The nonexistence of a distance-regular graph with intersection array {22,16,5;1,2,20}

Supalak Sumalroj   Chalermpong Worawannotai*

Department of Mathematics
Silpakorn University
Nakhon Pathom, Thailand

{sumalroj_s,worawannotai_c}@silpakorn.edu

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Abstract

We prove that a distance-regular graph with intersection array $\{22, 16, 5; 1, 2, 20\}$ does not exist. To prove this, we assume that such a graph exists and derive some combinatorial properties of its local graph. Then we construct a partial linear space from the local graph to display the contradiction.

Keywords: distance-regular graph; nonexistence; partial linear space

1 Introduction

One of the main problems in distance-regular graphs is to decide whether a distance-regular graph with a given intersection array exists. Brouwer, Cohen and Neumaier [3] have compiled a list of intersection arrays that passed known feasibility conditions, but the existence of the corresponding distance-regular graphs was unknown for many of those arrays. Since then the arrays from the list are studied and the existence and nonexistence of distance-regular graphs associated to many arrays from the list are proved [5, Section 17] but more than half are still unknown.

In this paper we investigate the intersection array $\{22, 16, 5; 1, 2, 20\}$ [3, pp. 427]. If a distance-regular graph with such array exists, then the number of vertices is $243 = 3^5$, which is relatively small, and the valency is 22. Moreover, the parameter μ equals 2, which is a very interesting case (it means that every two nonadjacent vertices have either 0 or 2 common neighbors). From [3] the spectrum of the graph is $22^1 7^{66} (-2)^{132} (-5)^{44}$ and the distribution diagram is shown in Figure 1.

*Corresponding author

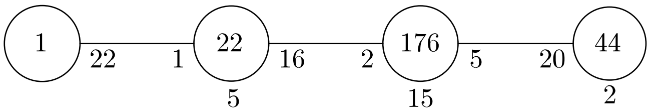

Figure 1: Distribution diagram for a distance-regular graph with intersection array {22,16,5;1,2,20}.

In addition, the distance-two graph is a strongly regular graph whose parameters are $(243, 176, 130, 120)$; according to Brouwer [2], it is unknown whether such a strongly regular graph exists. Incidentally, there is a very interesting strongly regular graph on 243 vertices, valency 22, and $\mu = 2$, the Berlekamp-Van Lint-Seidel graph, that corresponds to the ternary Golay code [1].

In this paper we prove, however, that a distance-regular graph with intersection array $\{22, 16, 5; 1, 2, 20\}$ does not exist. Our method for showing this is inspired by [4] where the author cleverly partitioned a local graph of a hypothetical distance-regular graph with intersection array $\{21, 16, 8; 1, 4, 14\}$ and constructed a partial linear space on the partition. The paper is organized as follows. In Section 2 we recall some definitions and properties of distance-regular graphs. In Section 3 we assume that such a distance-regular graph exists and derive some combinatorial properties of its local graph. Then we construct a partial linear space from the local graph to display the contradiction.

2 Preliminaries

A *simple graph* is a graph having no loops or parallel edges. All graphs we consider are simple. For any graph Γ, we identify Γ with its vertex set $V(\Gamma)$, and let $E(\Gamma)$ be its edge set. We denote the subgraph of Γ induced by a subset S of $V(\Gamma)$ by S itself. For a subset S of $V(\Gamma)$, the *neighborhood* of S in Γ, denoted by $N_\Gamma(S)$, is the set of all vertices in $\Gamma - S$ that are adjacent to at least one vertex of S. For a vertex x in Γ, the subgraph of Γ induced by the neighbors of x is called the *local graph* of Γ with respect to x. A walk $C = v_0 e_1 v_1 e_2 \dots e_{n-1} v_{n-1} e_n v_0$ is called a *cycle* if the edges $e_1, e_2, \dots, e_n$ and the vertices $v_0, v_1, \dots, v_{n-1}$ of C are distinct and C has at least 3 edges. A cycle C has *length* n if the number of edges of C is n. A *complete graph* is a simple graph in which any two distinct vertices are adjacent. A complete graph with n vertices is denoted by K_n.

For vertices u and v in Γ, the *distance* between u and v is the length of a shortest path between u and v in Γ. The *diameter* of Γ is the greatest distance between any pair of vertices in Γ. A *clique* of a graph Γ is a maximal complete subgraph of Γ. The *eigenvalues* of Γ are the eigenvalues of its adjacency matrix.

Let Γ be a connected graph with diameter d and a vertex set V. For $x \in V$ let $\Gamma_i(x)$ be the set of vertices at distance i from x. The graph Γ is called *distance-regular* if for all vertices x and y at distance i, the numbers $b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|$, $c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|$ and $a_i = |\Gamma_i(x) \cap \Gamma_1(y)|$ depend only on i. In particular, Γ is a regular graph of degree

$k = b_0$ and $c_i + a_i + b_i = k$ for all $0 \leqslant i \leqslant d$. The sequence $\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}$ is called the *intersection array* of Γ.

The following proposition gives an upper bound of the size of a clique of a distance-regular graph in terms of its smallest and largest eigenvalues.

Proposition 1. [3, Proposition 4.4.6] *Let Γ be a distance-regular graph of diameter $d \geqslant 2$ with eigenvalues $k = \theta_0 > \theta_1 > \cdots > \theta_d$. Then the size of a clique K in Γ is bounded by*

$$|K| \leqslant 1 - k/\theta_d.$$

An *incidence geometry* (P, L) consists of a set P whose elements are called *points* and a set L whose elements are called *lines* together with an *incidence relation* between points and lines, that is, a subset of $P \times L$. A *partial linear space* is an incidence geometry such that every pair of distinct points lie on at most one common line and every line has at least two points.

3 Main results

From now on we assume that Γ is a distance-regular graph with intersection array $\{22,16,5;1,2,20\}$. Then Γ has eigenvalues $22, 7, -2$ and -5. Fix a vertex x of Γ. Let $\Delta = \Gamma_1(x)$ be the subgraph of Γ induced by all vertices of Γ adjacent to x. Then Δ is a regular graph with 22 vertices and degree 5. The following results give some combinatorial properties of the local graph Δ.

Corollary 2. *Δ does not contain a complete subgraph K_i for all $i \geqslant 5$.*

Proof. By Proposition 1, the size of a clique in Γ is at most 5. Thus the size of a clique in Δ is at most 4. □

Lemma 3. *If Δ contains a cycle C of length 4, then the subgraph induced by C is a complete graph K_4.*

Proof. Suppose that Δ contains a cycle C of length 4. Suppose there exist vertices u and v of C that are not adjacent in Δ. Then the distance between u and v is 2 and there exist two distinct paths from u to v of length 2 in C and a path uxv in Γ which contradicts the fact that $c_2 = 2$. Thus any two distinct vertices of C are adjacent. Therefore the subgraph induced by C is a complete graph K_4. □

Lemma 4. *Each vertex in Δ is on at least two subgraphs K_3's of Δ.*

Proof. Suppose there exists a vertex $v \in \Delta$ which is on at most one subgraph K_3 of Δ. Let v_1, v_2, v_3, v_4 and v_5 be the distinct neighbors of v in Δ. Then there is at most one edge joining these neighbors of v. By Lemma 3, v is the only common neighbor of v_i and v_j for all $1 \leqslant i < j \leqslant 5$. Therefore the vertex set of Δ contains v, its neighbors, and at least $(3 \times 2) + (4 \times 3)$ vertices at distance 2 from v. Hence the number of vertices of Δ is at least 24, a contradiction. Therefore each vertex in Δ is on at least two subgraphs K_3's of Δ. □

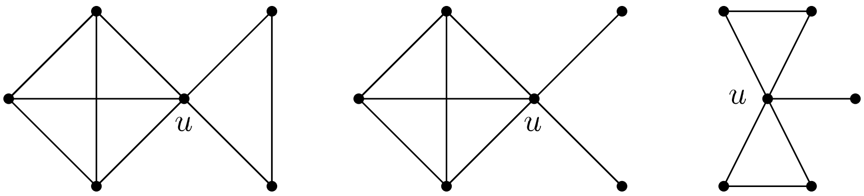

Figure 2: The 3 possibilities for the subgraph of Δ induced by a vertex u and its neighbors.

By Corollary 2 and Lemma 4, there are 3 possibilities for the subgraph of Δ induced by a vertex u and its neighbors as shown in Figure 2.

Lemma 5. Δ *contains a complete subgraph* K_4.

Proof. Suppose not. Then the subgraph of Δ induced by a vertex in Δ and its neighbors must be isomorphic to the graph on the right in Figure 1. Thus each vertex in Δ is on exactly two K_3's so $|\{(u, K_3)|K_3 \subseteq \Delta, u \in K_3\}| = 22 \times 2 = 44$. Since the number of vertices of K_3 is three, $3|44$, a contradiction. Thus Δ contains a complete subgraph K_4. □

Now we partition the vertex set of the local graph Δ. For the rest of the paper, fix a complete subgraph K on four vertices of Δ. Let $S = \Delta_1(K) = \{y \in \Delta - K | y$ is adjacent to some vertices in $K\}$ be the neighborhood of K in Δ and define $R = \Delta - K - S$.

Lemma 6. *K has size* 4*, S has size* 8*, and R has size* 10.

Proof. Clearly, $|K| = 4$. Let u_1, u_2, u_3 and u_4 be the vertices in K. Since Δ is a regular graph of degree 5, for each $1 \leqslant i \leqslant 4$ there exist two vertices in S which are adjacent to u_i. If u_i and u_j have a common neighbor s in S for some $1 \leqslant i < j \leqslant 4$, then by Lemma 3, s is adjacent to u_l for all $1 \leqslant l \leqslant 4$ and hence $\{s, u_1, u_2, u_3, u_4\}$ induces a K_5 in Δ which contradicts Corollary 2. Thus u_i and u_j have no common neighbors in S for all $1 \leqslant i < j \leqslant 4$. Therefore $|S| = 8$, and hence $|R| = |\Delta| - |K| - |S| = 22 - 4 - 8 = 10$. □

Let u_1, u_2, u_3 and u_4 be the vertices of K. For $1 \leqslant i \leqslant 4$ let s_{2i-1} and s_{2i} be the vertices of S which are adjacent to u_i.

Lemma 7. *The only possible edges in S are $s_{2i-1}s_{2i}$ for $1 \leqslant i \leqslant 4$. Moreover, the vertices s_{2i-1} and s_{2i} have no common neighbors in R.*

Proof. The result follows from Lemma 3. □

To further investigate the structure of R we define an incidence geometry $G = (R, S)$ where elements of R are regarded as points and elements of S are regarded as lines, and a point $r \in R$ is on a line $s \in S$ if and only if the vertices r and s are adjacent in Γ.

Lemma 8. *G is a partial linear space. Moreover each line in G is incident with at least* 3 *points.*

Proof. Suppose two distinct points r and r' of R are incident with two distinct lines s and s'. Then the vertices s, r, s' and r' form a cycle in Δ. By Lemma 3, the vertices s and s' are adjacent. Thus by Lemma 7 the vertices s and s' are adjacent to a common vertex u in K. Now u, s, r and s' form a cycle in Δ. By Lemma 3, the vertices u and r are adjacent, a contradiction. Thus every pair of distinct points lie on at most one common line.

By Lemma 7 and since Δ is a regular graph of degree 5, it follows that each vertex of S is adjacent to at least 3 vertices of R, that is, each line in S is incident with at least 3 points in R. Therefore G is a partial linear space. □

Lemma 9. *One of the following two conditions holds:*
1). The number of edges in S is 3*. The number of edges in R is* 12*. The number of edges between S and R is* 26*.*
2). The number of edges in S is 4*. The number of edges in R is* 13*. The number of edges between S and R is* 24*.*

Proof. First we will show that the subgraph induced by S contains at least 3 edges.

Without loss of generality, we may assume that s_7 and s_8 are not adjacent. Then s_7 and s_8 are lines of size 4 in G. By Lemma 7, the lines s_7 and s_8 have no common points.

Suppose that s_1 is a line of size 4 in G. Then s_1 and s_2 are not adjacent and hence s_2 is also a line of size 4 in G. By Lemma 7, the lines s_1 and s_2 have no common points. Since every pair of distinct points lie on at most one common line and $|R| = 10$, the line s_1 is incident with one point of s_7, one point of s_8 and other two points not on s_7 or s_8. Similarly, the line s_2 is incident with one point of s_7, one point of s_8 and two points not on s_1, s_7 or s_8. Thus G has more than 10 points, a contradiction. Therefore s_1 is a line of size 3 in G. Similarly, s_i is a line of size 3 in G for all $2 \leqslant i \leqslant 6$.

Thus s_{2i-1} is adjacent to s_{2i} for all $1 \leqslant i \leqslant 3$ and hence the subgraph induced by S contains at least 3 edges.

If S contains exactly 4 edges, then the number of edges between S and R is $3 \times 8 = 24$ and the number of edges in R is $(5 \times 10 - 24)/2 = 13$. If S contains exactly 3 edges, then the number of edges between S and R is $(3 \times 6) + (4 \times 2) = 26$ and the number of edges in R is $(5 \times 10 - 26)/2 = 12$. □

Lemma 10. *Each vertex in R has degree at least* 2 *in R. Moreover there are at least* 4 *vertices in R with degree* 2 *in R.*

Proof. If a vertex r in R is adjacent to 5 vertices in S, then r is adjacent to s_{2i-1} and s_{2i} for some $1 \leqslant i \leqslant 4$. The vertices r, s_{2i-1}, u_i and s_{2i} form a cycle in Δ. By Lemma 3, the vertices u_i and r are adjacent, a contradiction. Thus each vertex in R is adjacent to at most 4 vertices in S.

Suppose that there exists a vertex r_1 in R such that the number of edges from r_1 to S is 4. By Lemma 3, we may assume that r_1 is adjacent to s_1, s_3, s_5 and s_7. By Lemma 4 applied to r_1, there exist $i, j \in \{1, 3, 5, 7\}$, $i \neq j$, such that s_i and s_j are adjacent which contradicts Lemma 7. Thus there are no vertices in R which are adjacent to 4 vertices in S. That is each vertex in R has degree at least 2 in R.

If there are at most 3 vertices in R with degree 2 in R, then the number of edges between R and S is less than or equal to $(3 \times 3) + (7 \times 2) = 23$ which contradicts Lemma 9. Thus there are at least 4 vertices in R with degree 2 in R. $\square$

By Lemma 9 and Lemma 10, there are 8 possibilities for the degree sequence of R as shown in Table 1.

The number of vertices in the induced subgraph R with degree i				$\|E(R)\|$
$i=2$	$i=3$	$i=4$	$i=5$	
4	6	0	0	13
5	4	1	0	13
6	3	0	1	13
6	2	2	0	13
6	4	0	0	12
7	2	1	0	12
8	0	2	0	12
8	1	0	1	12

Table 1: The 8 possibilities for the degree sequence of R.

By Lemma 9, either $|E(R)| = 12$ or $|E(R)| = 13$. We now rule out both possibilities. We start with the latter.

Lemma 11. $|E(R)| \neq 13$.

Proof. Suppose that $|E(R)| = 13$. By Lemma 9, the subgraph induced by S contains 4 edges and the number of edges between S and R is 24. Thus each vertex in S is adjacent to 3 vertices in R. By Lemma 3 and Lemma 4, there are 8 distinct edges $e_1, e_2, ldots, e_8$ in R such that s_i is adjacent to both ends of e_i for $1 \leqslant i \leqslant 8$. Let $T = \{e_1, e_2, \ldots, e_8\}$.

Suppose that there exists a vertex $r \in R$ which has degree 5 in R. Let r_1, r_2, r_3, r_4 and r_5 be the distinct neighbors of r in R. Then for each $i \in \{1, 2, 3, 4, 5\}$, $rr_i \notin T$. Since R has 13 edges, $E(R) - \{rr_1, rr_2, rr_3, rr_4, rr_5\} = T$. By Lemma 4 applied to r, we may assume that r_1 and r_2 are adjacent. Thus $e_i = r_1r_2$ for some $1 \leqslant i \leqslant 8$. So the vertices s_i, r_1, r and r_2 form a cycle in Δ and hence r is adjacent to s_i, a contradiction. Therefore each vertex in R has degree at most 4 in R. By Lemma 10, each vertex in R is adjacent to 1, 2 or 3 vertices in S.

Now suppose that r is a vertex in R with degree 3 in R. Let $N_R(r) = \{r_1, r_2, r_3\}$. Without loss of generality, we may assume that $N_S(r) = \{s_1, s_3\}$.
<u>Case 1</u> : s_i and r_j are not adjacent for all $i \in \{1, 3\}$ and $j \in \{1, 2, 3\}$.

Then r_j and r_k are adjacent for all $1 \leqslant j < k \leqslant 3$ by Lemma 4 applied to r. By Lemma 3, the edges $rr_1, rr_2, rr_3, r_1r_2, r_1r_3, r_2r_3 \notin T$. Since R contains 13 edges,

$8 = |T| \leqslant |E(R) - \{rr_1, rr_2, rr_3, r_1r_2, r_1r_3, r_2r_3\}| = 7$, a contradiction. Thus Case 1 cannot occur.

<u>Case 2</u> : s_1 is adjacent to exactly one vertex in $\{r_1, r_2, r_3\}$.

Without loss of generality, we may assume that s_1 is adjacent to r_3. Then s_1 is not adjacent to r_1 and r_2. Since s_1 is adjacent to 3 vertices in R, there exists a vertex $r_4 \in R - \{r, r_1, r_2, r_3\}$ such that r_4 is adjacent to s_1. By Lemma 3, the vertex s_2 is not adjacent to r_i for $1 \leqslant i \leqslant 4$. Since s_2 is adjacent to 3 vertices in R, there exist $r_5, r_6, r_7 \in R - \{r, r_1, r_2, r_3, r_4\}$ such that r_5, r_6, r_7 are adjacent to s_2. Since R has 10 vertices, there exist $r_8, r_9 \in R - \{r, r_i | 1 \leqslant i \leqslant 7\}$. By Lemma 3, r_4 is not adjacent to r_i for $1 \leqslant i \leqslant 7$. By Lemma 10, r_4 is adjacent to r_8 and r_9. By Lemma 3, r_3 is not adjacent to r_i for $1 \leqslant i \leqslant 9$. Thus r_3 has degree 1 in R, a contradiction to Lemma 10. Hence Case 2 cannot occur.

<u>Case 3</u> : s_1 is adjacent to exactly two vertices in $\{r_1, r_2, r_3\}$.

Without loss of generality, we may assume that s_1 is adjacent to r_2 and r_3. Then s_1 is not adjacent to r_1. By Lemma 3, r_2 is adjacent to r_3, and s_3 is not adjacent to r_2 and r_3. By Case 2 applied to r and s_3, the vertex s_3 is not adjacent to r_1. By Lemma 3, r_1 is not adjacent to s_2 and s_4. So r_1 has at most two neighbors in S by Lemma 7 that is r_1 has degree at least 3 in R. By Lemma 3, r_1 is not adjacent to r_2 and r_3. Then there exist $r_4, r_5 \in R - \{r, r_1, r_2, r_3\}$ such that r_4, r_5 are adjacent to r_1. Since each vertex in R is adjacent to at least one vertex in S, we may assume that r_1 is adjacent to s_5. By Lemma 3, s_3 is not adjacent to r_4 and r_5. Since s_3 is adjacent to 3 vertices in R, there exist $r_6, r_7 \in R - \{r, r_1, r_2, r_3, r_4, r_5\}$ such that r_6, r_7 is adjacent to s_3. By Lemma 4 applied to s_3, the vertex r_6 is adjacent to r_7. By Lemma 3, s_4 is not adjacent to $r, r_1, r_2, r_3, r_6, r_7$, and s_4 is adjacent to at most one vertex in $\{r_4, r_5\}$. Since s_4 is adjacent to 3 vertices in R and $|R| = 10$, we may assume that s_4 is adjacent to r_4, r_8 and r_9 where $\{r_8, r_9\} = R - \{r, r_1, r_2, \ldots, r_7\}$. Then r_1 and r_8 are not adjacent; otherwise r_1, r_8, s_4 and r_4 form a cycle in Δ and hence r_1 is adjacent to s_4, a contradiction. Similarly, r_1 and r_9 are not adjacent. By Lemma 3, r_1 is not adjacent to r_6 and r_7. Thus r_1 has degree 3 in R. By Lemma 3, we may assume that r_1 is adjacent to s_7. By Case 1 and Case 2 appiled to r_1 and s_5, we may assume that s_5 is adjacent to r_4 and r_5. Then r_4 and r_5 are adjacent by Lemma 3. Since s_2 is adjacent to 3 vertices in R and by Lemma 3, s_2 is adjacent to one vertex in $\{r_4, r_5\}$, one vertex in $\{r_6, r_7\}$ and one vertex in $\{r_8, r_9\}$. Without loss of generality, we may assume that s_2 is adjacent to r_6 and r_8. Then s_2 and r_4 are not adjacent; otherwise s_2, r_4, s_4 and r_8 form a cycle in Δ and hence s_2 is adjacent to s_4, a contradiction. Thus s_2 is adjacent to r_5. The vertices s_7 and r_4 are not adjacent; otherwise the vertices s_7, r_4, s_5 and r_1 form a cycle in Δ and hence s_5 is adjacent to s_7, a contradiction. By Lemma 3, r_4 is not adjacent to s_6 and s_8. Thus r_4 has degree 3 in R. The vertex r_4 is not adjacent to r_2 and r_3; otherwise the vertices r_4, r_i, r and r_1 form a cycle in Δ where $i \in \{2, 3\}$ and hence r_4 is adjcent to r, a contradiction. The vertices r_4 and r_6 are not adjcent; otherwise the vertices r_4, r_6, s_3 and s_4 form a cycle in Δ and hence r_4 is adjcent to s_3, a contradiction. Similarly, r_4 is not adjacent to r_7. Hence r_4 is adjacent to either r_8 or r_9. The vertices r_4 and r_8 are not adjacent; otherwise r_4, r_8, s_2 and r_5 form a cycle in Δ and hence r_4 is adjacent to s_2, a contradiction. It follows that r_4

is adjacent to r_9. By Case 2 appiled to r_4 and s_4, the vertex s_4 is adjacent to r_5. Hence s_4 has degree more than 5 in Δ, a contradiction. Therefore Case 3 cannot occur.

By Case 1, Case 2 and Case 3, $|E(R)| \neq 13$. $\square$

Lemma 12. $|E(R)| \neq 12$.

Proof. Suppose that $|E(R)| = 12$. Then the subgraph induced by S contains 3 edges. Without loss of generality, we may assume that s_{2i-1} and s_{2i} are adjacent for $i \in \{1, 2, 3\}$ but s_7 and s_8 are not adjacent. By Lemma 9, the number of edges between S and R is 26. By Lemma 3 and Lemma 4, there are 10 distinct edges $e_1, e_2, \ldots, e_{10}$ in R such that s_i is adjacent to both ends of e_i for $1 \leqslant i \leqslant 6$, s_7 is adjacent to both ends of e_7 and e_8 and s_8 is adjacent to both ends of e_9 and e_{10}. Let $T = \{e_1, e_2, \ldots, e_{10}\}$. By similar arguments as in Lemma 11, each vertex in R has degree at most 4 in R.

Suppose that there exists a vertex r in R which has degree 4 in R. Let r_1, r_2, r_3 and r_4 be distinct neighbors of r in R. Since $|E(R) - T| = 2$, we may assume that $rr_1, rr_2 \in T$ and r is adjacent to s_7. By Lemma 3, r_1 is adjacent to r_2. By construction, $r_1r_2 \notin T$. Since rr_1 and rr_2 are two edges with both ends adjacent to s_7, it follows that $rr_3, rr_4 \notin T$. Hence $13 = |T \cup \{r_1r_2, rr_3, rr_4\}| \leqslant |E(R)| = 12$, a contradiction.

Thus there are no vertices in R which has degree 4 in R. By Table 1, there exist 6 vertices in R with degree 2 in R, and 4 vertices in R with degree 3 in R. By Lemma 8, each line in G is incident with at least 3 points. Since s_7 and s_8 are not adjacent, s_7 and s_8 are lines of size 4 in G. By Lemma 7, the lines s_7 and s_8 have no common points. Let the point set of G be $\{r_i | 1 \leqslant i \leqslant 10\}$ such that r_3, r_4, r_5, r_6 lie on s_7 and r_7, r_8, r_9, r_{10} lie on s_8. Note that any line other than s_7 and s_8 must be incidence with either r_1 or r_2. If r_1 lies on exactly 2 lines, then G has at most 7 lines, a contradiction. Since every vertex in R is adjacent to 2 or 3 vertices in S, r_1 lies on 3 lines in G. Similarly, r_2 lies on 3 lines in G. The points r_1 and r_2 are not on the same line; otherwise G has at most 7 lines, a contradiction. If there exist at least 3 points in s_7 each of which lies on exactly two lines, then G has at most 7 lines, a contradiction. So there are 2 points on the line s_7 which lie on exactly two lines. Similarly, there are 2 points on the line s_8 which lie on exactly two lines. Without loss of generality, we may assume that each of r_5, r_6, r_9 and r_{10} lies on exactly 2 lines and each of r_3, r_4, r_7 and r_8 lies on exactly 3 lines. Then there are 3 possibilities for the incidence geometry G on 10 points and 8 lines satisfying these properties as shown in Figure 3.

In each figure a pair of solid lines represents s_7 and s_8, and each pair of nonsolid lines of same style represents s_{2i-1} and s_{2i} for $1 \leqslant i \leqslant 3$. If a point r is on a line s_{2i-1} and a point r' is on a line s_{2i}, then the vertex r is not adjacent to r'; otherwise r, r', s_{2i} and s_{2i-1} form a cycle in Δ, and by Lemma 3, the point r is on both s_{2i-1} and s_{2i}, a contradiction. For convenience we call this the parallelity of lines.

In Figure 3a, by the parallelity of lines, the vertex r_3 is not adjacent to r_4, r_6, and the vertex r_5 is not adjacent to r_4. Suppose that the vertices r_5 and r_6 are adjacent. The vertices r_3 and r_5 are not adjacent; otherwise the vertices r_3, r_5, r_6 and s_7 form a cycle in Δ, and by Lemma 3, the vertices r_3 and r_6 are adjacent, a contradiction. The vertices r_4 and r_6 are not adjacent; otherwise the vertices r_4, r_6, r_5 and s_7 form a cycle in Δ, and by

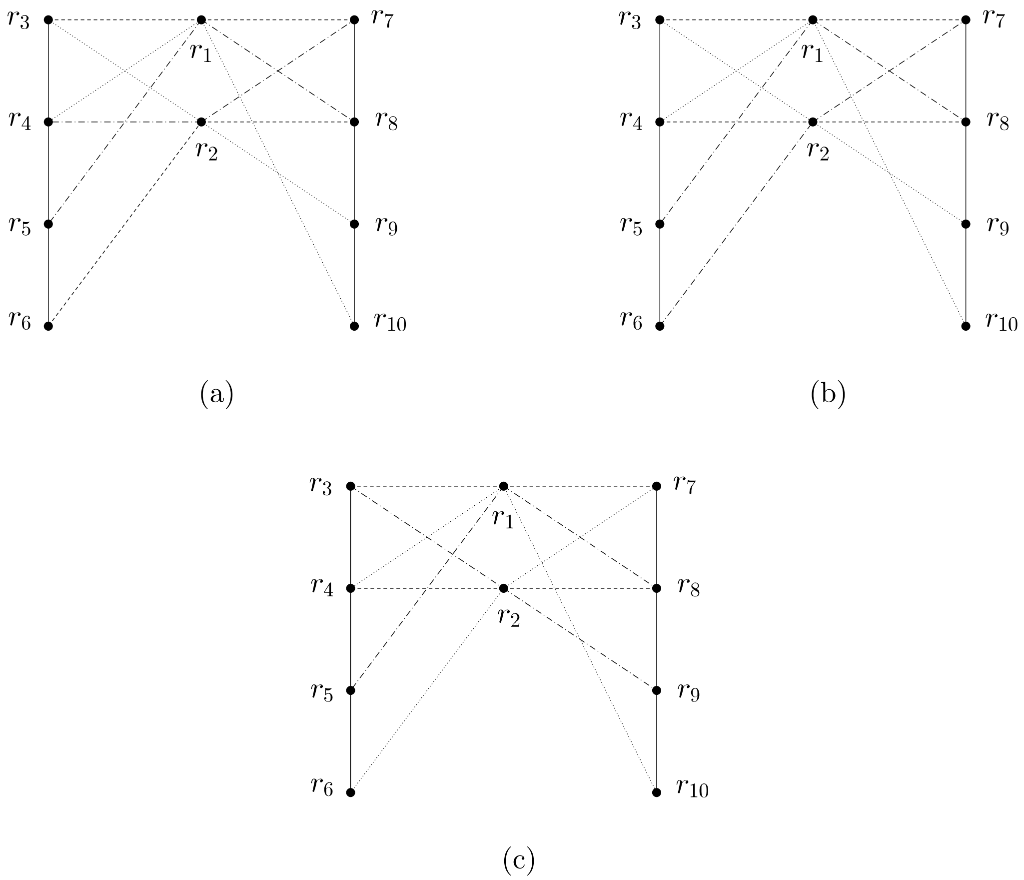

Figure 3: The 3 possibilities for the incidence geometry G.

Lemma 3, the vertices r_4 and r_5 are adjacent, a contradiction. Thus the vertex s_7 is on exactly one subgraph K_3 of Δ which contradicts Lemma 4. Hence the vertices r_5 and r_6 are not adjacent. The vertex r_6 is not adjacent to r_i for $i \in \{1, 2\}$; otherwise the vetices r_6, r_i, s_j and r_4 form a cycle in Δ where s_j is the line containing both r_i and r_4, and by Lemma 3, the point r_6 is on s_j, a contradiction. Since r_6 has degree 3 in R, the vertex r_6 is adjacent to 2 vertices u, v in $\{r_7, r_8, r_9, r_{10}\}$. Thus the vertices r_6, u, s_8 and v form a cycle in Δ, and by Lemma 3, the point r_6 is on s_8, a contradiction.

In Figure 3b, by the parallelity of lines, the vertex r_3 is not adjacent to r_4, and the vertex r_5 is not adjacent to r_6. Since r_2 has degree 2 in R, the vertex r_2 is adjacent to r_6 and r_9 by the parallelity of lines. The vertices r_4 and r_6 are not adjacent; otherwise the vertices r_4, r_6, r_2 and s_j forms a cycle in Δ where s_j is the line containing both r_2 and r_4, and by Lemma 3, the point r_6 is on s_j, a contradiction. Suppose that the vertices r_3 and r_5 are adjacent. The vertices r_3 and r_6 are not adjacent; otherwise the vertices r_3, r_6, s_7 and r_5 form a cycle in Δ, and by Lemma 3, the vertices r_5 and r_6 are adjacent, a contradiction. The vertices r_4 and r_5 are not adjacent; otherwise the vertices r_4, r_5, r_3 and s_7 form a cycle in Δ, and by Lemma 3, the vertices r_3 and r_4 are adjacent, a contradiction.

Hence the vertex s_7 is on exactly one subgraph K_3 of Δ which contradicts Lemma 4. Thus the vertices r_3 and r_5 are not adjacent. The vertex r_5 is not adjacent to r_i for $i \in \{1,2\}$; otherwise the vertices r_5, r_i, s_j and r_4 form a cycle in Δ where s_j is the line containing both r_i and r_4, and by Lemma 3, the point r_5 is on s_j, a contradiction. Since r_5 has degree 3 in R, the vertex r_5 is adjacent to 2 vertices u, v in $\{r_7, r_8, r_9, r_{10}\}$. Thus the vertices r_5, u, s_8 and v form a cycle in Δ, and by Lemma 3, the point r_5 is on s_8, a contradiction.

In Figure 3c, by the parallelity of lines, the vertex r_7 is not adjacent to r_8, r_{10}, and the vertex r_9 is not adjacent to r_8. Suppose that the vertices r_9 and r_{10} are adjacent. The vertices r_7 and r_9 are not adjacent; otherwise the vertices r_7, r_9, r_{10} and s_8 form a cycle in Δ, and by Lemma 3, the vertices r_7 and r_{10} are adjacent, a contradiction. The vertices r_8 and r_{10} are not adjacent; otherwise the vertices r_8, r_{10}, r_9 and s_8 form a cycle in Δ, and by Lemma 3, the vertices r_8 and r_9 are adjacent, a contradiction. Thus the vertex s_8 is on exactly one subgraph K_3 of Δ which contradicts Lemma 4. Hence the vertices r_9 and r_{10} are not adjacent. The vertex r_{10} is not adjacent to r_i for $i \in \{1,2\}$; otherwise the vertices r_{10}, r_i, s_j and r_8 form a cycle in Δ where s_j is the line containing both r_i and r_8, and by Lemma 3, the point r_{10} is on s_j, a contradiction. Since r_{10} has degree 3 in R, the vertex r_6 is adjacent to 2 vertices u, v in $\{r_3, r_4, r_5, r_6\}$. Thus the vertices r_{10}, u, s_7 and v form a cycle in Δ, and by Lemma 3, the point r_{10} is on s_7, a contradiction. Hence $|E(R)| \neq 12$. □

By Lemma 9, Lemma 11 and Lemma 12, we have our main result.

Theorem 13. *A distance-regular graph with intersection array* $\{22, 16, 5; 1, 2, 20\}$ *does not exist.*

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