On the resistance matrix of a graph

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Abstract

Let G be a connected graph of order n. The resistance matrix of G is defined as $R_G = (r_{ij}(G))_{n\times n}$, where $r_{ij}(G)$ is the resistance distance between two vertices i and j in G. Eigenvalues of R_G are called R-eigenvalues of G. If all row sums of R_G are equal, then G is called resistance-regular. For any connected graph G, we show that R_G determines the structure of G up to isomorphism. Moreover, the structure of G or the number of spanning trees of G is determined by partial entries of R_G under certain conditions. We give some characterizations of resistance-regular graphs and graphs with few distinct R-eigenvalues. For a connected regular graph G with diameter at least 2, we show that G is strongly regular if and only if there exist c_1, c_2 such that $r_{ij}(G) = c_1$ for any adjacent vertices $i, j \in V(G)$, and $r_{ij}(G) = c_2$ for any non-adjacent vertices $i, j \in V(G)$.

Keywords: Resistance distance; Resistance matrix; Laplacian matrix; Resistance-regular graph; R-eigenvalue

1 Introduction

All graphs considered in this paper are simple and undirected. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. The resistance distance is a

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distance function on graphs introduced by Klein and Randić [19]. Let G be a connected graph of order n. For two vertices i, j in G, the *resistance distance* between i and j, denoted by $r_{ij}(G)$, is defined to be the effective resistance between them when unit resistors are placed on every edge of G. The *resistance matrix* of G is defined as $R_G = (r_{ij}(G))_{n\times n}$. Eigenvalues of R_G are called *R-eigenvalues* of G. The resistance (distance) in graphs has been studied extensively [8,9,11,12,14,19-21,24-28]. Some properties of the determinant, minors and spectrum of the resistance matrix can be found in [3, 4, 6, 22, 24, 25].

It is known that $r_{ij}(G) \leqslant d_{ij}(G)$ ($d_{ij}(G)$ denotes the distance between i and j), with equality if and only if i and j are connected by a unique path [19]. Hence for a tree T, R_T is equal to the distance matrix of T. The determinant and the inverse of the distance matrix of a tree are given in [15, 16]. These formulas have been extended to the resistance matrix [3]. In [23], Merris gave an inequality for the spectrum of the distance matrix of a tree. This inequality also holds for the spectrum of the resistance matrix of any connected graph [24].

For a connected graph G of order n, let $D_i = \sum_{j=1}^{n} d_{ij}(G)$, $R_i = \sum_{j=1}^{n} r_{ij}(G)$. If $D_1 = D_2 = \cdots = D_n$, then G is called *transmission-regular* [1, 2]. Similar to transmission-regular graphs, we say that G is *resistance-regular* if $R_1 = R_2 = \cdots = R_n$.

In this paper, we show that R_G determines the structure of any connected graph G up to isomorphism. The structure of G or the number of spanning trees of G is determined by partial entries of R_G under certain conditions. We give some characterizations of resistance-regular graphs and graphs with few distinct R-eigenvalues. Applying properties of the resistance matrix, we obtain a characterization of strongly regular graphs via resistance distance.

2 Preliminaries

For a graph G, let A_G denote the adjacency matrix of G, and let D_G denote the diagonal matrix of vertex degrees of G. The matrix $L_G = D_G - A_G$ is called the *Laplacian matrix* of G.

The $\{1\}$*-inverse* of a matrix A is a matrix X such that $AXA = A$. If A is singular, then it has infinite many $\{1\}$-inverses [5, 11]. We use $A^{(1)}$ to denote any $\{1\}$-inverse of A. Let $(A)_{uv}$ or A_{uv} denote the (u, v)-entry of A.

Lemma 1. [5, 11] *Let G be a connected graph. If $L_G^{(1)}$ is a symmetric $\{1\}$-inverse of L_G, then $r_{uv}(G) = (L_G^{(1)})_{uu} + (L_G^{(1)})_{vv} - 2(L_G^{(1)})_{uv}$.*

For a real matrix A, the *Moore-Penrose inverse* of A is the unique real matrix A^+ such that $AA^+A = A$, $A^+AA^+ = A^+$, $(AA^+)^\top = AA^+$ and $(A^+A)^\top = A^+A$. Let I denote the identity matrix, and let $J_{m\times n}$ denote an $m \times n$ all-ones matrix.

Lemma 2. [18] *Let G be a connected graph of order n. Then $L_G^+ J_{n\times n} = J_{n\times n} L_G^+ = 0$, $L_G L_G^+ = L_G^+ L_G = I - \frac{1}{n} J_{n\times n}$.*

For a vertex u of a graph G, let $L_G(u)$ denote the principal submatrix of L_G obtained by deleting the row and column corresponding to u. By the Matrix-Tree Theorem [5], $L_G(u)$ is nonsingular if G is connected.

Lemma 3. *Let G be a connected graph of order n. Then $\begin{pmatrix} L_G(u)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}$ is a symmetric $\{1\}$-inverse of L_G, where u is the vertex corresponding to the last row of L_G.*

Proof. Suppose that $L_G = \begin{pmatrix} L_G(u) & x \\ x^\top & d_u \end{pmatrix}$, where d_u is the degree of u. Since G is connected, $L_G(u)$ is nonsingular. By using the Schur complement formula, we have $\operatorname{rank}(L_G) = \operatorname{rank}(L_G(u)) + \operatorname{rank}(d_u - x^\top L_G(u)^{-1}x) = n-1$. By $\operatorname{rank}(L_G(u)) = n-1$, we get $d_u = x^\top L_G(u)^{-1}x$. Then

$$L_G \begin{pmatrix} L_G(u)^{-1} & 0 \\ 0 & 0 \end{pmatrix} L_G = \begin{pmatrix} I & 0 \\ x^\top L_G(u)^{-1} & 0 \end{pmatrix} \begin{pmatrix} L_G(u) & x \\ x^\top & d_u \end{pmatrix} = L_G.$$

Hence $\begin{pmatrix} L_G(u)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ is a symmetric $\{1\}$-inverse of L_G. □

Lemma 4. [11] *Let $M = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}$ be a nonsingular matrix, and A is nonsingular. Then $M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}B^\top A^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}B^\top A^{-1} & S^{-1} \end{pmatrix}$, where $S = C - B^\top A^{-1}B$.*

For a connected graph G of order n, let $\tau_i = 2 - \sum_{j\in\Gamma(i)} r_{ij}(G)$, where $\Gamma(i)$ denotes the set of all neighbors of i. Let τ be be the $n\times 1$ vector with components $\tau_1,\ldots,\tau_n$.

Lemma 5. [3, 5] *Let G be a connected graph of order n, and let $X = (L_G + \frac{1}{n}J_{n\times n})^{-1}$, $\widetilde{X} = \operatorname{diag}(X_{11},\ldots,X_{nn})$. Then the following hold:*
(a) $\tau = L_G\widetilde{X}\mathbf{j} + \frac{2}{n}\mathbf{j}$, where $\mathbf{j}$ is an all-ones column vector.
(b) $R_G = \widetilde{X}J_{n\times n} + J_{n\times n}\widetilde{X} - 2X$.
(c) $L_G^+ = X - \frac{1}{n}J_{n\times n}$.

For a real symmetric matrix M of order n, let $\lambda_1(M) \geqslant \lambda_2(M) \geqslant \cdots \geqslant \lambda_n(M)$ denote the eigenvalues of M.

Lemma 6. [24] *Let G be a connected graph of order n. Then*

$$0 > -\frac{2}{\lambda_1(L_G)} \geqslant \lambda_2(R_G) \geqslant -\frac{2}{\lambda_2(L_G)} \geqslant \cdots \geqslant -\frac{2}{\lambda_{n-1}(L_G)} \geqslant \lambda_n(R_G).$$

The *Kirchhoff index* of G is defined as $\operatorname{Kf}(G) = \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n r_{ij}(G)$.

Lemma 7. [17, 29] *Let G be a connected graph of order n. Then*

$$\operatorname{Kf}(G) = n\sum_{i=1}^{n-1}\frac{1}{\lambda_i(L_G)}.$$

Lemma 8. [14, 24] *Let G be a connected graph of order n. Then*

$$\sum_{ij \in E(G)} r_{ij}(G) = n - 1.$$

3 Main results

All connected graphs in this section have at least two vertices. We first show that the structure of a connected graph is determined by its resistance matrix up to isomorphism, i.e., if two connected graphs have the same resistance matrix, then they are isomorphic.

Theorem 9. *For any connected graph G, the structure of G is determined by R_G up to isomorphism.*

Proof. By Lemma 3, the matrix $\begin{pmatrix} L_G(u)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ is a symmetric $\{1\}$-inverse of L_G, where u is the vertex corresponding to the last row of L_G. Since R_G is known, by Lemma 1, all entries of $L_G(u)^{-1}$ is known, i.e., $L_G(u)$ is determined by R_G. Since each row (column) sum of L_G is 0, L_G is determined by R_G. Hence G is determined by R_G up to isomorphism. □

A vertex of degree one is called a *pendant vertex*. For a vertex u of a connected graph G, let $R_G(u)$ denote the principal submatrix of R_G obtained by deleting the row and column corresponding to u. Next we show that $R_G(u)$ determines G up to isomorphism if u is not a pendant vertex, i.e., if u is not a pendant vertex of G, and H is a connected graph satisfying $R_H(v) = R_G(u)$ for some $v \in V(H)$, then H is isomorphic to G.

Theorem 10. *For a connected graph G, if u is a vertex of G with degree larger than one, then $R_G(u)$ determines G up to isomorphism.*

Proof. Without loss of generality, suppose that the first row of L_G corresponds to vertex u, and the last row of L_G corresponds to a vertex v. By Lemma 3, $\begin{pmatrix} L_G(v)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ is a symmetric $\{1\}$-inverse of L_G. Suppose that $L_G(v) = \begin{pmatrix} d_u & L_2 \\ L_2^\top & L_3 \end{pmatrix}$, where d_u is the degree of u. Let $S = L_3 - d_u^{-1} L_2^\top L_2$. By Lemma 4, we have

$$\begin{pmatrix} L_G(v)^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} d_u^{-1} + d_u^{-2} L_2 S^{-1} L_2^\top & -d_u^{-1} L_2 S^{-1} & 0 \\ -d_u^{-1} S^{-1} L_2^\top & S^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $R_G(u)$ is known, by Lemma 1, all entries of S^{-1} are known, i.e., S is determined by $R_G(u)$. Since $d_u > 1$ and $S = L_3 - d_u^{-1} L_2^\top L_2$, the following hold:

(1) For any vertex $i \in V(G) \setminus \{u, v\}$, i and u are adjacent if $(S)_{ii}$ is not an integer, are non-adjacent if $(S)_{ii}$ is an integer. Moreover, the degree of i is $d_i = \lceil (S)_{ii} \rceil$, where $\lceil (S)_{ii} \rceil$ is the smallest integer larger than or equal to $(S)_{ii}$.

(2) There exists a vertex $i \in V(G) \setminus \{u, v\}$ such that i and u are adjacent, and $d_u = (\lceil (S)_{ii} \rceil - (S)_{ii})^{-1}$.

(3) For any vertex $i, j \in V(G) \setminus \{u, v\}$, i and j are adjacent if $(S)_{ij} \leqslant -1$, are non-adjacent if $(S)_{ij} > -1$.

From (1)-(3) we know that G is determined by S up to isomorphism. Since S is determined by $R_G(u)$, $R_G(u)$ determines G up to isomorphism. □

Remark 1. Let P_n denote the path with n vertices. Let G be the graph obtained from P_n by attaching a pendant vertex u at a vertex of degree two, and let v be a pendant vertex of P_{n+1}. In this case, we have $R_G(u) = R_{P_{n+1}}(v)$ and G is not isomorphic to P_{n+1}. Hence the condition "u is a vertex of degree larger than one" in Theorem 10 is necessary.

Let $t(G)$ denote the number of spanning trees of a graph G. If V_1 and V_2 are disjoint subsets of $V(G)$, then we define $E(V_1, V_2) = \{ij \in E(G) : i \in V_1, j \in V_2\}$.

Theorem 11. *Let G be a connected graph whose vertex set has a partition $V(G) = V_1 \cup V_2 \cup \{u\}$, and $G - u$ has a unique perfect matching M satisfying $M \subseteq E(V_1, V_2)$. Let $R_G = \begin{pmatrix} R_1 & R_3 & a_1 \\ R_3^\top & R_2 & a_2 \\ a_1^\top & a_2^\top & 0 \end{pmatrix}$, where R_1 and R_2 are principal matrices of R_G corresponding to V_1 and V_2 respectively. Then $t(G)$ is determined by a_1, a_2 and R_3.*

Proof. Without loss of generality, suppose that the last row of L_G corresponds to the vertex u. By Lemma 3, $\begin{pmatrix} L_G(u)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ is a symmetric $\{1\}$-inverse of L_G. Since $G - u$ has a unique perfect matching M satisfying $M \subseteq E(V_1, V_2)$, $L_G(u)$ can be partitioned as $L_G(u) = \begin{pmatrix} L_1 & L_3 \\ L_3^\top & L_2 \end{pmatrix}$, where L_3 is an upper triangular matrix, L_1 and L_2 correspond to V_1 and V_2 respectively. Let $S = L_2 - L_3^\top L_1^{-1} L_3$. By Lemma 4, we have

$$\begin{pmatrix} L_G(u)^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L_1^{-1} + L_1^{-1} L_3 S^{-1} L_3^\top L_1^{-1} & -L_1^{-1} L_3 S^{-1} & 0 \\ -S^{-1} L_3^\top L_1^{-1} & S^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since a_1 and a_2 are known, by Lemma 1, all diagonal entries of $L_G(u)^{-1}$ are known. Since R_3 is also known, by Lemma 1, the matrix $A = -L_1^{-1} L_3 S^{-1}$ is known. Hence $\det(A) = \det(-L_3)[\det(L_1)\det(S)]^{-1}$ is determined by a_1, a_2 and R_3. Note that $-L_3$ is an upper triangular matrix and each diagonal entry of $-L_3$ is 1. So $\det(A) = [\det(L_1)\det(S)]^{-1}$. From the Matrix-Tree Theorem, we have $t(G) = \det(L_G(u)) = \det(L_1)\det(S)$. Hence $t(G)$ is determined by a_1, a_2 and R_3. □

Theorem 12. *Let G be a connected graph with n vertices. Then the following are equivalent:*
(1) G is resistance-regular.
(2) The spectral radius of R_G is

$$\lambda_1(R_G) = \frac{2\mathrm{Kf}(G)}{n}.$$

(3) The spectrum of R_G is

$$\lambda_1(R_G) = \frac{2\mathrm{Kf}(G)}{n},\ \lambda_i(R_G) = -\frac{2}{\lambda_{i-1}(L_G)},\ i = 2, \ldots, n.$$

(4) $X_{11} = \cdots = X_{nn}$, where $X = (L_G + \frac{1}{n}J_{n\times n})^{-1}$.
(5) $(L_G^+)_{11} = \cdots = (L_G^+)_{nn}$.
(6) For each $i \in V(G)$, we have $\sum_{j\in\Gamma(i)} r_{ij}(G) = 2 - \frac{2}{n}$, where $\Gamma(i)$ denotes the set of all neighbors of i.

Proof. By [22, Corollary 2.2], we have (1)$\Longleftrightarrow$(2).

(2)$\Longrightarrow$(3). The trace of R_G is

$$\sum_{i=1}^{n} \lambda_i(R_G) = \frac{2\mathrm{Kf}(G)}{n} + \sum_{i=2}^{n} \lambda_i(R_G) = 0.$$

By Lemmas 6 and 7, we have $\lambda_i(R_G) = -\frac{2}{\lambda_{i-1}(L_G)},\ i = 2, \ldots, n$.

(3)$\Longrightarrow$(2). Obviously.

(1)$\Longleftrightarrow$(4). By part (b) of Lemma 5, we have $R_G\mathbf{j} = n\widetilde{X}\mathbf{j} + (\sum_{i=1}^{n} X_{ii})\mathbf{j} - 2\mathbf{j}$, where $\mathbf{j}$ is an all-ones column vector. Hence G is resistance-regular if and only if $X_{11} = \cdots = X_{nn}$, where $X = (L_G + \frac{1}{n}J_{n\times n})^{-1}$.

By part (c) of Lemma 5, we have (4)$\Longleftrightarrow$(5).

(4)$\Longleftrightarrow$(6). By Lemma 5(a), (4) is equivalent to $\tau = \frac{2}{n}\mathbf{j}$; that is, $\sum_{j\in\Gamma(i)} r_{ij}(G) = 2 - \frac{2}{n}$ for any $i \in V(G)$. □

Remark 2. For any nonsingular matrix B, there exists polynomial $p(x)$ such that $B^{-1} = p(B)$ [7]. Hence $X = (L_G + \frac{1}{n}J_{n\times n})^{-1}$ is a polynomial in $L_G + \frac{1}{n}J_{n\times n}$. If G is a connected regular graph of degree r, then $J_{n\times n}$ is a polynomial in A_G (see [5, Theorem 6.12]). In this case, $X = (rI - A_G + \frac{1}{n}J_{n\times n})^{-1}$ is a polynomial in A_G. A graph G of order n is called *walk-regular*, if $(A_G^k)_{11} = \cdots = (A_G^k)_{nn}$ for any $k \geqslant 0$ [10]. For a connected walk-regular graph G of degree r, since $X = (rI - A_G + \frac{1}{n}J_{n\times n})^{-1}$ is a polynomial in A_G and $(A_G^k)_{11} = \cdots = (A_G^k)_{nn}$ for any $k \geqslant 0$, we have $X_{11} = \cdots = X_{nn}$. By Theorem 12, connected walk-regular graphs (including distance-regular graphs and vertex-transitive graphs) are resistance-regular.

Graphs with few distinct eigenvalues with respect to adjacency matrix and Laplacian matrix have interesting combinatorial properties [10, 13]. Next we consider graphs with few distinct R-eigenvalues.

Theorem 13. *A connected graph with two distinct R-eigenvalues is a complete graph.*

Proof. Let G be a connected graph of order n with two distinct R-eigenvalues $\lambda_1 > \lambda_2$. Since R_G is irreducible and nonnegative, λ_1 is simple. So $R_G - \lambda_2 I$ has rank 1. Since each diagonal entry of R_G is 0, we have $R_G - \lambda_2 I = -\lambda_2 J_{n\times n}$, $R_G = \lambda_2(I - J_{n\times n})$. Hence G is resistance-regular. By part (3) of Theorem 12, L_G has only one nonzero eigenvalue. So G is complete. □

A *strongly regular graph* with parameters (n, k, λ, μ) is a k-regular graph on n vertices such that for every pair of adjacent vertices there are λ vertices adjacent to both, and for every pair of non-adjacent vertices there are μ vertices adjacent to both. It is well known that a connected regular graph whose adjacency matrix has three distinct eigenvalues is strongly regular [10].

Theorem 14. *A resistance-regular graph with three distinct R-eigenvalues is strongly regular.*

Proof. Let G be a resistance-regular graph of order n with three distinct R-eigenvalues. By part (3) of Theorem 12, L_G has two distinct nonzero eigenvalues. Let $\mu_1 > \mu_2 > 0$ be two distinct nonzero eigenvalues of L_G. Since $(L_G - \mu_1 I)(L_G - \mu_2 I)$ has rank 1 and row sum $\mu_1\mu_2$, we have

$$(L_G - \mu_1 I)(L_G - \mu_2 I) = \frac{\mu_1\mu_2}{n} J_{n\times n},$$

$$L_G^2 - (\mu_1 + \mu_2)L_G + \mu_1\mu_2 I = \frac{\mu_1\mu_2}{n} J_{n\times n}. \tag{3.1}$$

By Lemma 2, we have

$$L_G L_G^+ = I - \frac{1}{n} J_{n\times n},\ L_G^2 L_G^+ = L_G(I - \frac{1}{n} J_{n\times n}) = L_G,\ J_{n\times n} L_G^+ = 0.$$

We multiply L_G^+ on both side of (3.1), then

$$[L_G^2 - (\mu_1 + \mu_2)L_G + \mu_1\mu_2 I]L_G^+ = \frac{\mu_1\mu_2}{n} J_{n\times n} L_G^+,$$

$$L_G - (\mu_1 + \mu_2)(I - \frac{1}{n} J_{n\times n}) + \mu_1\mu_2 L_G^+ = 0.$$

From part (5) of Theorem 12, we know that G is regular. Since G is a connected regular graph and L_G has two distinct nonzero eigenvalues, G is strongly regular. □

Theorem 15. *Let G be a connected regular graph with diameter at least 2. Then G is strongly regular if and only if there exist c_1, c_2 such that $r_{ij}(G) = c_1$ for any adjacent vertices $i, j \in V(G)$, and $r_{ij}(G) = c_2$ for any non-adjacent vertices $i, j \in V(G)$.*

Proof. Suppose that G has n vertices and m edges. We need to prove that G is strongly regular if and only if there exist c_1, c_2 such that

$$R_G = c_1 A_G + c_2(J_{n\times n} - I - A_G). \tag{3.2}$$

If G is strongly regular, then $r_{ij}(G)$ depends only on the distance between i and j (see [8, 20]), i.e., the equation (3.2) holds.

If (3.2) holds, then by Lemma 8, we have $c_1 = \frac{n-1}{m}$. Then $\sum_{j\in\Gamma(i)} r_{ij}(G) = \frac{(n-1)k}{m} = 2 - \frac{2}{n}$ for each $i \in V(G)$, where k is the degree of regular graph G. By parts (4) and (6) of

Theorem 12, there exists c_0 such that $c_0 = X_{11} = \cdots = X_{nn}$, where $X = (L_G + \frac{1}{n}J_{n\times n})^{-1} = (kI + \frac{1}{n}J_{n\times n} - A_G)^{-1}$. By part (b) of Lemma 5 and (3.2), we have

$$R_G = 2c_0 J_{n\times n} - 2X = c_2(J_{n\times n} - I) + (c_1 - c_2)A_G,$$

$$2c_0 J_{n\times n} X^{-1} - 2I = c_2(J_{n\times n} - I)X^{-1} + (c_1 - c_2)A_G X^{-1}. \tag{3.3}$$

Since G is regular, by the equation (3.3), there exist a_1, a_2, a_3 such that

$$(c_1 - c_2)A_G^2 + a_1 A_G = a_2 I + a_3 J_{n\times n}. \tag{3.4}$$

If $c_1 = c_2$, then by (3.2), we get $R_G = c_1(J_{n\times n} - I)$. In this case, R_G has two distinct eigenvalues. By Theorem 3.5, G is complete, a contradiction to that the diameter of G at least 2. Hence $c_1 \neq c_2$. By the equation (3.4), we know that there exist λ, μ such that $(A_G^2)_{ij} = \lambda$ for any adjacent vertices $i, j \in V(G)$, and $(A_G^2)_{ij} = \mu$ for any non-adjacent vertices $i, j \in V(G)$. Then G is a strongly regular graph with parameters (n, k, λ, μ). □

4 Concluding remarks

In this paper, the relationship between the graph structure and resistance matrix is studied, and some spectral properties of the resistance matrix are obtained. We list some problems as follows.

(1) For a connected graph G, the structure of G or $t(G)$ is determined by partial entries of R_G under certain conditions (see Theorems 10 and 11). Are there some other graph properties can be determined by partial entries of the resistance matrix?

(2) Some equivalent conditions for resistance-regular graphs are given in Theorem 12. From Remark 2, we know that connected walk-regular graphs are resistance-regular. It is natural to consider the problem"Which graphs are resistance-regular?". Note that a transmission-regular graph does not need to be a (degree) regular graph [1, 2]. Is there a nonregular resistance-regular graph?

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