_n(t))$ for $n = 1, \dots, 7$.

The purpose of this note is to present four different combinatorial interpretations for the coefficients γ_j that are consequences of results in the work of Liu [15], Dotsenko-Khoroshkin [5], Bershtein-Dotsenko-Khoroshkin [1], González D'León-Wachs [14] and González D'León [13]. We present now one of these combinatorial interpretations, whose proof will be given in Section 2.

A *planar leaf-labeled binary tree* with label set $[n]$ is a rooted tree (a priori without labels) in which the set of children of every internal node is a totally ordered set with exactly two elements (the left and right children) and where each leaf has been assigned a unique element from the set $[n]$. By a subtree in a rooted tree T we mean the rooted tree induced by the descendants of any node x of T , including and rooted at x . We say that a planar leaf-labeled binary tree with label set $[n]$ is *normalized* if in each subtree, the leftmost leaf is the one with the smallest label. We denote the set of normalized binary trees with label set $[n]$ by \mathbf{Nor}_n . All normalized trees with leaf labels in $[3]$ are illustrated in Figure 2.

A *right descent* in a normalized tree is an internal node that is the right child of its parent. For $T \in \mathbf{Nor}_n$ we define $\text{rdes}(T) := |\{\text{right descents of } T\}|$. A *double right descent* is a right descent whose parent is also a right descent. We denote by \mathbf{NDRD}_n the set of trees in \mathbf{Nor}_n with no double right descents.

Theorem 3. For $n \geq 1$ and $j \in \{0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$,

$$\gamma_j(T_n(t)) = |\{T \in \mathbf{NDRD}_n \mid \text{rdes}(T) = j\}|.$$

As it is illustrated in Figure 2, there are two trees in \mathbf{NDRD}_3 (for $n = 3$ it happens to be equal to \mathbf{Nor}_3) with $\text{rdes}(T) = 0$ and one with $\text{rdes}(T) = 1$, corresponding to $\gamma_0 = 2$ and $\gamma_1 = 1$ respectively.

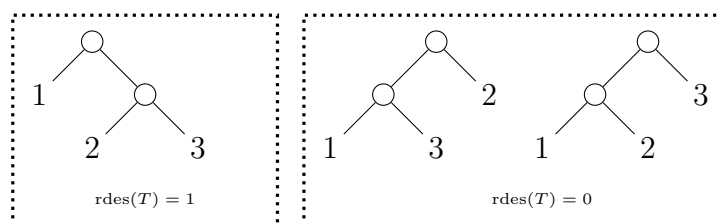


Figure 2: Set of trees in $\mathbf{NDRD}_3 = \mathbf{Nor}_3$

In Section 2 we provide the proof of Theorem 3 and an additional version of Theorem 3 also in terms of normalized trees but with a statistic different than rdes . In Section 3 we provide two additional versions of Theorem 3 in terms of the Stirling permutations introduced by Gessel and Stanley in [11]. In Section 4 we discuss a generalization of the γ -positivity of $T_n(t)$ to the positivity of certain symmetric function in the basis of elementary symmetric functions.

2 Combinatorial interpretations in terms of binary trees

Now we consider normalized trees T where every internal node x of T has been assigned an element $\mathbf{color}(x) \in \{0, 1\}$. We call an element of this set of trees a *bicolored normalized tree* on $[n]$. A *bicolored comb* is a bicolored normalized tree T satisfying the following coloring restriction:

(C) If x is a right descent of T then $\mathbf{color}(x) = 0$ and $\mathbf{color}(p(x)) = 1$.

We denote by \mathbf{Comb}_n the set of bicolored combs and by $\mathbf{Comb}_{n,i}$ the set of bicolored combs where i internal nodes have been colored 1 (and $n - 1 - i$ colored 0). Figure 3 illustrates the bicolored combs on $[3]$ grouped by the number of internal nodes that have been colored 1.

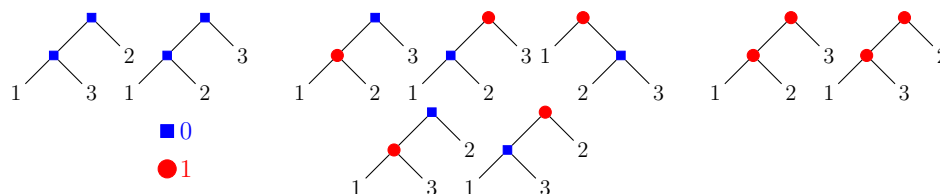


Figure 3: Set of bicolored combs on $[3]$

Denote by $\tilde{T} \in \mathbf{Nor}_n$ the underlying uncolored normalized tree associated to a tree $T \in \mathbf{Comb}_n$. Note that the coloring condition (C) implies that $\tilde{T} \in \mathbf{NDRD}_n$. Indeed, in a double right descent the coloring condition (C) cannot be satisfied since the parent of a double right descent is also a right descent. Also note that the monochromatic combs in $\mathbf{Comb}_{n,0}$ and $\mathbf{Comb}_{n,n-1}$ are just the traditional left combs that are described in [21] and that index a basis for the space $\mathcal{L}ie(n)$, the multilinear component of the free Lie algebra over \mathbb{C} on n generators (see [21] for details). Liu [15] and Dotsenko-Khoroshkin [5] independently proved a conjecture of Feigin regarding the dimension of the multilinear component $\mathcal{L}ie_2(n)$ of the free Lie algebra with two compatible brackets, a generalization of $\mathcal{L}ie(n)$.

Theorem 4 ([5, 15]). *For $n \geq 1$, $\dim \mathcal{L}ie_2(n) = |\mathcal{T}_n|$.*

In particular, the space $\mathcal{L}ie_2(n)$ has the decomposition

$$\mathcal{L}ie_2(n) = \bigoplus_{i=0}^{n-1} \mathcal{L}ie_2(n, i),$$

where the subspace $\mathcal{L}ie_2(n, i)$ is the component generated by certain “bracketed permutations” with exactly i brackets of one of the types. Liu finds the following formula for the dimension of $\mathcal{L}ie_2(n, i)$.

Theorem 5 ([15, Proposition 11.3]). *For $n \geq 1$ and $i \in \{0, 1, \dots, n - 1\}$,*

$$\dim \mathcal{L}ie_2(n, i) = |\mathcal{T}_{n,i}|.$$

In [1] Bershtein, Dotsenko and Khoroshkin found a basis for $\mathcal{L}ie_2(n, i)$ indexed by the elements of $\mathbf{Comb}_{n,i}$ giving the following alternative description for the dimension of $\mathcal{L}ie_2(n, i)$.

Theorem 6 ([1, Lemma 5.2]). For $n \geq 1$ and $i \in \{0, 1, \dots, n-1\}$,

$$\dim \mathcal{L}ie_2(n, i) = |\mathbf{Comb}_{n,i}|.$$

Corollary 7. For every $n \geq 1$ and $i \in \{0, \dots, n-1\}$,

$$|\mathbf{Comb}_{n,i}| = |\mathcal{T}_{n,i}|.$$

Problem 8. Find an explicit bijection $\mathbf{Comb}_{n,i} \rightarrow \mathcal{T}_{n,i}$, for every $n \geq 1$ and $i \in \{0, \dots, n-1\}$.

Theorem 9 (Theorem 3). For $n \geq 1$ and $j \in \{0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$,

$$\gamma_j(T_n(t)) = |\{T \in \mathbf{NDRD}_n \mid \text{rdes}(T) = j\}|.$$

Proof. First note that by the comments above if $T \in \mathbf{Comb}_n$ then its underlying uncolored tree $\tilde{T} \in \mathbf{NDRD}_n$.

For a tree $T \in \mathbf{Comb}_n$ call $\text{free}(T)$ the number of internal nodes that are not right descents and whose right child is a leaf. Then in \mathbf{NDRD}_n we have that

$$\text{free}(T) + 2\text{rdes}(T) = n - 1.$$

Over the set of bicolored combs with m free nodes there is a free action of $(\mathbb{Z}_2)^m$ by toggling the colors of the free nodes. Then there are 2^m bicolored combs with the same underlying tree $\tilde{T} \in \mathbf{NDRD}_n$. By Corollary 7 we can write $T_n(t)$ as

$$\begin{aligned} T_n(t) &= \sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i \\ &= \sum_{i=0}^{n-1} |\mathbf{Comb}_{n,i}| t^i \\ &= \sum_{T \in \mathbf{Comb}_n} t^{|\{x \text{ internal in } T \mid \text{color}(x)=1\}|} \\ &= \sum_{\mathfrak{T} \in \mathbf{NDRD}_n} \sum_{\substack{T \in \mathbf{Comb}_n \\ \tilde{T}=\mathfrak{T}}} t^{|\{x \text{ internal in } T \mid \text{color}(x)=1\}|} \\ &= \sum_{\mathfrak{T} \in \mathbf{NDRD}_n} t^{\text{rdes}(\mathfrak{T})} (1+t)^{\text{free}(\mathfrak{T})} \\ &= \sum_{\mathfrak{T} \in \mathbf{NDRD}_n} t^{\text{rdes}(\mathfrak{T})} (1+t)^{n-1-2\text{rdes}(\mathfrak{T})}. \end{aligned}$$

□

2.1 A second description in terms of normalized trees

In [14] the author and Wachs studied the relation between $\mathcal{L}ie_2(n, i)$ and the cohomology of the maximal intervals of a poset of weighted partitions. Using poset topology techniques they found an alternative description for the dimension of $\mathcal{L}ie_2(n, i)$.

Define the *valency* $v(x)$ of a node (internal or leaf) x of $T \in \mathbf{Nor}_n$ to be the minimal label in the subtree of T rooted at x . For an internal node x of T let $L(x)$ and $R(x)$ denote the left and right children of x respectively. A *Lyndon node* is an internal node x of T such that

$$v(R(L(x))) > v(R(x)). \quad (2.1)$$

A *Lyndon tree* is a normalized tree in which all its internal nodes are Lyndon. We denote $\text{nlyn}(T)$ the number of non-Lyndon nodes in T . A *double non-Lyndon node* is a non-Lyndon node that is the left child of its parent and its parent is also a non-Lyndon node. We denote the set of trees in \mathbf{Nor}_n with no double non-Lyndon nodes by \mathbf{NDNL}_n . A *bicolored Lyndon tree* is a bicolored normalized tree satisfying the coloring condition:

- (L) For every non-Lyndon node x of T then $\mathbf{color}(x) = 0$ and $\mathbf{color}(L(x)) = 1$.

The set of bicolored Lyndon trees is denoted \mathbf{Lyn}_n and the set of the ones with exactly i nodes with color 1 is denoted $\mathbf{Lyn}_{n,i}$.

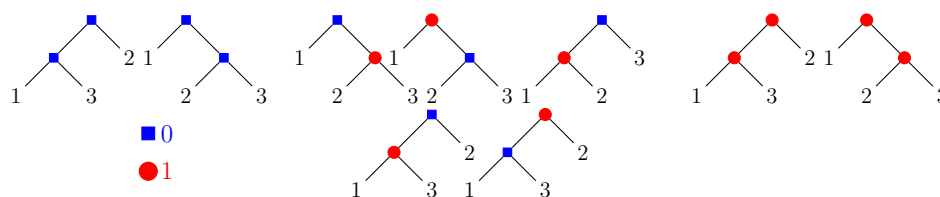


Figure 4: Set of bicolored Lyndon trees on $[3]$

Theorem 10 ([14, Section 5]). *For every $n \geq 1$ and $i \in \{0, \dots, n-1\}$,*

$$\dim \mathcal{L}ie_2(n, i) = |\mathbf{Lyn}_{n,i}|.$$

Hence,

$$|\mathbf{Lyn}_{n,i}| = |\mathcal{T}_{n,i}|.$$

The proof of the following theorem follows the same arguments of the proof of Theorem 3.

Theorem 11. *For $n \geq 1$ and $j \in \{0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$,*

$$\gamma_j(T_n(t)) = |\{T \in \mathbf{NDNL}_n \mid \text{nlyn}(T) = j\}|.$$

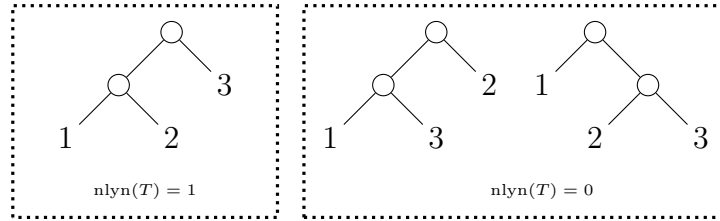


Figure 5: Set of trees in $\text{NDNL}_3 = \text{Nor}_3$

3 Combinatorial interpretation in terms of Stirling permutations

Consider now the set of multipermutations of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ such that all numbers between the two occurrences of any number m are larger than m . To this family belongs for example the permutation 12234431 but not 11322344 since 2 is less than 3 and 2 is between the two occurrences of 3. This family (denoted \mathcal{Q}_n) of permutations was introduced by Gessel and Stanley in [11] and the permutations in \mathcal{Q}_n are known as *Stirling Permutations*.

For a permutation $\theta = \theta_1\theta_2 \dots \theta_{2n}$ in \mathcal{Q}_n we say that the position i contains a *first occurrence* of a letter if $\theta_j \neq \theta_i$ for all $j < i$, otherwise we say that it contains a *second occurrence*. An *ascending adjacent pair* in θ is a pair (a, b) such that $a < b$ and in θ the second occurrence of a is the immediate predecessor of the first occurrence of b . An *ascending adjacent sequence (of length 2)* is a sequence $a < b < c$ such that (a, b) and (b, c) are both ascending adjacent pairs. For example, in $\theta = 13344155688776$ the ascending adjacent pairs are $(1, 5)$, $(5, 6)$ and $(3, 4)$ but the only ascending adjacent sequence is $1 < 5 < 6$. We denote NAAS_n the set of all Stirling permutations in \mathcal{Q}_n that do not contain ascending adjacent sequences. Similarly, a *terminally nested pair* in θ is a pair (a, b) such that $a < b$ and in θ the second occurrence of a is the immediate successor of the second occurrence of b . A *terminally nested sequence (of length 2)* is a sequence $a < b < c$ such that (a, b) and (b, c) are both terminally nested pairs. For example, in $\theta = 13443566518877$ the terminally nested pairs are $(1, 5)$, $(5, 6)$ and $(3, 4)$ but the only terminally nested sequence is $1 < 5 < 6$. We denote NTNS_n the set of all Stirling permutations in \mathcal{Q}_n that do not contain terminally nested sequences. For $\sigma \in \mathcal{Q}_n$, we denote $\text{aapair}(\sigma)$ the number of ascending adjacent pairs in σ and $\text{tnpair}(\sigma)$ the number of terminally nested pairs in σ .

The following result in [13] relates the statistics above in \mathcal{Q}_{n-1} with the ones previously discussed for Nor_n .

Proposition 12 ([13, Proposition 4.8]). *There is a bijection $\phi : \text{Nor}_n \rightarrow \mathcal{Q}_{n-1}$ such that for every $T \in \text{Nor}_n$,*

1. $\text{rdes}(T) = \text{tnpair}(\phi(T))$
2. $\text{nlyn}(T) = \text{aapair}(\phi(T))$

$$3. \phi(NDRD_n) = NAAS_{n-1}$$

$$4. \phi(NDNL_n) = NTNS_{n-1}.$$

Corollary 13. For $n \geq 1$ and $j \in \{0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$,

$$\begin{aligned} \gamma_j(T_n(t)) &= |\{T \in NTNS_{n-1} \mid \text{tnpair}(T) = j\}| \\ &= |\{T \in NAAS_{n-1} \mid \text{aapair}(T) = j\}|. \end{aligned}$$

Example 14. The Stirling permutations in \mathcal{Q}_2 are 1122, 1221 and 2211. In this particular case $\mathcal{Q}_2 = NAAS_2 = NTNS_2$ and the statistics in Table 2 imply that $\gamma_0 = 2$ and $\gamma_1 = 1$ are the γ -coefficients of the polynomial $T_3(t)$.

σ	tnpair	aapair
1122	0	1
1221	1	0
2211	0	0

Table 2: tnpair and aapair statistics in \mathcal{Q}_2 .

4 A comment about γ -positivity and e -positivity

Let $\mathbf{x} = x_1, x_2, \dots$ be an infinite set of variables and $\Lambda = \Lambda_{\mathbb{Q}}$ the ring of symmetric functions with rational coefficients on the variables \mathbf{x} , that is, the ring of power series on \mathbf{x} of bounded degree that are invariant under permutation of the variables.

Define $e_0 := 1$, for $n \geq 1$

$$e_n := \sum_{1 \leq i_1 < i_2 < \dots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n},$$

and for an integer partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ (i.e., a weakly decreasing finite sequence of positive integers) $e_\lambda := \prod_i e_{\lambda_i}$. We call e_λ the *elementary symmetric function* corresponding to the partition λ .

It is known that for $n \geq 0$, the set $\{e_\lambda \mid \lambda \vdash n\}$ is a basis for the n -th homogeneous graded component of Λ , where the grading is with respect to degree. See [16] and [19] for more information about symmetric functions.

Note that if we make the specialization $x_i \mapsto 0$ in Λ for all $i \geq 3$ then $e_1 \mapsto x_1 + x_2$, $e_2 \mapsto x_1 x_2$ and $e_i \mapsto 0$ for all $i \geq 3$. Thus for a partition λ of n (i.e., $\sum_i \lambda_i = n$) the symmetric function $e_\lambda \mapsto 0$ unless $\lambda = (2^j, 1^{n-2j})$ for some $j \in \mathbb{N}$. In that case,

$$e_{(2^j, 1^{n-2j})} \mapsto (x_1 x_2)^j (x_1 + x_2)^{n-2j}.$$

If we further replace $x_1 \mapsto 1$ and $x_2 \mapsto t$ we obtain

$$e_{(2^j, 1^{n-2j})} \mapsto t^j (1+t)^{n-2j}.$$

Let $\mathbf{x}^\mu := x_1^{\mu(1)} x_2^{\mu(2)} \dots$ and

$$F_{\text{MComb}_n}(\mathbf{x}) := \sum_{C \in \text{MComb}_n} \mathbf{x}^{\mu(C)}.$$

The following theorem is a consequence of the definition of a colored comb, the definition of the symmetric functions $e_i(\mathbf{x})$ and the observations above (see [13]).

Theorem 15 ([13]). *For $n \geq 1$*

$$F_{\text{MComb}_n}(\mathbf{x}) = \sum_{T \in \text{Nor}_n} e_{\lambda(T)}(\mathbf{x}).$$

Note that $F_{\text{MComb}_n}(1, t, 0, 0, \dots) = \sum_{C \in \text{Comb}_n} t^{\text{red } C} = T_n(t)$ and so Theorem 15 is a generalization of Theorem 3.

Remark 16. Versions of Theorem 15 can also be given in terms of a completely different type on the set Nor_n corresponding to a family of multicolored Lyndon trees and also in terms of colored Stirling permutations, see [13].

Acknowledgments

The author would like to thank Ira Gessel and an anonymous referee for pointing out the correct references for some of the classical results mentioned in this note.

References

- [1] M. Bershtein, V. Dotsenko, and A. Khoroshkin. Quadratic algebras related to the bi-Hamiltonian operad. *Int. Math. Res. Not. IMRN*, (24):Art. ID rnm122, 30, 2007.
- [2] P. Brändén. Sign-graded posets, unimodality of W -polynomials and the Charney-Davis conjecture. *Electron. J. Combin.*, 11(2):Research Paper 9, 15 pp. (electronic), 2004/06.
- [3] P. Brändén. Actions on permutations and unimodality of descent polynomials. *European J. Combin.*, 29(2):514–531, 2008.
- [4] P. Brändén. Unimodality, log-concavity, real-rootedness and beyond. *ArXiv preprint arXiv:1410.6601*, 2014.
- [5] V. V. Dotsenko and A. S. Khoroshkin. Character formulas for the operad of a pair of compatible brackets and for the bi-Hamiltonian operad. *Funktsional. Anal. i Prilozhen.*, 41(1):1–22, 96, 2007.
- [6] B. Drake. *An inversion theorem for labeled trees and some limits of areas under lattice paths*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)—Brandeis University.
- [7] Ö. Eğecioğlu and J. B. Remmel. Bijections for Cayley trees, spanning trees, and their q -analogues. *J. Combin. Theory Ser. A*, 42(1):15–30, 1986.

- [8] D. Foata and M.-P. Schützenberger. *Théorie géométrique des polynômes eulériens*. Lecture Notes in Mathematics, Vol. 138. Springer-Verlag, Berlin-New York, 1970.
- [9] D. Foata and V. Strehl. Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers. *Math. Z.*, 137:257–264, 1974.
- [10] S. R. Gal. Real root conjecture fails for five- and higher-dimensional spheres. *Discrete Comput. Geom.*, 34(2):269–284, 2005.
- [11] I. Gessel and R. P. Stanley. Stirling polynomials. *J. Combinatorial Theory Ser. A*, 24(1):24–33, 1978.
- [12] I. M. Gessel and S. Seo. A refinement of Cayley’s formula for trees. *Electron. J. Combin.*, 11(2):Research Paper 27, 23, 2004/06.
- [13] R. S. González D’León. On the free Lie algebra with multiple brackets. *arXiv preprint [arXiv:1408.5415](https://arxiv.org/abs/1408.5415)*, 2014.
- [14] R. S. González D’León and M. L. Wachs. On the (co) homology of the poset of weighted partitions. *arXiv preprint [arXiv:1309.5527](https://arxiv.org/abs/1309.5527)*, 2013.
- [15] F. Liu. Combinatorial bases for multilinear parts of free algebras with two compatible brackets. *J. Algebra*, 323(1):132–166, 2010.
- [16] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995.
- [17] A. Postnikov, V. Reiner, and L. Williams. Faces of generalized permutohedra. *Doc. Math.*, 13:207–273, 2008.
- [18] L. W. Shapiro, W. J. Woan, and S. Getu. Runs, slides and moments. *SIAM J. Algebraic Discrete Methods*, 4(4):459–466, 1983.
- [19] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.
- [20] R. P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.
- [21] M. L. Wachs. On the (co)homology of the partition lattice and the free Lie algebra. *Discrete Math.*, 193(1-3):287–319, 1998. Selected papers in honor of Adriano Garsia (Taormina, 1994).