Flag statistics from the Ehrhart h^*-polynomial of multi-hypersimplices

Guo-Niu Han

Institut de Recherche Mathématique Avancée
Université de Strasbourg and CNRS
France

guoniu.han@unistra.fr

Matthieu Josuat-Vergès*

Laboratoire d'Informatique Gaspard Monge
Université Paris-Est Marne-la-Vallée and CNRS
France

matthieu.josuat-verges@u-pem.fr

Submitted: Sep 2, 2015; Accepted: Mar 7, 2016; Published: Mar 18, 2016
Mathematics Subject Classifications: 05C88, 05C89

Abstract

It is known that the normalized volume of standard hypersimplices (defined as some slices of the unit hypercube) are the Eulerian numbers. More generally, a recent conjecture of Stanley relates the Ehrhart series of hypersimplices with descents and excedences in permutations. This conjecture was proved by Nan Li, who also gave a generalization to colored permutations. In this article, we give another generalization to colored permutations, using the flag statistics introduced by Foata and Han. We obtain in particular a new proof of Stanley's conjecture, and some combinatorial identities relating pairs of Eulerian statistics on colored permutations.

1 Introduction

A modern combinatorial definition of the Eulerian numbers $A_{n,k}$ is given by counting descents in permutations:

$$A_{n,k} := \#\{\sigma \in \mathfrak{S}_n : \operatorname{des}(\sigma) = k-1\}. \tag{1}$$

*The second author is supported by ANR CARMA (ANR-12-BS01-0017).

Foata suggested in [6] the problem that we describe below. It is known that the Eulerian numbers $A_{n,k}$ satisfy

$$\frac{A_{n,k}}{n!} = \mathrm{Vol}\big(\big\{v \in [0,1]^n \,:\, k-1 \leqslant \sum v_i \leqslant k\big\}\big),$$

this is essentially a calculation due to Laplace (see [6] for details). But the combinatorial definition can be easily translated in the following way:

$$\frac{A_{n,k}}{n!} = \mathrm{Vol}(\{v \in [0,1]^n \,:\, \mathrm{des}(v) = k-1\}).$$

The problem is to find a measure-preserving bijection between the two sets, to explain why they have the same volume. A simple solution was given by Stanley [12].

The set $\{v \in [0,1]^n \,:\, k \leqslant \sum v_i \leqslant k+1\}$ is in fact a convex integral polytope known as the hypersimplex, and in this context we can consider the h^*-polynomial, which is a generalization of the volume. This led to a recent conjecture by Stanley about the h^*-polynomial of the hypersimplex (more precisely, a partially open version of the hypersimplex), which was proved by Nan Li [10] in two different ways. It is remarkable that two Eulerian statistics are needed to state the conjecture, which says that the h^*-polynomial of a hypersimplex is the descent generating function for permutations with a given number of excedences. Nan Li also extended the result to colored permutations by considering the hypercube $[0,r]^n$ for some integer $r > 0$, and the polytopes $\{v \in [0,r]^n \,:\, k \leqslant \sum v_i \leqslant k+1\}$ are the multi-hypersimplices referred to in the title of this article. We would like to mention that besides Stanley's conjecture, some recent works deals with the geometry and combinatorics of hypersimplices, see [8, 11].

The goal of this article is to give another generalization of Stanley's conjecture to colored permutations. Our result is stated in terms of the flag descents and flag excedences in colored permutations, and relies on some related work by Foata and Han [5]. Our method gives in particular a new proof of Stanley's conjecture in the uncolored case. Our method can roughly be described as follows. We first consider the case of the half-open hypercube $[0,r)^n$, where an analog of Stanley's conjecture in terms of descents and inverse descents can be proved in a rather elementary way. We can relate the half-open hypercube $[0,r)^n$ with the usual hypercube $[0,r]^n$ via an inclusion-exclusion argument. Then, it remains only to prove an identity relating two generating functions for colored permutations.

This article is organized as follows. Section 2 contains some preliminaries. Our main results are Theorems 17 and 18, whose particular case $r = 1$ gives Stanley's conjecture. The core of the proof is in Sections 3 and 4, but it also relies on some combinatorial results on colored permutations which are in Sections 5, 6 and 7.

2 Triangulations of the unit hypercube

This section contains nothing particularly new, but we introduce some notation and background (see [1, 13]). Let $\mathcal{X} \subset \mathbb{R}^n$ be a convex polytope with integral vertices. The *Ehrhart* *polynomial* $E(\mathcal{X},t)$ is defined as the unique polynomial in t such that, for any integer $t > 0$,

$$E(\mathcal{X},t) = \#\big(t\mathcal{X} \cap \mathbb{Z}^n\big) \tag{2}$$

where $t\mathcal{X} := \{tx \,:\, x \in \mathcal{X}\}$. The h^**-polynomial* of $\mathcal{X}$ is defined as

$$E^*(\mathcal{X},z) := (1-z)^{n+1} \sum_{t \geqslant 0} E(\mathcal{X},t) z^t. \tag{3}$$

From a general result of Stanley [13], the series $E^*(\mathcal{X},z)$ is in fact a polynomial with positive integral coefficients. As it often happens, it is an interesting problem to find their combinatorial meaning for special polytopes. Perhaps the most basic example is the unit hypercube $[0,1]^n$ which has the nth Eulerian polynomial as h^*-polynomial, as will be detailed below. Although we do not use this language here, a general method to find the h^*-polynomial of a polytope is to use unimodular shellable triangulations (as was done for example in [10]) and it is essentially the idea behind what follows.

Definition 1. If $v = (v_1,\dots,v_n) \in \mathbb{R}^n$, let

$$\operatorname{des}(v) = \#\{i \,:\, 1 \leqslant i \leqslant n-1,\ v_i > v_{i+1}\}.$$

We define the *standardization* $\operatorname{std}(v)$ of v to be the unique permutation $\sigma \in \mathfrak{S}_n$ such that for all $i < j$ we have $v_i \leqslant v_j$ iff $\sigma_i < \sigma_j$. For each permutation $\sigma \in \mathfrak{S}_n$, let

$$\mathcal{S}_\sigma = \{v \in [0,1]^n \,:\, \operatorname{std}(v) = \sigma\}.$$

For example, one can check that if $x < y < z$, $\operatorname{std}(x,y,x,z,y,x,x) = 1527634$. Note that the unit hypercube $[0,1]^n$ is the disjoint union of the subsets $\mathcal{S}_\sigma$ for $\sigma \in \mathfrak{S}_n$. Geometrically, each $\mathcal{S}_\sigma$ is a unit simplex where some facets are removed. So it is not a polytope in the usual sense; we should call it a "partially open" polytope. But note that Equations (2) and (3) make sense even when $\mathcal{X}$ is not a polytope, so in particular $E^*(\mathcal{S}_\sigma,z)$ is well defined, and we have:

Lemma 2. $E^*(\mathcal{S}_\sigma,z) = z^{\operatorname{des}(\sigma^{-1})}$.

Proof. We can check that $v \in [0,1]^n$ is in $\mathcal{S}_\sigma$ if and only if $v_{\sigma^{-1}(1)} \leqslant \dots \leqslant v_{\sigma^{-1}(n)}$, and $v_{\sigma^{-1}(i)} < v_{\sigma^{-1}(i+1)}$ if $\sigma^{-1}(i) > \sigma^{-1}(i+1)$. The number $E(\mathcal{S}_\sigma,t)$ counts such sequences with the additional condition that all elements are integers between 0 and t. By defining

$$w_i = v_{\sigma^{-1}(i)} - \operatorname{des}(\sigma^{-1}(1),\dots,\sigma^{-1}(i)),$$

we have a bijection with integer sequences satisfying $0 \leqslant w_1 \leqslant \dots \leqslant w_n \leqslant t - \operatorname{des}(\sigma^{-1})$, so that

$$E(\mathcal{S}_\sigma,t) = \binom{n+t-\operatorname{des}(\sigma^{-1})}{n}.$$

The expansion

$$\sum_{t \geqslant 0} \binom{t}{n} z^t = \frac{z^n}{(1-z)^{n+1}}$$

permits to finish the proof. $\square$

Definition 3. The *Eulerian polynomials* are

$$A_n(z) := \sum_{\sigma \in \mathfrak{S}_n} z^{\operatorname{des}(\sigma^{-1})} = \sum_{\sigma \in \mathfrak{S}_n} z^{\operatorname{des}(\sigma)}.$$

Note that their coefficients are the numbers $A_{n,k}$ defined in Equation (1).

For example, $A_0(z) = A_1(z) = 1$, $A_2(z) = 1 + z$, $A_3(z) = 1 + 4z + z^2$, etc. Since the h^*-polynomial is additive with respect to disjoint union, we get from the previous lemma that

$$E^*([0,1]^n, z) = A_n(z).$$

Note that, since the Ehrhart polynomial of $[0,1]^n$ is clearly $(t+1)^n$, we have proved the classical identity:

$$A_n(z) = (1-z)^{n+1} \sum_{t \geqslant 0} (t+1)^n z^t. \tag{4}$$

Let us turn to the case of the half-open hypercube $[0,1)^n$. Note that we are now dealing with a half-open polytope, i.e., a polytope where some of the $(n-1)$-dimensional faces are removed. In this case, it is not *a priori* clear that the h^*-polynomial is a polynomial with nonnegative integral coefficients. We can decompose $[0,1)^n$ as the disjoint union of the polytopes:

$$\mathcal{T}_\sigma = \{v \in [0,1)^n \,:\, \operatorname{std}(v) = \sigma\}.$$

These are also simplices where some facets are removed, and we get:

Lemma 4. $E^*(\mathcal{T}_\sigma, z) = z^{\operatorname{des}(\sigma^{-1})+1}$.

Proof. It is similar to the one of Lemma 2. □

Thus, the half-open hypercube has the h^*-polynomial $zA_n(z)$. Besides, its Ehrhart polynomial is clearly $E([0,1)^n, t) = t^n$. Once again we get Identity (4), with an additional factor z.

Let us present another example of a h^*-polynomial that will be used in the sequel. It is presented in [14, Section 7.19] in the context of quasi-symmetric functions. Let λ be a Young diagram (we use the French notation). Let $\mathbb{Z}^\lambda$ (respectively, $\mathbb{R}^\lambda$) denote the set of fillings of λ with integers (respectively, real numbers). And let $\mathcal{Y}_\lambda$ denote the set of semi-standard fillings of λ by real numbers in $(0,1]$ where semi-standard mean weakly increasing in rows and strictly increasing in columns. Clearly, $\mathcal{Y}_\lambda$ is a (partially open) convex polytope in $\mathbb{R}^\lambda$. A semi-standard tableau with largest entry less than t is just an element of $\mathbb{Z}^\lambda \cap t\mathcal{Y}_\lambda$, so that

$$E(\mathcal{Y}_\lambda, t) = s_\lambda(1^t),$$

the Schur function s_λ where t variables are set to 1 and the others to 0. Let $SYT(\lambda)$ denote the set of standard tableaux of shape λ, and recall that a descent of a standard tableau is an entry i such that the entry $i+1$ is in an upper row. Let $\operatorname{des}(T)$ denote the number of descents of a standard tableau, then we have:

Proposition 5.

$$E^*(\mathcal{Y}_\lambda, z) = (1-z)^{n+1} \sum_{t\geqslant 0} s_\lambda(1^t) z^t = \sum_{T \in SYT(\lambda)} z^{\mathrm{des}(T)+1}. \tag{5}$$

Let us sketch the proof. The reading word $w(T)$ of a semi-standard tableau $T \in \mathcal{Y}_\lambda$ is defined by ordering its entries row by row, from left to right and from top to bottom. Then, the standardization $\mathrm{std}(T)$ is defined to be the unique standard tableau U of the same shape such that $\mathrm{std}(w(T)) = w(U)$. The set $\mathcal{Y}_\lambda$ is partitioned into the subsets

$$\mathcal{Y}_U = \{T \in \mathcal{Y}_\lambda \,:\, \mathrm{std}(T) = U\}$$

where $U \in SYT(\lambda)$. Now, the previous proposition is a consequence of the following:

Lemma 6. $E^*(\mathcal{Y}_U, z) = z^{\mathrm{des}(U)+1}$.

Proof. This is essentially the same as Proposition 4. □

It is in order to precise the link with quasi-symmetric functions mentionned above. Semi-standard tableaux of shape λ can be seen as (P,ω)-partitions in the sense of [14]. In general, (P,ω)-partitions are the integral points of a polytope called the order polytope, and quasi-symmetric functions are useful in this context to compute the h^*-polynomial.

3 The generalization of Stanley's bijection

In this section we adapt Stanley's bijection from [12] to the case of the half-open hypercube $[0,r)^n$ (for some integer $r > 0$), and r-colored permutations. Note that a similar generalization was given by Steingrímsson in [15, Section 4.4].

Definition 7. The set of *r-colored permutations* $\mathfrak{S}_n^{(r)}$ is the set of pairs (σ, c) where $\sigma \in \mathfrak{S}_n$, $c = (c_i)_{1\leqslant i\leqslant n}$, and $c_i \in \{0,1,\dots,r-1\}$ for all i. We define the *descent number* of a colored permutation as:

$$\mathrm{des}(\sigma, c) := \#\{i \,:\, c_i > c_{i+1}, \text{ or } c_i = c_{i+1} \text{ and } \sigma_i > \sigma_{i+1}\},$$

and we define the *flag descent number* [2, 5] as:

$$\mathrm{fdes}(\sigma, c) := r.\mathrm{des}(\sigma, c) + c_n.$$

The *flag Eulerian numbers* are defined by $A_{0,0}^{(r)} := 1$, $A_{n,0}^{(r)} := 0$ if $n > 0$, and

$$A_{n,k}^{(r)} := \#\big\{\sigma \in \mathfrak{S}_n^{(r)} \,:\, \mathrm{fdes}(\sigma) = k-1\big\}$$

if $n \geqslant 1$ and $1 \leqslant k \leqslant rn$.

We will not use here the group structure of colored permutations. Still, note that flag descents are indeed related with it [2].

To define our generalization of Stanley's bijection, let $(a_i)_{1\leqslant i\leqslant n} \in [0,r)^n$, and let $a_0 = 0$. Then the map $\phi((a_i)_{1\leqslant i\leqslant n}) = (b_i)_{1\leqslant i\leqslant n}$ is defined as follows:

$$b_i = \begin{cases} a_i - a_{i-1} & \text{if } a_{i-1} \leqslant a_i, \\ a_i - a_{i-1} + r & \text{if } a_{i-1} > a_i. \end{cases} \tag{6}$$

From this definition we get:

$$a_i = \sum_{j=1}^{i} b_j \mod r, \tag{7}$$

where the modulo means that we take the unique representative in $[0,r)$. In fact, it is elementary to check that Equations (6) and (7) define two inverse bijections from $[0,r)^n$ to itself.

Definition 8. If $v \in [0,r)^n$, we define $\mathrm{fdes}(v) = r.\mathrm{des}(v) + v_n$. If $1 \leqslant k \leqslant rn$, let

$$\mathcal{F}_{n,k}^{(r)} := \{v \in [0,r)^n \,:\, k-1 \leqslant \mathrm{fdes}(v) < k\},$$

and

$$\mathcal{A}_{n,k}^{(r)} := \{v \in [0,r)^n \,:\, k-1 \leqslant \sum v_i < k\}.$$

For each colored permutation (σ, c) we define the translated simplex:

$$\mathcal{T}_{(\sigma,c)} := c + \mathcal{T}_\sigma.$$

Also, the *colored standardization* of $v \in [0,r)^n$ is $\mathrm{cstd}(v) = (\sigma, c) \in \mathfrak{S}_n^{(r)}$ where $c_i = \lfloor v_i \rfloor$ and $\sigma = \mathrm{std}(v_1 \mod 1, \ldots, v_n \mod 1)$.

Lemma 9. $\lfloor \mathrm{fdes}(v) \rfloor = \mathrm{fdes}(\mathrm{cstd}(v))$.

Proof. This follows straighforwardly from the definitions. □

Lemma 10. $\mathcal{A}_{n,k}^{(r)} = \phi(\mathcal{F}_{n,k}^{(r)})$.

Proof. From (6) and keeping the notation we get

$$\sum_{i=1}^{n} b_i = r.\mathrm{des}(a_1, \ldots, a_n) + a_n = \mathrm{fdes}(a_1, \ldots, a_n)$$

and the result follows. □

Lemma 11. $E^*(\phi(\mathcal{T}_{(\sigma,c)}), z) = z^{\mathrm{des}(\sigma^{-1})+1}$.

Proof. Let $v \in \mathcal{T}_{(\sigma,c)}$, then the condition $v_i \leqslant v_{i+1}$ or $v_i > v_{i+1}$ only depends on (σ, c). So from its definition in (6), we see that the restriction of ϕ to $\mathcal{T}_{(\sigma,c)}$ is equal to an affine map that sends $\mathbb{Z}^n$ to itself. It follows that $\phi(\mathcal{T}_{(\sigma,c)})$ has the same h^*-polynomial as $\mathcal{T}_{(\sigma,c)}$. Besides, since $\mathcal{T}_{(\sigma,c)}$ is a translation of $\mathcal{T}_\sigma$ by an integer vector, they have the same h^*-polynomial, which is therefore $z^{\operatorname{des}(\sigma^{-1})+1}$ by Lemma 4. □

Proposition 12.

$$E^*(\mathcal{A}_{n,k}^{(r)}, z) = \sum_{\substack{(\sigma,c)\in\mathfrak{S}_n^{(r)} \\ \operatorname{fdes}(\sigma,c)=k-1}} z^{\operatorname{des}(\sigma^{-1})+1}.$$

Proof. Using Lemma 9, we get:

$$\mathcal{F}_{n,k}^{(r)} = \biguplus_{\substack{(\sigma,c)\in\mathfrak{S}_n^{(r)} \\ \operatorname{fdes}(\sigma,c)=k-1}} \mathcal{T}_{(\sigma,c)}.$$

From Lemma 10 and the fact that ϕ is a bijection, we have:

$$\mathcal{A}_{n,k}^{(r)} = \phi(\mathcal{F}_{n,k}^{(r)}) = \biguplus_{\substack{(\sigma,c)\in\mathfrak{S}_n^{(r)} \\ \operatorname{fdes}(\sigma,c)=k-1}} \phi(\mathcal{T}_{(\sigma,c)}).$$

From Lemma 11 and the fact that the h^*-polynomial is additive with respect to disjoint union, we get the result. □

4 From the half-open hypercube to the closed hypercube

Definition 13. The *multi-hypersimplices* are the polytopes defined by:

$$\mathcal{B}_{n,k}^{(r)} := \begin{cases} \left\{v \in [0,r]^n \;:\; k-1 \leqslant \sum v_i < k\right\} & \text{if } 1 \leqslant k < rn, \\ \left\{v \in [0,r]^n \;:\; k-1 \leqslant \sum v_i \leqslant k\right\} & \text{if } k = rn. \end{cases}$$

These polytopes form a particular class of the ones introduced by Lam and Postnikov [9] under the same name. The polytopes $\mathcal{B}_{n,k}^{(1)}$ are simply called the hypersimplices, and they can be described geometrically as truncated simplices (this fact is essentially due to Coxeter [4, Section 8.7]).

Note that

$$[0,r]^n = \biguplus_{1\leqslant k\leqslant rn} \mathcal{B}_{n,k}^{(r)}.$$

Proposition 14. *Let $B_{n,k}^{(r)}(z) = E^*(\mathcal{B}_{n,k}^{(r)}, z)$ and $A_{n,k}^{(r)}(z) = E^*(\mathcal{A}_{n,k}^{(r)}, z)$, with the convention that $A_{n,0}^{(r)}(z) = B_{n,0}^{(r)}(z) = \delta_{n0}$ and $A_{n,k}^{(r)}(z) = B_{n,k}^{(r)}(z) = 0$ if $k < 0$ or $k > rn$. Then:*

$$B_{n,k}^{(r)}(z) = \sum_{j=0}^{n} \binom{n}{j} (1-z)^j A_{n-j,k-rj}^{(r)}(z).$$

Proof. For each $\Delta \subset \{1,\dots,n\}$, let $\mathcal{H}_\Delta = \{v \in \mathcal{B}^{(r)}_{n,k} \,:\, v_i = r \text{ iff } i \in \Delta\}$. The sets $\mathcal{H}_\Delta$ form a partition of $\mathcal{B}^{(r)}_{n,k}$, so that

$$B^{(r)}_{n,k}(z) = \sum_{\Delta \subset \{1,\dots,n\}} E^*(\mathcal{H}_\Delta, z).$$

By removing the coordinates equal to r, we see that $\mathcal{H}_\Delta$ is in bijection with $\mathcal{A}^{(r)}_{n-j,k-rj}$ where $j = \#\Delta$. The bijection preserves integral points, and this holds with the convention that both polytopes are empty if $k - rj < 0$. Hence:

$$E^*(\mathcal{H}_\Delta, z) = (1-z)^j A^{(r)}_{n-j,k-rj}(z). \qquad \square$$

The previous proposition is conveniently rewritten in terms of the generating functions. Let

$$A^{(r)}(x,y,z) = \sum_{n\geqslant 0} \left(\sum_{k=0}^{rn} A^{(r)}_{n,k}(z) y^k \right) \frac{x^n}{n!}, \quad B^{(r)}(x,y,z) = \sum_{n\geqslant 0} \left(\sum_{k=0}^{rn} B^{(r)}_{n,k}(z) y^k \right) \frac{x^n}{n!},$$

then these two series are related as stated below.

Theorem 15. *The following identity holds:*

$$B^{(r)}(x,y,z) = e^{(1-z)y^r x} A^{(r)}(x,y,z).$$

Proof. From Proposition 14, we get:

$$\sum_{k\geqslant 0} y^k B^{(r)}_{n,k}(z) = \sum_{j=0}^{n} \binom{n}{j} ((1-z)y^r)^j \sum_{k\geqslant 0} y^{k-rj} A^{(r)}_{n-j,k-rj}(z)$$

and the result follows. $\square$

Together with Proposition 12, the relation in the previous theorem shows that a generalization of Stanley's conjecture can be obtained via an identity on generating functions. This identity will be presented in the next sections. We first need some definitions to state the result.

Definition 16 ([3, 5])**.** The *flag excedence number* of a colored permutation is:

$$\operatorname{fexc}(\sigma, c) := r.\#\big\{ i \in \{1,\dots,n\} \,:\, \sigma_i > i \text{ and } c_i = 0 \big\} + \sum_{i=1}^{n} c_i.$$

We also need another definition of flag descents, which is the one originally due to Foata and Han [5]:

$$\operatorname{fdes}^*(\sigma, c) := r.\operatorname{des}^*(\sigma, c) + c_1$$

where

$$\operatorname{des}^*(\sigma, c) := \#\{ i \,:\, c_i < c_{i+1}, \text{ or } c_i = c_{i+1} \text{ and } \sigma_i > \sigma_{i+1} \}.$$

In particular, let us mention that the statistics fexc and fdes* are equidistributed on $\mathfrak{S}_n^{(r)}$, see [5, Theorem 1.4]. We will also give a proof that fdes and fdes* are equidistributed in the next section.

Theorem 17.

$$B_{n,k}^{(r)}(z) = \sum_{\substack{(\sigma,c)\in\mathfrak{S}_n^{(r)} \\ \mathrm{fexc}(\sigma,c)=rn-k}} z^{\lceil \mathrm{fdes}^*(\sigma,c)/r \rceil}. \tag{8}$$

Proof. Let $C_{n,k}^{(r)}(z)$ denote the right-hand side of the equation, and we use the same convention as with $B_{n,k}^{(r)}(z)$ when $k \leqslant 0$. Let also $C_n^{(r)}(y,z) = \sum_{k=0}^{rn} C_{n,k}^{(r)}(z) y^k$, and

$$C^{(r)}(x,y,z) = \sum_{n\geqslant 0} C_n^{(r)}(y,z) \frac{x^n}{n!}.$$

From Proposition 12 and Theorem 24 in the sequel, we have

$$C^{(r)}(x,y,z) = e^{(1-z)y^r x} A^{(r)}(x,y,z).$$

Comparing with Theorem 15 shows that we have $B^{(r)}(x,y,z) = C^{(r)}(x,y,z)$, which proves the theorem. □

We have in fact another result, which is not trivially equivalent to the previous one:

Theorem 18.

$$B_{n,k}^{(r)}(z) = \sum_{\substack{(\sigma,c)\in\mathfrak{S}_n^{(r)} \\ \mathrm{fexc}(\sigma,c)=rn-k}} z^{\lceil \mathrm{fdes}(\sigma,c)/r \rceil}.$$

Proof. This is a consequence of the previous theorem, together with the bijection in Section 7. □

In view of the previous two theorems, one can ask whether the pairs $(\mathrm{fexc}, \mathrm{fdes})$ and $(\mathrm{fexc}, \mathrm{fdes}^*)$ are equidistributed. This is however not the case.

5 Chromatic descents

Definition 19. For a colored permutation (σ, c) we define its *chromatic descent number* as

$$\mathrm{cdes}(\sigma,c) := \mathrm{des}(\sigma) + \sum_{i=1}^{n} c_i.$$

We show that it is equidistributed with the flag descent number, via a bijection α. Let (σ, c) be a colored permutation. Let $\alpha(\sigma,c) = (\sigma, c')$ where

$$c'_i = \sum_{j=1}^{i} c_j + \mathrm{des}(\sigma_1,\dots,\sigma_i) \mod r.$$

Proposition 20. $\mathrm{fdes}(\sigma, c') = \mathrm{cdes}(\sigma, c)$.

Proof. Let $w_k = \sum_{j=1}^{k} c_j + \mathrm{des}(\sigma_1, \dots, \sigma_k)$ for $k = 1, \dots, n$ so that $c'_i = w_i \mod r$. Clearly, $w_1, \dots, w_n$ is a nondecreasing sequence and $w_n = \mathrm{cdes}(\sigma, c)$. We can write $w_n = qr + c'_n$ for a unique q. This integer q counts the number of positive multiples of r that are smaller than w_n. Using the fact that $w_k - w_{k-1} \leqslant r$ and $w_1 < r$, we have:

$$q = \#\{i \,:\, \exists k, w_{i-1} < kr \leqslant w_i\}.$$

To count the cardinality of this set, we distinguish two cases. If $w_{i-1} - w_i = r$, it means that $c_i = r - 1$ and $\sigma_{i-1} > \sigma_i$. From the definition of the bijection, this is equivalent to $c'_i = c'_{i+1}$ and $\sigma_{i-1} > \sigma_i$. Otherwise, $w_{i-1} - w_i < r$. We can see that this case is equivalent to $c_{i-1} > c_i$. Hence, we obtain $q = \mathrm{des}(\sigma, c')$, and $w_n = \mathrm{fdes}(\sigma, c')$. $\square$

It is also possible to define a bijection α^* by $\alpha^*(\sigma, c) = (\sigma, c'')$ where

$$c''_i = \sum_{j=i}^{n} c_j + \mathrm{des}(\sigma_i, \dots, \sigma_n) \mod r.$$

As in the case of the previous proposition, we can prove $\mathrm{fdes}^*(\sigma, c'') = \mathrm{cdes}(\sigma, c)$. In particular, it follows that fdes and fdes^* are equidistributed.

The bijection α only changes the colors c_i, and not the permutation σ, so we have:

$$\sum_{(\sigma,c)\in\mathfrak{S}_n^{(r)}} y^{\mathrm{fdes}(\sigma,c)} z^{\mathrm{des}(\sigma^{-1})} = \sum_{(\sigma,c)\in\mathfrak{S}_n^{(r)}} y^{\mathrm{cdes}(\sigma,c)} z^{\mathrm{des}(\sigma^{-1})}.$$

But the right-hand side clearly can be factorized, so that with the notation

$$A_n^{(r)}(y, z) = \sum_{k=0}^{rn} y^k A_{n,k}^{(r)}(z),$$

we have:

$$A_n^{(r)}(y, z) = \left(\frac{1 - y^r}{1 - y}\right)^n A_n^{(1)}(y, z). \tag{9}$$

A formula for the case $r = 1$ is given in the proposition below. This is in fact a particular case of a result of Garsia and Gessel [7, Theorem 2.3], but we also include a short proof based on the Robinson-Schensted correspondence.

Proposition 21. *For $r = 1$, we have:*

$$\frac{A_n^{(1)}(y, z)}{(1 - y)^{n+1}(1 - z)^{n+1}} = \sum_{i,j\geqslant 0} \binom{ij + n - 1}{n} y^i z^j.$$

Proof. Let $\mathrm{Par}(n)$ denote the set of integer partitions of n. By the Robinson-Schensted correspondence, we have:

$$\begin{aligned} A_n^{(1)}(y,z) &= \sum_{\sigma\in\mathfrak{S}_n} y^{\mathrm{des}(\sigma)+1} z^{\mathrm{des}(\sigma^{-1})+1} \\ &= \sum_{\lambda\in\mathrm{Par}(n)} \sum_{P,Q\in SYT(\lambda)} y^{\mathrm{des}(P)+1} z^{\mathrm{des}(Q)+1}. \end{aligned}$$

So, using Equation (5), we get:

$$\frac{A^{(1)}(y,z)}{(1-y)^{n+1}(1-z)^{n+1}} = \sum_{\lambda\in\mathrm{Par}(n)} \sum_{s,t\geqslant 0} s_\lambda(1^s)s_\lambda(1^t)y^s z^t.$$

By the Cauchy identity on Schur functions, we have

$$\sum_{\lambda\in\mathrm{Par}(n)} s_\lambda(1^s)s_\lambda(1^t) = [x^n]\Big(\frac{1}{1-x}\Big)^{st} = \binom{st+n-1}{n}.$$

This ends the proof. ☐

From Equation (9) and the previous proposition, we deduce:

Proposition 22.

$$\frac{A_n^{(r)}(y,z)}{(1-y^r)^n(1-y)(1-z)^{n+1}} = \sum_{i,j\geqslant 0} \binom{ij+n-1}{n} y^i z^j. \tag{10}$$

Note that another consequence of Equation (9), together with Equation (4), is the following (which is not a new result, see for example [2]).

Proposition 23.

$$\frac{A_n^{(r)}(y,1)}{(1-y^r)^n(1-y)} = \sum_{i\geqslant 1} i^n y^i. \tag{11}$$

6 Identities on bi-Eulerian generating functions

We keep the definition of $A_n^{(r)}(y,z)$ and $A^{(r)}(x,y,z)$ as before, but in this section we only need the formula in Equation (10). We recall that $C_{n,k}^{(r)}(y,z)$, $C_n^{(r)}(y,z)$, and $C(x,y,z)$ were defined in the proof of Theorem 17. The goal of this section is to prove the following relation between the two generating functions for colored permutations:

Theorem 24. $C^{(r)}(x,y,z) = e^{(1-z)y^r x} A^{(r)}(x,y,z)$.

Let us define

$$W_n(y,z) := \sum_{(\sigma,c)\in\mathfrak{S}_n^{(r)}} y^{\mathrm{fexc}(\sigma,c)} z^{\mathrm{fdes}^*(\sigma,c)}.$$

The particular case $q = 1$ of [5, Theorem 5.11], after an easy simplification, gives the following formula:

$$\sum_{n\geqslant 0} W_n(y,z)\frac{x^n}{(1-z^r)^n} = (1-z)\sum_{k\geqslant 0} z^k F_k(x,y), \tag{12}$$

where

$$F_k(x,y) = \frac{(1-xy^r)^{\lfloor k/r\rfloor}}{(1-x)^{\lfloor k/r\rfloor+1}}(1-y^r)\times\Big(\frac{1-y^r}{1-y} - \sum_{i=1}^{r} y^i \frac{(1-xy^r)^{\lfloor (k-i)/r\rfloor+1}}{(1-x)^{\lfloor (k-i)/r\rfloor+1}}\Big)^{-1}.$$

Next, we define a linear operator β on power series in z by $\beta(z^k) = z^{\lceil k/r\rceil}$. So:

$$\beta(z^k - z^{k+1}) = \begin{cases} 0 & \text{if } k \not\equiv 0 \mod r, \\ z^m - z^{m+1} & \text{if } k = rm. \end{cases}$$

From Equation (12), we get

$$\sum_{n\geqslant 0} \beta(W_n(y,z))\frac{x^n}{(1-z)^n} = \sum_{m\geqslant 0}(z^m - z^{m+1})F_{rm}(x,y), \tag{13}$$

and from the definition of F_k we get

$$\begin{aligned} F_{rm}(x,y) &= \frac{(1-xy^r)^m}{(1-x)^{m+1}}(1-y^r)\Big(\frac{1-y^r}{1-y} - \sum_{i=1}^{r} y^i \frac{(1-xy^r)^m}{(1-x)^m}\Big)^{-1} \\ &= \Big(\frac{(1-x)^{m+1}}{(1-xy^r)^m(1-y)} - \sum_{i=1}^{r} y^i \frac{1-x}{1-y^r}\Big)^{-1} \\ &= \frac{1-y}{1-x}\Big(\big(\frac{1-x}{1-xy^r}\big)^m - y\Big)^{-1}. \end{aligned}$$

From Equation (13) and the previous equation, and after the substitution $(x,y) \leftarrow (xy^r(1-z), y^{-1})$, we reach:

Theorem 25.

$$\sum_{n\geqslant 0} C_n^{(r)}(y,z)x^n = \frac{(1-z)(1-y)}{1-xy^r(1-z)}\sum_{m\geqslant 0} z^m \Big(1 - y\big(\frac{1-xy^r(1-z)}{1-x(1-z)}\big)^m\Big)^{-1}.$$

Besides, from Equation (10), we have:

$$\sum_{n\geqslant 0} x^n \frac{A_n^{(r)}(y,z)}{(1-y^r)^n(1-y)(1-z)^{n+1}} = \sum_{i,j\geqslant 0}\Big(\frac{1}{1-x}\Big)^{ij} y^i z^j = \sum_{j\geqslant 0} z^j\Big(1 - y\big(\frac{1}{1-x}\big)^j\Big)^{-1}.$$

After the substition $x \leftarrow x(1-y^r)(1-z)$, we obtain:

Theorem 26.

$$\sum_{n\geqslant 0} A_n^{(r)}(y,z)x^n = (1-y)(1-z)\sum_{m\geqslant 0} z^m\Big(1-y\big(\frac{1}{1-x(1-y^r)(1-z)}\big)^m\Big)^{-1}.$$

Proof of Theorem 24. Since Theorem 24 is a relation on exponential generating functions, it is convenient to use the Laplace transform. It sends a function $f(x)$ to

$$\mathcal{L}(f(x),x,s) = \int_0^\infty f(x)e^{-xs}\mathrm{d}x,$$

in particular,

$$\mathcal{L}\Big(\frac{x^k}{k!},x,s\Big) = \frac{1}{s^{k+1}}.$$

We have:

$$\begin{aligned}\mathcal{L}\big(e^{(1-z)y^r x}A^{(r)}(x,y,z),x,s\big) &= \int_0^\infty e^{(1-z)y^r x}A^{(r)}(x,y,z)e^{-xs}\mathrm{d}x\\ &= \int_0^\infty A^{(r)}(x,y,z)e^{(1-z)y^r x - xs}\mathrm{d}x\\ &= \mathcal{L}\big(A^{(r)}(x,y,z),x,s-(1-z)y^r\big).\end{aligned}$$

By Theorem 26, with $s' = s-(1-z)y^r$, the latter expression is equal to

$$\frac{(1-y)(1-z)}{s'}\sum_{m\geqslant 0} z^m\Big(1-y\big(\frac{1}{1-\frac{1}{s'}(1-y^r)(1-z)}\big)^m\Big)^{-1}.$$

Since

$$\frac{1}{1-\frac{1}{s'}(1-y^r)(1-z)} = \frac{1}{1-\frac{1}{s-(1-z)y^r}(1-y^r)(1-z)} = \frac{s-(1-z)y^r}{s-(1-z)},$$

we get:

$$\mathcal{L}\big(e^{(1-z)y^r x}A^{(r)}(x,y,z),x,s\big) = \frac{(1-y)(1-z)}{s-(1-z)y^r}\sum_{m\geqslant 0} z^m\Big(1-y\big(\frac{s-(1-z)y^r}{s-(1-z)}\big)^m\Big)^{-1}.$$

Besides, from Theorem 25, we also get:

$$\mathcal{L}\big(C^{(r)}(x,y,z),x,s\big) = \frac{(1-y)(1-z)}{s-(1-z)y^r}\sum_{m\geqslant 0} z^m\Big(1-y\big(\frac{s-(1-z)y^r}{s-(1-z)}\big)^m\Big)^{-1}.$$

So we have proved

$$\mathcal{L}\big(C^{(r)}(x,y,z),x,s\big) = \mathcal{L}\big(e^{(1-z)y^r x}A^{(r)}(x,y,z),x,s\big),$$

which completes the proof of Theorem 24. □

7 Another combinatorial model

We give in this section a bijective proof of

$$\sum_{(\sigma,c)\in\mathfrak{S}_n^{(r)}} y^{\mathrm{fexc}(\sigma,c)} z^{\lceil \mathrm{fdes}(\sigma,c)/r\rceil} = \sum_{(\sigma,c)\in\mathfrak{S}_n^{(r)}} y^{\mathrm{fexc}(\sigma,c)} z^{\lceil \mathrm{fdes}^*(\sigma,c)/r\rceil},$$

by defining an involution I on colored permutations such that

$$y^{\mathrm{fexc}(\sigma,c)} z^{\lceil \mathrm{fdes}(\sigma,c)/r\rceil} = y^{\mathrm{fexc}(I(\sigma,c))} z^{\lceil \mathrm{fdes}^*(I(\sigma,c))/r\rceil}.$$

Let $(\sigma,c) \in \mathfrak{S}_n^{(r)}$. We consider (σ,c) as a word whose successive letters are (σ_1,c_1), $(\sigma_2,c_2),\dots,$ (σ_n,c_n). Note that the pair (σ_i,c_i) is considered as a letter with color c_i. Then, we consider the unique factorization

$$(\sigma_1,c_1)\dots(\sigma_n,c_n) = B_1\dots B_m$$

where each block B_i contains letters of the same color, and m is minimal. The involution I is defined by permuting the blocks, following these two conditions:

- each zero colored block stays at the same location,
- each maximal sequence of nonzero colored blocks $B_j\dots B_k$ is replaced with $B_k\dots B_j$ (maximal means that B_{j-1} is zero colored or $j=1$, and B_{k+1} is zero colored or $k=n$).

For example, with $n=8$ and $r=3$:

$$(8,1)(2,0)(7,2)(1,2)(4,1)(3,0)(5,1)(6,1)$$

is sent to

$$(8,1)(2,0)(4,1)(7,2)(1,2)(3,0)(5,1)(6,1).$$

Lemma 27. $\mathrm{fexc}(\sigma,c) = \mathrm{fexc}(I(\sigma,c))$.

Proof. This is immediate, since the letters with color 0 are unchanged by I (and the sum of the colors is also preserved). ☐

Lemma 28. $\lceil \mathrm{fdes}(\sigma,c)/r\rceil = \lceil \mathrm{fdes}^*(I(\sigma,c))/r\rceil$.

Proof. We compute $\lceil \mathrm{fdes}(\sigma,c)/r\rceil$ on one side and $\lceil \mathrm{fdes}^*(I(\sigma,c))/r\rceil$ and the other side, by examining the different contributions to each quantity.

First, each pair of letters $(\sigma_i,c_i)(\sigma_{i+1},c_{i+1})$ where $\sigma_i > \sigma_{i+1}$ inside a given block B_j contribute by 1 to each side (since $c_i = c_{i+1}$ by definition of the blocks). It remains to consider the term $r\times\#\{i : c_i > c_{i+1}\} + c_n$ in the definition of fdes, and the term $r\times\#\{i : c_i < c_{i+1}\} + c_1$ in the definition of fdes*.

Let us write $B_i > B_{i+1}$ or $B_i < B_{i+1}$ to mean that the color of the block B_i is greater or smaller than that of B_{i+1} (by definition they cannot be equal). Let $j<k$ be such that

B_j and B_k are zero colored blocks, but $B_{j+1}, \dots, B_{k-1}$ are not. In the factor $B_j \dots B_k$ of (σ, c), there is a contribution

$$\#\{i \,:\, j \leqslant i < k \text{ and } B_i > B_{i+1}\}$$

to $\lceil \mathrm{fdes}(\sigma, c)/r \rceil$. But in the factor $B_j B_{k-1} \dots B_{j+1} B_k$ of $I(\sigma, c)$, there is the same contribution to $\lceil \mathrm{fdes}^*(I(\sigma, c))/r \rceil$.

Now, let B_j be the first zero colored block of (σ, c). If $j > 1$, the prefix $B_1 \dots B_j$ of (σ, c) contributes by

$$\#\{i \,:\, 1 \leqslant i < j \text{ and } B_i > B_{i+1}\}$$

to $\lceil \mathrm{fdes}(\sigma, c)/r \rceil$, and the prefix $B_{j-1} \dots B_1 B_j$ of $I(\sigma, c)$ contributes by

$$1 + \#\{i \,:\, j - 1 > i \geqslant 1 \text{ and } B_{i+1} < B_i\}$$

to $\lceil \mathrm{fdes}^*(I(\sigma, c))/r \rceil$ (the 1 come from the term c_1 in the definition of fdes^* since B_{j-1} is a nonzero colored block). The two numbers are easily seen to be equal.

Similarly, let B_k be the last zero colored block of (σ, c). If $k < m$, the suffix $B_k \dots B_m$ of (σ, c) contributes by

$$1 + \#\{i \,:\, k \leqslant i < m \text{ and } B_i > B_{i+1}\}$$

to $\lceil \mathrm{fdes}(\sigma, c)/r \rceil$ (the 1 come from the term c_n in the definition of fdes since B_m is a nonzero colored block), and the suffix $B_k B_m \dots B_{k+1}$ of $I(\sigma, c)$ contributes by

$$1 + \#\{i \,:\, m > i \geqslant k + 1 \text{ and } B_{i+1} < B_i\}$$

to $\lceil \mathrm{fdes}^*(I(\sigma, c))/r \rceil$ (the 1 come from the fact that $B_k < B_m$). The two numbers are easily seen to be equal.

Checking the respective definitions of fdes and fdes^*, we can see that what we have counted proves the proposition. $\square$

8 Formulas for the Ehrhart polynomials

Theorem 29. *The Ehrhart polynomial of $\mathcal{A}_{n,k}^{(r)}$ is:*

$$\sum_{j=0}^{\lfloor (k-1)/r \rfloor} (-1)^{j+1} \binom{n}{j} \binom{n - rtj + kt - t - 1}{n} - \sum_{j=0}^{\lfloor k/r \rfloor} (-1)^{j+1} \binom{n}{j} \binom{n - rtj + kt - 1}{n}.$$

Proof. Let CT_q denote the operator that gives the constant term of a Laurent series in q. We have:

$$\#\big(\mathbb{Z}^n \cap t\mathcal{A}_{n,k}^{(r)}\big) = \#\big\{v \in \{0, 1, \dots, rt - 1\}^n \,:\, kt - t \leqslant \sum v_i < kt\big\}$$

$$= \mathrm{CT}_q\Big([rt]_q^n([kt]_{q^{-1}} - [kt - t]_{q^{-1}})\Big)$$

$$= \mathrm{CT}_q\left(\frac{(1-q^{rt})^n(q^{-kt+t-1}-q^{-kt-1})}{(1-q)^n(1-q^{-1})}\right)$$

$$= \mathrm{CT}_q\left(\sum_{j=0}^{n}\binom{n}{j}(-1)^{j+1}q^{rtj+j}\frac{(q^{-kt+t}-q^{-kt})}{(1-q)^{n+1}}\right)$$

$$= \mathrm{CT}_q\left(\sum_{j=0}^{n}\sum_{i\geqslant 0}\binom{n}{j}\binom{n+i}{n}(-1)^{j+1}q^{rtj+j+i}(q^{-kt+t}-q^{-kt})\right)$$

$$= \sum_{j=0}^{n}\binom{n}{j}(-1)^{j+1}\left(\binom{n+kt+t-(rt+1)j}{n}-\binom{n+kt-(rt+1)j}{n}\right)$$

with the (unusual) convention that $\binom{n}{k}=0$ when $n<0$. With this convention, it is not clear that we have a polynomial in t. But we can improve the formula by keeping only some of the indices j, those appearing in the announced formula. Indeed both formulas are equal for large t, hence for every t since these are polynomials. ☐

Theorem 30. *The Ehrhart polynomial of $\mathcal{B}^{(r)}_{n,k}$ is:*

$$\sum_{j=0}^{\lfloor (k-1)/r\rfloor}(-1)^{j+1}\tbinom{n}{j}\tbinom{n-rtj-j+kt-t-1}{n}-\sum_{j=0}^{\lfloor k/r\rfloor}(-1)^{j+1}\tbinom{n}{j}\tbinom{n-rtj-j+kt-1}{n}.$$

Proof. This is similar to the previous proposition:

$$\#\big(\mathbb{Z}^n\cap t\mathcal{B}^{(r)}_{n,k}\big)=\#\big\{v\in\{0,1,\dots,rt\}^n\ :\ kt-t\leqslant\sum v_i<kt\big\}$$

$$= \mathrm{CT}_q\left([rt+1]_q^n([kt]_{q^{-1}}-[kt-t]_{q^{-1}})\right)$$

$$= \mathrm{CT}_q\left(\frac{(1-q^{rt+1})^n(q^{-kt+t}-q^{-kt})}{(1-q)^n(1-q^{-1})}\right)$$

$$= \mathrm{CT}_q\left(\sum_{j=0}^{n}\binom{n}{j}(-1)^{j+1}q^{rtj+j}\frac{(q^{-kt+t+1}-q^{-kt+1})}{(1-q)^{n+1}}\right)$$

$$= \mathrm{CT}_q\left(\sum_{j=0}^{n}\sum_{i\geqslant 0}\binom{n}{j}\binom{n+i}{n}(-1)^{j+1}q^{rtj+j+i}(q^{-kt+t+1}-q^{-kt+1})\right)$$

$$= \sum_{j=0}^{n}\binom{n}{j}(-1)^{j+1}\left(\binom{n-rtj-j+kt-t-1}{n}-\binom{n-rtj-j+kt-1}{n}\right).$$

As in the previous case, the formula is obtained with the convention that $\binom{n}{k}=0$ when $n<0$, but is true in general. ☐

We also obtain a formula for the flag Eulerian numbers.

Theorem 31. *The flag Eulerian number is:*

$$A_{n,k}^{(r)} = \sum_{j=0}^{\lfloor (k-1)/r \rfloor} \binom{n}{j} (-1)^{j+1} (k-rj-1)^n - \sum_{j=0}^{\lfloor k/r \rfloor} \binom{n}{j} (-1)^{j+1} (k-rj)^n.$$

Proof. The number $\mathrm{Vol}(\mathcal{A}_{n,k}^{(r)}) = \frac{1}{n!} A_{n,k}^{(r)}$ can be obtained as the dominant coefficient of the Ehrhart polynomial. From the exact formula we have just obtained, this dominant coefficient is:

$$\sum_{j=0}^{\lfloor (k-1)/r \rfloor} \binom{n}{j} \frac{(-1)^{j+1}}{n!} (k-rj-1)^n - \sum_{j=0}^{\lfloor k/r \rfloor} \binom{n}{j} \frac{(-1)^{j+1}}{n!} (k-rj)^n.$$

This is the announced formula up to the normalization factor $n!$. This could also be obtained from (11). □

Note that the particular case $r = 1$ gives a well-known formula:

$$\begin{aligned} A_{n,k} &= \sum_{j=0}^{k} \binom{n}{j} (-1)^{j+1} ((k-j)^n - (k-j+1)^n) \\ &= \sum_{j=1}^{k+1} \binom{n}{j-1} (-1)^{j} (k-j+1)^n - \sum_{j=0}^{k} \binom{n}{j} (-1)^{j+1} (k-j+1)^n \\ &= \sum_{j=0}^{k} \binom{n+1}{j} (-1)^{j} (k-j+1)^n. \end{aligned}$$

9 Bijective problems

In this article we have obtained a combinatorial interpretation of $E^*(\mathcal{B}_{n,k}^{(r)}, z)$ which differs from the one previously obtained by Li [10]. It would be interesting to have a bijective proof that the two results are equivalent. For convenience, let us state Li's result here. We make the convention that $\sigma(0) = \sigma^{-1}(0) = 0$ for each permutation $\sigma \in \mathfrak{S}_n$.

Definition 32 (Li [10])**.** The statistic $\mathrm{cover}(\sigma)$ of a permutation σ is defined by

$$\mathrm{cover}(\sigma) := \#\big\{ i \,:\, 1 \leqslant i \leqslant n \text{ and } \sigma^{-1}(i-1) + 1 < \sigma^{-1}(i) \big\},$$

and the statistic $\mathrm{cef}(\sigma, c)$ of a colored permutation (σ, c) is defined by

$$\mathrm{cef}(\sigma, c) := \#\big\{ i \,:\, 1 \leqslant i \leqslant n,\ c_i > 0, \text{ and } \sigma(i-1) + 1 = \sigma(i) \big\}.$$

Although we have used slightly different conventions, it is easily seen that Theorem 7.3 from [10] can be stated as follows.

Theorem 33 (Li [10]). *We have:*

$$E^*(\mathcal{B}_{n,k}^{(r)}, z) = \sum_{\substack{(\sigma,c)\in\mathfrak{S}_n^{(r)}\\ \mathrm{cdes}(\sigma,c)=rn-k}} z^{\mathrm{cover}(\sigma)+\mathrm{cef}(\sigma,c)}.$$

From Theorem 18 and Theorem 33, we obtain the equality

$$\sum_{\substack{(\sigma,c)\in\mathfrak{S}_n^{(r)}\\ \mathrm{fexc}(\sigma,c)=k}} z^{\lceil \mathrm{fdes}(\sigma,c)/r\rceil} = \sum_{\substack{(\sigma,c)\in\mathfrak{S}_n^{(r)}\\ \mathrm{cdes}(\sigma,c)=k}} z^{\mathrm{cover}(\sigma)+\mathrm{cef}(\sigma,c)},$$

which appeals for a bijective proof. Some numerical values are given in Figure 1. The case $r = 1$ might be already an interesting problem to begin with.

The second problem is to find a bijective or combinatorial proof of Theorem 24, i.e., of the relation:

$$e^{zy^r x}\left(1+\sum_{n\geqslant 1}\sum_{(\sigma,c)\in\mathfrak{S}_n^{(r)}} y^{rn-\mathrm{fexc}(\sigma,c)} z^{\lceil \mathrm{fdes}(\sigma,c)/r\rceil}\frac{x^n}{n!}\right)$$
$$= e^{y^r x}\left(1+\sum_{n\geqslant 1}\sum_{(\sigma,c)\in\mathfrak{S}_n^{(r)}} y^{\mathrm{fdes}(\sigma,c)+1} z^{\mathrm{des}(\sigma^{-1})+1}\frac{x^n}{n!}\right).$$

Acknowledgement

We thank Dominique Foata for his corrections and comments on this article.

References

[1] M. Beck and S. Robins: Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra. Springer-Verlag, New York, 2007.

[2] E. Bagno and R. Biagioli: Colored-descent representations for complex reflection groups. Israel J. Math. 160 (2007), 317–348.

[3] E. Bagno and D. Garber: On the excedance number of colored permutation groups. Sém. Lothar. Combin. 53 (2006), Article B53f.

[4] H.S.M. Coxeter: Regular polytopes. Dover, New York, 1973.

[5] D. Foata and G.-N. Han: The decrease value theorem with an application to permutation statistics. Adv. in Appl. Math. 46 (2011), 296–311.

[6] D. Foata: Distributions eulériennes et mahoniennes sur le groupe des permutations. Higher Combinatorics (M. Aigner, ed.), Reidel, Dordrecht/Boston, 1977, pp. 27–49.

[7] A. Garsia and I. Gessel: Permutation statistics and partitions. Adv. Math. 31 (1979), 288–305.

$r = 1$

$k\backslash n$	1	2	3	4	5	6	7
1	1	z	z	z	z	z	z
2	.	1	z^2+3z	z^3+6z^2+4z	$6z^3+15z^2+5z$	$z^4+21z^3+29z^2+6z$	$8z^4+56z^3+49z^2+7z$
3	.	.	1	$5z^2+6z$	$z^4+19z^3+36z^2+10z$	$z^5+34z^4+141z^3+111z^2+15z$	$29z^5+300z^4+582z^3+259z^2+21z$
4	.	.	.	1	z^3+15z^2+10z	$22z^4+133z^3+127z^2+20z$	$z^6+83z^5+673z^4+1154z^3+470z^2+35z$
5	.	.	.	.	1	$7z^3+35z^2+15z$	$8z^5+209z^4+596z^3+343z^2+35z$
6	.	.	.	.	.	1	$z^4+28z^3+70z^2+21z$
7	.	.	.	.	.	.	1

$r = 2$

$k\backslash n$	1	2	3	4	5	6
1	z	z	z	z	z	z
2	1	z^2+2z	$4z^2+3z$	z^3+10z^2+4z	$6z^3+20z^2+5z$	$z^4+21z^3+35z^2+6z$
3	.	$3z$	z^3+9z^2+6z	$14z^3+37z^2+10z$	$6z^4+84z^3+101z^2+15z$	$z^5+76z^4+336z^3+225z^2+21z$
4	.	1	$9z^2+7z$	$z^4+30z^3+68z^2+16z$	$45z^4+291z^3+260z^2+30z$	$27z^5+609z^4+1561z^3+742z^2+50z$
5	.	.	z^2+6z	$26z^3+70z^2+19z$	$z^5+90z^4+505z^3+415z^2+45z$	$140z^5+1883z^4+3874z^3+1567z^2+90z$
6	.	.	1	$5z^3+40z^2+16z$	$75z^4+490z^3+440z^2+51z$	$z^6+260z^5+3095z^4+5982z^3+2310z^2+126z$
7	.	.	.	$5z^2+10z$	$20z^4+256z^3+305z^2+45z$	$216z^5+2926z^4+6041z^3+2450z^2+141z$
8	.	.	.	1	$z^4+49z^3+126z^2+30z$	$71z^5+1546z^4+3941z^3+1870z^2+126z$
9	.	.	.	.	z^3+15z^2+15z	$7z^5+379z^4+1526z^3+987z^2+90z$
10	.	.	.	.	1	$28z^4+259z^3+322z^2+50z$
11	.	.	.	.	.	$7z^3+35z^2+21z$
12	.	.	.	.	.	1

$r = 3$

$k\backslash n$	1	2	3	4
1	z	z	z	z
2	z	z^2+2z	$4z^2+3z$	z^3+10z^2+4z
3	1	$2z^2+3z$	z^3+12z^2+6z	$14z^3+41z^2+10z$
4	.	z^2+4z	$3z^3+21z^2+10z$	$z^4+50z^3+100z^2+20z$
5	.	$3z$	$3z^3+25z^2+12z$	$4z^4+100z^3+174z^2+31z$
6	.	1	z^3+21z^2+12z	$6z^4+135z^3+230z^2+40z$
7	.	.	$9z^2+10z$	$4z^4+128z^3+235z^2+44z$
8	.	.	z^2+6z	$z^4+82z^3+186z^2+40z$
9	.	.	1	$30z^3+110z^2+31z$
10	.	.	.	$5z^3+40z^2+20z$
11	.	.	.	$5z^2+10z$
12	.	.	.	1

$r = 4$

$k\backslash n$	1	2	3
1	z	z	z
2	z	z^2+2z	$4z^2+3z$
3	z	$2z^2+3z$	z^3+12z^2+6z
4	1	$3z^2+4z$	$3z^3+24z^2+10z$
5	.	$2z^2+5z$	$6z^3+37z^2+15z$
6	.	z^2+4z	$7z^3+45z^2+18z$
7	.	$3z$	$6z^3+45z^2+19z$
8	.	1	$3z^3+37z^2+18z$
9	.	.	z^3+21z^2+15z
10	.	.	$9z^2+10z$
11	.	.	z^2+6z
12	.	.	1

Figure 1: Some values of $E^*(\mathcal{B}^{(r)}_{n,k}, z)$

[8] T. Hibi, N. Li and H. Ohsugi: The face vector of a half-open hypersimplex. J. Integer Seq. 18(6) (2015), Article 15.6.6, 12 pp.

[9] T. Lam and A. Postnikov: Alcoved Polytopes I. Discrete Comput. Geom. 38(3) (2007), 453–478.

[10] N. Li: Ehrhart h^*-vectors of hypersimplices. Discrete Comput. Geom. 48(4) (2012), 847–878.

[11] H. Ohsugi and K. Shibata: Roots of the Ehrhart polynomial of hypersimplices. Preprint (2013). arXiv:1304.7587v1

[12] R.P. Stanley: Eulerian partitions of a unit hypercube. Higher Combinatorics (M. Aigner, ed.), Reidel, Dordrecht/Boston, 1977, p. 49.

[13] R.P. Stanley: Decompositions of rational convex polytopes. Ann. Discrete Math. 6 (1980), 333–342.

[14] R.P. Stanley: Enumerative combinatorics, Volume 2. Cambridge University Press, 1999.

[15] E. Steingrímsson: Permutation statistics of indexed permutations. European J. Combin. 15(2) (1994), 187–205.