Subspaces intersecting each element of a regulus in one point, André-Bruck-Bose representation and clubs

Michel Lavrauw* Corrado Zanella†

Dipartimento di Tecnica e Gestione dei Sistemi Industriali
Università di Padova
Stradella S. Nicola 3
36100 Vicenza, Italy

{michel.lavrauw,corrado.zanella}@unipd.it

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Abstract

In this paper results are proved with applications to the orbits of $(n-1)$-dimensional subspaces disjoint from a regulus $\mathcal{R}$ of $(n-1)$-subspaces in $\mathrm{PG}(2n-1,q)$, with respect to the subgroup of $\mathrm{PGL}(2n,q)$ fixing $\mathcal{R}$. Such results have consequences on several aspects of finite geometry. First of all, a necessary condition for an $(n-1)$-subspace U and a regulus $\mathcal{R}$ of $(n-1)$-subspaces to be extendable to a Desarguesian spread is given. The description also allows to improve results in [2] on the André-Bruck-Bose representation of a q-subline in $\mathrm{PG}(2,q^n)$. Furthermore, the results in this paper are applied to the classification of linear sets, in particular clubs.

Keywords: club; linear set; subplane; André-Bruck-Bose representation; Segre variety

1 Introduction

The $(n-1)$-dimensional projective projective space over the field F is denoted by $\mathrm{PG}(n-1,F)$ or $\mathrm{PG}(n-1,q)$ if F is the finite field of order q (denoted by $\mathbb{F}_q$). The set of nonzero elements of a field F will be denoted by F^*, and similarly, the set of nonzero vectors of a vector space V by V^*. If L is an extension field $\mathbb{F}_q$, then the projective space defined by

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the $\mathbb{F}_q$-vector space induced by L^d is also denoted by $\mathrm{PG}_q(L^d)$. For a (sets of) subspace(s) R of a vector space or a projective space, the notation $\langle R \rangle$ is used to denote the subspace generated by (the elements of) R. In case there is any ambiguity about the coefficient field, then the notation $\langle R \rangle_q$ will be used, to denote that the considered subspace is the one generated over $\mathbb{F}_q$. In this case the terminology of $\mathbb{F}_q$-*span* will sometimes be used. For example, if S is a set of two points on the projective line $\mathrm{PG}(1, q^2)$, then $\langle S \rangle_q$ denotes the $\mathbb{F}_q$-subline defined by S, while $\langle S \rangle_{q^2}$ coincides with the whole projective line $\mathrm{PG}(1, q^2)$.

For further notation and general definitions employed in this paper the reader is referred to [9, 11, 13].

For more information on Desarguesian spreads see [1].

This paper is structured as follows. In Section 2 subspaces which intersect each element of a regulus in one point are studied and a result from [4] is generalised. Section 3 contains one of the main results of this paper, determining the order of the normal rational curves obtained from n-dimensional subspaces on an external $(n-1)$-dimensional subspace with respect to a regulus in $\mathrm{PG}(2n-1, q)$, obtained from a point and a subline after applying the field reduction map to $\mathrm{PG}(1, q^n)$. This leads to a necessary condition on the existence of a Desarguesian spread containing a subspace and regulus (Corollary 7). The André-Bruck-Bose representation of sublines and subplanes of a finite projective plane is studied[1] in Section 4 and improvements are obtained with respect to the known results [3, 14, 16, 2]. The results from the first sections of this paper are then applied to the classification problem for clubs of rank three in $\mathrm{PG}(1, q^n)$ in Section 5. A study of the incidence structure of the clubs in $\mathrm{PG}(1, q^n)$ after field reduction yields to a partial classification, concluding that the orbits of clubs under $\mathrm{PGL}(2, q^n)$ are at least $k-1$, where k stands for the number of divisors of n. The paper concludes with an appendix discussing a result motivated by Burau [4] for the complex numbers: the result is extended to general algebraically closed fields; a new proof is provided; and counterexamples are given to some of the arguments used in the original proof.

2 Subspaces intersecting each element of a regulus in one point

Let $\mathcal{R}$ be a regulus of subspaces in a projective space and let S be any subspace of $\langle \mathcal{R} \rangle$. Questions about the properties of the set of intersection points, which for reasons of simplicity of notation we will denote by $S \cap \mathcal{R}$, often turn up while investigating objects in finite geometry. If S intersects each element of the regulus $\mathcal{R}$ in a point, then the intersection $S \cap \mathcal{R}$ is a normal rational curve, see Lemma 1. This was already pointed out in [4, p.173] with a proof originally intended for complex projective spaces, but actually holding in a more general setting. The notation of [4] will be partly adopted.

The Segre variety representing the Cartesian product $\mathrm{PG}(n, F) \times \mathrm{PG}(m, F)$ in $\mathrm{PG}((n+1)(m+1)-1, F)$ is denoted by $\mathcal{S}_{n,m,F}$. It is well known that $\mathcal{S}_{n,m,F}$ contains two families $\mathcal{S}^{I}_{n,m,F}$ and $\mathcal{S}^{II}_{n,m,F}$ of maximal subspaces of dimensions n and m, respectively. When

[1] A different study of $\mathbb{F}_{q^k}$-sublines and $\mathbb{F}_{q^k}$-subplanes of $\mathrm{PG}(2, q^n)$ in this representation can be found in [15].

convenient, the notation S^I or S^{II} will be used for a subspace belonging to the first or second family. The points of $\mathcal{S}_{n,m,F}$ may be represented as one-dimensional subspaces spanned by rank one $(m+1)\times(n+1)$ matrices. This is the standard example of a regular embedding of product spaces, see [17]. Note that in the finite case it is possible to embed product spaces in projective spaces of smaller dimension (see e.g. [7]). A regulus $\mathcal{R}$ of $(n-1)$-dimensional subspaces can also be defined as $\mathcal{S}^I_{n-1,1,F}$.

Lemma 1. *Let $n > 1$ be an integer, and F a field. Let S_t be a t-subspace of $\mathrm{PG}(2n-1, F)$ intersecting each $S^I \in \mathcal{S}^I_{n-1,1,F}$ in precisely one point. Define $\Phi = S_t \cap \mathcal{S}_{n-1,1,F}$, and assume $\langle\Phi\rangle = S_t$. Then $|F| \geqslant t$ and the following properties hold.*

(i) The set Φ is a normal rational curve of order t.

(ii) Let $\Xi^I \in \mathcal{S}^I_{n-1,1,F}$. Then the set $S(\Phi, \Xi^I)$ of the intersections of Ξ^I with all transversal lines l^{II} such that $l^{II} \cap \Phi \neq \emptyset$ is a normal rational curve of order t or $t-1$ if $|F| = t$, and of order $t-1$ if $|F| > t$.

(iii) If Φ is contained in a subvariety $\mathcal{S}_{t-1,1,F}$ of $\mathcal{S}_{n-1,1,F}$, then homogeneous coordinates can be chosen such that Φ is represented parametrically by

$$\left\langle \begin{pmatrix} y_0^t & y_0^{t-1}y_1 & \cdots & y_0 y_1^{t-1} \\ y_0^{t-1}y_1 & y_0^{t-2}y_1^2 & \cdots & y_1^t \end{pmatrix} \right\rangle, \quad (y_0, y_1) \in (F^2)^*, \tag{1}$$

and $S(\Phi, \Xi^I)$, for z_0, z_1 depending only on Ξ^I, by

$$\left\langle \begin{pmatrix} y_0^{t-1}z_0 & y_0^{t-2}y_1 z_0 & \cdots & y_1^{t-1}z_0 \\ y_0^{t-1}z_1 & y_0^{t-2}y_1 z_1 & \cdots & y_1^{t-1}z_1 \end{pmatrix} \right\rangle, \quad (y_0, y_1) \in (F^2)^*. \tag{2}$$

Proof. (i), (iii) The proof in [4, Sect.41 no.3], which is offered for $F = \mathbb{C}$, works exactly the same provided that $|F| > t$ or, more generally, that Φ is contained in some subvariety $\mathcal{S}_{t-1,1,F}$ of $\mathcal{S}_{n-1,1,F}$. In case $|F| \leqslant t$, the size of Φ being $|F| + 1$ implies $|F| = t$, so Φ is just a set of $t + 1$ independent points in a subspace isomorphic to $\mathrm{PG}(t, t)$, hence Φ is a normal rational curve of order t.

(ii) The case $|F| > t$ is proved in [4] immediately after the corollary at p. 175. If $|F| \leqslant t$, then $|F| = t$ and two cases are possible. If Φ is contained in some $\mathcal{S}_{t-1,1,F} \subseteq \mathcal{S}_{n-1,1,F}$, Burau's proof is still valid as was mentioned in case (ii); so, $S(\Phi, \Xi^I)$ is a normal rational curve of order $t - 1 = |F| - 1$. Otherwise $S(\Phi, \Xi^I)$ is an independent $(t + 1)$-set, hence a normal rational curve of order $|F|$. □

Remark 2. If $|F| = t$ both cases in Lemma 1 (ii) can occur. The following two examples use the Segre embedding $\sigma = \sigma_{t-1,1,F}$ of the product space $\mathrm{PG}(t-1, t)\times\mathrm{PG}(1, t)$ in $\mathrm{PG}(2t-1, t)$. Let $\{s_0, s_1, \ldots, s_t\}$ be the set of points on $\mathrm{PG}(1, t)$ and suppose $\{r_0, r_1, \ldots, r_t\}$ is a set of $t+1$ points in $\mathrm{PG}(t-1, t)$. Put $\Xi^I = \sigma(\mathrm{PG}(1, t) \times s_0)$ and $\Phi := \{\sigma(r_i \times s_i) : i = 0, 1, \ldots, t\}$. Then Φ consists of $t + 1$ points on the Segre variety $\mathcal{S}_{t-1,1,F}$. Depending on the set $\{r_0, r_1, \ldots, r_t\}$ one obtains the two cases described in Lemma 1 (ii).

a. *If* $\{r_0, r_1, \ldots, r_t\}$ *is a frame of a hyperplane of* $\mathrm{PG}(t-1,t)$ *then* Φ *generates a* t*-dimensional subspace of* $\mathrm{PG}(2t-1,t)$ *intersecting* $\mathcal{S}_{t-1,1,F}$ *in* Φ *and* $S(\Phi, \Xi^I)$ *is a normal rational curve of order* $t-1$.

b. *If* $\{r_0, r_1, \ldots, r_t\}$ *generates* $\mathrm{PG}(t-1,t)$ *then* Φ *generates a* t*-dimensional subspace of* $\mathrm{PG}(2t-1,t)$ *intersecting* $\mathcal{S}_{t-1,1,F}$ *in* Φ *and* $S(\Phi, \Xi^I)$ *is a normal rational curve of order* t.

Remark 3. *By (1) and (2), the map* $\alpha : \Phi \to S(\Phi, \Xi^I)$ *defined by the condition that* X *and* X^α *are on a common line in* $\mathcal{S}^{II}_{n-1,1,F}$ *is related to a projectivity between the parametrizing projective lines. Such an* α *is also called a projectivity.*

3 The order of normal rational curves contained in $\mathcal{S}_{n-1,1,q}$

Here $n \geqslant 2$ is an integer. The field reduction map $\mathcal{F}_{m,n,q}$ from $\mathrm{PG}(m-1,q^n)$ to $\mathrm{PG}(mn-1,q)$ will also be denoted by $\mathcal{F}$. If S is a set of points, in $\mathrm{PG}(m-1,q^n)$, then $\mathcal{F}(S)$ is a set of subspaces, whose union, as a set of points will be denoted by $\tilde{\mathcal{F}}(S)$. The $\mathbb{F}_{q^h}$-span of a subset b of $\mathrm{PG}(d,q^n)$ is denoted by $\langle b \rangle_{q^h}$.

Proposition 4. *Let* b *be a* q*-subline of* $\mathrm{PG}(1,q^n)$*, and let* Θ *be a point of* $\mathrm{PG}(1,q^n)$*. Let* $(1,\zeta)$ *and* $(1,\zeta')$ *be homogeneous coordinates of* Θ *with respect to two reference frames for* $\langle b \rangle_{q^n}$*, each of which consists of three points of* b*. Then* $\mathbb{F}_q(\zeta) = \mathbb{F}_q(\zeta')$.

Proof. Homogeneous coordinates of a point in both reference frames, say (x_0, x_1) and (x'_0, x'_1), are related by an equation of the form $\rho(x'_0\ x'_1)^T = A(x_0\ x_1)^T$, $\rho \in \mathbb{F}^*_{q^n}$, $A \in \mathrm{GL}(2,q)$. Hence $(\rho\ \rho\zeta')^T = A(1\ \zeta)^T$ and this implies $\zeta' \in \mathbb{F}_q(\zeta)$. The proof of $\zeta \in \mathbb{F}_q(\zeta')$ is similar. □

By Proposition 4, given a q-subline b in a finite projective space $\mathrm{PG}(d,q^n)$ and a point $\Theta \in \langle b \rangle_{q^n}$, with homogeneous coordinates $(1,\zeta)$ with respect to a reference frame of $\langle b \rangle_{q^n}$ consisting of three points of b, the *degree* of Θ over b, denoted by $[\Theta : b]$, is well-defined in terms of the field extension degree as follows: $[\Theta : b] = [\mathbb{F}_q(\zeta) : \mathbb{F}_q]$.

This $[\Theta : b]$ also equals the minimum integer m such that a subgeometry $\Sigma \cong \mathrm{PG}(d,q^m)$ exists containing both b and Θ.

Proposition 5. *Any* n*-subspace of* $\mathrm{PG}(2n-1,q)$ *containing an* $(n-1)$*-subspace* $S^I \in \mathcal{S}^I_{n-1,1,q}$ *intersects* $\mathcal{S}_{n-1,1,q}$ *in the union of* S^I *and a line in* $\mathcal{S}^{II}_{n-1,1,q}$.

Theorem 6. *Let* b *be a* q*-subline of* $\mathrm{PG}(1,q^n)$*, and* $\Theta \notin b$ *a point of* $\mathrm{PG}(1,q^n)$*. Then in* $\mathrm{PG}(2n-1,q)$ *any* n*-subspace* $\mathcal{H}$ *containing* $\mathcal{F}(\Theta)$ *intersects the Segre variety* $\mathcal{S}_{n-1,1,q} = \tilde{\mathcal{F}}(b)$*, in a normal rational curve whose order is* $\min\{q, [\Theta : b]\}$.

Proof. Set $L = \mathbb{F}_{q^n}$, $F = \mathbb{F}_q$. Without loss of generality, $\mathrm{PG}(2n-1,q) = \mathrm{PG}_q(L^2)$, $\mathcal{F}(b) = \{L(x,y) \mid (x,y) \in (F^2)^*\}$[2], and $\Theta = L(1,\xi)$ with $[F(\xi) : F] = [\Theta : b]$. The

[2] For $x, y \in L$, $F(x,y) = \langle (x,y) \rangle_q$, and $L(x,y) = \langle (x,y) \rangle_{q^n}$.

n-subspace $\mathcal{H}$ intersects $L(1,0)$ in one point Y of the form $Y = F(\theta, 0)$, $\theta \in L^*$. For any $x \in F$, seeking for the intersection $\langle \mathcal{F}(\Theta), Y\rangle_q \cap L(x,1)$, or

$$\langle L(1,\xi), F(\theta,0)\rangle_q \cap L(x,1)$$

gives two equations in $\alpha, \beta \in L$:

$$\alpha + \theta = \beta x, \quad \alpha\xi = \beta,$$

whence $\beta = \theta(x-\xi^{-1})^{-1}$. The intersection point is then $F\left(x\theta(x-\xi^{-1})^{-1}, \theta(x-\xi^{-1})^{-1}\right)$. So, for $\Xi = L(0,1)$, the set of the intersections of Ξ with all lines in $\mathcal{S}^{II}_{n-1,1,q}$ which meet $\mathcal{H}$ is

$$S(\mathcal{H} \cap \mathcal{S}_{n-1,1,q}, \Xi) = \{F(0, \theta(x-\xi^{-1})^{-1}) \mid x \in \mathcal{F}_q\} \cup \{F(0,\theta)\}.$$

This $S(\mathcal{H} \cap \mathcal{S}_{n-1,1,q}, \Xi)$ is obtained by inversion from the line joining the points $F(0,\theta^{-1})$ and $F(0, \theta^{-1}\xi^{-1})$. By [10, Theorem 5], $\mathcal{C}_Y$ is a normal rational curve of order $\delta' = \min\{q, [F(\xi^{-1}) : F]-1\} = \min\{q, [\Theta : b]-1\}$. Now apply lemma 1 for $S_t = \langle \mathcal{H} \cap \mathcal{S}_{n-1,1,q}\rangle_q$:if $t \geqslant q$, then $t = q$ and $\delta' = q$ or $\delta' = q-1$, so $[\Theta : b] \geqslant q$ and $t = \min\{q, [\Theta : b]\}$. If on the contrary $t < q$, then $t - 1 = \delta' = [\Theta : b] - 1$, so $t = [\Theta : b]$ and $t = \min\{q, [\Theta : b]\}$ again. □

An important consequence of the above result answers the question of the existence of a Desarguesian spread containing a given regulus $\mathcal{R}$ and a subspace disjoint from $\mathcal{R}$.

Corollary 7. *If a regulus $\mathcal{R} = \mathcal{S}_{n-1,1,q}$ and an $(n-1)$-dimensional subspace U, disjoint from $\mathcal{R}$, in $\mathrm{PG}(2n-1,q)$ are contained in a Desarguesian spread then there is an integer c such that any n-subspace $\mathcal{H}$ containing U intersects $\mathcal{R}$ in a normal rational curve of order c.*

The following remark illustrates that this necessary condition is not always satisfied.

Remark 8. For $n > 2$ by using the package FinInG [5] of GAP [6] examples can be given of $(n-1)$-subspaces disjoint from $\mathcal{S}_{n-1,1,q}$ contained in n-subspaces intersecting the Segre variety in normal rational curves of distinct orders. We include one explicit example. Let $q = 4$, $\mathbb{F}_q = \mathbb{F}_2(\omega)$, with $\omega^2+\omega+1 = 0$. Let $\mathcal{R}$ be the regulus of 3-dimensional subspaces of $\mathrm{PG}(7,4)$ obtained from the standard subline $\mathrm{PG}(1,q)$ in $\mathrm{PG}(1,q^4)$, and put

$$\begin{aligned} S_3 := \langle &(1,0,0,0,\omega^2,1,0,1), (0,1,0,0,1,\omega^2,0,\omega^2), \\ &(0,0,1,0,0,\omega,1,\omega), (0,0,0,1,\omega^2,\omega^2,\omega,1)\rangle. \end{aligned}$$

Then S_3 is a three-dimensional subspace disjoint from the regulus $\mathcal{R}$. Moreover, the 4-dimensional subspace $\langle S_3, (1,0,0,0,0,0,0,0)\rangle$ intersects the regulus $\mathcal{R}$ in a normal rational curve of order 4, while the 4-dimensional subspace $\langle S_3, (0,1,0,\omega^2,0,0,0,0)\rangle$ intersects $\mathcal{R}$ in a conic.

4 André-Bruck-Bose representation

The André-Bruck-Bose representation of a Desarguesian affine plane of order q^n is related to the image of $\mathrm{PG}(2, q^n)$, under the field reduction map $\mathcal{F}$, by means of the following straightforward result.

Proposition 9. *Let $\mathcal{D}$ be the Desarguesian spread in $\mathrm{PG}(3n-1, q)$ obtained after applying the field reduction map $\mathcal{F}$ to the set of points of $\mathrm{PG}(2, q^n)$, l_∞ a line in $\mathrm{PG}(2, q^n)$, and $\mathcal{K}$ a $(2n)$-subspace of $\mathrm{PG}(3n-1, q)$, containing the spread $\mathcal{F}(l_\infty)$. Take $\mathrm{PG}(2, q^n) \setminus l_\infty$ and $\mathcal{K} \setminus \langle \mathcal{F}(l_\infty) \rangle_q$ as representatives of $\mathrm{AG}(2, q^n)$ and $\mathrm{AG}(2n, q)$, respectively. Then the map $\varphi : \mathrm{AG}(2, q^n) \to \mathrm{AG}(2n, q)$ defined by $\varphi(X) = \mathcal{F}(X) \cap \mathcal{K}$ for any $X \in \mathrm{AG}(2, q^n)$ is a bijection, mapping lines of $\mathrm{AG}(2, q^n)$ into n-subspaces of $\mathrm{AG}(2n, q)$ whose $(n-1)$-subspaces at infinity belong to the spread $\mathcal{F}(l_\infty)$.*

The notation in Proposition 9 is assumed to hold in the whole section. The following result improves [2, Theorems 3.3 and 3.5], by determining the order of the involved normal rational curves.

Theorem 10. *Let b be a q-subline of $\mathrm{PG}(2, q^n)$, not contained in l_∞. Set $\Theta = \langle b \rangle_{q^n} \cap l_\infty$. Then the André-Bruck-Bose representation $\varphi(b \setminus l_\infty)$ is the affine part of a normal rational curve whose order is $\delta = \min\{q, [\Theta : b]\}$. More precisely, if $\delta = 1$, then $\varphi(b \setminus l_\infty)$ is an affine line; if $\delta > 1$, then $b \cap l_\infty = \emptyset$, and $\varphi(b)$ is a normal rational curve with no points at infinity.*

Proof. The intersection $\mathcal{H} = \langle \mathcal{F}(b) \rangle_q \cap \mathcal{K}$ is an n-space containing $\mathcal{F}(\Theta)$, and contained in the span of the Segre variety $\mathcal{S}_{n-1,1,q} = \tilde{\mathcal{F}}(b)$, as defined at the start of Section 3. The result follows from Proposition 5 and Theorem 6. □

The results in [2, Theorems 3.3 and 3.5] also characterize the normal rational curves arising from q-sublines in $\mathrm{AG}(2, q^n)$.

In [3, 14, 16] for $n = 2$ and [2, Theorem 3.6 (a)(b)] for any n the André-Bruck-Bose representation of a q-subplane tangent to a line at the infinity is described. Further properties are stated in the following theorem:

Theorem 11. *Let B be a q-subplane of $\mathrm{PG}(2, q^n)$ that is tangent to l_∞ at the point T. Let b be a line of B not through T, $\Theta = \langle b \rangle_{q^n} \cap l_\infty$, and $\delta = \min\{q, [\Theta : b]\}$. Then there are a normal rational curve $\mathcal{C}_0$ of order δ in the n-subspace $\varphi(\langle b \rangle_{q^n})$, a normal rational curve $\mathcal{C}_1 \subset \mathcal{F}(T)$ of order δ', with*

$$\delta' \begin{cases} = [\Theta : b] - 1 & \text{for } q > [\Theta : b] \\ \in \{q-1, q\} & \text{otherwise,} \end{cases} \tag{3}$$

and a projectivity $\kappa : \mathcal{C}_0 \to \mathcal{C}_1$ (in the sense of Remark 3), such that $\varphi(B \setminus l_\infty)$ is the ruled surface union of all lines XX^κ for $X \in \mathcal{C}_0$.

Proof. By Theorem 10, $\mathcal{C}_0 := \varphi(b)$ is a normal rational curve of order δ in the n-subspace $\varphi(\langle b\rangle_{q^n} \setminus l_\infty)$, and for any $P = \varphi(X) \in \mathcal{C}_0$, the subline TX of B corresponds to an affine line PP^κ with $P^\kappa \in \mathcal{F}(T)$ at infinity. Define $\mathcal{C}_1 = \{P^\kappa \mid P \in \mathcal{C}_0\}$.

By the field reduction map $\mathcal{F} = \mathcal{F}_{3,n,q}$, the subplane B is mapped to $\mathcal{F}(B)$ which is the set of all maximal subspaces of the first family in $\mathcal{S}_{n-1,2,q} \subset \mathrm{PG}(3n-1,q)$. Considering $\mathbb{F}_{q^n}$ as an $\mathbb{F}_q$-vector space, the homomorphism

$$\mathbb{F}_{q^n} \times \mathbb{F}_q^3 \to \mathbb{F}_{q^n} \otimes \mathbb{F}_q^3 \;:\; (\lambda, v) \mapsto \lambda \otimes v$$

corresponds to a projective embedding $g : \mathrm{PG}(n-1,q) \times B \to \mathcal{S}_{n-1,2,q}$ whose image is $\mathcal{S}_{n-1,2,q}$, and such that $\mathcal{F}(X) = (\mathrm{PG}(n-1,q) \times X)^g$ for any point X in B. It holds $\varphi(B \setminus l_\infty) = \mathcal{S}_{n-1,2,q} \cap \mathcal{K} \setminus \mathcal{F}(T)$. For any point U in B define

$$\kappa_U : (X,Y)^g \in \mathcal{S}_{n-1,2,q} \mapsto (X,U)^g \in \mathcal{F}(U).$$

Note that for any $Y \in B$, the restriction of κ_U to $\mathcal{F}(Y)$ is a projectivity. For any $U \in b$, using the notation from Lemma 1 it holds $\mathcal{C}_0^{\kappa_U} = S(\mathcal{C}_0, \mathcal{F}(U))$, and as a consequence, $\mathcal{C}_0^{\kappa_U}$ is a normal rational curve of order δ' as in (3). Now, since for any $P \in \mathcal{C}_0$, say $P = (X_P, Y_P)^g$, the points P, P^κ and P^{κ_T} are on the plane $(X_P \times B)^g \in \mathcal{S}^{II}_{n-1,2,q}$, and $P^\kappa, P^{\kappa_T} \in \mathcal{F}(T)$, it follows that $P^\kappa = P^{\kappa_T}$. It also follows that $\mathcal{C}_1 = \mathcal{C}_0^{\kappa_U \kappa_T} = S(\mathcal{C}_0, \mathcal{F}(U))^{\kappa_T}$, and hence $\mathcal{C}_1$ is a normal rational curve of order δ' as in (3). Finally, $\kappa_U : \mathcal{C}_0 \to S(\mathcal{C}_0, \mathcal{F}(U))$ is a projectivity as defined in Remark 3, and hence so is κ. □

5 On the classification of clubs

An $\mathbb{F}_q$-*club* (or simply a club) in $\mathrm{PG}(1,q^n)$ is an $\mathbb{F}_q$-linear set of rank three, having a point of weight two, called the *head* of the club. An $\mathbb{F}_q$-club has q^2+1 points, and the non-head points have weight one. From now on it will be assumed that $n > 2$. The next proposition is a straightforward consequence of the representation of linear sets as projections of subgeometries [12, Theorem 2].

Proposition 12. *Let L be an $\mathbb{F}_q$-club in $\mathrm{PG}(1,q^n) \subset \mathrm{PG}(2,q^n)$. Then there are a q-subplane Σ of $\mathrm{PG}(2,q^n)$, a q-subline b in Σ, and a point $\Theta \in \langle b\rangle_{q^n} \setminus b$, such that L is the projection of Σ from the center Θ onto the axis $\mathrm{PG}(1,q^n)$.*

As before the notation $\mathcal{F}$ and $\tilde{\mathcal{F}}$ is used, where $\mathcal{F} = \mathcal{F}_{2,n,q}$ denotes the field reduction map from $\mathrm{PG}(1,q^n)$ to $\mathrm{PG}(2n-1,q)$.

Proposition 13. *Let L be an $\mathbb{F}_q$-club of $\mathrm{PG}(1,q^n)$ with head Υ. Then $\tilde{\mathcal{F}}(L)$ contains two collections of subspaces, say F_1 and F_2, satisfying the following properties.*

(i) The subspaces in F_1 are $(n-1)$-dimensional, are pairwise disjoint, and any subspace in F_1 is disjoint from $\mathcal{F}(\Upsilon)$.

(ii) Any subspace in F_2 is a plane and intersects $\mathcal{F}(\Upsilon)$ in precisely a line.

(iii) Any point of $\mathcal{F}(\Upsilon)$ belongs to exactly $q+1$ planes in F_2.

(iv) If L is not isomorphic to $\mathrm{PG}(1,q^2)$, and l is any line of $\mathrm{PG}(2n-1,q)$ contained in $\tilde{\mathcal{F}}(L)$, then l is contained in $\mathcal{F}(\Upsilon)$ or in a subspace in $F_1 \cup F_2$.

Proof. The assumptions imply the existence of Σ and a q-subline b in Σ as in Proposition 12. The assertions are a consequence of the fact that $\tilde{\mathcal{F}}(\Sigma)$ is a Segre variety $\mathcal{S}_{n-1,2,q}$ in $\mathrm{PG}(3n-1,q)$. Let

$$p_1 : \mathrm{PG}(2,q^n) \setminus \Theta \to \mathrm{PG}(1,q^n)$$

be the projection with center Θ, associated with

$$p_2 : \mathrm{PG}(3n-1,q) \setminus \mathcal{F}(\Theta) \to \mathrm{PG}(2n-1,q).$$

The collections F_1 and F_2 are defined as follows:

$$F_1 = \{\mathcal{F}(p_1(X)) \mid X \in \Sigma \setminus b\} = \mathcal{F}(L) \setminus \mathcal{F}(\Upsilon), \quad F_2 = \{p_2(V^{II}) \mid V^{II} \in \tilde{\mathcal{F}}(\Sigma)^{II}\}.$$

The assertion *(i)* is straightforward, as well as $\dim(V) = 2$ for any $V \in F_2$. For any $V^{II} \in \tilde{\mathcal{F}}(\Sigma)^{II}$, the intersection $V^{II} \cap \langle \tilde{\mathcal{F}}(b) \rangle_q$ is a line, and this with $p_2^{-1}(\mathcal{F}(\Upsilon)) = \langle \tilde{\mathcal{F}}(b) \rangle_q \setminus \mathcal{F}(\Theta)$ implies the second assertion in *(ii)*. Next, let P be a point in $\mathcal{F}(\Upsilon)$. A plane $V = p_2(V^{II})$ contains P if, and only if, V^{II} intersects the n-subspace $\langle \mathcal{F}(\Theta), P \rangle_q$, that is, V^{II} intersects the normal rational curve $\mathcal{S}_{n-1,2,q} \cap \langle \mathcal{F}(\Theta), P \rangle_q$; this implies *(iii)*.

Assume that a line $l \subset \tilde{\mathcal{F}}(L)$ exists which is neither contained in $\mathcal{F}(\Upsilon)$, nor in a $T \in F_1 \cup F_2$. Let Q be a point in $l \setminus \mathcal{F}(\Upsilon)$, and let $V \in F_2$ such that $Q \in V$. It holds $L = \mathcal{B}(V)$. Then $\mathcal{B}(l)$ is a q-subline of L. Suppose that a line l' in V exists such that $\mathcal{B}(l') = \mathcal{B}(l)$. Since $\mathcal{B}(Q) \neq \mathcal{B}(Q')$ for any $Q' \in V$, $Q' \neq Q$, the line l' contains Q. Then l, l' are two distinct transversal lines in $\mathcal{B}(l)^{II}$, a contradiction. Hence $\mathcal{B}(l') \neq \mathcal{B}(l)$ for any line l' in V, that is, $\mathcal{B}(l)$ is a so-called *irregular subline* [8]. By [8, Corollary 13], no irregular subline exists in L, and this contradiction implies *(iv)*. □

Proposition 14. *Let L be an $\mathbb{F}_q$-club with head Υ. Let Θ be the point and b be the subline as defined in Proposition 12. Then for any point X in $\mathcal{F}(\Upsilon)$, the intersection lines of $\mathcal{F}(\Upsilon)$ with any q distinct planes in F_2 containing X span an s-dimensional subspace, where*

(i) $s = [\Theta : b] - 1$ if $q > [\Theta : b]$;

(ii) $s \in \{q-1, q\}$ if $q \leqslant [\Theta : b]$.

Proof. Let p_2 be the projection map as defined in the proof of Proposition 13, $X = p_2(P)$, and $\mathcal{H} = \langle \mathcal{F}(\Theta), P \rangle_q$. For any plane $V = p_2(V^{II})$, it holds $X \in V$ if, and only if $V^{II} \cap \mathcal{H} \neq \emptyset$. The intersection $\mathcal{H} \cap \tilde{\mathcal{F}}(b)$ is a normal rational curve of order $\min\{q, [\Theta : b]\}$ (cf. Theorem 6). Let $V_0 = p_2(V_0^{II})$ be the unique plane of F_2 through X distinct from the q planes chosen in the assumptions (cf. Proposition 13). Let $Q = \tilde{\mathcal{F}}(b) \cap V_0^{II}$; $\mathcal{B}(Q)$ is an $(n-1)$-subspace of $\tilde{\mathcal{F}}(b)^I$. Such $\mathcal{B}(Q)$ is mapped onto $\mathcal{B}(X) = \mathcal{F}(\Upsilon)$ by p_2. Assume $V_i = p_2(V_i^{II})$, $i = 1, 2, \ldots, q$, are the q planes chosen in the assumptions. Any V_i^{II},

$i = 1, 2, \dots, q$, intersects $\mathcal{H}$, hence $V_i^{II} \cap \mathcal{B}(Q)$ is the intersection of $\mathcal{B}(Q)$ with a transversal line of $\tilde{\mathcal{F}}(b)$ intersecting the normal rational curve $\mathcal{H} \cap \tilde{\mathcal{F}}(b)$. By Lemma 1 *(ii)*, the set

$$S = \{V_i^{II} \cap \mathcal{B}(Q) \mid i = 1, 2, \dots, q\} \cup \{Q\}$$

is a normal rational curve of order s where s takes the values as stated in (i) and (ii). Since $V_i \cap \mathcal{F}(\Upsilon)$ is the line through X and a point of $p_2(S)$, distinct from X, the span of the intersection lines is the same as the span of $p_2(S)$. □

Theorem 15. *Let $\mathcal{I}_{n,q}$ be the set of integers h dividing n and such that $1 < h < q$. For any $h \in \mathcal{I}_{n,q}$, let L_h be the linear set obtained by projecting a q-subplane Σ of* $\mathrm{PG}(2, q^n)$ *from a point Θ_h collinear with a q-subline b in Σ and such that $[\Theta_h : b] = h$. Then the set $\Lambda = \{L_h \mid h \in \mathcal{I}_{n,q}\}$ contains $\mathbb{F}_q$-clubs in* $\mathrm{PG}(1, q^n)$ *all belonging to distinct orbits under* $\mathrm{PGL}(2, q^n)$.

Proof. If n is odd, then no club is isomorphic to $\mathrm{PG}(1, q^2)$. So, by Proposition 13 *(iv)*, the families F_1 and F_2 are uniquely determined. The thesis is a consequence of Proposition 14, taking into account that if L and L' are projectively equivalent, then $\tilde{\mathcal{F}}(L)$ and $\tilde{\mathcal{F}}(L')$ are projectively equivalent in $\mathrm{PG}(2n-1, q)$.

In order to deal with the case n even, it is enough to show that in Λ at most one club is isomorphic to $\mathrm{PG}(1, q^2)$. So assume $L_h \cong \mathrm{PG}(1, q^2)$. Then $\tilde{\mathcal{F}}(L_h)$ has a partition $\mathcal{P}_1$ in $(n-1)$-subspaces, and a partition $\mathcal{P}_2$ in 3-subspaces. From [8, Lemma 11] it can be deduced that any line contained in $\tilde{\mathcal{F}}(L_h)$ is contained in an element of $\mathcal{P}_1$ or $\mathcal{P}_2$. The intersections of a subspace U of a family $\mathcal{P}_i$ with the elements of the other family form a line spread of U. Hence all planes in F_2 are contained in 3-subspaces of $\mathcal{P}_2$, and all planes of F_2 through a point X in $\mathcal{F}(\Upsilon)$ meet $\mathcal{F}(\Upsilon)$ in the same line. By Proposition 14 this implies $h = 2$. □

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A Appendix: On a result in [4]

In [4, p.175] the following result *(Korollar)* is stated for $F=\mathbb{C}$.

Corollary 16. *Let F be an algebraically closed field. If an s-subspace S_s of $\mathrm{PG}(2s-1,F)$ meets all $S^I \in \mathcal{S}^I_{s-1,1,F}$ only in points, then such points span S_s.*

In [4] the previous result is seemingly proved using methods valid in any field with enough elements. However such a generalisation would contradict Theorem 3.3. In the opinion of the authors the proof in [4] is obtained using an erroneous argument. As a matter of fact, it is claimed in the proof at page 174 that the assumption $\langle\Phi\rangle = S_s$ is not used. However the contradiction $S_s \subset \langle \mathcal{S}_{s-2,1,\mathbb{C}}\rangle$ is inferred from $\Phi \subset \mathcal{S}_{s-2,1,\mathbb{C}}$.

A further counterexample, which exists whenever a hyperbolic quadric $Q^+(3,F)$ in a three-dimensional projective space admits an external line (a condition which is not met when the field F is algebraically closed) is the following. If ℓ is the line corresponding to the two-dimensional vector space $\langle e_1\rangle \otimes \langle e'_1, e'_2\rangle$ and m is a line external to the hyperbolic quadric obtained by the intersection of the Segre variety $\mathcal{S}_{2,1,F}$ with the 3-space corresponding to the vector space $\langle e_2, e_3\rangle \otimes \langle e'_1, e'_2\rangle$, then the 3-dimensional subspace $\langle \ell, m\rangle$ intersects $\mathcal{S}_{2,1,F}$ in the line ℓ belonging to $\mathcal{S}^{II}_{2,1,F}$.

For the sake of completeness, a proof for corollary 16 is given.

Proof of corollary 16. Define

$$S_t = \langle S_s \cap \mathcal{S}_{s-1,1,F}\rangle,\ t = \dim S_t \tag{4}$$

and suppose $t < s$. It is proved in [4, p.173 (6)] that $S_t \subset \langle \mathcal{S}_{t-1,1,F}\rangle$ for some $\mathcal{S}_{t-1,1,F} \subset \mathcal{S}_{s-1,1,F}$.

Note that $S_s \cap \langle \mathcal{S}_{t-1,1,F}\rangle = S_t$; otherwise, comparing dimensions, S_s would intersect each $S^I \in \mathcal{S}_{t-1,1,F}$ in more than one point. Now choose

- a subspace $S_{s-t-1} \subset S_s$ such that $S_{s-t-1} \cap \langle \mathcal{S}_{t-1,1,F}\rangle = \emptyset$;
- a Segre variety $\mathcal{S}_{s-t-1,1,F} \subset \mathcal{S}_{s-1,1,F}$, such that $\langle \mathcal{S}_{s-t-1,1,F}\rangle \cap \langle \mathcal{S}_{t-1,1,F}\rangle = \emptyset$;
- two distinct $A^I, B^I \in \mathcal{S}^I_{s-t-1,1,F}$.

Since $\langle \mathcal{S}_{s-t-1,1,F}\rangle$ and $\langle \mathcal{S}_{t-1,1,F}\rangle$ are complementary subspaces of $\langle \mathcal{S}_{s-1,1,F}\rangle$, a projection map

$$\pi : \langle \mathcal{S}_{s-1,1,F}\rangle \setminus \langle \mathcal{S}_{t-1,1,F}\rangle \to \langle \mathcal{S}_{s-t-1,1,F}\rangle$$

is defined by $\pi(P) = \langle P \cup \mathcal{S}_{t-1,1,F}\rangle \cap \langle \mathcal{S}_{s-t-1,1,F}\rangle$.

Now suppose $\pi(S_{s-t-1}) \cap \mathcal{S}_{s-t-1,1,F} = \emptyset$. In $\langle \mathcal{S}_{s-t-1,1,F}\rangle$ consider

- the regulus $\mathcal{R}$ corresponding to $\mathcal{S}^I_{s-t-1,1,F}$, and the projectivity $\kappa : A^I \to B^I$ such that, for any $P \in A^I$, the line $\langle P, \kappa(P)\rangle$ belongs to $\mathcal{S}^{II}_{s-t-1,1,F}$;
- the regulus $\mathcal{R}'$ containing A^I, B^I and $\pi(S_{s-t-1})$, and the projectivity $\kappa' : A^I \to B^I$ such that, for any $P \in A^I$, the line $\langle P, \kappa'(P)\rangle$ is a transversal line of $\mathcal{R}'$.

Since F is an algebraically closed field, $\kappa'^{-1} \circ \kappa$ has a fixed point P. Therefore $\kappa(P) = \kappa'(P)$, so $\mathcal{R}$ and $\mathcal{R}'$ have a common transversal. This contradicts $\pi(S_{s-t-1}) \cap \mathcal{S}_{s-t-1,1,F} = \emptyset$. So, a point $P \in S_{s-t-1}$ exists such that $\pi(P) \in \mathcal{S}_{s-t-1,1,F}$.

Next, let $C^I \in \mathcal{S}^I_{s-1,1,F}$ be such that $\pi(P) \in C^I$, and Q the point in $\langle \mathcal{S}_{t-1,1,F}\rangle$ such that Q, P, and $\pi(P)$ are collinear. If $Q \in S_t$, then $\pi(P) \in S_s$, a contradiction; also $Q \in C^I$ leads to a contradiction (since it implies $P \in C^I$). So $Q \notin S_t \cup C^I$ and by a dimension argument two points $Q_1 \in C^I \setminus S_t$ and $Q_2 \in S_t \setminus C^I$ exist such that Q, Q_1 and Q_2 are collinear: they are on the unique line through Q meeting both $C^I \cap \langle \mathcal{S}_{t-1,1,F}\rangle$ and a $(t-1)$subspace of S_t disjoint from C^I.

The plane $\langle P, Q_1, Q_2\rangle$ contains the lines $PQ_2 \subset S_s$ and $\pi(P)Q_1 \subset \mathcal{S}_{s-1,1,F}$ which meet outside $\langle \mathcal{S}_{t-1,1,F}\rangle$. This is again a contradiction. □