Graphs with induced-saturation number zero

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Abstract

Given graphs G and H, G is H-saturated if H is not a subgraph of G, but for all $e \notin E(G)$, H appears as a subgraph of $G + e$. While for every $n \geqslant |V(H)|$, there exists an n-vertex graph that is H-saturated, the same does not hold for induced subgraphs. That is, there exist graphs H and values of $n \geqslant |V(H)|$, for which every n-vertex graph G either contains H as an induced subgraph, or there exists $e \notin E(G)$ such that $G + e$ does not contain H as an induced subgraph. To circumvent this Martin and Smith [12] make use of a generalized notion of "graph" when introducing the concept of induced saturation and the induced saturation number of graphs. This allows for edges that can be included or excluded when searching for an induced copy of H, and the induced saturation number is the minimum number of such edges that are required.

In this paper, we show that the induced saturation number of many common graphs is zero. This yields graphs that are H-induced-saturated. That is, graphs such that no induced copy of H exists, but adding or deleting any edge creates an induced copy of H. We introduce a new parameter for such graphs, $\text{indsat}^*(n, H)$, which is the minimum number of edges in an H-induced-saturated graph. We

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provide bounds on $\text{indsat}^*(n, H)$ for many graphs. In particular, we determine $\text{indsat}^*(n, H)$ completely when H is the paw graph $K_{1,3} + e$, and we determine $\text{indsat}^*(n, K_{1,3})$ within an additive constant of four.

1 Background and Introduction

1.1 Background and Definitions

A well-known graph parameter is the *saturation number*, defined for a graph H and a whole number n as the minimum number of edges in a graph G on n vertices such that H is not a subgraph of G, but H occurs if any edge is added to G. Formally,

$$\text{sat}(n, H) = \min\{|E(G)| : \text{G has } n \text{ vertices}, H \not\subseteq G, \text{ and } \forall e \notin E(G), H \subseteq G + e\}.$$

Determining the saturation number for a given graph H has proven, in general, quite difficult. For more information on the saturation number, see the dynamic survey of Faudree, Faudree, and Schmitt [6].

A natural attempt at defining an induced variant of graph saturation would be to state that an n-vertex graph G is H-induced-saturated if G is H-free (that is, without H as an induced subgraph) and for all $e \notin E(G)$, $G + e$ contains H as an induced subgraph. Unfortunately, this is not always well-defined. That is, there exist graphs H and values of $n \geqslant |V(H)|$ for which every n-vertex graph G either contains H as induced subgraph, or there exists $e \notin E(G)$ such that $G+e$ is H-free. A simple example is $n = 4$ and $H = K_{1,3}$.

In this paper, we consider a variant of the saturation number introduced by Martin and Smith in 2012 that looks for induced copies of H, and considers deleting as well as adding edges. To create a well-defined parameter, Martin and Smith [12] make use of *trigraphs*, objects also used by Chudnovsky and Seymour in their structure theorems on claw-free graphs [3].

Definition 1.1. A *trigraph* T is a quadruple $(V(T), E_B(T), E_W(T), E_G(T))$, where $V(T)$ is the vertex set and the other three elements partition $\binom{V(T)}{2}$ into a set $E_B(T)$ of black edges, a set $E_W(T)$ of white edges, and a set $E_G(T)$ of gray edges. These can be thought of as edges, nonedges, and potential edges, respectively. For any $e \in E_B(T) \cup E_W(T)$, let T_e denote the trigraph where e is changed to a gray edge, i.e. $T_e = (V(T), E_B(T) - e, E_W(T) - e, E_G(T) + e)$.

A *realization* of T is a graph $G = (V(G), E(G))$ with $V(G) = V(T)$ and $E(G) = E_B(T) \cup S$ for some $S \subseteq E_G(T)$. Let $\mathcal{R}(T)$ be the family of graphs that are a realization of T.

A trigraph T is *H-induced-saturated* if no realization of T contains H as an induced subgraph, but H occurs as an induced subgraph of some realization whenever any black or white edge of T is changed to gray.

The *induced saturation number* of a graph H with respect to n, written $\text{indsat}(n, H)$, is the minimum number of gray edges in an H-induced-saturated trigraph with n vertices.

Formally,

$$\text{indsat}(n, H) = \min\{|E_G(T)| : |V(T)| = n, \forall G \in \mathcal{R}(T), H \not\leqslant G, \\ \text{and } \forall e \in E_B(T) \cup E_W(T), H \leqslant G', \text{ where } G' \in \mathcal{R}(T_e)\}.$$

Notice that a trigraph with $E_G(T) = \emptyset$ has a unique realization, so if $\text{indsat}(n, H) = 0$, there is a graph G that has no induced copy of H yet adding or removing any edge creates an induced copy of H. We will call such a graph *H-induced-saturated*.

The *complement* of a trigraph T, denoted $\overline{T}$, is the trigraph with $V(\overline{T}) = V(T)$, $E_B(\overline{T}) = E_W(T)$, $E_W(\overline{T}) = E_B(T)$, and $E_G(\overline{T}) = E_G(T)$.

1.2 Notation

A trivial component of a graph is an isolated vertex. For a graph G, we use $v(G)$ for the number of vertices and $e(G)$ for the number of edges in G. We let P_n denote the path on n vertices and C_n the cycle on n vertices. K_n is the complete graph on n vertices, and for $k \geqslant 2$, $K_{a_1,\ldots,a_k}$ is the complete multipartite graph with parts of size $a_1, \ldots, a_k$. $K_{1,3}^+$ is the paw, which is obtained by adding a single edge to $K_{1,3}$. For a set $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S, and if $S = \{v_1, \ldots, v_p\}$, we will sometimes write $G[v_1, \ldots, v_p]$. For a vertex $v \in V(G)$, $N_G(v)$ (or $N(v)$, if G is clear from context) is the set of neighbors of v in G, and $N[v] = N(v) \cup \{v\}$. We use $\deg_G(v)$ or $\deg(v)$ to denote the degree of v, that is, $|N(v)|$. In a trigraph, the black (resp. gray) degree of a vertex is the number of black (resp. gray) edges incident to that vertex. We say a set S of vertices *dominates* G, and we call S a *dominating set*, if every vertex of $G - S$ is adjacent to some vertex in S; if $S = \{v\}$, we say v is a *dominating vertex*. Similarly, a vertex u *dominates* a vertex set S if u is adjacent to every vertex in S. For an integer n, we let $[n] = \{1, \ldots, n\}$.

If G and H are graphs, then $G \cup H$ denotes their disjoint union. The graph $G \vee H$, known as the *join* of G and H, is obtained by taking disjoint copies of G and H and making each vertex $u \in V(G)$ adjacent to each vertex $v \in V(H)$. The *Cartesian product* of G and H, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$, where (u_1, u_2) is adjacent to (v_1, v_2) if either $u_1 = v_1$ and u_2 is adjacent to v_2 in H, or if $u_2 = v_2$ and u_1 is adjacent to v_1 in G.

Other notation will be defined as it is used, or see [13] for any undefined terms.

1.3 Observations and Previous Results

By definition, the only trigraphs on fewer than $v(H)$ vertices that are H-induced-saturated are those in which all edges are gray. Thus we will usually assume that $n \geqslant v(H)$ when we discuss $\text{indsat}(n, H)$.

The following theorem summarizes the results of Martin and Smith [12]:

Theorem 1.2. *Let H be a graph.*

- *For all $n \geqslant v(H)$, $\text{indsat}(n, H) \leqslant \text{sat}(n, H)$. By [9], $\text{sat}(n, H) = O(n)$, so in particular $\text{indsat}(n, H) = O(n)$.*

- *For all* $n \geqslant m \geqslant 3$, $\mathrm{indsat}(n, K_m) = \mathrm{sat}(n, K_m)$. *(Note that* $\mathrm{sat}(n, K_m)$ *was determined by Erdős, Hajnal, and Moon in [5].)*
- *For all* $n \geqslant m \geqslant 2$, *and for* $e \in E(K_m)$, $\mathrm{indsat}(n, K_m - e) = 0$. *In particular, for all* $n \geqslant 3$, $\mathrm{indsat}(n, P_3) = 0$.
- *For all* $n \geqslant 4$, $\mathrm{indsat}(n, P_4) = \left\lceil \frac{n+1}{3} \right\rceil$.

We also make the following observation, which was stated in [12] for $H = P_4$:

Observation 1.3. *A trigraph* T *is* H*-induced-saturated if and only if* $\overline{T}$ *is* $\overline{H}$*-induced-saturated. In particular,* $\mathrm{indsat}(n, H) = \mathrm{indsat}(n, \overline{H})$.

Proof. Suppose a trigraph T has a realization G such that H is an induced subgraph of G. Then $\overline{H}$ is an induced subgraph of $\overline{G}$. Using the definition of $\overline{T}$, $\overline{G}$ is a representation of $\overline{T}$. It follows that a trigraph T is H-induced-saturated if and only if $\overline{T}$ is $\overline{H}$-induced-saturated. □

1.4 Minimally H-induced-saturated Graphs

In this paper we show that $\mathrm{indsat}(n, H)$ is zero for several graphs, which as noted above, means that there exists a graph that is H-induced-saturated. This leads to the natural question: What is the minimum number of edges in such a graph?

Definition 1.4. For a graph H and whole number n with $\mathrm{indsat}(n, H) = 0$, we define

$$\mathrm{indsat}^*(n, H) := \min\{e(G) : v(G) = n \text{ and } G \text{ is } H\text{-induced-saturated}\}.$$

We say a graph G on n vertices with $\mathrm{indsat}^*(n, H)$ edges is **minimally H-induced-saturated**.

By Observation 1.3, the *maximum* number of edges in an n-vertex H-induced-saturated graph is $\binom{n}{2} - \mathrm{indsat}^*(n, \overline{H})$.

The paper proceeds as follows. In Section 2, we characterize the $K_{1,3}^+$-induced-saturated graphs, which in turn completely determines $\mathrm{indsat}^*(n, K_{1,3}^+)$. Section 3 contains a construction that shows that $\mathrm{indsat}(n, H) = 0$ when H is a star, and gives bounds on $\mathrm{indsat}^*(n, H)$ for stars. Section 4 studies $\mathrm{indsat}^*(n, K_{1,3})$ in detail, showing that the upper bound given in Section 3 is correct for $K_{1,3}$. In Section 5, we turn our attention to C_4, proving that $\mathrm{indsat}(n, C_4) = 0$ and giving an upper bound on $\mathrm{indsat}^*(n, C_4)$. We also use the complement of C_4 to show that $\mathrm{indsat}(n, kK_2) = 0$ with a constant upper bound on $\mathrm{indsat}^*(n, kK_2)$. Section 6 gives an upper bound on $\mathrm{indsat}^*(n, H)$ when H is an odd cycle or a minor modification of an even cycle. Finally, in Section 7, we introduce the induced-saturation number of a family of graphs and show that while every graph in a family may have induced-saturation number zero, the family itself could have nonzero induced-saturation number.

2 The Paw

In this section we will prove the following theorem, which will allow us to completely determine $\text{indsat}^*(n, K_{1,3}^+)$ for $n \geqslant 7$.

Theorem 2.1. *A graph is $K_{1,3}^+$-induced-saturated if and only if it is as described in Construction 2.2.*

The graphs in Construction 2.2 do not exist for $n < 7$, and as Theorem 2.1 shows that these are the only $K_{1,3}^+$-induced-saturated graphs, it follows that $\text{indsat}(n, K_{1,3}^+)$ is nonzero for $n \in \{4, 5, 6\}$. The exact values for such n are provided in Table 1. In this table, we see that for $n \in \{5, 6\}$, there are $K_{1,3}^+$-induced-saturated trigraphs on n vertices that have only one gray edge. This establishes $\text{indsat}(n, K_{1,3}^+) = 1$ for such n. For $n = 4$, we have a 4-vertex, $K_{1,3}^+$-induced-saturated trigraph with two gray edges. To show that $\text{indsat}(4, K_{1,3}^+) = 2$, we argue that any 4-vertex trigraph with only one gray edge is not $K_{1,3}^+$-induced-saturated.

Suppose T is a $K_{1,3}^+$-induced-saturated trigraph on 4 vertices with one gray edge. Clearly, T has at least two black edges, otherwise changing a white edge to gray does not result in a realization with an induced $K_{1,3}^+$. Now suppose T has no white edges. Since it has precisely one gray edge, its black edges form $K_4 - e$, and changing the black edge whose endpoints have black degree three to a gray edge does not result in a realization with an induced $K_{1,3}^+$. Next, suppose T has at least two white edges. Since $K_{1,3}^+$ has precisely two nonedges, changing a black edge to gray does not result in a realization with an induced $K_{1,3}^+$, unless T already had such a realization. Therefore T has precisely one white edge. If the gray edge of T is incident to the white edge, then $K_{1,3}^+$ is a realization, so the black edges induce C_4. Since $C_4 \not\subseteq K_{1,3}^+$, changing the white edge to gray does not create an induced $K_{1,3}^+$. Thus, this trigraph does not exist, and $\text{indsat}(4, K_{1,3}^+) = 2$.

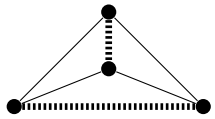
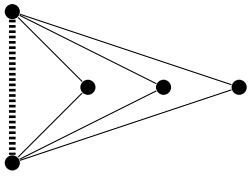
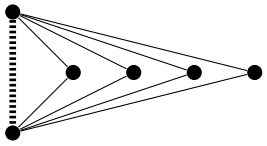
	trigraph
$\text{indsat}(4, K_{1,3}^+) = 2$	
$\text{indsat}(5, K_{1,3}^+) = 1$	
$\text{indsat}(6, K_{1,3}^+) = 1$	

Table 1: Values of $\text{indsat}(n, K_{1,3}^+)$ for $4 \leqslant n \leqslant 6$ and trigraphs realizing those values

Having established $\text{indsat}(n, K_{1,3}^+)$ for small values of n, we now present our construction.

Construction 2.2. Let G be a graph with at most one trivial component, where each nontrivial component is complete multipartite. Further, each nontrivial component has at least three parts, at most one of which contains only one vertex, and the remainder of which have order at least three.

Notice that graphs such as those in Construction 2.2 exist for all $n \geqslant 7$. In particular, any complete tripartite graph with parts of the appropriate order will suffice.

Proposition 2.3. *The graph G in Construction 2.2 is $K_{1,3}^+$-induced-saturated.*

Proof. Since $K_{1,3}^+$ is not an induced subgraph of a complete multipartite graph, G contains no induced $K_{1,3}^+$. Suppose we add an edge xy such that x and y are in distinct components, say F_x and F_y, respectively. Since at least one of these components, say F_x, has at least three parts, x is in some triangle xab in F_x. Because y is in a different component, y is adjacent to x but not a or b. Thus $\{x, y, a, b\}$ induces a $K_{1,3}^+$.

Suppose we add an edge xy such that x and y are in the same component. Then in particular, they are in the same part. This part has at least two vertices, so by construction it has at least three vertices; choose z distinct from x and y from this part, and let a be in another part of the component. Then $\{x, y, z, a\}$ induces a $K_{1,3}^+$.

Suppose we delete an edge xy. Then x and y were in different parts of one component, say F. As F is complete multipartite with at least three parts, there exists a vertex z in a third part of that component. Since at most one part has only one vertex, there is a vertex a in the same part as either x or y; say x. Then $\{x, y, z, a\}$ induces a $K_{1,3}^+$. □

The induced-saturation number $\text{indsat}(n, K_{1,3}^+)$ is an immediate corollary of this result.

Corollary 2.4. *For $n \geqslant 7$,* $\text{indsat}(n, K_{1,3}^+) = 0$.

Since $\text{indsat}(n, K_{1,3}^+) = 0$ for $n \geqslant 7$, we turn our attention to Theorem 2.1, which will allow us to determine $\text{indsat}^*(n, K_{1,3}^+)$. To prove the theorem, we begin by making several observations.

Lemma 2.5. *Let G be a $K_{1,3}^+$-induced-saturated graph. Then G has the following properties:*

(a) Every edge of G is in a triangle.

(b) The neighborhood of any vertex of G is a complete multipartite graph.

(c) Given any non-isolated vertex $v \in V(G)$, the set $N(x) \setminus N[v]$ is the same for every $x \in N(v)$. Call this set $S(v)$. If $S(v)$ is not empty, then $S(v)$ is an independent set.

Proof. Lemma 2.5(a) holds because deleting any edge in G creates an induced $K_{1,3}^+$. As a consequence, any vertex has either degree zero or degree at least two.

Since G does not contain an induced $K_{1,3}^+$, the neighborhood of any vertex cannot contain an induced copy of $K_2 \cup K_1$. This is equivalent to the neighborhood being a complete multipartite graph. This gives us Lemma 2.5(b).

To prove Lemma 2.5(c), suppose there exists $x \in N(v)$ that has a neighbor not in $N[v]$. (If no such x exists, the claim holds with $S = \emptyset$.) Let $S := N(x) \setminus N[v]$. If $G[S]$ has an edge ss', then $G[v, x, s, s']$ is a $K_{1,3}^+$. Since G is $K_{1,3}^+$-induced-saturated, we conclude that S is independent.

By Lemma 2.5(a), there exists $y \in N(v) \cap N(x)$. If any element $s \in S$ is not adjacent to y, then $G[v, x, y, s]$ is a $K_{1,3}^+$ with s as the pendant vertex. Therefore, $S \subseteq N(y)$, but also $N(y) \setminus N[v] \subseteq S$ or else we would have a $K_{1,3}^+$. Because $N(v)$ is complete multipartite by Lemma 2.5(b), every vertex in $N(v) \setminus \{x, y\}$ is adjacent to x or y. By symmetry, we conclude that for every $z \in N(v)$, $N(z) \setminus N[v] = S$. □

We proceed to the proof of Theorem 2.1.

Proof of Theorem 2.1. Let G be a $K_{1,3}^+$-induced-saturated graph. It is clear that G has at most one nontrivial component, since adding an edge between two isolated vertices does not create an induced $K_{1,3}^+$. We now show that every nontrivial component of G is a complete multipartite graph. Let v be a non-isolated vertex in G and let S be the set given by Lemma 2.5(c). By Lemmas 2.5(b) and 2.5(c), $G[N[v] \cup S]$ is a complete multipartite graph, with v and S sharing a part. So, we need only show $N[v] \cup S$ is a component of G. If not, then there exists some vertex $s \in S$ with a neighbor $t \notin N[v] \cup S$ since we have included the neighborhood of every $x \in N[v]$ and S is an independent set. If there exists an edge xy in $G[N(v)]$, then $G[x, y, s, t]$ is a $K_{1,3}^+$, so $N(v)$ is an independent set. This violates Lemma 2.5(a).

Now, by Lemma 2.5(a), every nontrivial component of G must have at least three parts. Next, we show that no part in any component of G has order two, and each component has at most one parts of order one. Suppose x and y either make up a part of order two, or are each a part of order one in a component F. Then $\{x, y\}$ dominates $F \setminus \{x, y\}$, and so x and y do not appear together in an induced claw, so adding or deleting the edge xy does not create an induced paw. Hence, G being $K_{1,3}^+$-induced-saturated implies that it must be as described by Construction 2.2. □

Corollary 2.6. *For $n \geqslant 7$, let $n \equiv r \mod 7$, where $0 \leqslant r \leqslant 6$. Then*

$$\operatorname{indsat}^*(n, K_{1,3}^+) = \begin{cases} \dfrac{15}{7}n & \text{if } r = 0 \\ 15\lfloor n/7 \rfloor + 4(r-1) & \text{if } r \neq 0 \end{cases}.$$

Proof. Let G be a minimally $K_{1,3}^+$-induced-saturated graph on n vertices. From Theorem 2.1, each nontrivial component of G is a complete multipartite graph with at least three parts. If some nontrivial component F of G has more than three parts, then we form a $K_{1,3}^+$-induced-saturated graph with strictly fewer edges by dropping edges between two

of the parts and forming a single larger part. Thus each nontrivial component of G is tripartite.

The number of edges of a complete tripartite graph on m vertices with parts of size s, t, and $m-(s+t)$ is given by $(m-[s+t])(s+t)+st$. Given the constraints $s \geqslant 1, t \geqslant 3$, and $m \geqslant t$, we see that $(m-(s+t))(s+t)$ is minimized when $s+t$ is minimized, i.e. $s+t=4$; also st is minimized when $s+t$ is minimized. Therefore, $K_{1,3,m-4}$ obtains the smallest number of edges among all complete tripartite graphs on m vertices.

Now, we may assume G has components $F_0, F_1, \ldots, F_k$ with $v(F_0) \in \{0,1\}$ and for $i>0$, $F_i = K_{1,3,n_i-4}$, where $v(F_0)+\sum_{i=1}^{k} n_i = n$. Then:

$$e(G) = \sum_{i=1}^{k} e(F_i) = \sum_{i=1}^{k} (4n_i - 13) = 4n - 13k - 4v(F_0).$$

Clearly, this is minimized by taking k as big as possible and, subject to this, $v(F_0)=1$. That is, we take $k = \lfloor n/7 \rfloor$ and

$$v(F_0) = \begin{cases} 0 & \text{if 7 divides } n \\ 1 & \text{else.} \end{cases}$$ □

Observation 2.7. *Given H for which* $\mathrm{indsat}^*(n,H)$ *is defined for all sufficiently large n, the function* $\mathrm{indsat}^*(n,H)$ *is not necessarily monotone in n. In particular, from Corollary 2.6 we see for any integer $k \geqslant 2$,* $\mathrm{indsat}^*(7k, K_{1,3}^+) < \mathrm{indsat}^*(7k+2, K_{1,3}^+) < \mathrm{indsat}^*(7k-1, K_{1,3}^+)$. *This is a similarity between minimal induced saturation and saturation: as noted in [6], the function* $\mathrm{sat}(n,H)$ *is not necessarily monotone in n for fixed H.*

3 Stars

Recall that $K_{1,2} = P_3$, and $\mathrm{indsat}(n, P_3) = 0$ for $n \geqslant 3$, as established in [12]. In this section we provide a construction extending this result, to show that for fixed $k \geqslant 2$ and n sufficiently large, $\mathrm{indsat}(n, K_{1,k+1}) = 0$. Additionally, our construction, together with a simple argument, determines $\mathrm{indsat}^*(n, K_{1,k+1})$ within a factor of two. The main result of this section is Theorem 3.4, which gives the bounds on $\mathrm{indsat}^*(n, K_{1,k+1})$. The case when $k=2$, which refers to the graph $K_{1,3}$, commonly known as the claw, will be addressed in further detail in Section 4.

Construction 3.1. Fix $k \geqslant 2$ and $n \geqslant 3^k$. Let z, R be positive integers such that $n = z3^k + R$ with $0 \leqslant R < 3^k$. Let H be the graph $K_3^1 \,\square\, K_3^2 \,\square \cdots \square\, K_3^k$, where K_3^i denotes a single copy of K_3. In other words, $V(H) = \{(\alpha_1, \ldots, \alpha_k) : \alpha_i \in [3]\}$, and $(\alpha_1, \ldots, \alpha_k)(\beta_1, \ldots, \beta_k) \in E(H)$ if and only if $|\{i : \alpha_i \neq \beta_i\}| = 1$. Let H' be such that $V(H')$ is the disjoint union of $V(H)$ and $V(K_R)$, and $E(H') = E(K_R) \cup E(H) \cup \{v\alpha : v \in V(K_R), \alpha \in V(H),$ and $\alpha = (\alpha_1, 1, 1, \ldots, 1)$ where $\alpha_1 \in [3]\}$. Let G be the disjoint union of $z-1$ copies of H and a single copy of H'.

Proposition 3.2. *The graphs in Construction 3.1 are $K_{1,k+1}$-induced-saturated.*

Proof. Given fixed n and k, let G and R be as defined in Construction 3.1. Let F denote the subgraph of H' isomorphic to H. Suppose that G contains an induced $K_{1,k+1}$ with center x, and suppose first that x is in a copy of H. Since $V(G)$ can be represented by k-dimensional vectors as described in the construction, any set of $k+1$ neighbors of x contains two vertices with vectors that differ in exactly one coordinate. Thus, x cannot have $k+1$ neighbors which form an independent set, and H is $K_{1,k+1}$-free.

If H' contains an induced $K_{1,k+1}$, then x cannot be in the K_R as the neighborhood of x would be a clique. Thus, x is in F. If this induced $K_{1,k+1}$ contains no vertices from the copy of K_R, then the above argument produces a contradiction. Thus, this $K_{1,k+1}$ contains a vertex from the copy of K_R, and without loss of generality, we may assume that x is represented by $(1,1,\ldots,1)$ in F. Consequently, our $K_{1,k+1}$ has exactly one vertex in K_R, but then contains no vertices of the form $(\alpha_1,1,1,\ldots,1)$ other than x. Hence, x has at most $k-1$ other neighbors from F in this copy of $K_{1,k+1}$ from F. So G is $K_{1,k+1}$-free.

It is clear that every vertex in a copy of H (or in F) is the center of an induced $K_{1,k}$. Thus, if we add an edge between two components of G, one component must be a copy of H, and we obtain an induced $K_{1,k+1}$. Thus, it remains to consider adding an edge within a component. Note that by the construction of H', the only possible way to add an edge is within F, which is isomorphic to H. So, it suffices to consider adding an edge to a copy of H. Suppose we add the edge uv. Without loss of generality, we may assume that u is represented by $(1,1,\ldots,1)$. Since u and v were not adjacent in H, their corresponding vectors must differ in at least two coordinates, say the first and second. As a consequence, v is adjacent to neither y nor w, where $y \in \{(2,1,1,\ldots,1),(3,1,1,\ldots,1)\}$ and $w \in \{(1,2,1,1,\ldots,1),(1,3,1,1,\ldots,1)\}$. Thus, $\{u,v,w,y\}$ is an induced $K_{1,3}$ centered at u. To this set we add vertices $\alpha^3,\alpha^4,\ldots,\alpha^k$, where α^i has all coordinates equal to 1 except that the ith coordinate is either 2 or 3. This induces $K_{1,k+1}$.

Lastly, suppose we remove an edge uv. There are three cases to consider. The first case is if uv is in a copy of H (or in F). Here, we may assume $u=(2,1,1,\ldots,1)$ and $v=(3,1,1,\ldots,1)$. The second case is if both u and v are in K_R. The last case is if only endpoint, say v, is in K_R. Here, we may again assume that $u=(2,1,1,\ldots,1)$. In all three cases, $(1,1,\ldots,1)$ together with u,v, and the vertices $\alpha^2,\ldots,\alpha^k$ defined above induce a $K_{1,k+1}$. This completes the proof. □

Corollary 3.3. *For fixed $k \geqslant 2$ and $n \geqslant 3^k$,* $\operatorname{indsat}(n,K_{1,k+1}) = 0$.

Theorem 3.4. *For $n \geqslant 2\cdot 3^k$ and $k \geqslant 2$, there exist constants $c_1 = c_1(k)$ and $c_2 = c_2(k)$ such that $n\frac{k}{2} - c_1 \leqslant \operatorname{indsat}^*(n,K_{1,k+1}) \leqslant nk + c_2$.*

Proof. Given fixed n and k, let G and R be as defined in Construction 3.1.

We establish $e(G)$ by considering vertex degrees. The component H' has at most $2\cdot 3^k$ vertices, and so (trivially) at most $\binom{2\cdot 3^k}{2}$ edges. The remaining vertices, of which there are at most $n-3^k$, all have degree $2k$ for a contribution of at most $(n-3^k)k$ edges. All told, $e(G) \leqslant nk - k\cdot 3^k + \binom{2\cdot 3^k}{2}$.

To show the lower bound, suppose that G is a $K_{1,k+1}$-induced-saturated graph. Let $S = \{x \in V(G)\colon \deg(x) \leqslant k-1\}$. We claim that $|S| \leqslant k$.

If $|S| > k$, then there exist $x, y \in S$ such that $xy \notin E(G)$. Let G' denote $G + xy$. As G was $K_{1,k+1}$-induced-saturated, G' must contain an induced $K_{1,k+1}$, using the edge xy with either x or y as the center of this $K_{1,k+1}$. However, as both x and y are adjacent to at most $k-1$ vertices in G, this cannot happen. So $|S| \leqslant k$, as claimed.

Observe:

$$e(G) \geqslant \frac{1}{2}\left(k(n-|S|) + \sum_{x \in S} \deg(x)\right) \geqslant \frac{nk}{2} - \frac{k^2}{2}.$$

This establishes the lower bound. □

It worth noting that we can extend Construction 3.1, as any graph formed as a Cartesian product of exactly k cliques, each of size at least three, is $K_{1,k+1}$-induced-saturated.

4 The Claw

For sufficiently large n, Theorem 3.4 states that $\text{indsat}^*(n, K_{1,k+1})$ is linear in n, and in particular, we know the coefficient within a factor of two. In this section, we will prove the following theorem, which agrees with the upper bound given in Theorem 3.4 and further narrows the range of possible values of $\text{indsat}^*(n, K_{1,3})$.

Theorem 4.1. *For $n \geqslant 9$, if $n \neq 14, 17$, then $2n - 2 \leqslant \text{indsat}^*(n, K_{1,3}) \leqslant 2n + 2$.*

Values of $\text{indsat}(n, K_{1,3})$ were determined for $4 \leqslant n \leqslant 10$ by computer search[1] and are listed in Table 2, along with trigraphs that achieve the minimum number of gray edges. This together with Corollary 3.3 determines $\text{indsat}(n, K_{1,3})$ for all n. We now turn our attention to $\text{indsat}^*(n, K_{1,3})$.

In order to prove Theorem 4.1, we first prove a series of lemmas.

Lemma 4.2. *Let G be a $K_{1,3}$-induced-saturated graph. Then G has*

1. *at most one isolated vertex,*
2. *no vertices of degree one,*
3. *at most one vertex of degree two, and*
4. *at most two vertices of degree three.*

Furthermore, if G has an isolated vertex v, then $\delta(G - v) \geqslant 4$.

Proof. Let G be a $K_{1,3}$-induced-saturated graph. Observe that if we had two isolated vertices, then adding the edge between them would not yield a $K_{1,3}$. Also, any edge of G lies in a triangle, so there are no vertices of degree one.

Suppose that u and v are vertices of degree two. Since every edge lies in a triangle the neighbors of u are adjacent, as are the neighbors of v. Thus, if u and v are not

[1] A program was written in C++ and is available at http://www.math.unl.edu/~s-sbehren7/main/Data.html.

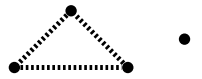
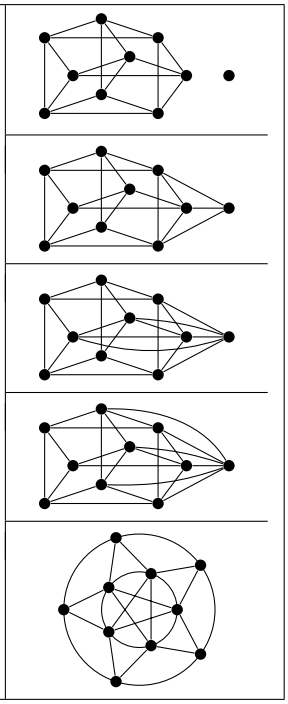
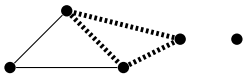
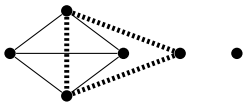
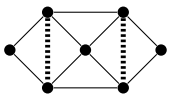
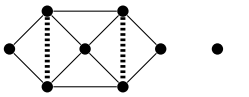
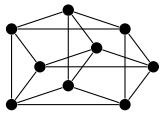
$\text{indsat}(4, K_{1,3}) = 3$		$\text{indsat}(10, K_{1,3}) = 0$	
$\text{indsat}(5, K_{1,3}) = 3$			
$\text{indsat}(6, K_{1,3}) = 3$			
$\text{indsat}(7, K_{1,3}) = 2$			
$\text{indsat}(8, K_{1,3}) = 2$			
$\text{indsat}(9, K_{1,3}) = 0$			

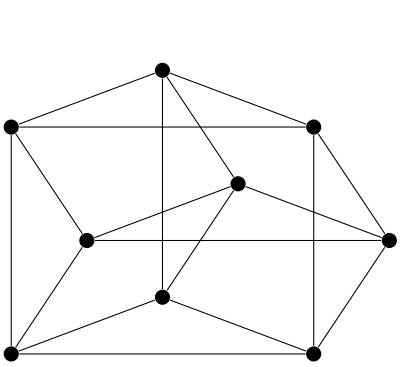
Table 2: Values of $\text{indsat}(n, K_{1,3})$ for $4 \leqslant n \leqslant 10$ along with trigraphs realizing those values. All $K_{1,3}$-induced-saturated graphs for $n = 9$ and $n = 10$ are shown.

adjacent, adding the edge uv does not create an induced $K_{1,3}$. If u and v are adjacent, then $N[u] = N[v] = \{u, v, w\}$ for some w. However, removing uw does not create an induced $K_{1,3}$ as v would have to have been its center. So G has at most one vertex of degree two.

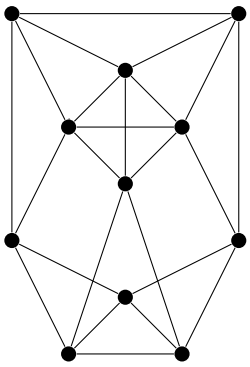
To prove (4), suppose u is a vertex of degree three with neighbors u_1, u_2, u_3. Since every edge is in a triangle, we may assume that $u_1u_2, u_2u_3 \in E(G)$. *Case 1:* $u_1u_3 \notin E(G)$. Then adding u_1u_3 creates an induced $K_{1,3}$ centered at either u_1 or u_3; say u_1. Then u_1 has two nonadjacent neighbors x and y that are distinct from u_2 and u_3. However, $\{u, u_1, x, y\}$ induces a $K_{1,3}$ in G, a contradiction. *Case 2:* $u_1u_3 \in E(G)$. In particular, every vertex of degree three in G is contained in a K_4. Let v be another vertex of degree three. By the above, $N[v]$ induces K_4. If $uv \notin E(G)$, then adding uv does not create an induced $K_{1,3}$. Thus, u and v are adjacent, and consequently the only vertices of degree three are contained in $N[u]$.

If we remove uu_1, then an induced $K_{1,3}$ exists, centered at either u_2 or u_3. So at least one of them has degree at least four, say u_3. Similarly, removing uu_3 creates an induced $K_{1,3}$ centered at either u_1 or u_2 so that at least one of them has degree at least four. In any case, at most two vertices in $N[u]$, and as a result in G, have degree three. Thus, (4) holds.

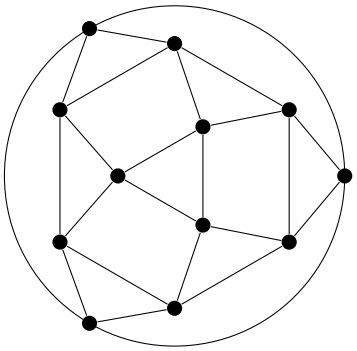
If G has an isolate, u, and another vertex v with $\deg(v) \leqslant 2$, then adding uv cannot create an induced $K_{1,3}$ unless $\deg(v) = 2$. In this case, the neighbors of v cannot be adjacent, however every edge of G must be in a triangle, a contradiction. □



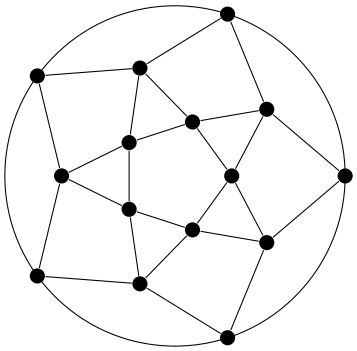
(a) $H = K_3 \square K_3$, 9 vertices



(b) Graph J on 11 vertices



(c) Graph K on 12 vertices



(d) Graph L on 15 vertices

Figure 1: These graphs are $K_{1,3}$-induced-saturated.

Lemma 4.3. *If G is a graph where the neighborhood of every vertex induces $2K_2$, then G is $K_{1,3}$-induced-saturated.*

Proof. Since no vertex has three independent neighbors, G contains no induced $K_{1,3}$. Suppose we delete an edge xy. Since every edge is in a triangle, say xyz, deleting xy leaves z as the center of a $K_{1,3}$ with leaves x, y, and any other neighbor of z. If we add an edge between two vertices with no common neighbors, then we take the new edge together with two nonadjacent neighbors of one of the vertices and find a $K_{1,3}$. Therefore it suffices to consider adding an edge xy, where x and y share a neighbor. Let $N(x) = \{u_1, u_2, v_1, v_2\}$ with $u_1u_2, v_1v_2 \in E(G)$, and suppose $u_1 \in N(y)$. Then $u_2 \notin N(y)$ otherwise $N(u_2)$ would contain a P_3 and not be $2K_2$. Similarly, both v_1 and v_2 cannot be in $N(y)$. So we may assume $v_2 \notin N(y)$. Then upon adding xy, $\{x, y, u_2, v_2\}$ induces a $K_{1,3}$. □

Lemma 4.4. *Let G be a graph with at most one isolated vertex, where each nontrivial component is one of the graphs in Figure 1. Then G is $K_{1,3}$-induced-saturated.*

Proof. By inspection, the graph in Figure 1b is $K_{1,3}$-induced-saturated, and since the graphs in Figures 1a, 1c, and 1d have the property that the neighborhood of every vertex induces $2K_2$, they are $K_{1,3}$-induced-saturated by Lemma 4.3.

Now let G be a graph with at most one isolated vertex and each of the remaining components are one of the graphs from Figure 1. Since each nontrivial component of G is $K_{1,3}$-induced-saturated, we only need to consider adding an edge xy between components. When we add the edge xy, at least one of x and y must be in a nontrivial component, say x. By inspection we see every vertex in every graph of Figure 1 has two nonadjacent neighbors, and in particular, this holds for x. Thus, x together with these two neighbors and y induce a $K_{1,3}$. Therefore, G is $K_{1,3}$-induced-saturated. □

We now can prove Theorem 4.1.

Proof of Theorem 4.1. Lemma 4.2 together with the degree-sum formula gives us the lower bound of $2n-2$. To obtain the upper bound, we contruct graphs using H, J, K, and L from Figure 1. These constructions will be based on the residue class of n modulo 3.

Case 4.4.1. $n \equiv 0 \mod 3$, $n \geqslant 9$

Use $\lfloor n/9 \rfloor - 1$ copies of H, together with one copy of H, K, or L, for a graph with $2n$ edges.

Case 4.4.2. $n \equiv 1 \mod 3$, $n \geqslant 10$

Use an isolated vertex with a graph from Case 4.4.1 for a graph with $2n-2$ edges.

Case 4.4.3. $n \equiv 2 \mod 3$, $n \geqslant 20$ or $n = 11$.

If $n = 11$, the graph J suffices. If $n > 11$, then take J and a construction from Case 4.4.1. This achieves $2n+2$ edges. □

It is worth noting that for $n \equiv 1 \mod 3$, $n \geqslant 10$, $\text{indsat}^*(n, K_{1,3}) = 2n-2$. Additionally, a detailed case analysis of degree sequences of $K_{1,3}$-induced-saturated graphs shows that $\text{indsat}^*(n, K_{1,3}) \geqslant 2n$ for $n \not\equiv 1 \mod 3$. In particular, this implies $\text{indsat}^*(n, K_{1,3}) = 2n$ for $n \equiv 0 \mod 3$, $n \geqslant 9$, and $2n \leqslant \text{indsat}^*(n, K_{1,3}) \leqslant 2n+2$ for $n \equiv 2 \mod 3$, $n \geqslant 20$.

5 C_4 and Matchings

In this section we show that the induced saturation number of C_4 is zero for sufficiently large n, and we show that $(5/2)n \leqslant \text{indsat}^*(n, C_4) \leqslant (7/64)n^2 + o(n)$ Additionally, together with Observation 1.3 and the fact that $\overline{C_4} = 2K_2$, we use C_4-induced-saturated graphs to show that the induced saturation number of kK_2 is zero for n sufficiently large and $k \geqslant 2$. We then show that $\text{indsat}^*(n, kK_2)$ is constant in terms of n.

Construction 5.1. For $j \geqslant 5$ and $k \geqslant 2$, let I_j^k be the following graph. Begin with k copies of a wheel with j spokes. Label the wheels $W^1, \ldots, W^k$, and label the vertices of W^i so that its center is w_0^i, and the outer cycle of W^i is $w_1^i, \ldots, w_j^i$. For each i and p with $1 \leqslant i, p \leqslant k$ and $i \neq p$, add the following edges: $w_\ell^i w_\ell^p$ for each $\ell \in [j]$; $w_\ell^i w_{\ell+1}^p$ for each $\ell \in [j-1]$; and $w_j^i w_1^p$.

I_5^2 is the icosahedron, shown in Figure 2. The icosahedron can be thought of as two wheels with 5 spokes whose outer-cycle vertices are joined by a zig-zag pattern (as described precisely in Construction 5.1). Construction 5.1 generalizes the icosahedron by allowing the number of wheels and the length of their outer cycles to vary.

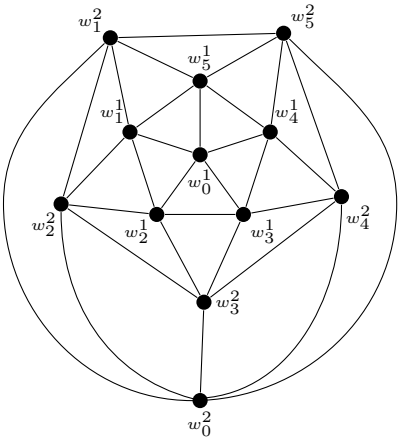


Figure 2: The icosahedron graph.

Proposition 5.2. *For $j \in \{5, 6, 7\}$, and $k \geqslant 2$, I_j^k is C_4-induced-saturated.*

Proof. We first show that I_j^k does not contain an induced C_4. Suppose to the contrary that it does. Since a single wheel does not contain an induced C_4, this C_4 must contain vertices from at least two different wheels. Suppose that w_0^p is in this C_4. Recall that w_0^p is the center of wheel W^p. Then, this C_4 must contain w_r^p and w_s^p such that $|s - r| \geqslant 2$. However, since $|s-r| \geqslant 2$, w_r^p and w_s^p contain no common neighbors outside of W^p. Thus, all four vertices of this induced C_4 must be inside of W^p, a contradiction. So our induced C_4 contains no centers of wheels.

If this C_4 contains exactly three vertices from a single W^p, then they must be consecutive along their cycle. That is, C_4 contains w_s^p, w_{s+1}^p, and w_{s+2}^p. However, as above, w_s^p and w_{s+2}^p have no common neighbors outside of W^p. Thus, our induced C_4 contains at most two vertices from each W^p.

If this C_4 contains exactly two vertices from a single W^p, then by the same arguments used above, they must be adjacent in W^p, say w_s^p and w_{s+1}^p. No vertex of the form w_s^q, with $q < p$, or w_{s+1}^r, with $r > p$, can be in our C_4, as either produces a triangle with w_s^p and w_{s+1}^p.

Now, w_{s+1}^p must have another neighbor in our C_4. Suppose it is in W^t. If $t > p$, then it must be w_{s+2}^t by the above. However, the only common neighbors w_{s+2}^t and w_s^p have are of the form w_{s+1}^q where $q > p$, a contradiction. So $t < p$, and the other neighbor of w_{s+1}^p is w_{s+1}^t. Again though, the only common neighbors of w_s^p and w_{s+1}^t are either of the form w_{s+1}^q where $q > p$, or w_s^r where $r < p$. In either case, we have a contradiction to the above. Thus, our C_4 has exactly one vertex from each wheel.

Suppose our induced C_4 contains the vertices $w^p_{t_1}, w^q_{t_2}, w^r_{t_3}, w^s_{t_4}$. If $|\{t_1, t_2, t_3, t_4\}| \leqslant 2$, then we have a triangle, a contradiction. If $|\{t_1, t_2, t_3, t_4\}| = 4$, then some vertex is not adjacent to two of the others, a contradiction. So $|\{t_1, t_2, t_3, t_4\}| = 3$, and two vertices have the same subscript. We may assume that it is $w^p_{t_1}, w^q_{t_2} = w^q_{t_1}$, and that $p < q$. Then, $w^p_{t_1}$ must have a neighbor not adjacent to $w^q_{t_1}$ in this C_4, say it is $w^r_{t_3}$. However, in order for this to be possible, we must have $t_3 = t_1 + 1$ and $p < r < q$. Thus, $w^s_{t_4}$ is adjacent to both $w^r_{t_1+1}$ and $w^q_{t_1}$. However, since t_4 must be distinct from both t_1 and $t_1 + 1$, this cannot happen, a contradiction. So I^k_j is C_4-free.

We now must show that adding and removing edges from I^k_j creates an induced C_4. We begin by considering removal of edges. The edges of I^k_j can be classified into three types: edges that are on the outer cycle of a wheel, edges that are incident to the center of a wheel, and edges that are incident to two different wheels. If we remove an edge that is on the outer cycle of a wheel, say $w^i_\ell w^i_{\ell+1}$ where $\ell \neq j$, then the vertices $w^i_0, w^i_\ell, w^i_{\ell+1}$, and $w^p_{\ell+1}$, where $p \neq i$, create an induced C_4. If $\ell = j$, then removing $w^i_j w^i_1$ creates a C_4 induced by the vertices w^i_j, w^i_1, w^i_0, and w^p_1, again where $p \neq i$. If we remove an edge that is incident to the center of a wheel, say $w^i_0 w^i_\ell$, then $w^i_0, w^i_{\ell-1}, w^i_\ell$, and $w^i_{\ell+1}$ induce a C_4. Now, for $i \neq p$ and $\ell \in [j]$, we have the edge $w^i_\ell w^p_\ell$. Removing this edge creates a C_4 induced by the vertices $w^i_{\ell-1}, w^i_\ell, w^p_\ell$, and $w^p_{\ell+1}$, if $\ell \neq 1$, or by the vertices w^i_j, w^i_1, w^p_1, and w^p_2 if $\ell = 1$. When $\ell \in [j-1]$, the edge $w^i_\ell w^p_{\ell+1}$ exists in I^k_j; removing it creates a C_4 induced by $w^i_\ell, w^p_\ell, w^p_{\ell+1}$, and w^i_ℓ. Finally, removing the edge $w^i_j w^p_1$ from I^k_j yields a C_4 induced by w^i_j, w^p_j, w^p_1, and w^i_1.

To show that adding edges creates an induced C_4, we consider the different types of edges that are missing from I^k_j. Adding an edge within one wheel (say W^m) is simply adding a chord $w^m_i w^m_p$ to a 5-, 6-, or 7-cycle. If $p \neq i+2$ or $j = 5$, then this chord creates an induced 4-cycle. If $p = i+2$ and $j = 6$ or $j = 7$, then if $m \neq k$, $w^m_i w^\ell_{i+1} w^\ell_{i+2} w^m_{i+2} w^m_i$ is an induced 4-cycle, where $\ell > m$, and if $m = k$, then $w^m_i w^m_{i+1} w^\ell_{i+1} w^\ell_i w^m_i$ is an induced C_4.

Now suppose we add an edge between wheels, say W^m and W^ℓ, where we may assume $m < \ell$. If the new edge is between the centers of these wheels, that is, $w^m_0 w^\ell_0$, then $w^m_0 w^\ell_0 w^\ell_1 w^m_1 w^m_0$ is an induced C_4. If it is from the center of W_m to a vertex on the cycle of W^ℓ, say w^ℓ_i, then $w^m_0 w^\ell_i w^\ell_{i+1} w^m_{i+1} w^m_0$ is an induced C_4; a similar cycle is also created if the new edge is $w^\ell_0 w^m_i$. Finally, if we add an edge $w^m_i w^\ell_p$, note that w^m_i is not adjacent to at least one of w^m_p and w^m_{p-1}; label this vertex u. Since u is adjacent to w^ℓ_p, the vertices w^m_0, w^m_i, w^ℓ_p, and u induce a C_4. □

Proposition 5.2 implies that for many values of n, $\mathrm{indsat}(n, C_4) = 0$. In fact, this is the case for all $n \geqslant 12$. To show this, we use the following proposition regarding kK_2. While we only employ the proposition in the case $k = 2$, the more general statement which we present is not difficult.

Proposition 5.3. *Let $s := (s_1, \ldots, s_n)$ be a sequence of positive integers. Let G be a graph with vertex set $\{v_1, \ldots v_n\}$, and let G_s be the graph obtained from G by replacing each vertex v_i with an independent set of order s_i and each edge with a complete bipartite graph between the corresponding independent sets. For $k \geqslant 2$, G is kK_2-induced-saturated if and only if G_s is kK_2-induced-saturated.*

Note that the graph G_s constructed in this proposition is in fact the blow-up of G, as used in [10].

Proof. For each vertex $v_i \in V(G)$, let V_i be the independent set in G_s that corresponds to it. We will call this collection of vertices in G_s that replaces a single vertex in G a *part*.

Note that no induced matching in G_s uses two vertices from the same part, and the same holds if we add or remove a single edge from G_s. We claim that if w_i and w_j are vertices from different parts V_i and V_j, respectively, of G_s, then G_s (or $G_s + w_iw_j$ or $G_s - w_iw_j$) contains an induced matching if and only if G (resp. $G + v_iv_j$, or $G - v_iv_j$) contains an induced matching M. Suppose M_s is such an induced matching in G_s (or $G_s + w_iw_j$ or $G_s - w_iw_j$). Then each vertex in M_s comes from a different part of G_s (resp. $G_s + w_iw_j$ or $G_s - w_iw_j$), and thus they correspond to distinct vertices in $V(G)$. This is an induced matching in G.

If G (or $G + v_iv_j$ or $G - v_iv_j$) has an induced matching M, then when the graph is expanded, no new adjacencies have been added between the parts corresponding to the endpoints of vertices in M (except for w_iw_j in the case of $G + v_iv_j$). Thus, we can find an induced matching in G_s (resp. $G_s + w_iw_j$ or $G_s - w_iw_j$). This shows that if G_s is kK_2-induced-saturated, then so is G.

To show that if G is kK_2-induced-saturated, then so is G_s, it remains to consider adding edges between vertices in one part of G_s. First we note that G has no dominating vertex. Indeed, if u is a dominating vertex, then deleting an edge incident to u, say uw, does not create an induced $2K_2$, let alone an induced kK_2, as u dominates $N_G(w)$.

Now, suppose we add w_iw_i' to G_s, in the part V_i corresponding to v_i. Since v_i is not dominating, there exists w not adjacent to v_i. Since G is kK_2-induced-saturated, $G + v_iw$ contains an induced matching $M = \{v_iw, x_2y_2, \ldots, x_ky_k\}$. Then
$M_s = \{w_iw_i', X_2Y_2, \ldots, X_kY_k\}$ is an induced matching in $G_s + w_iw_i'$, where X_j and Y_j are vertices in the parts corresponding to x_j and y_j, respectively. □

Corollary 5.4. *For $n \geqslant 12$,* $\mathrm{indsat}(n, C_4) = 0$.

Proof. Applying Observation 1.3 to case $k = 2$ in Proposition 5.3, allows us to begin with a graph that is C_4-induced-saturated, replace a single vertex with a clique of any order, replace the affected edges with complete bipartite graphs, and produce another graph that is C_4-induced-saturated. Thus, beginning with I_5^2, applying these operations obtains C_4-induced-saturated graphs for all values of $n \geqslant 12$. □

For $4 \leqslant n \leqslant 10$, a computer search showed $\mathrm{indsat}(n, C_4) > 0$. At this time, the value of $\mathrm{indsat}(11, C_4)$ is yet unknown. We now turn our attention to $\mathrm{indsat}^*(n, C_4)$.

Theorem 5.5. *For sufficiently large n,* $(5/2)n \leqslant \mathrm{indsat}^*(n, C_4) \leqslant (7/64)n^2 + o(n)$.

Proof. To prove the lower bound we show that $\delta(G) \geqslant 5$. Suppose G is a C_4-induced-saturated graph. Let $x \in V(G)$, and let $H := G[N(x)]$. Since deleting any edge produces an induced C_4, every edge is the diagonal of a C_4 and $\deg(x) \geqslant 3$. In particular, there exist $v_1, v_2, v_3 \in V(H)$ such that v_1v_3 is not an edge, but v_1v_2 and v_2v_3 are edges. Now, $G - xv_1$ contains an induced C_4 that contains both x and v_1, but not v_3. If v_2 is not in

this C_4, then there exist two other vertices distinct from v_1, v_2, v_3 in H. Thus, $\deg(x) \geqslant 5$. If v_2 is in this C_4, then there exists $v_4 \in V(H)$ distinct from v_1, v_2, v_3 such that v_1v_4 is an edge, but v_2v_4 is not. By a similar argument, considering $G - xv_3$ gives at least one additional vertex in H distinct from v_1, v_2, v_3, v_4. So in any case, $\deg(x) \geqslant 5$, and as x was arbitrary, $\delta(G) \geqslant 5$. Thus, provided $n \geqslant 12$, $\text{indsat}^*(n, C_4) \geqslant (5/2)n$.

To prove the upper bound, we choose $n \geqslant 56$ and create a graph G of order n. Let $r \equiv n \mod 8$, where $0 \leqslant r \leqslant 7$. Set $k = \lfloor n/8 \rfloor$ so that $k \geqslant r$ and $v(I_7^k) = 8k$. If $r = 0$, choose $G = I_7^k$. If $r > 0$, we create G by adding r vertices to I_7^k. Recall, as discussed after Proposition 5.3, by replacing the vertices of I_7^k with cliques, and its edges with complete bipartite graphs, we preserve the property of being C_4-induced-saturated. Accordingly, using the notation of Construction 5.1, we replace $w_0^1, \ldots, w_0^r$ with copies of K_2 and make each new vertex adjacent to the neighborhood of the vertex it replaces.

Now we determine $e(G)$. The first r wheels have 22 edges, and the rest have 14. Between any two wheels there are 14 edges. So $e(G) = 14\left[\binom{k}{2} + k\right] + 8r$. Since $r \in [0, 7]$ and $k = \lfloor n/8 \rfloor$, $e(G) \leqslant \frac{7}{64}n^2 + \frac{7}{8}n + 56$. □

5.1 Matchings

Another graph that is C_4-induced-saturated is the join $I_5^2 \vee K_{n-12}$. Observation 1.3 implies that the complement of this graph is $2K_2$-induced-saturated. We can further generalize this to get a kK_2-induced-saturated graph for any $k \geqslant 2$.

Proposition 5.6. *Let $\overline{I_5^2}$ be the complement of the icosahedron. For fixed $k \geqslant 2$ and $n \geqslant 12(k-1)$, the graph $(k-1)\overline{I_5^2} + (n - 12(k-1))K_1$ is kK_2-induced-saturated. Thus, for $n \geqslant 12(k-1)$, $\text{indsat}(n, kK_2) = 0$.*

Proof. By Proposition 5.2 and Observation 1.3, the complement of an icosahedron is $2K_2$-induced-saturated. Let G denote $(k-1)\overline{I_5^2} + (n - 12(k-1))K_1$. Clearly, G contains $(k-1)K_2$ as an induced subgraph, but no induced kK_2. If we add or delete any edge inside a component, or add an edge among the isolates, we create an induced kK_2. Note that every vertex v in $\overline{I_5^2}$ is in an induced copy of $K_2 + K_1$ where v is the isolate. Thus, adding any edge with an endpoint in a copy of $\overline{I_5^2}$ creates an induced kK_2. □

Corollary 5.7. *For $n \geqslant 12(k-1)$, $\text{indsat}^*(n, kK_2) \leqslant 36(k-1)$.*

In particular, for fixed k, $\text{indsat}^*(n, kK_2)$ is constant.

6 Other Cycles and Generalizations of Cycles

In this section we provide a construction proving that odd cycles also have induced saturation number zero for n sufficiently large. As it is already known that $\text{indsat}(n, C_3) = \text{sat}(n, C_3)$ [12], we only consider odd cycles of length at least five. Additionally, this construction is also H-induced-saturated when H is a modification of an even cycle as described below.

Let C'_{2k} denote a cycle of length $2k$ with a pendant vertex, and $\hat{C}_{2k}$ denote an even cycle with a chord between two vertices at distance 2 from each other (sometimes called a triangle chord or hop).

For a given k and $n \geqslant (k+1)^2+2$, we can write n as $(k+1)t-s$ where t and s are integers with $t \geqslant k+2$ and $0 \leqslant s \leqslant t-3$. In particular, we choose $t = \lceil \frac{n}{k+1} \rceil$. Using this expression for n, we give the following construction.

Construction 6.1. For $k \geqslant 3$ and $n \geqslant (k+1)^2+2$, let $n = (k+1)t-s$, where $t = \lceil \frac{n}{k+1} \rceil \geqslant k+2$ and $0 \leqslant s \leqslant t-3$. Let $G_{n,k}$ be formed from the Cartesian product $K_{k+1} \,\square\, K_t$ by removing s vertices from one copy of K_t.

Proposition 6.2. *If $H \in \{C_{2k-1}, C'_{2k}, \hat{C}_{2k}\}$, then the graph $G_{n,k}$ in Construction 6.1 is H-induced-saturated.*

Proof. Let $G_{n,k}$ be as described in Construction 6.1. We first show that $G_{n,k}$ is H-free for $H \in \{C_{2k-1}, C'_{2k}, \hat{C}_{2k}\}$. Any induced subgraph of $G_{n,k}$ that is triangle-free has at most two vertices from any copy of K_{k+1} or K_t. Since $2k-1$ is odd, an induced C_{2k-1} contains precisely one vertex v from some copy of K_{k+1}. Then the neighbors of v must be in the same copy of K_t, which means they form a triangle. Thus, $G_{n,k}$ has no induced odd cycle larger than a triangle. Since $\hat{C}_{2k}$ contains C_{2k-1} as an induced subgraph, neither C_{2k-1} nor $\hat{C}_{2k}$ are induced subgraphs of $G_{n,k}$. As C'_{2k} contains $K_{1,3}$ as an induced subgraph, but $G_{n,k}$ does not, $G_{n,k}$ has no induced C'_{2k}.

In the remainder of this proof we view the vertices of $K_{k+1} \,\square\, K_t$ as a $k+1$ by t grid with vertices $v_{i,j}$ for $1 \leqslant i \leqslant k+1$ and $1 \leqslant j \leqslant t$. The vertices in each row induce K_t and those in each column induce K_{k+1}. Note that $v_{i,j}$ is adjacent to $v_{m,p}$ if and only if $i=j$ or $m=p$. To create $G_{n,k}$ from this, we remove s vertices from the last row of the grid. In particular, we remove $v_{k+1,t-s+1}, v_{k+1,t-s+2}, \ldots, v_{k+1,t}$. Thus, we can partition the vertices of $G_{n,k}$ into the following three sets: $R_1 = \{v_{i,j} : 1 \leqslant i \leqslant k,\, 1 \leqslant j \leqslant t-s\}$, $R_2 = \{v_{k+1,j} : 1 \leqslant j \leqslant t-s\}$, and $R_3 = \{v_{i,j} : 1 \leqslant i \leqslant k,\, t-s < j \leqslant t\}$. Recall that $t-s \geqslant 3$, $t \geqslant k+2$, and $k \geqslant 3$.

In the following arguments, we will use the fact that as $t-s \geqslant 3$, $v_{1,1}, v_{1,2}, v_{1,3} \in R_1$ and $v_{k+1,1}, v_{k+1,2}, v_{k+1,3} \in R_2$. Furthermore, we will use the path on $2k-6$ vertices induced by the set $A = \{v_{p,p+1}, v_{p,p+2} : 2 \leqslant p \leqslant k-2\}$.

Case 6.2.1. Adding edges

We first consider adding an edge to $G_{n,k}$. Observe that all the vertices in R_1 are interchangeable by switching rows and columns, and the same holds for the vertices in R_2 and R_3. So without loss of generality, there are five kinds of edges that can be added: an edge contained in R_1, contained in R_3, or between any two parts.

Suppose we add an edge contained in R_1. Without loss of generality, assume it is $v_{1,1}v_{k,2}$. Let $B_1 = \{v_{1,1}, v_{k,2}, v_{k+1,1}, v_{k+1,3}\}$. Then the vertex set $A \cup B_1 \cup \{v_{k,k}\}$ induces C_{2k-1}, the set $A \cup B_1 \cup \{v_{k,k}, v_{1,2}\}$ induces $\hat{C}_{2k}$, and the set $A \cup B_1 \cup \{v_{k-1,k}, v_{k-1,k+1}, v_{1,k+1}\}$ induces C'_{2k} with pendant $v_{k,2}$.

Suppose we add an edge contained in R_3. Without loss of generality, assume it is $v_{1,t}v_{k,t-1}$. Let $B_2 = \{v_{1,t}, v_{k,t-1}, v_{k-1,k}, v_{k-1,t}\}$. Then the vertex set $A \cup B_2 \cup \{v_{k,3}\}$ induces

C_{2k-1}, the set $A \cup B_2 \cup \{v_{k,2,}, v_{k,3}\}$ induces $\hat{C}_{2k}$, and the set $A \cup B_2 \cup \{v_{1,2}, v_{k+1,2}, v_{k+1,3}\}$ induces C'_{2k} with pendant $v_{k,t-1}$.

Suppose we add an edge between R_1 and R_2. Without loss of generality, assume it is $v_{1,1}v_{k+1,2}$. Let $B_3 = \{v_{1,1}, v_{k+1,2}, v_{k-1,1}, v_{k-1,k}\}$. Then the vertex set $A \cup B_3 \cup \{v_{k+1,3}\}$ induces C_{2k-1}, the set $A \cup B_3 \cup \{v_{k+1,3}, v_{1,2}\}$ induces $\hat{C}_{2k}$, and the set $A \cup B_3 \cup \{v_{k,3}, v_{k,k+1}, v_{1,k+1}\}$ induces C'_{2k} with pendant $v_{k+1,2}$.

Suppose we add an edge between R_1 and R_3. Without loss of generality, assume it is $v_{1,1}v_{k,t}$. Let $B_4 = \{v_{1,1}, v_{k,t}, v_{k+1,1}, v_{k+1,3}\}$. Then the vertex set $A \cup B_4 \cup \{v_{k,k}\}$ induces C_{2k-1}, the set $A \cup B_4 \cup \{v_{k,k}, v_{1,t}\}$ induces $\hat{C}_{2k}$, and the set $A \cup B_4 \cup \{v_{k-1,k}, v_{k-1,k+1}, v_{1,k+1}\}$ induces C'_{2k} with pendant $v_{k,t}$.

Suppose we add an edge between R_2 and R_3. Without loss of generality, assume it is $v_{1,t}v_{k+1,1}$. Let $B_5 = \{v_{1,t}, v_{k+1,1}, v_{1,k}, v_{k,3}\}$. Then the vertex set $A \cup B_5 \cup \{v_{k,1}\}$ induces C_{2k-1}, the set $A \cup B_5 \cup \{v_{k,1}, v_{k,2}\}$ induces $\hat{C}_{2k}$, and the set $A \cup B_5 \cup \{v_{k,2}, v_{k-1,2}, v_{k-1,t}\}$ induces C'_{2k} with pendant $v_{k+1,1}$.

Case 6.2.2. Deleting edges

There are several types of edges that we can delete within the categories of "vertical" and "horizontal" edges. There are three types of vertical edges: edges with both endpoints in R_1, both in R_3, and between R_1 and R_2. There are four types of horizontal edges: edges with both endpoints in R_1, both in R_2, both in R_3, and between R_1 and R_3.

We first consider deleting vertical edges. Suppose we delete an edge contained in R_1 or an edge between R_1 and R_2. If the former, assume it is $v_{1,1}v_{k,1}$, and let $B_6 = \{v_{1,1}, v_{k,1}, v_{k-1,3}, v_{1,k}\}$. If the latter, assume it is $v_{1,1}v_{k+1,1}$, and let
$B_6 = \{v_{1,1}, v_{k+1,1}, v_{k-1,3}, v_{1,k}\}$. Then in either case, the vertex set $A \cup B_6 \cup \{v_{k-1,1}\}$ induces C_{2k-1}, and the set $A \cup B_6 \cup \{v_{k-1,1}, v_{k-1,2}\}$ induces $\hat{C}_{2k}$. If we deleted the edge $v_{1,1}v_{k,1}$, then set $A \cup B_6 \cup \{v_{k+1,1}, v_{k+1,2}, v_{k-1,2}\}$ induces C'_{2k} with pendant $v_{k,1}$. If we deleted the edge $v_{1,1}v_{k+1,1}$, then the set $A \cup B_6 \cup \{v_{k,1}, v_{k,2}, v_{k-1,2}\}$ induces C'_{2k} with pendant $v_{k+1,1}$.

Suppose we delete an edge contained in R_3. Without loss of generality, assume it is $v_{1,t}v_{k,t}$. Let $B_7 = \{v_{1,t}, v_{k,t}, v_{1,k}, v_{k-1,t}\}$. Then the vertex set $A \cup B_7 \cup \{v_{k,3}\}$ induces C_{2k-1}, the set $A \cup B_7 \cup \{v_{k,2}, v_{k,3}\}$ induces $\hat{C}_{2k}$, and the set $A \cup B_7 \cup \{v_{k+1,2}, v_{k+1,3}, v_{k-1,2}\}$ induces C'_{2k} with pendant $v_{k,t}$.

We now consider deleting horizontal edges. Suppose we delete an edge contained in R_1 or contained in R_2. If the former, assume it is $v_{1,1}v_{1,3}$, and let $B_8 = \{v_{1,1}, v_{1,2}, v_{1,3}, v_{k-1,k}\}$. If the latter, assume it is $v_{k+1,1}v_{k+1,3}$, and let $B_8 = \{v_{k+1,1}, v_{k+1,2}, v_{k+1,3}, v_{k-1,k}\}$. Then in either case, the vertex set $A \cup B_8 \cup \{v_{k-1,1}\}$ induces C_{2k-1}, the set $A \cup B_8 \cup \{v_{k-1,1}, v_{k,1}\}$ induces $\hat{C}_{2k}$, and the set $A \cup B_8 \cup \{v_{k,2}, v_{k,k+1}, v_{k-1,k+1}\}$ induces C'_{2k} with pendant $v_{1,1}$ when deleting $v_{1,1}v_{1,3}$, and pendant $v_{k+1,1}$ when deleting $v_{k+1,1}v_{k+1,3}$.

Suppose we delete an edge between R_1 and R_3. Without loss of generality, assume it is $v_{1,3}v_{1,t}$. Let $B_9 = \{v_{1,3}, v_{1,t}, v_{1,1}, v_{k-1,k}\}$. Then the vertex set $A \cup B_9 \cup \{v_{k-1,t}\}$ induces C_{2k-1}, the set $A \cup B_9 \cup \{v_{k-1,t-1}, v_{k-1,t}\}$ induces $\hat{C}_{2k}$, and the set $A \cup B_9 \cup \{v_{k+1,1}, v_{k+1,2}, v_{k-1,2}\}$ induces C'_{2k} with pendant $v_{1,t}$.

Lastly, suppose we delete an edge contained in R_3. Without loss of generality, assume it is $v_{1,t-1}v_{1,t}$. Let $\tilde{A} = \{v_{p,t-k+p}, v_{p,t-k+p+1} : 2 \leqslant p \leqslant k-2\}$, and let $\tilde{B} =$

$\{v_{1,t-1}, v_{1,t}, v_{1,1}, v_{k-1,t-k+2}\}$. Since $t \geqslant k+2$, note that $t-k+2 \geqslant 4$. Then the vertex set $\tilde{A} \cup \tilde{B} \cup \{v_{k-1,t}\}$ induces C_{2k-1}, the set $\tilde{A} \cup \tilde{B} \cup \{v_{k-1,t}, v_{k,t}\}$ induces $\hat{C}_{2k}$, and the set $\tilde{A} \cup \tilde{B} \cup \{v_{k+1,1}, v_{k+1,2}, v_{k-1,2}\}$ induces C'_{2k} with pendant $v_{1,t}$.

This completes the proof that $G_{n,k}$ is H-induced-saturated for $H \in \{C_{2k-1}, \hat{C_{2k}}, C'_{2k}\}$. □

Corollary 6.3. *For all $k \geqslant 3$, if $n \geqslant (k+1)^2 + 2$ and $H \in \{C_{2k-1}, C'_{2k}, \hat{C}_{2k}\}$, then* $\mathrm{indsat}(n, H) = 0$.

In the following discussion assume $H \in \{C_{2k-1}, C'_{2k}, \hat{C}_{2k}\}$. Using Construction 6.1 we obtain an upper bound on $\mathrm{indsat}^*(n, H)$ with order of magnitude n^2, which is trivial. We can improve this order of magnitude slightly in the case when $\lceil \sqrt{n} \rceil$ is not prime. To do so we note that if n can be written as a product of two integers s and t that are both at least k, then the graph $K_s \,\square\, K_t$ is H-induced-saturated.

Proposition 6.4. *Fix $k \geqslant 3$ and choose n such that $n^{1/4} \geqslant k+1$.*
For $H \in \{C_{2k-1}, C'_{2k}, \hat{C}_{2k}\}$, if $\lceil \sqrt{n} \rceil$ has a proper divisor $t \geqslant 3$, then $\mathrm{indsat}^*(n, H) \leqslant cn^{7/4} + O(n^{3/2})$ *for some constant c.*

Proof. As noted above, the Cartesian product of two sufficiently large cliques is H-induced-saturated. So, consider $G := K_{\lceil \sqrt{n} \rceil / t} \,\square\, K_{t\lceil \sqrt{n} \rceil}$. Simple computation shows $n \leqslant v(G) \leqslant n + 2\sqrt{n} + 1$. So, $v(G)$ can be written as $n+s$, where $0 \leqslant s \leqslant 2\sqrt{n}+1 \leqslant t\sqrt{n} - 3$, as $t \geqslant 3$. Let G' be obtained from G by removing s vertices from a single copy of $K_{3\lceil \sqrt{n} \rceil}$ as in Construction 6.1. An argument similar to that in Proposition 6.2 shows that G' is H-induced-saturated. Observe:

$$e(G') \leqslant t\lceil \sqrt{n} \rceil \binom{(1/t)\lceil \sqrt{n} \rceil}{2} + (1/t)\lceil \sqrt{n} \rceil \binom{t\lceil \sqrt{n} \rceil}{2} = \frac{\lceil \sqrt{n} \rceil^2}{2} \left(\left(t + \frac{1}{t} \right) \lceil \sqrt{n} \rceil \right) - 2 \right).$$

Since t divides $\lceil \sqrt{n} \rceil$, $t \leqslant \sqrt{\lceil \sqrt{n} \rceil} \leqslant c' n^{1/4}$ for some $c' > 1$. Using this and $\lceil \sqrt{n} \rceil \leqslant \sqrt{n} + 1$ gives $e(G') \leqslant \frac{c'}{2} n^{7/4} + O(n^{3/2})$. □

Considering odd cycles points out another property of the induced saturation number. That is, if $\mathrm{indsat}(n, H) = 0$ for a particular n, it is not necessarily the case that $\mathrm{indsat}(k, H) = 0$ for all $k > n$. For example, Construction 6.1 shows $\mathrm{indsat}(n, C_5) = 0$ for $n = 9$ and $n \geqslant 12$. However, a computer search showed that for $n = 10$ and $n = 11$, we have $\mathrm{indsat}(n, C_5) > 0$. (A C_5-induced-saturated trigraph on 10 vertices with one gray edge is shown in Figure 3, so that $\mathrm{indsat}(10, C_5) = 1$.)

7 Families of Graphs

In this section we extend the definition of induced saturation to families of graphs in the natural way, and show the surprising result that a family $\mathcal{F}$ can consist entirely of graphs with induced-saturation number 0, yet satisfy $\mathrm{indsat}(n, \mathcal{F}) > 0$.

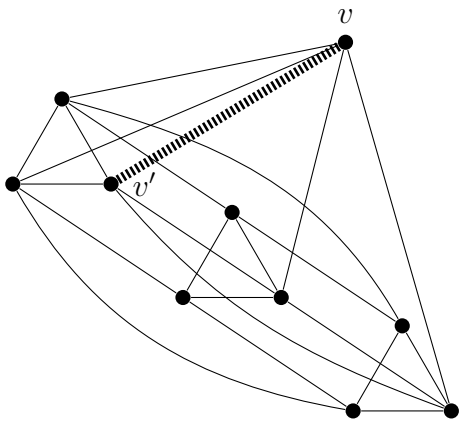


Figure 3: This trigraph, with the gray edge vv', is a C_5-induced-saturated trigraph.

Definition 7.1. For a family $\mathcal{F}$ of graphs, a trigraph T is **$\mathcal{F}$-induced-saturated** if no realization of T contains any member of $\mathcal{F}$ as an induced subgraph, but whenever any black or white edge of T is turned to gray, some member of $\mathcal{F}$ occurs as an induced subgraph of some realization.

The **induced saturation number** of $\mathcal{F}$ with respect to n, written $\text{indsat}(n, \mathcal{F})$, is the minimum number of gray edges in an $\mathcal{F}$-induced-saturated trigraph with n vertices.

For any family $\mathcal{F}$ containing all graphs on k vertices, $\text{indsat}(n, \mathcal{F}) = \binom{n}{2}$.

Construction 6.1 and Proposition 6.2 demonstrate that for any family $\mathcal{F}$, all of whose elements are odd cycles of length at least five, even cycles with a pendant, or even cycles with a triangle chord, $\text{indsat}(n, \mathcal{F}) = 0$ for n sufficiently large. However, we could have $\text{indsat}(n, \mathcal{F}) \neq 0$ even if there is some $G \in \mathcal{F}$ such that $\text{indsat}(n, G) = 0$ as demonstrated in Proposition 7.2 below. One may suspect this is because of the presence of P_4, which has nonzero induced-saturation number, yet it is also possible for a family $\mathcal{F}$ to consist of graphs that each individually have induced saturation number zero, while the induced saturation number of $\mathcal{F}$ is nonzero. We provide an example of this in Proposition 7.3.

Proposition 7.2. *For all n,* $\text{indsat}(n, \{2K_2, P_4, C_4\}) \neq 0$.

Proof. The graphs that contain no induced $2K_2, P_4$, or C_4 are precisely the *threshold graphs* [4]. These graphs are characterized in a second way: they are constructed by iteratively adding a vertex to a graph either as an isolate or a dominating vertex. Thus, an n-vertex threshold graph can be represented as a string of n symbols from $\{-, +\}$ as follows: on the vertex set $V = \{v_1, \ldots, v_n\}$, for every $i > j$, $v_i v_j$ is an edge if and only if the ith symbol in the string is $+$.

We claim that for any threshold graph G with at least one edge, there exists $e \in E(G)$ such that $G - e$ is also threshold. Let $\pi = s_1, \ldots, s_n$ be a string of symbols from $\{-, +\}$ representing G. Suppose there exists $i \in [n-1]$ such that $s_i = -$ and $s_{i+1} = +$, and let i be minimal with this property. Then the graph $G' = G - v_i v_{i+1}$ is represented by the symbol list $\pi' = s_1 \ldots s_{i-1} s_{i+1} s_i s_{i+1} \ldots s_n$, so G' is threshold. If no such index i exists,

then π is a list consisting only of $+$, so G is the complete graph K_n; however, $K_n - e$ is also threshold.

Thus, for any graph G with no induced $2K_2$, P_4, or C_4, there exists an edge $e \in G$ such that $G - e$ also has no induced $2K_2$, P_4, or C_4. It follows that $\text{indsat}(n, \{2K_2, P_4, C_4\}) \neq 0$. □

The family of *split graphs* is another family of graphs that can be characterized by a set of forbidden induced subgraphs. A split graph is a graph whose vertex set can be partitioned into a clique and an independent set. Földes and Hammer [7] showed that a graph is a split graph if and only if it contains no induced $2K_2$, C_4, or C_5.

Proposition 7.3. *For all n,* $\text{indsat}(n, \{2K_2, C_4, C_5\}) \neq 0$.

Proof. Since adding or deleting an edge between the clique part and the independent set of a split graph still results in a split graph, it follows that $\text{indsat}(n, \{2K_2, C_4, C_5\}) \neq 0$. □

We have shown that $\text{indsat}(n, 2K_2)$, $\text{indsat}(n, C_4)$, and $\text{indsat}(n, C_5)$ are all equal to zero for sufficiently large n. Thus, this example shows that even though every graph in a family has induced-saturation number zero, the family itself may not have induced-saturation number zero.

Other families characterized by a (not necessarily finite) family of forbidden induced subgraphs include perfect graphs [2], trivially perfect graphs [14], [8], interval graphs [11], and line graphs [1]. It would be interesting to determine $\text{indsat}(n, \mathcal{F})$ and $\text{indsat}^*(n, \mathcal{F})$, if it exists, for these families. We suspect that doing so will be much more difficult than for threshold and split graphs, as the families of forbidden graphs are significantly more complicated.

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References

[1] L.W. Beineke. Characterizations of derived graphs. *J. Combinatorial Theory* 9 (1970), 129-135

[2] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas. The strong perfect graph theorem. *Ann. of Math.* (2) 164 (2006), no. 1, 51-229.

[3] M. Chudnovsky, P. Seymour. The structure of claw-free graphs. *Surveys in Combinatorics* 2005, 153 - 171.

[4] V. Chvátal, P.L. Hammer. Aggregation of inequalities in integer programming. *Studies in integer programming (Proc. Workshop, Bonn, 1975)*, pp. 145-162. Ann. of Discrete Math., Vol. 1, North-Holland, Amsterdam, 1977.

[5] P. Erdős, A. Hajnal, J.W. Moon. A problem in graph theory. *Amer. Math. Monthly* 71 (1964) 1107-1110.

[6] J. Faudree, R. Faudree, J. Schmitt. A survey of minimum saturated graphs. *Electron. J. Combin.* (2011) Dynamic Survey 18.

[7] S. Földes, P.L. Hammer. On split graphs and some related questions. *Problémes Combinatoires et Théorie Des Graphes,* 138-140, Orsay, France, 1976, Colloques internationaux C.N.R.S. 260

[8] M.C. Golumbic. Trivially perfect graphs. *Discrete Math.* 24 (1978), no. 1, 105-107.

[9] L. Kaszonyi, Z. Tuza Z. Saturated graphs with minimal number of edges. *J. Graph Theory* 10 (1986), 203-210

[10] J. Komlós, G.N. Sárközy, E. Szemerédi. Blow-up lemma. *Combinatorica* 17 (1997), 109-127

[11] C.G. Lekkerkerker, J.Ch. Boland. Representation of a finite graph by a set of intervals on the real line. *Fund. Math.* 51 1962/1963 45-64.

[12] R. Martin, J. Smith. Induced saturation number. *Discrete Math.* 312 (2012), no. 21, 3096–3106.

[13] D. West. *Introduction to Graph Theory*, 2nd. ed., Prentice Hall, Upper Saddle River, NJ, 2001.

[14] E.S. Wolk. The comparability graph of a tree. *Proc. Amer. Math. Soc.* 13 1962 789-795.