Polynomials defined by tableaux and linear recurrences

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Abstract

We show that several families of polynomials defined via fillings of diagrams satisfy linear recurrences under a natural operation on the shape of the diagram. We focus on *key polynomials*, (also known as Demazure characters), and *Demazure atoms*. The same technique can be applied to Hall–Littlewood polynomials and dual Grothendieck polynomials.

The motivation behind this is that such recurrences are strongly connected with other nice properties, such as interpretations in terms of lattice points in polytopes and divided difference operators.

Keywords: Linear recurrences, Schur polynomials, Key polynomials, Demazure characters, Demazure atoms, Schubert polynomials, Grothendieck polynomials, Hall–Littlewood polynomials, Young tableaux.

Contents

1 Introduction

Using a similar technique as in [Ale14], we provide a framework for showing that under certain conditions, polynomials encoding statistics on certain tableaux, or fillings of diagrams, satisfy a linear recurrence. We prove that several of the classical polynomials from representation theory fall into this category, such as (skew) Schur polynomials and Hall–Littlewood polynomials.

The main concern in this paper are the so called *key polynomials*, indexed by integer compositions, and *atoms*. The key polynomials are natural, non-symmetric generalizations of Schur polynomials and are specializations of the non-symmetric integer-form Macdonald polynomials, see [Mas09] for details.

Let λ be a fixed diagram shape, (a partition shape, skew shape, *etc.*) and let $P_{k\lambda}(\mathbf{x})$, $k = 1, 2, \dots$ be a sequence of polynomials which are generating functions of fillings of shape $k\lambda$. For partitions, $k\lambda$ is simply elementwise multiplication by k. There are several reasons why one would be interested in showing that a such sequence satisfies a linear recurrence:

1. To obtain hints about the existence or non-existence of formulas of certain type. For example, the Weyl determinant formula for Schur polynomials implies that the ordinary Schur polynomials satisfy a linear recurrence.

2. To obtain evidence for alternative combinatorial interpretations of the tableaux involved. For example, the skew Schur polynomials can be obtained as lattice points in certain marked order polytopes, called Gelfand–Tsetlin polytopes. Such a polytope interpretation implies the existence of a linear recurrence relation.

3. To prove polynomiality in k of the number of fillings of shape $k\lambda$.

4. To obtain results about asymptotics. For example, in [Ale12] we used such recurrences to give a new proof of a classical result on asymptotics of eigenvalues of Toeplitz matrices.

In the last section, we provide several examples of polynomials that satisfy such linear recurrences. We also sketch two additional proofs in the case of key polynomials, to illustrate that several nice properties imply the existence of a linear recurrence relation. These methods are based on a lattice-point representation and an operator characterization of the key polynomials. There is no straightforward way to check if a family of polynomials have such characterizations, but it is easy to generate computer evidence that a sequence of polynomials satisfy a linear recurrence. Thus, proving the existence or non-existence of linear recurrence relations is an informative step towards alternative combinatorial descriptions of the family of polynomials.

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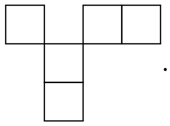
2 Diagrams and fillings

A *diagram* D is a subset of $\{(i,j) : i,j \geqslant 1\}$ which is realized as an arrangement of *boxes*, with a box at (i,j) for every (i,j) in D. Here, i refers to the *row* and j is the *column* of box (i,j) and we draw diagrams in the English notation. For example, $D = \{(1,1),(1,3),(1,4),(2,2),(3,2)\}$ is shown as



A filling is said to have l rows, if row l contains a box, but every row below row l is box-free.

Given an integer composition $\alpha = (\alpha_1, \dots, \alpha_l)$, the *diagram of shape* α, D_α, is given by $D_\alpha = \{(i,j) : 1 \leqslant j \leqslant \alpha_i, 1 \leqslant i \leqslant l\}$. If $\beta = (\beta_1, \dots, \beta_l)$ is another integer composition such that $\alpha \supseteq \beta$, that is, $\alpha_i \geqslant \beta_i$ for all i, then the *diagram of shape* α/β is given by the set-theoretic difference $D_\alpha \setminus D_\beta$, and is denoted $D_{\alpha/\beta}$. Finally, if D is a diagram, let kD be the diagram obtained from D by repeating each column in D k times, that is,

$$kD = \bigcup_{(i,j)\in D} \{(i, kj-k+1), (i, kj-k+2), \dots, (i, kj)\}.$$

Note that $kD_{\alpha/\beta} = D_{k\alpha/k\beta}$.

2.1 Fillings

A *filling* of a diagram is a map $T : D \to \mathbb{N}$, which we represent by writing $T(i,j)$ in the box (i,j). For example,

$$\begin{array}{cccccc} \boxed{1} & \boxed{8} & & & & \\ \times & \times & \boxed{9} & \boxed{1} & & \\ \times & \times & \boxed{5} & & & \\ \times & & & & & \\ \times & \times & \times & \boxed{9} & \boxed{2} & \boxed{7} \end{array} \tag{1}$$

is a filling of the diagram with shape $(2,4,3,1,6)/(0,2,2,1,3)$, where the entries marked $\times$ correspond to boxes in D_β. The shape of a filling refers to the shape of the underlying diagram.

The jth column in a diagram D with l rows has a shape defined as the integer composition $(s_1, \dots, s_l)$, where $s_i = 1$ if $(i,j) \in D$ and 0 otherwise. Thus, if α is an integer composition with only 0 and 1 as parts, then the first column of D_α has shape α. Whenever α is an integer partition, D_α is called a *Young diagram* and any filling of a Young diagram is called a *tableau*. A filling with shape α/β where both α and β are partitions, is called a *skew tableau*.

Given a diagram or a filling, we can *duplicate* or *delete* columns. For example, deleting the fourth column and duplicating the third column two times in the filling in Eq. (1)

results in the filling

$$\begin{array}{ccccccc} \boxed{1} & \boxed{8} & & & & & \\ \times & \times & \boxed{9} & \boxed{9} & \boxed{9} & & \\ \times & \times & \boxed{5} & \boxed{5} & \boxed{5} & & \\ \times & & & & & & \\ \times & \times & \times & \times & \times & \boxed{2} & \boxed{7} \end{array} \quad . \tag{2}$$

Note that if the original filling T has shape $D_{\alpha/\beta}$, then duplication and deletion on T will result in some T' of shape $D_{\alpha'/\beta'}$. This is straightforward to prove.

2.2 Column-closed families of fillings

In most applications, one is interested in a restricted family of fillings, perhaps tableaux or skew tableaux, together with some conditions on the numbers that appear in the boxes. Note that a filling T can be viewed as a concatenation of its columns, some of which might be empty. Obviously, T can only be expressed in one such way if we require that the last (rightmost) column is non-empty.

Let $(C_1, \dots, C_l)$ be a filling with columns $C_1, \dots, C_l$ and let $(m_1C_1, \dots, m_lC_l)$ denote the filling with m_1 copies of the column C_1, followed by m_2 copies of C_2 and so on. Finally, the concatenation, $\parallel$, of two fillings $(C_1, \dots, C_l)$ and $(C'_1, \dots, C'_{l'})$ is simply given by

$$(C_1, \dots, C_l) \parallel (C'_1, \dots, C'_{l'}) = (C_1, \dots, C_l, C'_1, \dots, C'_{l'}).$$

Definition 1. A family of fillings, $\mathcal{T}$, is said to be *weakly column-closed* if

$$(C_1, \dots, C_i, \dots, C_l) \in \mathcal{T} \text{ if and only if } (m_1C_1, \dots, m_lC_l) \in \mathcal{T} \tag{3}$$

holds for every combination of integers m_i where $m_i \geqslant 1$. The family $\mathcal{T}$ is said to be *strictly column-closed* if Eq. (3) holds for every combination where $m_i \geqslant 0$. That is, the family is closed under deletion of *any* column.

Less formally, $\mathcal{T}$ is weakly column-closed if it is closed under column duplication, and reduction of *duplicate* columns. The family is strictly column closed if it, in addition, is closed under removal of *any* column.

Combinatorial objects would be less interesting if it were not for combinatorial statistics. A *combinatorial statistic* on a family $\mathcal{T}$ is a map $\sigma : \mathcal{T} \to \mathbb{N}^s$. We will study a special type of statistics on fillings:

Definition 2. A combinatorial statistic σ on a weakly column-closed family $\mathcal{T}$ is *affine* if

$$\sigma(m_1C_1, \dots, m_lC_l) = A + S_1m_1 + S_2m_2 + \cdots + S_lm_l$$

for all choices of $m_i \geqslant 1$, where A and S_i are vectors in $\mathbb{N}^s$. Similarly, σ defined on a strictly column-closed family $\mathcal{T}$ is *linear* if

$$\sigma(m_1C_1, \dots, m_lC_l) = S_1m_1 + S_2m_2 + \cdots + S_lm_l$$

for all choices of $m_i \geqslant 0$. Note that this is equivalent with the statement that $\sigma(T_1 \parallel T_2) = \sigma(T_1) + \sigma(T_2)$ for every pair T_1 and T_2 of fillings such that $T_1 \parallel T_2$ is in $\mathcal{T}$.

Note that the statistic given by $w(T) = (w_1, w_2, \dots, w_n)$ where w_i is the number of boxes filled with i in T is a linear statistic. This is usually called the *weight* of T. Finally, two statistics $\sigma_1 : \mathcal{T} \to \mathbb{N}^{s_1}$ and $\sigma_2 : \mathcal{T} \to \mathbb{N}^{s_2}$ can be combined into a new statistic σ in the obvious manner as $\sigma(T) = (\sigma_1(T), \sigma_2(T))$, which maps to $\mathbb{N}^{s_1+s_2}$.

3 Properties of linear recurrences

We first recall some basic facts about linear recurrences. This can be seen as analogous to the theory of linear differential equations.

A sequence $\{a_k(\mathbf{x})\}_{k=0}^{\infty}$ of functions are said to satisfy a *linear recurrence* of length r if there are functions $c_1(\mathbf{x}), \dots, c_r(\mathbf{x})$ such that

$$a_k(\mathbf{x}) + c_1(\mathbf{x})a_{k-1}(\mathbf{x}) + \cdots + c_r(\mathbf{x})a_{k-r}(\mathbf{x}) \equiv 0 \tag{4}$$

for all integers $k \geqslant r$. The polynomial (in t)

$$\chi(t) = t^k + c_1(\mathbf{x})t^{k-1} + \cdots + c_{r-1}(\mathbf{x})t + c_r(\mathbf{x})$$

is called the *characteristic polynomial* of the recursion. If the characteristic polynomial factors as $(t-\rho_1)^{m_1} \cdots (t-\rho_r)^{m_r}$, where all $\rho_i(\mathbf{x})$ are *distinct*, then one can express $a_k(\mathbf{x})$ as

$$a_k(\mathbf{x}) = \sum_{l=1}^{r} (\rho_l(\mathbf{x}))^k \sum_{j=0}^{m_l-1} g_{lj}(\mathbf{x})k^j \tag{5}$$

for some functions $g_{li}(\mathbf{x})$, that only depend on the *initial conditions*, that is, the functions $a_0(\mathbf{x})$ to $a_{r-1}(\mathbf{x})$. In the other direction, any sequence of functions which are of the form given in Eq. (5) satisfy a linear recurrence with $\chi(t)$ as characteristic polynomial. Notice that the c_i are elementary symmetric polynomials in the ρ_i, with some signs.

From now on, we are only concerned about sequences where the $a_k(\mathbf{x})$ and $\rho_j(\mathbf{x})$ are polynomials, which implies that the $c_i(\mathbf{x})$ are polynomials and the $g_{li}(\mathbf{x})$ are rational functions. Let $a_k(\mathbf{x})$ and $b_k(\mathbf{x})$ be sequences of polynomials with characteristic polynomials given by $\prod_i (t-\rho_i(\mathbf{x}))^{p_i}$ and $\prod_i (t-\rho_i(\mathbf{x}))^{q_i}$ respectively, where some of the p_i or q_i may be zero. Then, as sequences for $k = 0, 1, \dots,$

- $h(\mathbf{x})a_k(\mathbf{x})$ satisfy the same linear recurrence as $a_k(\mathbf{x})$, where $h(\mathbf{x})$ is any polynomial,
- $a_k(\mathbf{x}) + b_k(\mathbf{x})$ satisfy a linear recurrence with characteristic polynomial given by

  $$\prod_i (t-\rho_i(\mathbf{x}))^{\max(p_i,q_i)}$$

  In particular, if both $a_k(\mathbf{x})$ and $b_k(\mathbf{x})$ have characteristic polynomials with simple roots, then so does $a_k(\mathbf{x}) + b_k(\mathbf{x})$.

- $a_k(\mathbf{x}) \cdot b_k(\mathbf{x})$ satisfy a linear recurrence with characteristic polynomial given by

$$\prod_{\substack{i,j \\ p_i \geqslant 1,\ q_j \geqslant 1}} (t - \rho_i(\mathbf{x})\rho_j(\mathbf{x}))^{p_i+q_j-1}.$$

  Example:

$$a_k(\mathbf{x}) = (1+k^3)(5x)^k, \qquad b_k(\mathbf{x}) = (2+k^2-k^4)(2x-1)^k$$

  satisfy linear recurrences with characteristic polynomials $(t-5x)^4$ and $(t-(2x-1))^5$ respectively. The product, $a_k(\mathbf{x})b_k(\mathbf{x}) = (1+k^3)(2+k^2-k^4)(10x^2-5x)^k$ satisfies a linear recurrence with characteristic polynomial $(t-(10x^2-5x))^8$.

  However, if $\rho_{i_1}(\mathbf{x})\rho_{j_1}(\mathbf{x}) = \rho_{i_2}(\mathbf{x})\rho_{j_2}(\mathbf{x})$ for some $(i_1, j_1) \neq (i_2, j_2)$, some roots of the characteristic equation can be removed — the statement in full generality is left as an exercise. We will only need the following special case: if p_i and q_i are at most 1, that is, the characteristic polynomials have simple roots, then $a_k(\mathbf{x}) \cdot b_k(\mathbf{x})$ also has a characteristic polynomial with simple roots.

  Proof: Using Eq. (5) we may write

$$a_k(\mathbf{x}) = \sum_{i=1}^{r} g_i(\mathbf{x})\rho_i(\mathbf{x})^k \qquad \text{and} \qquad b_k(\mathbf{x}) = \sum_{j=1}^{s} h_j(\mathbf{x})\theta_j(\mathbf{x})^k,$$

  where ρ_i and θ_j are the roots of the respective characteristic polynomials. The product $a_k b_k$ is now of the form

$$\sum_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}} g_i(\mathbf{x})h_j(\mathbf{x}) \left(\rho_i(\mathbf{x})\theta_j(\mathbf{x})\right)^k,$$

  where the coefficients $g_i(\mathbf{x})h_j(\mathbf{x})$ are *independent of* k. Hence $\{a_k(\mathbf{x})b_k(\mathbf{x})\}_{k=0}^{\infty}$ also satisfies a linear recurrence with a characteristic polynomials with *simple roots*.

- $a_{sk}(\mathbf{x})$ with s a fixed positive integer satisfy a linear recurrence with characteristic polynomial given by

$$\prod_i (t - \rho_i(\mathbf{x})^s)^{p_i}.$$

The proofs for these statements follow from writing $a_k(\mathbf{x})$ and $b_k(\mathbf{x})$ in the form Eq. (5) and examining the expressions above. Note that if $a_k(\mathbf{x})$ and $b_k(\mathbf{x})$ have characteristic polynomials with simple roots, then so does $h(\mathbf{x}) \cdot a_k(\mathbf{x})$, $a_k(\mathbf{x}) + b_k(\mathbf{x})$, $a_k(\mathbf{x}) \cdot b_k(\mathbf{x})$ and $a_{sk}(\mathbf{x})$.

Finally, the definition of a sequence satisfying a linear recurrence in Eq. (4) does not provide an easy method to check for a linear recurrence if the c_i are unknown. A useful shortcut might then be the following observation: a sequence $\{a_k(\mathbf{x})\}_{k=0}^{\infty}$ satisfies a linear recurrence of length r if and only if the following $r \times r$-determinant vanishes for all $k \geqslant r-1$:

$$\begin{vmatrix} a_k & a_{k-1} & \dots & a_{k-r+1} \\ a_{k+1} & a_k & \dots & a_{k-r+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+r-1} & a_{k+r-2} & \dots & a_k \end{vmatrix}.$$

This classical trick can be found in *e.g.* [Lyn57].

3.1 Tableaux and linear recurrences

Lemma 3. *Let $\mathcal{T}$ be a weakly column-closed family of fillings and $T = (C_1, \dots, C_l)$ is a fixed filling in $\mathcal{T}$, where no adjacent columns are equal. Let $\sigma : \mathcal{T} \to \mathbb{N}^n$ be a linear combinatorial statistic such that*

$$\sigma(a_1C_1, \dots, a_lC_l) = a_1S_1 + a_2S_2 + \dots + a_lS_l.$$

Define the sequence of polynomials (which depend on T)

$$F_k(\mathbf{z}) = \sum_{\substack{a_i \geqslant 1 \\ a_1+a_2+\dots+a_l=k}} \mathbf{z}^{\sigma(a_1C_1,\dots,a_lC_l)} \text{ and } F_0(\mathbf{z}) = (-1)^{l+1}. \tag{6}$$

Then $\{F_k(\mathbf{z})\}_{k=0}^{\infty}$ satisfy a linear recurrence, with characteristic polynomial

$$(t - \mathbf{z}^{S_1})(t - \mathbf{z}^{S_2}) \cdots (t - \mathbf{z}^{S_l}). \tag{7}$$

Proof. Note that the definition of $F_k(\mathbf{z})$ implies that $F_k(\mathbf{z}) \equiv 0$ whenever $1 \leqslant k < l$, and that $F_l(\mathbf{x}) = \mathbf{z}^{S_1+\dots+S_l}$. These are l conditions, so it remains to show that the linear recurrence given implicitly by Eq. (7) and these l *initial values* defines a sequence that agrees with the definition in Eq. (6).

Any tableau of the form $(a_1C_1, \dots, a_lC_l)$ where $a_i \geqslant 1$ and $a_1 + a_2 + \dots + a_l > l$, must have some $a_i \geqslant 2$. Thus, this tableau can be constructed from some $(a_1C_1, \dots, (a_i - 1)C_i, \dots, a_lC_l)$ by duplicating column C_i. However, there might be several ways to do this. Via an inclusion-exclusion argument, it is straightforward to show that

$$F_{k+l}(\mathbf{z}) - (\mathbf{z}^{S_1} + \dots + \mathbf{z}^{S_l})F_{k+l-1}(\mathbf{z}) + \dots + (-1)^l(\mathbf{z}^{S_1} \cdots \mathbf{z}^{S_l})F_k(\mathbf{z}) \equiv 0 \tag{8}$$

for all $k \geqslant 0$. Note that the coefficients are the elementary symmetric polynomials, evaluated at $\mathbf{z}^{S_1}, \dots, \mathbf{z}^{S_l}$, so factoring the characteristic polynomial gives exactly the expression in (7). Note that the case $k = 0$ in Eq. (8) together with the l conditions implies that $F_0(\mathbf{z})$ must be equal to $(-1)^{l+1}$ since for $k = 0$, Eq. (8) reduces to

$$\mathbf{z}^{S_1+\dots+S_l} + (-1)^l(\mathbf{z}^{S_1} \cdots \mathbf{z}^{S_l})F_0(\mathbf{z}) \equiv 0. \qquad \square$$

Lemma 4. *Let $\mathcal{T}$ be a weakly column-closed family of fillings and let the $T(\mathbf{a}) \in \mathcal{T}$ be the fillings parametrized by integer vectors $\mathbf{a} = (a_{11}, \dots, a_{1,l_1}, \dots, a_{m,l_m})$ as*

$$\begin{aligned} T(\mathbf{a}) = T_1 \parallel (a_{11}C_{11}, \dots, a_{1l_1}C_{1l_1}) \parallel T_2 \parallel (a_{21}C_{21}, \dots, a_{2l_2}C_{2l_2}) \parallel \cdots \\ \parallel T_m \parallel (a_{m1}C_{m1}, \dots, a_{m,l_m}C_{m,l_m}) \parallel T_{m+1} \end{aligned}$$

where each T_i is some fixed (possibly empty) filling and no adjacent columns in each $(C_{i1}, \dots, C_{il_i})$ are equal. Furthermore, let $\sigma : \mathcal{T} \to \mathbb{N}^n$ be an affine combinatorial statistic such that

$$\sigma(T(\mathbf{a})) = A + a_{11}S_{11} + \cdots + a_{m,l_m}S_{m,l_m}.$$

Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a fixed integer composition and define the polynomial

$$G_\alpha(\mathbf{z}) = \sum_{\substack{a_{ij} \geqslant 1 \\ a_{i1}+a_{i2}+\cdots+a_{il_i}=\alpha_i}} \mathbf{z}^{\sigma(T(\mathbf{a}))} \text{ and } G_0(\mathbf{z}) = -(-1)^{l_1+l_2+\cdots+l_m}\mathbf{z}^A$$

where the sum is over all vectors $\mathbf{a}$ such that $a_{ij} \geqslant 1$, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant \alpha_i$ and $a_{i1} + \cdots + a_{il_i} = \alpha_i$. Then

$$G_{k\alpha}(\mathbf{z}) = \mathbf{z}^A \prod_{i=1}^{m} F^i_{k\alpha_i}(\mathbf{z}) \text{ where } F^i_k(\mathbf{z}) = \sum_{\substack{c_i \geqslant 1 \\ c_1+\cdots+c_{l_i}=k}} \mathbf{z}^{c_1S_{i1}+\cdots+c_{l_i}S_{il_i}}, \quad F_0(\mathbf{z}) = -(-1)^{l_i}. \tag{9}$$

Proof. This follows by simply substituting the definition of $F^i_k(\mathbf{z})$ in the product and recognizing the expression for σ. $\square$

Note that the integer composition α should not be confused with some shape of a tableau. The composition α rather serves as the number of columns there are in each of the m "blocks" of columns in $T(\mathbf{a})$. The functions $G_{k\alpha}(\mathbf{x})$ can now be seen as the generating functions of σ, as the block sizes grows linearly with k, and each block i consists of column fillings with columns from $\{C_{i1}, \dots, C_{il_i}\}$, each present at least once.

However, note that if all T_i are empty fillings (no columns), then $G_{k\alpha}(\mathbf{x})$ can be seen as generating function for fillings of shape kD for some *fixed* diagram D as in Fig. 1. In the proof of Proposition 6, the relation between α and D is explained in more detail.

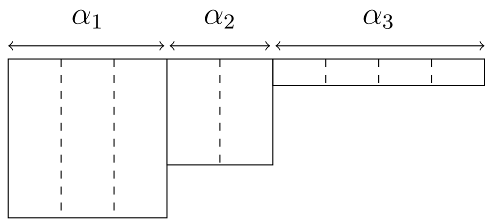


Figure 1: The role of α. Here, all T_i are empty.

Corollary 5. *The sequence* $\{G_{k\alpha}(\mathbf{z})\}_{k=0}^{\infty}$ *satisfies a linear recurrence with characteristic polynomial given by*

$$\prod_{\substack{1\leqslant j_1\leqslant l_1\\ 1\leqslant j_2\leqslant l_2\\ \vdots\\ 1\leqslant j_m\leqslant l_m}} \left(t-\mathbf{z}^{\alpha_1 S_{1j_1}}\mathbf{z}^{\alpha_2 S_{2j_2}}\cdots\mathbf{z}^{\alpha_m S_{m,j_m}}\right). \tag{10}$$

Furthermore, if σ *is linear, then Eq.* (10) *can be expressed as*

$$\prod_{\substack{1\leqslant j_1\leqslant l_1\\ 1\leqslant j_2\leqslant l_2\\ \vdots\\ 1\leqslant j_m\leqslant l_m}} \left(t-\mathbf{z}^{\sigma(\alpha_1 C_{1j_1},\alpha_2 C_{2j_2},\ldots,\alpha_m C_{m,j_m})}\right). \tag{11}$$

Multiple roots can be disregarded if $\{S_{i1},\ldots,S_{il_i}\}$ *are all distinct for every* i*.*

Proof. Each $F_k^i(\mathbf{z})$ in Eq. (9) can be seen as generated by a *linear* statistic $\sigma'(T)=\sigma(T)-A$. It follows from Lemma 3 that for each i, the sequence $\{F_k^i(\mathbf{z})\}_{k=0}^{\infty}$ satisfies a linear recurrence with characteristic polynomial given by

$$(t-\mathbf{z}^{S_{i1}})(t-\mathbf{z}^{S_{i2}})\cdots(t-\mathbf{z}^{S_{il_i}}), \qquad 1\leqslant i\leqslant m.$$

Using the last property of linear recurrences given in the beginning of Section 3, we have that $\{F_{k\alpha_i}^i(\mathbf{z})\}_{k=0}^{\infty}$ also satisfies a linear recurrence, now with characteristic polynomial

$$(t-\mathbf{z}^{\alpha_i S_{i1}})(t-\mathbf{z}^{\alpha_i S_{i2}})\cdots(t-\mathbf{z}^{\alpha_i S_{il_i}}), \qquad 1\leqslant i\leqslant m. \tag{12}$$

Since $G_{k\alpha}(\mathbf{z})$ is a product of the $F_k^i(\mathbf{z})$, we repeatedly use the third property in Section 3 to obtain a characteristic polynomial $\{G_{k\alpha}(\mathbf{z})\}_{k=0}^{\infty}$. The factor $\mathbf{z}^A$ does not depend on k, so it does not affect the characteristic polynomial. To be more precise, the roots of the characteristic polynomial for $G_{k\alpha}(\mathbf{z})$ are given by all ways of selecting one root for each i from Eq. (12). This procedure gives the expression in Eq. (10).

Eq. (11) follows from linearity of σ together with the definition of σ. Note that the value of $\sigma(\alpha_1 C_{1j_1},\alpha_2 C_{2j_2},\ldots,\alpha_m C_{m,j_m})$ is defined *via linearity* of σ, the tableau we evaluate σ on *might not be in* $\mathcal{T}$ (if some $\alpha_i=0$) unless this family is strictly column closed.

The final statement regarding simple roots follows from the proof of the third item in Section 3. □

So far, we have only treated generating functions of subsets of tableaux where the columns are from a specified subset and each column appears at least once. We will now treat the case where only the family of fillings and the diagram shape define the generating function. To do that, it is natural to restrict ourself to a special type of families of fillings.

A family $\mathcal{T}$ is said to be *partially ordered* if every filling $T\in\mathcal{T}$ satisfies the following two properties:

- if two columns in T are identical, then all columns in between are also identical to these two.
- if two columns C_1 and C_2 are different and C_1 appears to somewhere the left C_2, then C_1 never appears to the right of C_2 in some other filling $T' \in \mathcal{T}$.

This is basically stating that the columns satisfies a partial order, with the relation "C_1 appears to the left of C_2 in some filling". For example, fillings such that every row is weakly decreasing (or increasing) are partially ordered.

Being column-closed and partially ordered are two different concepts — the first property only regards *identical* columns, while the second property tells something about the relationship between *different* columns in the fillings in $\mathcal{T}$.

Proposition 6. *Let $\mathcal{T}$ be a partially ordered and weakly column-closed family of fillings and let σ be an affine statistic defined on $\mathcal{T}$. Let D be a fixed diagram and define the polynomials $H_D(\mathbf{z})$ as*

$$H_D(\mathbf{z}) = \sum_{T \in \mathcal{T}(D,n)} \mathbf{z}^{\sigma(T)}$$

where $\mathcal{T}(D,n)$ is the set of all fillings in $\mathcal{T}$ with shape D and for every box (i,j) in such a filling, we have $1 \leqslant T(i,j) \leqslant n$. Then $\{H_{kD}(\mathbf{z})\}_{k=1}^{\infty}$ satisfies a linear recurrence. Furthermore, if σ is linear, then the characteristic polynomial of the recurrence is given by

$$\prod_{T} \left(t - \mathbf{z}^{\sigma(T)}\right) \tag{13}$$

where T runs over all tableaux of shape D such that any adjacent columns of same shape in T are identical, and each T can be obtained from some $\mathcal{T}(kD,n)$ by deleting some columns. Note that T might *not* itself be an element in $\mathcal{T}$. *However, if $\mathcal{T}$ is strictly column closed, then each such T is in $\mathcal{T}(D,n)$.*

Proof. Note that every column in kD has the same shape as some column in D. Since we may only fill boxes with entries from $[n]$, there is a finite number of columns that can appear in $\mathcal{T}(kD,n)$. Furthermore, since $\mathcal{T}$ is partially ordered, there is a *finite* number of lists of columns, $(C_1, C_2, \dots, C_l)$, such that all C_i are different and C_i never appears to the right of C_j in some filling in $\mathcal{T}$, whenever $i < j$. Thus, for every k, every filling in $\mathcal{T}(kD,n)$ can be obtained in a unique way from such a list, by duplicating some columns in that list. Hence, $H_{kD}(\mathbf{z})$ can be expressed as a finite sum over such lists $(C_1, C_2, \dots, C_l)$, where each term corresponds to fillings T of shape kD where each column in T is in the list and every column in the list appears at least once in T.

More specifically, let the diagram D be the concatenation $D = (\alpha_1 s_1, \alpha_2 s_2, \dots, \alpha_m s_m)$ where the s_i are column shapes and $s_i \neq s_{i+1}$, and we use the same notation as for filled columns. Then every filling in $\mathcal{T}(kD,n)$ can be obtained in a unique way as

$$(a_{11}C_{11}, \dots, a_{1l_1}C_{1l_1}) \parallel (a_{21}C_{21}, \dots, a_{2l_2}C_{2l_2}) \parallel \cdots \parallel (a_{m1}C_{m1}, \dots, a_{m,l_m}C_{m,l_m})$$

where each C_{ij} has shape s_i and $a_{i1} + \cdots + a_{i,l_i} = \alpha_i$. Hence, $H_{kD}(\mathbf{x}, \mathbf{t})$ can be expressed as a sum over polynomials of the same form as $G_{k\alpha}$ in Lemma 4. Corollary 5 tells us that each $G_{k\alpha}$ satisfy a linear recurrence, so the sum of such sequences must too. This proves the first statement in the proposition.

The second statement follows from Corollary 5 and observing that the S_{ij} in Eq. (10) can be replaced by $\sigma(C_{ij})$, since σ is linear. The observation that it is enough to only consider tableaux where adjacent columns of same shape have identical fillings is a consequence of the combinatorial interpretation of the $F^i_{k\alpha_i}(\mathbf{z})$ in Lemma 4: a block of size $k\alpha_i$ must have α_i copies of some column if k is sufficiently large and now a similar inclusion-exclusion reasoning apply as in Lemma 3. $\square$

Corollary 7. *Let (σ, τ) be an affine statistic, such that the restriction to σ is linear and $\sigma(C_1) \neq \sigma(C_2) \Rightarrow C_1 \neq C_2$ for any pair of columns that appear in a filling in $\mathcal{T}$. Then the characteristic polynomial in* (13) *can be taken to have only simple roots.*

Proof. In the proposition above, $H_{kD}(\mathbf{z})$ is expressed as a finite sum of polynomials of type $G_{k\alpha}$ in Lemma 4. Thus, if all these $G_{k\alpha}$ have simple roots in their respective characteristic polynomial, so does the sum. This is the second property in the overview of linear recurrences in Section 3.

Furthermore, each $G_{k\alpha}$ is expressed as a product of even simpler polynomials, as in Eq. (9). Again, it suffices to show that each of these $F^i_{k\alpha}$ in such a product has a characteristic polynomial with simple roots.

Going back to Lemma 3 and (7), we see that the property $\sigma(C_1) \neq \sigma(C_2) \Rightarrow C_1 \neq C_2$ implies that the values of the S_i in (7) are all distinct. This proves that the characteristic polynomial has simple roots. $\square$

We end this section with a simple example illustrating Proposition 6.

Example 8. Consider the shape

$$D = \ydiagram{2,1}.$$

The column-closed family $\mathcal{T}$ we consider consists of all fillings with columns $\begin{smallmatrix}1\\1\end{smallmatrix}$, $\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}2\\2\end{smallmatrix}$, 1 and 2 such that both rows are weakly increasing. The statistic in our example is $\mathbf{z}^{\sigma(T)} = \mathbf{x}^{w(T)} y^{\mathrm{one}(T)}$, where $\mathrm{one}(T)$ counts the number of columns of type $\begin{smallmatrix}1\\1\end{smallmatrix}$. The possible fillings for D is then

$$\underset{x_1^3 y}{\begin{ytableau}1&1\\1\end{ytableau}}, \quad \underset{x_1^2 x_2 y}{\begin{ytableau}1&2\\1\end{ytableau}}, \quad \underset{x_1^2 x_2}{\begin{ytableau}1&1\\2\end{ytableau}}, \quad \underset{x_1 x_2^2}{\begin{ytableau}1&2\\2\end{ytableau}}, \quad \underset{x_2^3}{\begin{ytableau}2&2\\2\end{ytableau}}$$

with the corresponding value of $\mathbf{z}^{\sigma(T)}$ written below. Hence,

$$H_D(\mathbf{x}, y) = x_1^3 y + x_1^2 x_2 y + x_1^2 x_2 + x_1 x_2^2 + x_2^3$$

and $H_{2D}(\mathbf{x}, y)$ are given by a similar sum over all fillings of the diagram with shape

$$2D = \ydiagram{4,2}.$$

Note that σ is a linear statistic, so Proposition 6 gives that $\{H_{kD}(\mathbf{x}, y)\}_{k=0}^{\infty}$ satisfies a linear recurrence with characteristic polynomial

$$\prod_{T \in \mathcal{T}(D,2)} (t - \mathbf{x}^{w(T)} y^{\mathrm{one}(T)}) = (t - x_1^3 y)(t - x_1^2 x_2 y)(t - x_1^2 x_2)(t - x_1 x_2^2)(t - x_2^3).$$

There are no multiple roots that can be omitted — all roots are simple in our example.

4 Augmented fillings

This section introduces the diagram fillings that are responsible for key polynomials, Demazure atoms and Hall–Littlewood polynomials. We follow the terminology in [HLMvW11, Mas09], with a few minor modifications.

Let $\beta = (\beta_1, \dots, \beta_n)$ be a list of n different positive integers and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a weak integer composition, that is, a vector with non-negative integer entries. An *augmented filling* of shape α and *basement* β is a filling of a Young diagram of shape $(\alpha_1, \dots, \alpha_n)$ with positive integers, augmented with a zeroth column filled from top to bottom with $\beta_1, \dots, \beta_n$.

Definition 9. Let T be an augmented filling. Two boxes a, b, are *attacking* if $T(a) = T(b)$ and the boxes are either in the same column, or they are in adjacent columns, with the rightmost box in a row strictly below the other box.

$$\begin{array}{c}\boxed{a}\\ \vdots\\ \boxed{b}\end{array} \quad \text{or} \quad \begin{array}{cc}\boxed{a} & \\ \vdots & \\ \boxed{} & \!\!\boxed{b}\end{array}$$

A filling is *non-attacking* if there are no attacking pairs of boxes.

Definition 10. Let T be an augmented filling with weakly decreasing rows. A *coinversion triple of type A* is an arrangement of boxes, a, b, c, located such that a is immediately to the left of b, and c is somewhere below b in the same column, and the row containing a and b is at least as long as the row containing c and $T(a) \geqslant T(c) \geqslant T(b)$.

Similarly, a *coinversion triple of type B* is an arrangement of boxes, a, b, c, located such that a is immediately to the left of b, and c is somewhere above a in the same column, and the row containing a and b is *strictly* longer than the row containing c and $T(a) \geqslant T(c) \geqslant T(b)$.

$$\text{Type } A\text{:} \quad \begin{array}{cc}\boxed{a} & \!\!\boxed{b}\\ & \vdots\\ & \boxed{c}\end{array} \qquad \text{Type } B\text{:} \quad \begin{array}{cc}\boxed{c} & \\ \vdots & \\ \boxed{a} & \!\!\boxed{b}\end{array} \tag{14}$$

Warning! This definition is slightly different from what is stated in [HLMvW11]. However, the definitions coincides whenever the rows in the filling are weakly decreasing and we are only concerned with that special case.

Definition 11. A *semi-standard augmented filling*, (SSAF) of *shape* α and *basement* β is an augmented filling of shape α and basement β with weakly decreasing rows and no coinversion triples (every triple is an *inversion* triple).

Note that this definition implies that there are no attacking boxes in an SSAF. In particular, two entries in the same column must be different in order to be non-attacking.

Example 12. Here is an example of a semi-standard augmented filling, with basement $(1, 3, 2, 5, 4)$.

1				
3	3	$\underline{1}$	1	
2	2			
5	5	5	5	
4	4	$\underline{4}$	$\underline{3}$	2

We can for example check the underlined entries for the type B coinversion triple condition — since $4 \leqslant 1 \leqslant 3$ is *not* true, they do not form such a triple. It is left as an exercise to check that no other triples are coinversion triples.

Lemma 13. *The family of semi-standard augmented fillings is a weakly column-closed and partially ordered family.*

Proof. It suffices to show that duplication of a column in a SSAF T does not introduce any coinversion triples and it is enough to check that there are no coinversion triples in adjacent and identical columns.

Assume that a, b, c form a coinversion triple, as in Eq. (14) (either type). Since the columns are identical, $T(a) = T(b)$ which implies $T(a) = T(c) = T(b)$. However, this is means that two boxes in the same column are identical, so they are attacking. This contradicts the fact that the filling is a SSAF. $\square$

Let $\text{SSAF}(\beta, \alpha)$ be the set of all semi-standard augmented fillings with basement β and shape α. Note that $\text{SSAF}(\beta, \alpha)$ is a finite set. Given an augmented filling T, let $w(T) = (w_1, \dots, w_n)$ where w_i counts the number of boxes with content i *not including the basement*. The *generalized Demazure atoms* are defined as

$$\mathcal{A}_{\beta,\alpha}(\mathbf{x}) = \sum_{T \in \text{SSAF}(\beta,\alpha)} \mathbf{x}^{w(T)}. \tag{15}$$

The special case when $\beta_i = i$ corresponds to the ordinary Demazure atoms, introduced by Lascoux and Schützenberger in [LS90] under the name *standard bases*.

Let $\text{NAWF}(\alpha)$ denote the set of all non-attacking augmented fillings of shape α with weakly decreasing rows and basement given by $\beta_i = i$. The non-symmetric, integral form Hall–Littlewood polynomials, $E_\alpha(x_1, \dots, x_n)$, may be defined as

$$E_\alpha(\mathbf{x}; t) = \sum_{T \in \text{NAWF}(\alpha)} \mathbf{x}^{w(T)} t^{\text{coinv}(T)} (1-t)^{\text{dn}(T)} \tag{16}$$

where $\text{coinv}(T)$ is the number of coinversion triples in T and $\text{dn}(T)$ is the number of pairs of adjacent boxes, (i, j) and $(i, j+1)$, such that $T(i, j) \neq T(i, j+1)$ (different neighbors). This formula was first given in [HLMvW11]. It is straightforward to show that the NAWF form a weakly column-closed and partially ordered family. They show that the ordinary Hall–Littlewood polynomials $P_\mu(\mathbf{x};t)$ can be expressed as

$$P_\mu(\mathbf{x};t) = \sum_{\substack{\gamma \\ \mu=\lambda(\gamma)}} E_\gamma(\mathbf{x};t) \tag{17}$$

where $\lambda(\gamma)$ is the unique integer partition that is obtained from the weak integer composition γ by sorting the parts in decreasing order.

Lemma 14. *The statistics* dn *and* coinv *are affine statistics.*

Proof. It follows immediately from the definition of dn that if $\mathrm{dn}(C_1,\dots,C_l) = A$, then $\mathrm{dn}(m_1C_1,\dots,m_lC_l) = A$ for all $m_i \geqslant 1$, so this is affine. The fact that coinv is affine is also quite straightforward and is left as an exercise to the reader. □

Using Proposition 6, Lemma 13 and Corollary 7, we have the following result:

Corollary 15. *The sequences $\mathcal{A}_{\beta,k\alpha}(\mathbf{x})$ and $E_{k\alpha}(\mathbf{x};t)$ for $k = 1,2,\dots$ satisfy linear recurrences with simple roots.*

Note that Eq. (17) implies that the ordinary Hall–Littlewood P polynomials satisfy a linear recurrence. These are usually (see [Mac95]) defined as

$$P_\lambda(\mathbf{x};t) = \left(\prod_{i\geqslant 0}\prod_{j=1}^{m_\lambda(i)} \frac{1-t}{1-t^j}\right) \sum_{\sigma\in S_n} \sigma\left(x_1^{\lambda_1}\cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\right), \tag{18}$$

where $\lambda = (\lambda_1,\dots,\lambda_n)$, some parts might be zero, $m_\lambda(i)$ denotes the number of parts of λ equal to i, and σ acts on the indices of the variables.

Observe that from this definition, it is quite clear that $\{P_{k\lambda}(\mathbf{x};t)\}_{k=1}^\infty$ satisfies a linear recurrence, since for a fixed σ in (18), the expression is of the form $g(\mathbf{x};t)\sigma(\mathbf{x})^\lambda$ where g is independent under $\lambda \mapsto k\lambda$. Now compare with Eq. (5) above.

4.1 Key tableaux and key polynomials

Let $\alpha = (\alpha_1,\dots,\alpha_n)$ be a weak integer composition. To any such composition, construct a composition with unique entries, β, and a partition λ as follows: Create an augmented Young diagram with shape α and fill the zeroth column with the numbers $1,\dots,n$ decreasingly from top to bottom. Remove all rows for which $\alpha_i = 0$ and sort the remaining rows according to the number of boxes, in a decreasing manner. If two rows have the same number of boxes, preserve the relative order[1]. The resulting diagram has the shape of a partition, λ, which we denote $\lambda(\alpha)$, and the zeroth column will be our basement $\beta(\alpha)$. It is easy to show that this process can be reversed, that is, to any pair (β,λ), there is a corresponding α. Finally, note that $\beta(k\alpha) = \beta(\alpha)$ (multiplying α with a positive constant preserves the relative order of the parts) and $\lambda(k\alpha) = k\lambda(\alpha)$ for non-negative integers k.

[1] Although, in this paper, this convention will not be important.

This correspondence is illustrated in Eq. (19), for $\alpha = (0, 2, 3, 4, 2, 0, 1)$ and the tuple $\beta = (4, 5, 6, 3, 1)$, $\lambda = (4, 3, 2, 2, 1)$.

$$\begin{array}{lllll} 7 & & & & \\ 6 & \square & \square & & \\ 5 & \square & \square & \square & \\ 4 & \square & \square & \square & \square \\ 3 & \square & \square & & \\ 2 & & & & \\ 1 & \square & & & \end{array} \quad \longleftrightarrow \quad \begin{array}{lllll} 4 & \square & \square & \square & \square \\ 5 & \square & \square & \square & \\ 6 & \square & \square & & \\ 3 & \square & \square & & \\ 1 & \square & & & \end{array} \tag{19}$$

The *key polynomials* generalize the Schur polynomials and are indexed by integer compositions. They can be defined as

$$\mathcal{K}_\alpha(\mathbf{x}) = \sum_{T \in \text{SSAF}(\beta(\alpha), \lambda(\alpha))} x^{w(T)}. \tag{20}$$

Note that the key polynomials are a subset of the generalized Demazure atoms. This motivates the definition of a *key tableau* as a semi-standard augmented filling of partition shape, and we let $\text{KTAB}(\beta, \lambda) = \text{SSAF}(\beta, \lambda)$ to emphasize that this subset is of special interest. Note that we only need to be concerned about coinversion triples of type A since we are dealing with partition shapes.

Given a column $(\beta_1, \dots, \beta_n)$ and a set of entries $\{c_1, \dots, c_n\}$, there is at most one way to arrange the entries $c_1, \dots, c_n$ in a column next to β such that the result fulfills all properties of a key tableau. First of all, if there is such an arrangement, the rows must be decreasing. This implies that for some enumeration of the c_i, we must have that $\beta_i \geqslant c_i$ for all i, by having β_i to the left of c_i in the filling.

Secondly, the lack of coinversion triples in a key tableau implies that the order of the entries in the second column is unique; if (a, b, c) is a coinversion triple of type A, then switch the entries in box b and c to create an inversion triple. This defines a total order among the elements in the second column, so there can be at most one filling where the second column (as a set) consists of the entries $c_1, \dots, c_n$.

The following lemma shows that under the obvious condition that if $\beta_i \leqslant c_i$ for all i, then the c_i can be arranged in such a way that the columns form a proper key tableau with basement β. We prove a stronger statement:

Lemma 16. *Let T be a Young diagram of partition shape $1 + \lambda = (1 + \lambda_1, 1 + \lambda_2, \dots)$, filled with positive integers in such a way that each row is weakly decreasing, each column contains unique entries and the first column is given by β. Then each column in T can be sorted in a unique way such that the result is a key tableau of with shape λ and basement β.*

Proof. We do the proof in several steps. The first case we cover is the case when all parts in λ have the same size, that is, all columns of T have the same height. It is easy to see that in this case, we only need to show the statement for two columns. Thus, assume $(\beta_1, \dots, \beta_n)$ and $(c_1, \dots, c_n)$ are given, with $\beta_i \geqslant c_i$ for all i.

We now perform the following sorting procedure on the second column. Let c_i be the largest entry in the second column such that $\beta_1 \geqslant c_i$, and transpose c_1 and c_i. Since $c_i \geqslant c_1$, the rows are still weakly decreasing after this transposition. Note that β_1 and c_i cannot be involved in a coinversion triple later on: if there is some c_j such that $\beta_1 \geqslant c_j > c_i$, then the maximality of the choice of c_i is violated.

We now proceed recursively on the remaining entries of the two columns, $(\beta_2, \dots, \beta_n)$ and $(c_2, \dots, c_n)$ where we have performed a transposition in the second column.

To handle tableaux with more than two columns, simply apply the permutation that takes the original column c to the result on all subsequent columns. The result will now still be a tableau with weakly decreasing rows, but the first two columns do not contain any inversion triples. Proceed with the same method on column 2 and 3, then 3 and 4, and so on.

Note that if $c_1 < c_2 < \cdots < c_r \leqslant \beta_i$ for all i, and $c_r < c_j$ for all $j > r$, then the second column after the above procedure will end in the sequence $c_r, c_{r-1}, \dots, c_1$, reading from top to bottom. Thus, to turn an arbitrarily shaped tableau T into a key tableau, we first augment each column with negative integers such that all columns have the same height, and a new entry on row i will get the value $-i$. After performing the sorting procedure, the above observation implies that we can remove all boxes with negative entries from the result and recover a key tableau with the same shape as T. □

Note that Lemma 16 implies that if T is a key tableau, then one can remove any column from it, reorder the entries in each column and obtain another key tableau. In some sense, key tableaux behave similarly to a strictly column-closed family of tableaux.

Example 17. Here we illustrate the sorting procedure described in Lemma 16. We start with the tableau T which is then augmented with negative integers.

$$T = \begin{array}{|c|c|c|c|}\hline 8 & 5 & 4 & 1 \\ \hline 4 & 3 & 2 & 2 \\ \hline 6 & 6 & 5 & \multicolumn{1}{c}{} \\ \cline{1-3} 7 & 4 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|c|c|}\hline 8 & 5 & 4 & 1 \\ \hline 4 & 3 & 2 & 2 \\ \hline 6 & 6 & 5 & -3 \\ \hline 7 & 4 & -4 & -4 \\ \hline \end{array}.$$

The second column is then sorted and the entries in the other columns are permuted in the same fashion. Two more steps are performed to sort the remaining two columns.

$$\begin{array}{|c|c|c|c|}\hline 8 & 6 & 5 & -3 \\ \hline 4 & 4 & -4 & -4 \\ \hline 6 & 5 & 4 & 1 \\ \hline 7 & 3 & 2 & 2 \\ \hline \end{array} \quad \xrightarrow{\text{3rd column}} \quad \begin{array}{|c|c|c|c|}\hline 8 & 6 & 5 & -3 \\ \hline 4 & 4 & 4 & 1 \\ \hline 6 & 5 & 2 & 2 \\ \hline 7 & 3 & -4 & -4 \\ \hline \end{array} \quad \xrightarrow{\text{4th column}} \quad \begin{array}{|c|c|c|c|}\hline 8 & 6 & 5 & 2 \\ \hline 4 & 4 & 4 & 1 \\ \hline 6 & 5 & 2 & -3 \\ \hline 7 & 3 & -4 & -4 \\ \hline \end{array}$$

Removing the boxes with negative entries now yield a proper key tableaux with the same shape and basement as T.

Remark 18. Note that Lemma 16 does not generalize to arbitrary semi-standard augmented fillings. For example, it is impossible to remove the first column in Example 12 and reorder the entries in the remaining non-basement columns into a valid SSAF — the 1s always appear in some attacking configuration.

4.2 Key polynomial recurrence

We are now ready to state one of the main result of this paper.

Theorem 19. *The sequence of polynomials $\{\mathcal{K}_{k\alpha}(\mathbf{x})\}_{k=1}^{\infty}$ satisfies a linear recurrence with*

$$\prod_{T}(t-\mathbf{x}^T)$$

as characteristic polynomial, where the product is taken over all key tableaux of shape α such that columns of equal height have the same filling and multiple roots in the product are ignored.

Proof. This follows almost immediately from Proposition 6, except that KTAB is not a strictly column-closed family. However, Lemma 16 implies that the tableaux that appear in the product Eq. (13), can be rearranged to key tableaux, while preserving the weight of the tableau. $\square$

4.3 A final note on linear statistics

Suppose there is strong evidence that a family of polynomials are generated by a combinatorial statistic on fillings. That is,

$$p_D(\mathbf{x}) = \sum_{T\in D}\mathbf{x}^{\sigma(T)}$$

for some *unknown* statistic σ, but there are other means of producing the $p_D(\mathbf{x})$.

As a concrete example, let $\sigma : \mathcal{T} \to \mathbb{N}$ and suppose $p_D(s) = s^2 + s$ is (conjecturally) given as the sum is over the two fillings

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}. \tag{21}$$

Furthermore, suppose that $p_{2D}(s) = s^4 + s^3 + s^2$ is given as the sum is over the three fillings

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 2 & 2 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}. \tag{22}$$

Now, we cannot say which monomial corresponds to which filling. However, suppose that σ has the property that it is *linear with respect to k-duplication of* every *column*, meaning that $\sigma(kT) = k\sigma(T)$, where kT is the filling with k copies of every column in T. Then the 2-duplication of the fillings in (21), which are the first and last filling in (22), must produce the polynomial $s^4 + s^2$. We can then deduce that the middle filling in (22) is responsible for s^3 in $p_{2D}(s)$.

As illustrated, linearity of a combinatorial statistic can imply exact values of the statistic on some of the fillings. This information can the be used to guess a combinatorial description of σ. For example, the classical charge statistic (on SSYTs) that is responsible for generating the Koskta–Foulkes polynomials (see [LS78]) has the property that $\operatorname{charge}(kT) = k \operatorname{charge}(T)$ for positive integers k.

5 The polytope side

In this section, we show that the integer point transform of polytopes with the *integer decomposition property*, (IDP), satisfy a linear recurrence. In particular, this can be used to give an alternate proof of Theorem 19.

An *integral polytope* is the convex hull of a finite set of integer points in $\mathbb{R}^d$. The k-dilation of a polytope $\mathcal{P}$ is defined as $k\mathcal{P} = \{k\mathbf{x} : \mathbf{x} \in \mathcal{P}\}$ where k is a non-negative integer, and it is easy to see that this is an integral polytope if $\mathcal{P}$ is. Furthermore, a polytope $\mathcal{P}$ is said to have the *integer decomposition property* if for every integer $k \geqslant 1$, every lattice point $\mathbf{x} \in k\mathcal{P} \cap \mathbb{Z}^d$ can be expressed as $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k$ with $\mathbf{x}_i \in \mathcal{P} \cap \mathbb{Z}^d$. Note that only integral polytopes can have the integer decomposition property and that every face of a polytope with IDP is also a polytope with the IDP.

Examples of polytopes with IDP include all lattice polygons in the plane and order polytopes defined in [Sta01].

The following proposition shows that certain polynomials obtained from polytopes satisfy a linear recurrence. The argument is very similar to that in Lemma 3.

Proposition 20. *Let $\mathcal{P}$ be an integrally closed polytope in $\mathbb{R}^d$ and let $p_k(\mathbf{z})$ be the polynomial defined as*

$$p_k(\mathbf{z}) = \sum_{\mathbf{x} \in k\mathcal{P} \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{x}}. \tag{23}$$

Then the sequence $p_j(\mathbf{z})$ for $j = 0, 1, \dots$ satisfies a linear recurrence with characteristic polynomial given by

$$\prod_{\mathbf{x} \in \mathcal{P} \cap \mathbb{Z}^d} (t - \mathbf{z}^{\mathbf{x}}).$$

Proof. Since $\mathcal{P}$ has the IDP, one can easily show that every lattice point in $k\mathcal{P}$ can be expressed as a sum of a lattice point in $(k-1)\mathcal{P}$ plus a lattice point in $\mathcal{P}$. Therefore

$$p_k(\mathbf{z}) - \left(\sum_{\mathbf{x} \in \mathcal{P} \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{x}} \right) p_{k-1}(\mathbf{t})$$

is a polynomial with only *negative* coefficients corresponding to points in $k\mathcal{P}$ that are expressible in more than one way as $\mathbf{x} + \mathbf{y}$ with $\mathbf{x}$ in $(k-1)\mathcal{P} \cap \mathbb{Z}^d$ and $\mathbf{y}$ in $\mathcal{P} \cap \mathbb{Z}^d$. Hence

$$p_k(\mathbf{z}) - \left(\sum_{\mathbf{x} \in \mathcal{P} \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{x}} \right) p_{k-1}(\mathbf{z}) + \left(\sum_{\mathbf{x} \neq \mathbf{y} \in \mathcal{P} \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{x}} \cdot \mathbf{z}^{\mathbf{y}} \right) p_{k-2}(\mathbf{z})$$

is again a polynomial with *positive* coefficient corresponding to lattice points in $k\mathcal{P}$ expressible in at least three different ways. Repeating this argument using the principle of inclusion-exclusion then yield the desired formula. $\square$

The polynomial defined in Eq. (23) for $k = 1$ is commonly known as the *integer-point transform* of $\mathcal{P}$.

The intersection of two faces of a polytope is also a face (of possibly lower dimension) of the polytope. This enables us to generalize Proposition 20 slightly:

Corollary 21. *Let $\mathcal{P}$ be a polytope with the integer decomposition property and let $F_1, \dots, F_l$ be fixed faces of $\mathcal{P}$. Let $H = \cup_i F_i$ and define the sequence of polynomials*

$$p_{k,H}(\mathbf{z}) = \sum_{\mathbf{x} \in kH \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{x}}. \tag{24}$$

Then the sequence $p_{j,H}(\mathbf{z})$ for $j = 0, 1, \dots$ satisfies a linear recurrence with characteristic polynomial given by

$$\prod_{\mathbf{x} \in H \cap \mathbb{Z}^d} (t - \mathbf{z}^{\mathbf{x}}).$$

Proof. Let F be a face of P with IDP and define

$$p_{k,F}(\mathbf{z}) = \sum_{\mathbf{x} \in kF \cap \mathbb{Z}^d} \mathbf{z}^{\mathbf{x}}. \tag{25}$$

Proposition 20 implies that $\{p_{k,F}(\mathbf{z})\}_{k=0}^{\infty}$ satisfies a liner recurrence with a characteristic polynomial $\prod_{\mathbf{x} \in F \cap \mathbb{Z}^d} (t - \mathbf{z}^{\mathbf{x}})$. Using the notation in Eq. (24), note that

$$p_{k,F_1 \cup F_2}(\mathbf{z}) = p_{k,F_1}(\mathbf{z}) + p_{k,F_2}(\mathbf{z}) - p_{k,F_1 \cap F_2}(\mathbf{z})$$

by applying inclusion-exclusion. The right-hand side is a sum of polynomials of the form Eq. (25), where each term fulfills a linear recursion. We can therefore conclude that $p_{k,F_1 \cup F_2}(\mathbf{z})$ satisfies a linear recursion with characteristic polynomial $\prod_{\mathbf{x} \in (F_1 \cup F_2) \cap \mathbb{Z}^d} (t - \mathbf{z}^{\mathbf{x}})$.

The generalization from two faces to l faces follows from the exact same type of argument. $\square$

In [KST10], it was proven that key polynomials (Demazure characters) can be expressed as (a certain specialization of) the integer point transform of a union of faces of a Gelfand–Tsetlin polytope. Such polytopes are known to have the integer decomposition property, see *e.g.* [Ale14], so Corollary 21 implies a weaker version of Theorem 19. Note that the linear recurrences allow us to define $\mathcal{K}_{0\alpha}(\mathbf{x})$ and it follows from the polyhedral complex interpretation that this always evaluates to 1 (there is exactly one lattice point in the union of faces with dilation 0, namely the origin). It would be interesting to see a direct proof of this fact without using the polytope interpretation.

After extensive computer experimentation, it is hard not to ask the following question:

Question 22. Does the polynomial $k \mapsto \mathcal{K}_{k\alpha}(1^n)$ always have non-negative coefficients?

The case when α is a partition corresponds to a Schur polynomial and it is known that

$$\mathrm{s}_\lambda(1^n) = \prod_{1 \leqslant i < j \leqslant n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

where n is the number of variables. This gives a positive answer to the question in this case.

6 The operator side

Some families of polynomials, such as the Schubert and key polynomials can be defined via *divided difference operators*. Let s_i denote the transposition $(i, i+1)$ and let such transpositions act on $\mathbb{Z}[x_1, \dots,]$ by permuting the indices of the variables. Define the divided difference operators

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}, \quad \pi_i = \partial_i x_i.$$

Given a permutation $\omega \in S_n$, it can be expressed as a product of transpositions, $\omega = s_{i_1} \cdots s_{i_l}$. When the length l is minimal, we say that $i_1 i_2 \dots i_l$ is a *reduced word* of ω. Then, let $\partial_\pi = \partial_{i_1} \cdots \partial_{i_l}$ and $\pi_\omega = \pi_{i_1} \cdots \pi_{i_l}$. It can be shown that these operators does not depend on the choice of the reduced word.

The key polynomials may now be defined [RS95] as $\mathcal{K}_\alpha(\mathbf{x}) = \pi_{u(\alpha)} x^{\lambda(\alpha)}$, where $\lambda(\alpha)$ is the partition obtained by sorting the parts of α in decreasing order and $u(\alpha)$ is a permutation that sorts α into a partition shape. That this indeed is equivalent to the definition above was proved in [Mas09]. We will now give yet another proof that the key polynomials satisfy linear recurrences. First, note that $\mathbf{x}^{k\lambda}$ is a geometric series as $k = 0, 1, \dots$ and thus satisfy a linear recurrence with characteristic polynomial $t - \mathbf{x}^\lambda$. Now note that if $\{f_k(\mathbf{x})\}_{k=0}^\infty$ satisfy a linear recurrence, then so does $\{\partial_i f_k(\mathbf{x})\}_{k=0}^\infty$ and $\{\pi_i f_k(\mathbf{x})\}_{k=0}^\infty$. The result is now a consequence of induction.

The *Schubert polynomials*, $\mathfrak{S}_\omega(\mathbf{x})$, indexed by permutations in S_n, are defined in a similar fashion,

$$\mathfrak{S}_\omega(\mathbf{x}) = \partial_{(w^{-1} w_0)} x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1$$

where ω_0 is the longest permutation in S_n, namely $(n, n-1, \dots, 1)$ in one-line notation. Using a similar reasoning as for the key polynomials, one can produce sequences of Schubert polynomials that satisfy linear recurrences.

7 Appendix: Some families of column-closed fillings

In this section, we review some common families of column-closed fillings, related combinatorial statistics, and generating functions over subsets of such families. Some statements here are well-known or very easy to show, so we present them without proof. Proposition 6 implies that all the polynomials we define below satisfy linear recurrences.

7.1 Flagged skew semi-standard Young tableaux

Let λ and μ be partitions with at most l parts, such that $\lambda \supseteq \mu$. Let $\textsc{ssyt}(\lambda/\mu, n)$ be the set of fillings of $D_{\lambda/\mu}$ with entries in $[n]$, such that each row is weakly increasing and each column is strictly increasing. Then for every l and n, the families

$$\bigcup_{\lambda \supseteq \mu} \textsc{ssyt}(\lambda/\mu, n) \text{ and } \bigcup_{\lambda} \textsc{ssyt}(\lambda, n)$$

are strictly column-closed families, where the unions are taken over shapes with at most l rows. On any filling T with entries in $[n]$, we define the statistic $w(T) : \mathcal{T} \to \mathbb{N}^n$ such that if $w(T) = (w_1, \dots, w_n)$, then w_i is the number of boxes in T filled with i. It is evident that w is a linear statistic.

Finally, the skew Schur polynomials in n variables, indexed by skew partition shapes λ/μ, are defined as

$$\mathrm{s}_{\lambda/\mu}(\mathbf{x}) = \sum_{T \in \textsc{ssyt}(\lambda/\mu, n)} \mathbf{x}^{w(T)}.$$

Even more general, let $\lambda \supseteq \mu$ be a shape with at most l rows and let a and b be increasing sequences of integers of length l, such that $a_i \leqslant b_i$ for all i. Let $\textsc{ssyt}(\lambda/\mu, a, b, n) \subseteq \textsc{ssyt}(\lambda/\mu, n)$ be the subset of fillings T, such that $a_i \leqslant T(i,j) \leqslant b_i$ for every box $(i,j) \in D_{\lambda/\mu}$. Then for each n,

$$\bigcup_{\lambda \supseteq \mu} \textsc{ssyt}(\lambda/\mu, a, b, n)$$

is a strictly column-closed family, where the union is taken over all $\lambda \supseteq \mu$ with at most l rows. The *row-flagged* Schur polynomials, $s_{\lambda/\mu,a,b}(\mathbf{x})$ in n variables are defined as

$$\mathrm{s}_{\lambda/\mu,a,b}(\mathbf{x}) = \sum_{T \in \textsc{ssyt}(\lambda/\mu,a,b,n)} \mathbf{x}^{w(T)},$$

see *e.g.* [Wac85] as a reference. Proposition 6 now implies that $\{\mathrm{s}_{k\lambda/k\mu,a,b}(\mathbf{x})\}_{k=0}^{\infty}$ satisfies a linear recurrence, with characteristic polynomial (removing multiple roots)

$$\prod_{T \in \textsc{ssyt}(\lambda/\mu,a,b,n)} (t - \mathbf{x}^{w(T)}).$$

7.2 Symplectic fillings

The following definition is taken from [Kin76] and these polynomials are related to representations of $Sp(2n)$. The *symplectic Schur polynomials*, $\mathrm{sp}_\lambda(\mathbf{x})$, in the variables $x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}$ are defined via fillings of the Young diagram λ using the alphabet $1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}$ such that rows are weakly increasing, columns are strictly increasing, and entries in row i are greater than or equal to i. Then, for a partition λ with at most n parts,

$$\mathrm{sp}_\lambda(\mathbf{x}) = \sum_{T \in \textsc{spyt}(\lambda)} \mathbf{x}^{w(T)} \mathbf{x}^{-\overline{w}(T)}$$

where $w(T)$ is the weight only counting unbarred entries and $\overline{w}(T)$ only counts the barred entries. It is quite clear that the symplectic Young tableaux form a strictly column-closed family, and that the statistics w and $\overline{w}$ are linear. Consequently, $\{\mathrm{sp}_{k\lambda}\}_{k=1}^{\infty}$ satisfies a linear recurrence for every fixed partition λ.

7.3 Set-valued tableaux and reverse plane partitions

The *Grothendieck polynomials*[2] $\mathrm{G}_\lambda(\mathbf{x})$ can be defined (see [Buc02]) as

$$\mathrm{G}_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SVT}(\lambda)} (-1)^{|T|-|\lambda|} \mathbf{x}^{w(T)}$$

where the sum is taken over *set-valued Young tableaux*. These are defined as fillings of a diagram of shape λ, but now each box contains a *set* of natural numbers. For two such sets A, B we have $A < B$ if $\max A < \min B$ and similar for $A \leqslant B$. With this notation, $\mathrm{SVT}(\lambda)$ is the set of all set-valued tableaux (subsets of $[n]$) such that rows are weakly increasing, and columns are strictly increasing. Here, the ith component of $w(T)$ is now the total number of sets where i appears, and $|T|$ is the sum over all cardinalities of the sets in the boxes. Note that the lowest-degree part of $\mathrm{G}_\lambda(\mathbf{x})$ is the usual Schur polynomial $\mathrm{s}_\lambda(\mathbf{x})$.

There is also an operator definition of the more general Grothendieck polynomials which are indexed by permutations and similar to the Schubert polynomials and introduced by Lascoux and Schützenberger in 1982.

To show that $\mathrm{G}_\lambda(\mathbf{x})$ satisfy a linear recurrence, one needs to use the more general version of Lemma 4, since the family of set-valued Young tableaux is not a weakly column-closed family; only columns where each set is a singleton can be duplicated. However, we note that the family is partially ordered and that every tableau $T \in \mathrm{SVT}(k\lambda)$ contains duplicate columns for every k sufficiently large.

Proposition 23. *The sequence* $\mathrm{G}_{k\lambda}(\mathbf{x})$*,* $k = 1, 2, \dots,$ *satisfies a linear recurrence.*

Proof. Let $\mathcal{T}$ be the set of fillings in $\cup_k \mathrm{SVT}(k\lambda)$. As in Proposition 6, since $\mathcal{T}$ is partially ordered, there is a finite set of fillings in $\mathcal{T}$ where all columns are different. Thus, $\mathrm{G}_{k\lambda}(\mathbf{x})$ can be written as a *finite* sum over lists of columns $(C_1, \dots, C_l)$ where the C_i are the columns that appear in the filling at least once. Thus, we have the refinement

$$\mathrm{G}_{k\lambda}(\mathbf{x}) = \sum_{C=(C_1,\dots,C_l)} G_{k,C}(\mathbf{x}) \tag{26}$$

where C corresponds to the T_i and C_{ij} in Lemma 4, and $G_{k,C}(\mathbf{x})$ are polynomials of the type $G_{k\alpha}$ in Eq. (9).

Note that any column containing a box with a set of cardinality at least two can only appear once in a filling, and can therefore only contribute to the constant factor corresponding to $\mathbf{z}^A$ in Eq. (9). In other words, the only columns that affect the linear

[2]These are called the *stable* Grothendieck polynomials.

recurrence are columns with singleton sets, *i.e.* columns that appear in ordinary semi-standard Young tableaux of shape λ. Therefore, Corollary 5 implies that $\mathrm{G}_{k\lambda}(\mathbf{x})$ satisfies a linear recurrence with characteristic polynomial

$$\prod_{T \in \mathrm{SSYT}(\lambda,n)} (t - \mathbf{x}^{w(T)}),$$

where multiple roots can be ignored according to Corollary 7. $\square$

The fact that the Grothendieck polynomials satisfy linear recurrences can also be proved using a divided difference operator definition, similar to the Schubert polynomials.

Lam and Pylyavskyy [LP07] proved that the *dual stable Grothendieck polynomials*, $\mathrm{g}_\lambda(\mathbf{x})$ in n variables can be defined as

$$\mathrm{g}_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{RPP}(\lambda,n)} \mathbf{x}^{ev(T)}$$

where $\mathrm{RPP}(\lambda, n)$ is the set of *reverse plane partitions* of shape λ, that is, fillings of λ with numbers in $[n]$ such that rows and columns are weakly decreasing. The statistic $ev(T)_i$ is the total number of columns where i appears. From the definition of ev, it is clear that it is a linear statistic — it is obvious that $ev(T_1 \parallel T_2)_i = ev(T_1)_i + ev(T_2)_i$.

Furthermore, the family of reverse plane partitions is partially ordered since rows in such fillings are weakly decreasing. It is also strictly column-closed since removing any column preserves the property of rows and columns being decreasing. Consequently, we get a linear recurrence in this case for $\mathrm{g}_{k\lambda}(\mathbf{x})$, with characteristic polynomial

$$\prod_{T \in \mathrm{RPP}(\lambda,n)} (t - \mathbf{x}^{ev(T)}).$$

Since the value of $ev(C)$ does not uniquely determine the column C in general, we cannot apply Corollary 7. Thus we might need some multiple roots in this characteristic polynomial.

7.4 A note on Jack and Macdonald polynomials

A consequence of satisfying a linear recurrence is that the sequence of polynomials must satisfy a linear recurrence under every specialization of the variables. In particular, if we pick a specialization such that all roots of the characteristic polynomial become equal to 1, then the resulting sequence is a polynomial. For example, $k \mapsto \mathcal{K}_{k\alpha}(1^n)$ is a polynomial in k.

This observation allows us to deduce that the Jack polynomials $J_\lambda(\mathbf{x}, a)$ do *not* satisfy a linear recurrence for general values of a, since the sequences $J_{k\lambda}(1^n, a)$ are not of the form given in Eq. (5). This observation holds for both standard normalizations of Jack polynomials.

It follows that there are no linear recursions for Macdonald polynomials either, since the Jack polynomials are a specialization of the Macdonald polynomials.

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