On Erdős-Sós conjecture for trees of large size

Agnieszka Görlich and Andrzej Żak*

Faculty of Applied Mathematics
AGH University of Science and Technology
Kraków, Poland

forys@agh.edu.pl, zakandrz@agh.edu.pl

Submitted: Jul 13, 2015; Accepted: Mar 3, 2016; Published: Mar 18, 2016
Mathematics Subject Classifications: 05C35

Abstract

Erdős and Sós conjectured that every graph G of average degree greater than $k-1$ contains every tree of size k. Several results based upon the number of vertices in G have been proved including the special cases where G has exactly $k+1$ vertices (Zhou), $k+2$ vertices (Slater, Teo and Yap), $k+3$ vertices (Woźniak) and $k+4$ vertices (Tiner). We further explore this direction. Given an arbitrary integer $c \geqslant 1$, we prove Erdős-Sós conjecture in the case when G has $k+c$ vertices provided that $k \geqslant k_0(c)$ (here $k_0(c) = c^{12}\text{polylog}(c)$). We also derive a corollary related to the Tree Packing Conjecture.

1 Introduction

A set of (simple) graphs $G_1, G_2, \ldots, G_q$ are said to *pack into a complete graph* K_n (in short pack) if $G_1, G_2, \ldots, G_q$ can be found as pairwise edge-disjoint subgraphs in K_n. Many classical problems in Graph Theory can be stated as packing problems. In particular, H is a subgraph of G if and only if H and the complement of G pack.

Erdős and Sós conjectured that every graph G with average degree greater than $k-1$ contains every tree with k edges. This conjecture has been restated by Woźniak [16] as follows.

Conjecture 1. *Suppose that G is a graph with n vertices and T is any tree with k edges. If $|E(G)| < \frac{n(n-k)}{2}$, then G and T pack (into the complete graph K_n).*

Ajtai, Komlós, Simonovits and Szemerédi have announced a proof of Conjecture 1 for sufficiently large k. There are many partial results concerning this conjecture. They have been obtained either for some special families of graphs [2, 5, 6, 15] or for some

*The authors were partially supported by the Polish Ministry of Science and Higher Education.

special families of trees [7, 11, 12] or else for certain values of the parameters k and n. In particular, the cases where n is equal to $k+1$, $k+2$, $k+3$, or $k+4$ were proved by Zhou [17], by Slater, Teo, and Yap [13], by Woźniak [16], and by Tiner [14], respectively. We extend these results to $n = k + c$ for any c, provided k is sufficiently large.

Theorem 2. *Let c be a positive integer and let $k_0(c) = \gamma c^{12} \ln^4 c$ where γ is some universal sufficiently large constant. Then for every $t = 1, \dots, c$ and for every integer $k \geqslant k_0(c)$ the following holds. If T is a tree with k edges and G is a graph on $k+t$ vertices with $|E(G)| < \frac{t(k+t)}{2}$, then T and G pack into K_{k+t}.*

Another famous tree packing conjecture (TPC) posed by Gyárfás [9] states that any set of $n-1$ trees $T_n, T_{n-1}, \dots, T_2$ such that T_i has i vertices pack into K_n. In [8] Bollobás suggested the following weakening of TPC

Conjecture 3. *For every $c \geqslant 1$ there is an $n(c)$ such that if $n \geqslant n(c)$, then any set of c trees $T_1, T_2, \dots, T_c$ such that T_i has $n-(i-1)$ vertices pack into K_n.*

Bourgeois, Hobbs and Kasiraj [4] showed that any three trees T_n, T_{n-1}, T_{n-2} pack into K_n. Recently, Balogh and Palmer [3] proved that any set of $t = \frac{1}{10} n^{1/4}$ trees $T_1, \dots, T_t$ such that no tree is a star and T_i has $n-i+1$ vertices pack into K_n. We obtain the following corollary of Theorem 2:

Corollary 4. *Let c be a positive integer and let $n_0(c) = \gamma c^{12} \ln^4 c$ where γ is some universal sufficiently large constant. If $n \geqslant n_0(c)$, then any set of c trees $T_1, T_2, \dots, T_c$, such that T_i has $n - 2(i-1)$ vertices pack into K_n.*

Proof. The proof is by induction on c. For $c = 1$ the statement is obvious. So fix some $c > 1$ and assume that the statement is true for $c-1$. Let $T_1, T_2, \dots, T_c$ be any set of c trees such that T_i has $n-2(i-1)$ vertices. By the induction hypothesis $T_1, T_2, \dots, T_{c-1}$ pack into K_n. Let G be a graph with $V(G) = V(K_n)$ and $E(G) = \bigcup_{i=1}^{c-1} E(T_i)$. Clearly,

$$|E(G)| \leqslant (c-1)n < \frac{(2c-1)n}{2}.$$

Furthermore, T_c has $n-(2c-1)$ edges. Thus, by Theorem 2, G and T_c pack, which completes the proof of the corollary. □

The notation is standard. In particular $|V(G)|$ is called the order of G and $|E(G)|$ is called the size of G. Furthermore, $d_G(v)$ (abbreviated to $d(v)$ if no confusion arises) denotes the degree of a vertex v in G, $\delta(G)$ and $\Delta(G)$ denote the minimum and the maximum degree of G, respectively. $N_G(v)$ denotes the set of neighbors of v and, for a subset of vertices W, $N_G(W) = \bigcup_{w \in W} N(w) \setminus W$ and $N_G[W] = N_G(W) \cup W$.

2 Preliminaries

In the proof we refine the approach of Alon and Yuster from [1]. However, we apply it in a slightly different way as we choose random subsets B_i (to be defined later) in a denser graph.

We write $\mathrm{Bin}(p, n)$ for the binomial distribution with n trials and success probability p. Let $X \in \mathrm{Bin}(n, p)$. We will use the following two versions of the Chernoff bound which follows from formulas (2.5) and (2.6) from [10] by taking $t = 2\mu - np$ and $t = np - \mu/2$, respectively.

If $\mu \geqslant E[X] = np$ then

$$Pr[X \geqslant 2\mu] \leqslant \exp(-\mu/3) \tag{1}$$

On the other hand, if $\mu \leqslant E[X] = np$ then

$$Pr[X \leqslant \mu/2] \leqslant \exp(-\mu/8). \tag{2}$$

Proposition 5. *Let G be a graph with n vertices and at most m edges. Let $V(G) = \{v_1, \ldots, v_n\}$ with $d(v_1) \geqslant d(v_2) \geqslant \cdots \geqslant d(v_n)$. Then*

$$d(v_i) \leqslant \frac{2m}{i}.$$

Proof. The proposition is true because

$$2m \geqslant \sum_{j=1}^{n} d(v_j) \geqslant \sum_{j=1}^{i} d(v_j) \geqslant id(v_i). \qquad \square$$

The following technical lemma is the main tool in the proof. A version of it appeared in [1].

Lemma 6. *Let G be a graph with n vertices and at most m edges. Let $V(G) = \{v_1, \ldots, v_n\}$ with $d(v_1) \geqslant d(v_2) \geqslant \cdots \geqslant d(v_n)$. Let A_i, $i = 1, \ldots, n$, be any subsets of $V(G)$ with the additional requirement that if $u \in A_i$ then $d(u) < a$. For $i = 1, \ldots, n$ let B_i be a random subset of A_i where each vertex of A_i is independently selected to B_i with probability $p < 1/a$. Let*

$$C_i = \left(\bigcup_{j=1}^{i-1} B_j\right) \cap N(v_i),$$

$$D_i = B_i \setminus \left(\bigcup_{j=1}^{i-1} N[B_j]\right).$$

Then

1. *$Pr\left[|C_i| \geqslant 4mp\right] \leqslant \exp(-2mp/3)$ for $i = 1, \ldots, n$*
2. *$Pr\left[|D_i| \leqslant \frac{p|A_i|}{2e}\right] \leqslant \exp\left(\frac{-p|A_i|}{8e}\right)$ for $i = 1, \ldots, \lfloor 1/(ap) \rfloor$.*

Proof. Fix some vertex $v_i \in V(G)$.

Consider the first part of the lemma. If $d(v_i) \leqslant 2mp$ then the probability is zero because $|C_i| \leqslant |N(v_i)| = d(v_i)$. So we may assume that $d(v_i) > 2mp$. For $u \in N(v_i)$ the probability that $u \in B_j$ is at most p (it is either p if $u \in A_j$ or 0 if $u \notin A_j$.) Thus $Pr[u \in C_i] \leqslant (i-1)p$. By Proposition 5, $i \leqslant 2m/d(v_i)$. Hence,

$$Pr[u \in C_i] \leqslant \frac{2mp}{d(v_i)}.$$

Observe that $|C_i|$ is a sum of $d(v_i)$ independent indicator random variables each of which has success probability at most $\frac{2mp}{d(v_i)}$. Thus, the expectation of $|C_i|$ is at most $2mp$. Therefore, by (1), the probability of $|C_i|$ being larger than $4mp$ satisfies

$$Pr[|C_i| \geqslant 4mp] \leqslant \exp\left(-2mp/3\right).$$

Consider now the second part of the lemma. Observe that for $u \in A_i$, the probability that $u \in B_i$ is p. On the other hand, for any j, the probability that $u \notin N[B_j]$ is at least $1 - ap$. Indeed, $u \in N[B_j]$ if and only if $u \in B_j$ or one of its neighbors belongs to B_j. Since $u \in A_i$, it has at most $a-1$ neighbors. Hence, the probability that $u \in N[B_j]$ is at most ap. Therefore, as long as $i \leqslant 1/(ap)$,

$$Pr[u \in D_i] \geqslant p(1-ap)^{i-1} \geqslant \frac{p}{e}.$$

Observe that $|D_i|$ is a sum of $|A_i|$ independent indicator random variables, each having success probability at least $\frac{p}{e}$. Therefore the expectation of $|D_i|$ is at least $\frac{p|A_i|}{e}$. By (2), the probability that $|D_i|$ falls below $\frac{p|A_i|}{2e}$ satisfies

$$Pr\left[|D_i| \leqslant \frac{p|A_i|}{2e}\right] \leqslant \exp\left(-\frac{p|A_i|}{8e}\right). \qquad \square$$

3 Proof of Theorem 2

The proof is by induction on t. By Zhou's result the theorem holds for $t = 1$. So fix some t, $2 \leqslant t \leqslant c$, and assume that the statement is true for $t-1$. Let G' be a (bipartite) graph that arises from T by adding a set I' of $t-1$ isolated vertices. Thus $|V(G)| = |V(G')|$. Clearly, G' and G pack if and only if T and G pack.

Let $V(G) = \{v_1, \ldots, v_n\}$ where $d_G(v_i) \geqslant d_G(v_{i+1})$ and $V(G') = \{v'_1, \ldots, v'_n\}$ where $d_{G'}(v'_i) \geqslant d_{G'}(v'_{i+1})$. Since $|E(G)| < tn/2$, we have

$$\delta(G) \leqslant t-1 \tag{3}$$

Suppose first that there is a vertex $v \in V(G)$ with $d_G(v) \geqslant t + \frac{k-1}{2}$. Clearly,

$$|E(G-v)| = |E(G)| - d_G(v) < \frac{t(k+t)}{2} - t - \frac{k-1}{2} = \frac{(t-1)(k+t-1)}{2}.$$

Thus, by the induction hypothesis, $G - v$ and T pack. Therefore, G and T pack as well.

Hence, we may assume that

$$\Delta(G) \leqslant t - 1 + \frac{k}{2} \tag{4}$$

Let $S_i \subset V(G) \setminus N[v_i]$ with the assumption that if $u \in S_i$ then $d_G(u) < 5c$.

Claim 7. $|S_i| \geqslant \frac{n}{4} + t$

Proof. By (4) each vertex of G has at least

$$k + t - 1 - (t - 1 + k/2) = k/2$$

non-neighbors. Suppose that α vertices of G have degree greater than or equal to $5c$. Thus

$$cn > 2|E(G)| = \sum_{i=1}^{n} d(v_i) \geqslant \alpha \cdot 5c,$$

and so $\alpha \leqslant \frac{n}{5}$. Therefore

$$|S_i| \geqslant k/2 - \frac{n}{5} \geqslant n/4 + t. \qquad \square$$

Now, we divide the proof into two cases depending whether $\Delta(T) < 60cn^{3/4}$ or $\Delta(T) \geqslant 60cn^{3/4}$.

3.1 Case $\Delta(T) < 60cn^{3/4}$

Recall that $S_i \subset V(G) \setminus N[v_i]$ with the assumption that if $u \in S_i$ then $d_G(u) < 5c$.

For $i = 1, \ldots, n$ let B_i be a random subset of S_i where each vertex of S_i is independently selected to B_i with probability

$$p = \frac{n^{-3/4}}{15 \cdot 10^2 c^3} \tag{5}$$

Let

$$C_i = \left(\bigcup_{j=1}^{i-1} B_j \right) \cap N(v_i),$$

$$D_i = B_i \setminus \left(\bigcup_{j=1}^{i-1} N[B_j] \right).$$

Claim 8. *The following conditions hold simultaneously with positive probability:*

1. $|C_i| \leqslant \frac{n^{1/4}}{750c^2}$ *for* $i = 1, \ldots, n$
2. $|D_i| \geqslant 3$ *for* $i = 1, \ldots, \lfloor 300c^2 n^{3/4} \rfloor$.

Proof. Recall that $|E(G)| < \frac{tn}{2} \leqslant \frac{cn}{2}$. Thus, by Lemma 6,

$$Pr\left[|C_i| \geqslant \frac{n^{1/4}}{750c^2}\right] \leqslant \exp\left(\frac{-n^{1/4}}{4500c^2}\right) < \frac{1}{2n}.$$

Furthermore, by Claim 7,

$$3 \leqslant \frac{n^{1/4}}{12 \cdot 10^3 ec^3} = \frac{n^{-3/4} \cdot (n/4)}{15 \cdot 10^2 c^3 \cdot 2e} < \frac{p|S_i|}{2e}.$$

Hence, by Lemma 6 (with $a = 5c$ and $A_i = S_i$), for each $i \leqslant \lfloor 1/(ap) \rfloor = \lfloor 300c^2 n^{3/4} \rfloor$

$$\begin{aligned} Pr[|D_i| \leqslant 3] \leqslant Pr\left[|D_i| \leqslant \frac{p|S_i|}{2e}\right] &\leqslant \exp\left(-\frac{p|S_i|}{8e}\right) \\ &\leqslant \exp\left(-\frac{n^{1/4}}{48 \cdot 10^3 ec^3}\right) < \frac{1}{600c^2 n^{3/4}}. \end{aligned}$$

Thus, by the union bound, each part of the lemma holds with probability greater than 1/2. Hence both hold with positive probability. □

Therefore, we may fix sets $B_1, \ldots, B_n$ satisfying all the conditions of Claim 8 with respect to the cardinalities of the sets C_i and D_i. We construct a packing $f : V(G) \to V(G')$ in three stages. At each point of the construction, some vertices of $V(G)$ are *matched* to some vertices of $V(G')$, while the other vertices of $V(G)$ and $V(G')$ are yet unmatched. Initially, all vertices are unmatched. We always maintain the packing property, that is for any $u, v \in V(G)$ if $uv \in E(G)$ then $f(u)f(v) \notin E(G')$.

In Stage 1 we match certain number of vertices of G that have the largest degrees. After this stage, by the assumption that $\Delta(G') \leqslant 60cn^{3/4}$, both G and G' do not have unmatched vertices of high degree (vertices of high degree are the main obstacle in packing). This fact enables us to complete the packing in Stages 2 and 3.

Stage 1 Let x be the largest integer such that $d_G(v_x) \geqslant \frac{n^{1/4}}{300c}$. Thus, by Proposition 5,

$$x \leqslant 300c^2 n^{3/4} \tag{6}$$

This stage is done repeatedly for $i = 1, \ldots, x$ and throughout it we maintain the following two invariants

1. At iteration i we match v_i with some vertex $f(v_i)$ of G' such that $d_{G'}(f(v_i)) \leqslant 3$.

2. Furthermore, we match all yet unmatched neighbors of $f(v_i)$ to some vertices of B_i (this way all neighbors of $f(v_i)$ in G' are matched to vertices of $\bigcup_{j=1}^{i} B_j$).

To see that this is possible, consider the i'th iteration of Stage 1 where v_i is some yet unmatched vertex of G. Let Q' be the set of all yet unmatched vertices of G' having degree less than or equal to 3. Note that, by Proposition 5, the number of vertices of degree less than or equal to 3 in G' is at least $n/2$. Hence,

$$|Q'| \geqslant n/2 - 4(i-1) \geqslant n/2 - 4x \geqslant n/2 - 1200c^2 n^{3/4} \geqslant n/3.$$

Let X be the set of already matched neighbors of v_i and let $Y' = N_{G'}(f(X))$. Thus, the valid choice for $f(v_i)$ would be a vertex of $Q' \setminus Y'$. To see that such a choice is possible, it is enough to show that $|Q'| > |Y'|$. Let $X = X_1 \cup X_2$ with $X_1 \subseteq \{v_1, \ldots, v_{i-1}\}$ and $X_2 \subseteq B_1 \cup \cdots \cup B_{i-1}$. Hence $X_1 \leqslant x$ and $|X_2| = |C_i|$. Thus, by the first invariant of Stage 1, and by (6) and Claim 8

$$\begin{aligned}|Q'| - |Y'| &\geqslant n/3 - 3|X_1| - \Delta(G')|X_2| \geqslant n/3 - 3x - 60cn^{3/4}|C_i| \\ &\geqslant n/3 - 900c^2n^{3/4} - 60cn^{3/4}\frac{n^{1/4}}{750c^2} \\ &= n/3 - 6n/(75c) - 900c^2n^{3/4} > n/4 - 900c^2n^{3/4} > 0.\end{aligned}$$

In order to maintain the second invariant it remains to match the yet unmatched neighbors of $f(v_i)$ with vertices from B_i. Let R' be the set of neighbors of $f(v_i)$ in G' that are still unmatched. Recall that $|R'| \leqslant 3$. We have to match vertices of R' with some vertices of B_i. Since $D_i = B_i \setminus \left(\bigcup_{j=1}^{i-1} N[B_j]\right)$, a valid choice of such vertices is by taking an $|R'|$-subset of D_i. By Claim 8 and by (6), $|D_i| \geqslant 3$ for $i = 1, \ldots, x$. Furthermore, since each $v \in D_i$ satisfies $d_G(v) < 5c \leqslant d_G(v_x)$, $D_i \cap \{v_1, \ldots, v_{i-1}\} = \emptyset$. Thus, all vertices of D_i are still unmatched. Hence, such a choice is possible.

Stage 2 Let M_1 and M_1' be the set of matched vertices of G and G' after Stage 1, respectively. Clearly $|M_1| = |M_1'| \leqslant 4x < n/9$. Hence $G' - M_1'$ has an independent set J' with $|J'| \geqslant 4n/9$.

In Stage 2 we match the vertices from $V(G') \setminus (M_1' \cup J')$, one by one, with some $n - |M_1| - |J'|$ vertices from $V(G) \setminus M_1$. Suppose that $v' \in V(G') \setminus (M_1' \cup J')$ is still unmatched. Let Q be the set of all yet unmatched vertices of G. Clearly, $|Q| \geqslant |J'| \geqslant 4n/9$ since the vertices of J' remain unmatched in every step of Stage 2. Let X' be the set of already matched neighbors of v'. Let $Y = N_G(f^{-1}(X'))$. Thus, the valid choice for $f^{-1}(v')$ would be a vertex of $Q \setminus Y$. To see that such a choice is possible we will prove that $|Q \setminus Y| > 0$.
Recall that

$$|X'| \leqslant 60cn^{3/4}.$$

What is more, by the second invariant of Stage 1, the neighbors of each $f(v_i)$, $i = 1, \ldots, x$, are already matched. Hence,

$$X' \subset V(G') \setminus \{f(v_1), \ldots, f(v_x)\}$$

and so

$$f^{-1}(X') \subset V(G) \setminus \{v_1, \ldots, v_x\}.$$

Thus, by the definition of x, for each $u' \in X'$ we have

$$\left|N_G(f^{-1}(u'))\right| \leqslant \frac{n^{1/4}}{300c}. \tag{7}$$

Therefore,

$$|Q \setminus Y| \geqslant |Q| - |X'| \frac{n^{1/4}}{300c} \geqslant 4n/9 - 60cn^{3/4} \frac{n^{1/4}}{300c} > 0.$$

Stage 3 Let M_2 and M_2' be the sets of matched vertices of G and G' after Stage 2, respectively. In order to complete a packing of G and G', it remains to match the vertices of $V(G) \setminus M_2$ with the vertices of J'. Consider a bipartite graph B whose sides are $V(G) \setminus M_2$ and J'. For two vertices $u \in V(G) \setminus M_2$ and $v' \in J'$, we place an edge $uv' \in E(B)$ if and only if it is possible to match u with v' (by this we mean that mapping u to v' will not violate the packing property). Thus u is not allowed to be matched to at most $d_G(u)\Delta(G')$ vertices of J'. Hence

$$d_B(u) \geqslant |J'| - \frac{n^{1/4}}{300c} 60cn^{3/4} > |J'| - 2n/9 \geqslant |J'|/2.$$

Now we will evaluate $d_B(v')$. We define X' and Y in the same way as in Stage 2. Then (7) holds again. Hence, v' is not allowed to be matched to at most $\Delta(G')\frac{n^{1/4}}{300c}$ vertices of $V(G) \setminus M_2$. Thus,

$$d_B(v') \geqslant |J'| - \frac{n^{1/4}}{300c} 60cn^{3/4} \geqslant |J'|/2.$$

Therefore, by Hall's Theorem there is a perfect matching in B, and so a packing of G and G'.

3.2 Case $\Delta(T) \geqslant 60cn^{3/4}$

In this case we will follow the ideas from the previous subsection. However, the key difference is that now both G and G' may have vertices of high degrees. Because of this obstacle, a packing has two more stages at the beginning. After a preparatory Stage 1, in Stage 2 we match the vertices of G that have high degrees with vertices of G' that have small degrees. Then in Stage 3, we match the vertices of G' having high degree. This stage is very similar to Stage 1 from the previous subsection, but with the change of the role of G and G'. Finally, we complete the packing in Stages 4 and 5, which are analogous to Stages 2 and 3 from the previous subsection.

Let

$$q = \frac{n^{1/4}}{59c}. \tag{8}$$

Let $P' \subseteq N_{G'}(v_1')$ be the set of neighbors of v_1' such that each vertex in P' has degree at most q in G', and every neighbor different from v_1' of every vertex from P' has degree at most q in G'.

Claim 9. $|P'| > cn^{3/4}$.

Proof. Note that every vertex $v' \in N_{G'}(v'_1) \setminus P'$ has the property that $d_{G'}(v') > q$ or v' has a neighbor $w' \neq v'_1$ such that $d_{G'}(w') > q$. Therefore,

$$n = |V(G')| > (\Delta(G') - |P'|)q \geqslant (60cn^{3/4} - |P'|)\frac{n^{1/4}}{59c},$$

and the statement follows. □

We construct a packing $f : V(G) \to V(G')$ in five stages. At each point of the construction, some vertices of $V(G)$ are *matched* to some vertices of $V(G')$, while the other vertices of $V(G)$ and $V(G')$ are yet unmatched. Initially, all vertices are unmatched.

Stage 1. We first match v_n with v'_1, i.e. $f(v_n) = v'_1$. Next we match the neighbors of v_n with $d_G(v_n)$ vertices from I'. This is possible since, by (3), $d_G(v_n) = \delta(G) \leqslant t - 1 = |I'|$. Moreover, since I' is a set of isolated vertices, this maping does not violate the packing property.

Stage 2. Let z be the largest integer such that $d_G(v_z) \geqslant n^{1/4}$. Since $|E(G)| < cn/2$, by Proposition 5

$$z \leqslant cn^{3/4}. \tag{9}$$

This stage is done repeatedly for $i = 1, \ldots, z$ and throughout it we maintain the following invariants:

1. At iteration i we match v_i (if it is not matched in Stage 1) with some vertex $f(v_i)$ of G' such that $f(v_i) \in P' \cup I'$.

2. Furthermore, we also make sure that all neighbors of $f(v_i)$ in G' are matched to vertices of $S_i \cup \{v_n\}$.

Note that because G' is acyclic and since there are no edges (in G) between v_i and $S_i \cup \{v_n\}$ for those v_i that are non-neighbors of v_n, such a mapping does not violate the packing property.

To see that this mapping is possible, consider the i'th iteration of Stage 2, where v_i is a vertex of G with $d_G(v_i) \geqslant n^{1/4} \geqslant 5c$. In particular $v_i \notin \bigcup_{j=1}^{i-1} S_j$. Thus, if v_i is already matched, then it was matched in Stage 1 and so $f(v_i) \in I'$. Then, the second invariant of Stage 2 is automatically preserved because $f(v_i)$ is isolated.

Therefore, we may assume that v_i is yet unmatched. In this case we may take $f(v_i)$ to be any vertex of P'. Indeed, note that $|P'| \geqslant z$ and before iteration i, the number of already matched vertices of P' was at most $i - 1$.

Furthermore, observe that since v'_1 is a common neighbor of all $f(v_j)$, $j = 1, \ldots, i$, at iteration i the overall number of matched vertices is at most

$$\delta(G) + 1 + iq \leqslant t + zq \leqslant t + n/59. \tag{10}$$

Let R' be the set of neighbors of $f(v_i)$ in G' that are still unmatched. Note that R' contains all neighbors of $f(v_i)$ apart from v'_1. Thus, in order to maintain the second invariant, it suffices to match vertices of R' with some vertices of S_i. Note that by the choice of P' and since v_1' is already matched, $|R'| \leqslant q-1$. Let Q be the set of yet unmatched vertices of S_i. By Claim 7 and formula (10),

$$|Q| \geqslant n/4 + t - (t + n/59) \geqslant \frac{n^{1/4}}{59c} > q - 1.$$

Hence, this is possible.

Before we describe Stage 3, we need some preparations. Let M_2 be the set of all vertices of G that were matched in Stage 1 or 2. Similarly, let M_2' be the set of all vertices of G' that were matched in Stage 1 or 2. Recall that

$$|M_2| = |M_2'| \leqslant t + zq \leqslant t + n/59. \tag{11}$$

Let $H = G[V \setminus M_2]$ be a subgraph of G induced by yet unmatched vertices. Similarly let $H' = G'[V' \setminus M_2']$. Note that since G' is acyclic and by the construction of Stages 1 and 2,

$$d_{G'}(v') \leqslant d_{H'}(v') + 1 \text{ for each } v' \in V' \setminus M_2'. \tag{12}$$

Let $V(H') = \{w_1', \ldots, w_r'\}$ with $d_{H'}(w_1') \geqslant d_{H'}(w_2') \geqslant \cdots \geqslant d_{H'}(w_r')$. By (11),

$$r \geqslant n - (t + n/59) > 3n/4. \tag{13}$$

Let y be the largest integer such that $d_{H'}(w_y') \geqslant 360\sqrt{n}$. Then, by Proposition 5,

$$y \leqslant \frac{2n}{360\sqrt{n}} = \frac{\sqrt{n}}{180}. \tag{14}$$

For each $1 \leqslant i \leqslant r$ we define a set $S_i' \subseteq V(H') \setminus N_{H'}[w_i']$ to be a largest independent set of vertices but with the additional requirement that each $w' \in S_i'$ has $d_{H'}(w') < 180$.

Claim 10. *$|S_i'| \geqslant n/10$ for $i \geqslant 1$.*

Proof. Note that each w_i' has at least

$$r - d_{H'}(w_i') - 1 \geqslant r - d_{G'}(w_i') - 1 \geqslant r - d_{G'}(v_2') - 1 \geqslant r - \frac{n}{2} - 1 \geqslant \frac{3}{4}n - \frac{n}{2} - 1 = \frac{n}{4} - 1$$

non-neighbors. Since H' is a forest, the subgraph of H' induced by all non-neighbors of w_i' has an independent set of cardinality at least $\frac{n/4-1}{2} > n/9$. Let α be the number of vertices of H' that have degree greater than or equal to 180. Thus

$$2n > \sum_{j=1}^{r} d_{H'}(w_j') \geqslant \alpha \cdot 180,$$

and so $\alpha \leqslant \frac{n}{90}$. Therefore

$$|S_i'| \geqslant n/9 - \frac{n}{90} = n/10. \qquad \square$$

For $i = 1, \ldots, r$ let B'_i be a random subset of S'_i where each vertex of S'_i is independently selected to B'_i with probability $1/\sqrt{n}$. Let

$$C'_i = \left(\bigcup_{j=1}^{i-1} B'_j\right) \cap N(w'_i),$$
$$D'_i = B'_i \setminus \left(\bigcup_{j=1}^{i-1} N_{H'}[B'_j]\right).$$

Claim 11. *The following conditions hold simultaneously with positive probability:*
1. $|C'_i| \leqslant 4\sqrt{n}$ *for* $i = 1, \ldots, r$
2. $|D'_i| \geqslant \frac{\sqrt{n}}{20e}$ *for* $i = 1, \ldots, y$.

Proof. Clearly, $|E(H')| < n$. By Lemma 6 (with $m = n$, $p = 1/\sqrt{n}$ and $A_i = S'_i$ and $a = 180$),

$$Pr\left[|C'_i| \geqslant 4\sqrt{n}\right] \leqslant \exp(-2\sqrt{n}/3) < \frac{1}{2n} \leqslant \frac{1}{2r}.$$

Furthermore, by Claim 10,

$$\frac{\sqrt{n}}{20e} = \frac{(1/\sqrt{n})(n/10)}{2e} \leqslant \frac{p|S'_i|}{2e}.$$

Hence, by the second part of Lemma 6 and by (14), for $i \leqslant y \leqslant \lfloor \sqrt{n}/180 \rfloor$ we have

$$Pr\left[|D'_i| \leqslant \frac{\sqrt{n}}{20e}\right] \leqslant Pr\left[|D'_i| \leqslant \frac{p|S'_i|}{2e}\right] \leqslant \exp\left(-\frac{p|S'_i|}{8e}\right)$$
$$\leqslant \exp\left(-\frac{(1/\sqrt{n})(n/10)}{8e}\right) = \exp\left(-\frac{\sqrt{n}}{80e}\right) < \frac{90}{\sqrt{n}} \leqslant \frac{1}{2y}.$$

Thus, by the union bound, each part of the lemma holds with probability greater than 1/2. Hence both hold with positive probability. □

Now we are in the position to describe the next stages of a packing. By Claim 11 we may fix independent sets $B'_1, \ldots, B'_r$ satisfying all the conditions of Claim 11 with respect to the cardinalities of the sets C'_i and D'_i. Let $W = \{v_1, \ldots, v_z\}$. Recall that

$$\Delta(G - W) < n^{1/4}. \tag{15}$$

Stage 3 This stage is done repeatedly for $i = 1, \ldots, y$ and throughout it we maintain the following (similar to those from Stage 2) invariants

1. At iteration i we match $w'_i \in V(H')$ with some yet unmatched vertex $u = f^{-1}(w'_i)$ of H such that $d_G(u) \leqslant 2c$.

2. Furthermore, we match all yet unmatched neighbors in H of $f^{-1}(w'_i)$ to vertices of B'_i (this way all neighbors of $f^{-1}(w'_i)$ in H are matched to vertices of $\bigcup_{j=1}^{i} B'_j$).

To see that this is possible, consider the i'th iteration of Stage 3 where w'_i is some yet unmatched vertex of H'. Let Q be the set of all yet unmatched vertices of G having degree less than or equal to $2t$. Note that, by Proposition 5, the number of vertices of degree less than or equal to $2t$ in G is at least $n/2$. Hence, by (11) and (14), and since $t \leqslant c$,

$$|Q| \geqslant n/2 - |M_2| - 2ty \geqslant n/2 - t - n/59 - c\sqrt{n}/90 > n/4. \tag{16}$$

Let X' be the set of already matched neighbors (in G') of w'_i and let $Y = N_G(f^{-1}(X'))$. Thus, the valid choice for $f^{-1}(w'_i)$ would be a vertex of $Q \setminus Y$. Let $X' = X'_1 \cup X'_2 \cup X'_3$ such that $X'_1 \subset M'_2 = V(G') \setminus V(H')$, $X'_2 \subset \{w'_1, \ldots, w'_{i-1}\}$ and $X'_3 \subset \bigcup_{j=1}^{i-1} B'_i$. By (12), $|X'_1| \leqslant 1$. Moreover if $v' \in X'_1$ then, by the second invariant of Stage 2, $v' \notin N_{G'}(v'_1)$. Thus $v' = v'_1$ or v' is at distance 2 from v'_1. Hence, either $f^{-1}(v') = v_n$ or $f^{-1}(v')$ belongs to some set S_j, $j \in \{1, \ldots, z\}$. Therefore, $d_G(f^{-1}(v')) \leqslant 5c$. Furthermore, $|X'_2| \leqslant i - 1$ and, by Claim 11, $|X'_3| \leqslant 4\sqrt{n}$. Hence, by (15) and by the first invariant of Stage 3,

$$|Y| \leqslant 5c|X'_1| + 2c|X'_2| + |X'_3| \cdot n^{1/4} < n/4 \tag{17}$$

Therefore, by (16), $|Q \setminus Y| > 0$.

In order to maintain the second invariant we have to match yet unmatched neighbors of $f^{-1}(w'_i)$ with some vertices of B'_i. Let R be the set of neighbors of $f^{-1}(w'_i)$ in G that are still unmatched. Recall that by the first invariant (of Stage 3) $|R| \leqslant 2c$. Since $D'_i = B'_i \setminus \left(\bigcup_{j=1}^{i-1} N[B'_j]\right)$, a natural choice of such vertices is taking an $|R|$-subset of D'_i. However, unlike in Stage 1 in the previous subsection, this subset cannot be chosen arbitrarily because of the existence of possible edges between vertices from $P'' := N_{G'}(P') \setminus \{v'_1\}$ and D'_i. For this reason, we have to match the vertices from R carefully. We match them, one by one, with some vertices from D'_i in the following way. Suppose that $v \in R$ is yet unmatched. Let D' be the set of yet unmatched vertices of D'_i. Since each $w' \in D'_i$ satisfies $d_{H'}(w') < 180 \leqslant 360\sqrt{n}$, $D'_i \cap \{w'_1, \ldots, w'_{i-1}\} = \emptyset$. Hence,

$$|D'| \geqslant |D'_i| - |R| \geqslant \sqrt{n}/(20e) - 2c. \tag{18}$$

Let X_2 be the set of those already matched neighbors u of $v \in R$ which satisfy $f(u) \in P''$. Let $Y'_2 = N_{G'}(f(X_2))$. Thus, the valid choice for $f(v)$ would be a vertex from $D' \setminus Y'_2$. Recall, that by the definition of z, $|X_2| \leqslant d_G(v) \leqslant n^{1/4}$. Furthermore, by the definition of P', $N_{G'}(f(u)) \leqslant q$. Thus, by (8) and (18),

$$|D' \setminus Y'_2| > \sqrt{n}/(20e) - 2c - \sqrt{n}/59 > 0.$$

Thus, an appropriate choice for $f(v)$ is possible.

Stage 4 Let M_3 be the set of matched vertices of G after Stage 3. Similarly, let M'_3 be the set of matched vertices of G' after Stage 3. Note that, by (14) and (11),

$$|M_3| = |M'_3| \leqslant |M_2| + (2c+1)y \leqslant t + n/59 + (2c+1)\sqrt{n}/180 < n/4 \tag{19}$$

Let $W' = \{w'_1, \ldots, w'_y\} \cup \{v'_1\}$. By (12),

$$\Delta(G' - W') \leqslant \Delta(H' - W') + 1 \leqslant 360\sqrt{n} + 1. \tag{20}$$

Furthermore, $|V(G') \setminus M'_3| \geqslant n - n/4 = 3n/4$. Thus $G' - M'_3$ has an independent set J' with $|J'| \geqslant 3n/8$. Let $K' = V(G') \setminus (J' \cup M'_3)$.

In Stage 4 we match vertices from K' one by one, with arbitrary yet unmatched vertices of G. Suppose that $v' \in K'$ is still unmatched. Let Q be the set of all yet unmatched vertices of G. Clearly, $|Q| \geqslant |J'| \geqslant 3n/8$ since the vertices of J' remain unmatched in every step of Stage 4. Let X' be the set of already matched neighbors of v'. Let $Y = N_G(f^{-1}(X'))$. Thus, the valid choice for $f^{-1}(v')$ would be a vertex of $Q \setminus Y$. By (20), $|X'| \leqslant 360\sqrt{n} + 1$. Furthermore, by the second invariant of Stage 2,

$$X' \subseteq V(G') \setminus \{f(v_1), \ldots, f(v_z)\}$$

and so

$$f^{-1}(X') \subseteq V(G) \setminus W.$$

Hence, by (15),

$$|Y| \leqslant |X'| \cdot n^{1/4} < 3n/8.$$

Hence

$$|Q \setminus Y| > 0,$$

and so the choice for $f^{-1}(v')$ is possible.

Stage 5 Let M_4 and M'_4 be the sets of matched vertices of G and G', respectively, after Stage 4. In order to complete a packing of G and G' it remains to match the vertices of J' with the yet unmatched vertices of G. Consider a bipartite graph B whose sides are $V(G) \setminus M_4$ and J'. For two vertices $u \in V(G) \setminus M_4$ and $v' \in J'$, we place an edge $uv' \in E(B)$ if and only if it is possible to match u with v' (by this we mean that mapping u to v' will not violate the packing property). Recall that, by (15), $d_G(u) \leqslant n^{1/4}$. Moreover, by the construction of Stage 1 and by the second invariant of Stage 3, $f(N_G(u)) \subset V(G') \setminus W'$. Thus, by (20), u is not allowed to be matched to at most $n^{1/4}\,(360\sqrt{n} + 1)$ vertices of J'. Therefore,

$$d_B(u) \geqslant |J'| - n^{1/4}\left(360\sqrt{n} + 1\right) > |J'| - 3n/16 \geqslant |J'|/2.$$

Similarly, by (20), $d_{G'}(v') \leqslant 360\sqrt{n} + 1$. Moreover, by the second invariant of Stage 2, $f^{-1}(N_{G'}[v']) \subset V(G) \setminus W$. Therefore, by (15),

$$d_B(v') \geqslant |J'| - n^{1/4}\left(360\sqrt{n} + 1\right) > |J'|/2.$$

Therefore, by Hall's Theorem there is a perfect matching in B, and so a packing of G and G'. This completes the inductive step, and so the theorem is proved. □

Acknowledgements

We thank the reviewers for carefully reading our manuscript and for giving detailed comments and suggestions that have been helpful to improve the manuscript.

References

[1] N. Alon and R. Yuster. The Turán number of sparse spanning graphs. *J. Comb. Theory Ser. B*, 103(3):337–343, 2013.

[2] S. Balasubramanian and E. Dobson. Constructing trees in graphs with no $K_{2,s}$. *J. Graph Theory*, 56:301–310, 2007.

[3] J. Balogh and C. Palmer. On the Tree Packing Conjecture. *SIAM J. Discrete Math.*, 27(4):1995–2006, 2013.

[4] B.A. Bourgeois, A.M. Hobbs, and J. Kasiraj. Packing trees in complete graphs. *Discrete Math.*, 67:27–42, 1987.

[5] E. Dobson. Constructing trees in graphs whose complement has no $K_{2,s}$. *Combin. Probab. Comput.*, 11:343–347, 2002.

[6] N. Eaton and G. Tiner. On the Erdős-Sós conjecture for graphs having no path with $k+4$ vertices. *Discrete Math.*, 313:1621–1629, 2013.

[7] G. Fan and L. Sun. The Erdős-Sós conjecture for spiders. *Discrete Math.*, 307:3055–3062, 2007.

[8] R.L. Graham, M. Grötschel, and L. Lovász, Eds. *Handbook of combinatorics.* Vol. 1,2. Elsevier Science B.V., Amsterdam, 1995.

[9] A. Gyárfás and J. Lehel. Packing trees of different order into K_n. In *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Colloq. Math. Soc. János Bolyai*, volume 18 pages 463–469, North-Holland, Amsterdam, 1978.

[10] S. Janson, T. Łuczak, and A. Ruciński. Random graphs. *Wiley-Interscience Series in Discrete Mathematics and Optimization*, Wiley-Interscience, New York, 2000.

[11] A. McLennan. The Erdős-Sós conjecture for trees of diameter four. *J.Graph Theory*, 49(4):291–301, 2005.

[12] A.F. Sidorenko. Asymptotic solution for a new class of forbidden graphs. *Combinatorica*, 9(2):207–215, 1989.

[13] P. Slater, S. Teo, and H. Yap. Packing a Tree with a Graph of the Same Size. *J. Graph Theory*, 9:213–216, 1985.

[14] G. Tiner. On the Erdős-Sós Conjecture for Graphs on $n = k+3$ Vertices. *Ars Combin.*, 95:143–150, 2010.

[15] J.F. Saclé, M. Woźniak. The Erdős-Sós Conjecture for graphs without C_4. *J. Combin. Theory Ser. B*, 50:367–372, 1997.

[16] M. Woźniak. On the Erdős-Sós Conjecture. *J. Graph Theory*, 21:229–234, 1996.

[17] B. Zhou. A note on the Erdős-Sós Conjecture. *Acta Math. Sci.*, 4:287–289, 1984.