Chip-firing game and a partial Tutte polynomial for Eulerian digraphs

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Abstract

The Chip-firing game is a discrete dynamical system played on a graph, in which chips move along edges according to a simple local rule. Properties of the underlying graph are of course useful to the understanding of the game, but since a conjecture of Biggs that was proved by Merino López, we also know that the study of the Chip-firing game can give insights on the graph. In particular, a strong relation between the partial Tutte polynomial $T_G(1, y)$ and the set of recurrent configurations of a Chip-firing game (with a distinguished sink vertex) has been established for undirected graphs. A direct consequence is that the generating function of the set of recurrent configurations is independent of the choice of the sink for the game, as it characterizes the underlying graph itself. In this paper we prove that this property also holds for Eulerian directed graphs (digraphs), a class on the way from undirected graphs to general digraphs. It turns out from this property that the generating function of the set of recurrent configurations of an Eulerian digraph is a natural and convincing candidate for generalizing the partial Tutte polynomial

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$T_G(1, y)$ to this class. Our work also gives some promising directions of looking for a generalization of the Tutte polynomial to general digraphs.

Keywords. Chip-firing game, critical configuration, Eulerian digraph, feedback arc set, recurrent configuration, reliability polynomial, Sandpile model, Tutte polynomial.

1 Introduction

The Tutte polynomial is an important polynomial in two variables which is defined on undirected graphs [26]. It has many interesting properties and applications. Its evaluations at some specific points count various combinatorial objects such as vertex colorings, spanning trees, spanning subgraphs, acyclic orientations, etc [5, 12, 14, 15, 17, 25]. There is an evident interest in looking for analogues of the Tutte polynomial for directed graphs (digraphs) and some other objects [7, 11, 13]. These attempts share properties of the Tutte polynomial. Nevertheless, they are not natural extensions of the Tutte polynomial in the sense that one does not know a conversion from the properties of these polynomials to those of the Tutte polynomial, in particular how to get back to the Tutte polynomial on undirected graphs from these polynomials. For this reason the authors of [7] asked for a natural generalization of Tutte polynomial for digraphs.

The evaluation of the Tutte polynomial $T_G(x, y)$ at $x = 1$ is important since it has a strong connection to the reliability polynomial that is studied in network theory. In this paper we present a polynomial that can be considered as a natural generalization of $T_G(1, y)$ for the class of Eulerian digraphs. An Eulerian digraph is a strongly connected digraph in which each vertex has equal in-degree and out-degree, or equivalently there is a closed path covering all arcs. A connected undirected graph can be regarded as an Eulerian digraph by replacing each edge e by two reverse arcs a and b which have the same endpoints as e. When considering undirected graphs seen as Eulerian digraphs in that way, we will see that we get back to the partial Tutte polynomial $T_G(1, y)$, which is a new and relevant feature.

This work is based on an idea conjectured by Biggs and proved by Merino López, that the generating function of the set of recurrent configurations of the Chip-firing game of an undirected graph is equal to the partial Tutte polynomial $T_G(1, y)$ [1, 19]. Based on a discrete dynamical system, this construction defines a polynomial that characterizes the graph supporting the dynamic. It is not straightforward to generalize those ideas to the class of Eulerian digraphs, but the results we will develop give a promising direction for further extensions.

The Chip-firing game is a discrete dynamical system which is defined on a directed graph (digraph) G. The game is initialized by some chips stored on the vertices of G. If a vertex has at least one out-going arc and has as many chips as its out-degree, it can distribute its chips to its neighbors by sending one chip along each out-going arc to the corresponding vertex. The game playing with this rule is called *Chip-firing game (CFG)*. A distribution of chips on vertices is called a *configuration* of G. Stabilizing a configuration is the process of repeatedly applying the rule whenever it is applicable. This process may be infinite. If the process is finite, we call the game *convergent*. It is known that the game either plays forever or converges to a unique stable configuration, in which the rule is no longer applicable [3, 4, 16]. The unique stable configuration (if it exists) is called *stabilization* of the game. In this paper we are interested in the digraphs with a global sink, *i.e.* a vertex s of out-degree 0 and for any other vertex v there is a path from v to s. The game on such digraphs is always convergent for any initial configuration. For a strongly connected digraph (without global sink) the game plays forever if the number of chips in the initial configuration is sufficient, but we can make the game always convergent by choosing a vertex s and removing all out-going arcs of s. The vertex s becomes a global sink of the new graph. The game now is played on that new graph, and therefore it is always convergent. In this paper we study properties of such games, which are independent of the choice of s, and provide clues to define a natural analogue of the Tutte polynomial, for the class of Eulerian digraphs.

The Dollar game is a variant of CFG on undirected graphs in which a particular vertex is called *sink*, and this vertex can only be fired if all other vertices are *stable* (not firable) [2]. In this model the number of chips stored in the sink may be negative. This definition leads naturally to the notion of *recurrent configurations* (originally called *critical configurations*) that are stable, and unchanged under firing at the sink and taking the stabilization of the resulting configuration. The Dollar game can be defined for Eulerian digraphs with the same definition, *i.e.* some vertex is chosen to be the sink that can only be fired if all other vertices are stable [16]. Although the notion of recurrent configurations is originally defined on the Dollar game, it can be defined in an equivalent way on the Chip-firing game. Since we work only with the Chip-firing game throughout this paper, we will use the Chip-firing game to define recurrent configurations. This important notion has a generalization to the digraphs with a global sink [16].

The set of recurrent configurations of a CFG with a sink on an undirected graph has many interesting properties, such as it is an Abelian group with the addition defined by vertex-wise addition of chip content followed by stabilization, and its cardinality is equal to the number of spanning trees of the support graph, *etc.* A direct consequence of the result of Merino López for proving Biggs' conjecture is that the generating function of recurrent configurations of a CFG with a sink is independent of the chosen sink, and thus characterizes the support graph. This fact is definitely not trivial, and opened a new direction for studying graphs using the Chip-firing game as a tool [8, 20].

A lot of properties of recurrent configurations on undirected graphs can be extended to Eulerian digraphs without difficulty. However the situation is different when one tries to extend the sink-independence property of the generating function to a larger class of graphs, in particular to Eulerian digraphs, mainly because a natural definition of the Tutte polynomial for digraphs is unknown, even for Eulerian digraphs. In this paper we develop a combinatorial approach, based on a level-preserving bijection between two sets of recurrent configurations with respect to two different sinks, to show that this sink-independence property also holds for Eulerian digraphs. This bijection provides new insights into the groups of recurrent configurations.

It turns out from the sink-independence property of the generating function, that this latter is a characteristic of the support Eulerian digraph, and we can denote it by $\mathcal{T}_G(y)$ regardless of the sink. We will see that evaluations of $\mathcal{T}_G(y)$ can be considered as extensions of $T_G(1, y)$ to Eulerian digraphs, which make us believe that the polynomial $\mathcal{T}_G(y)$ is a natural generalization of $T_G(1, y)$. Furthermore, the most important feature is that $\mathcal{T}_G(y)$ and $T_G(1, y)$ are equal on undirected graphs. It requires to be inventive to discover which objects the evaluations of $\mathcal{T}_G(y)$ counts, and we hope that further properties will be found. The class of Eulerian digraphs is in-between undirected and directed graph, and following the track we develop in this paper, we propose some conjectures that would be promising directions of looking for a natural generalization of $T_G(x, y)$ to general digraphs.

The paper is divided into the following sections. Section 2 recalls the definition of the Chip-firing game on digraphs and some known results about the recurrent configurations on a digraph with global sink and on an Eulerian digraph with a sink. Section 3 establishes the sink-independence of the generating function of recurrent configurations in the case of Eulerian digraphs. The Tutte polynomial generalization is presented in Section 4, and Section 5 hints at continuations of the present work.

2 Chip-firing game and recurrent configurations

All graphs in this paper are assumed to be multi-digraphs without loops. Throughout this paper an undirected graph is regarded as a digraph by replacing each edge e by two reverse arcs that have the same endpoints as e. Graphs with loops will be considered in Section 4. In this section we recall the definition of the Chip-firing game and present some known results about recurrent configurations of a CFG on a graph with a global sink and on an Eulerian graph with a sink, followed by straightforward considerations on the number of chips stored on vertices of recurrent configurations. All graphs in this section are assumed to be connected.

2.1 Chip-firing game

For a digraph $G = (V, A)$ and an arc $e \in A$, we denote by e^- and e^+ the tail and head of e, respectively. For a vertex v let $\deg^+_G(v)$ denote the number of arcs e with $e^- = v$ and $\deg^-_G(v)$ denote the number of arcs e with $e^+ = v$. For two vertices $v, w \in V$, let $\deg_G(v, w)$ denote the number of arcs from v to w in G. A configuration c on G is a map from V to $\mathbb{N} = \{0, 1, 2, \dots\}$. A vertex v is *firable* in c if and only if $c(v) \geqslant \deg^+_G(v) > 0$. A configuration c is called *stable* if there is no firable vertex in c. Firing a firable vertex v is the process that decreases $c(v)$ by $\deg^+_G(v)$ and increases each $c(w)$ with $w \neq v$ by $\deg_G(v, w)$. A sequence $(v_1, v_2, \cdots, v_k)$ of vertices of G is called a *firing sequence* of a configuration c if starting from c we can consecutively fire the vertices $v_1, v_2, \cdots, v_k$ in this order. Applying the firing sequence leads to configuration d and we write $c \xrightarrow{v_1, v_2, \cdots, v_k} d$, or $c \xrightarrow{*} d$ without specifying the firing sequence. A sink is a vertex of out-degree 0. A vertex s is called *global sink* if it is a sink and from every other vertex there is a directed path leading to s. Note that if G has a global sink then the sink is unique.

Lemma 1. *[3, 4, 16] Suppose that G has a global sink. For any initial configuration c the game converges to a unique stable configuration, denoted by $\bar{c}$. Let $\mathfrak{f}$ and $\mathfrak{g}$ be two firing sequences of c such that $c \xrightarrow{\mathfrak{f}} \bar{c}$ and $c \xrightarrow{\mathfrak{g}} \bar{c}$. Then for any vertex v the number of times v occurs in $\mathfrak{f}$ is the same as in $\mathfrak{g}$.*

The configuration $\bar{c}$ in the lemma is called the *stabilization* of c.

2.2 Recurrent configurations on a digraph with a global sink

The following is simple but very important, and will often be used without explicit reference.

Lemma 2. *Let $G = (V, A)$ be a graph with a global sink. For two configurations c and d, we denote by $c + d$ the configuration given by $(c + d)(v) = c(v) + d(v)$ for any $v \in V$. Then $\overline{c + d} = \overline{\bar{c} + d}$.*

For two sets X and Y we denote by Y^X the set of maps from X to Y. In the rest of this subsection we work with a graph $G = (V, A)$ which has a global sink s. The definition of recurrent configurations is based on the convergence of the game, which is ensured if G has a global sink. Since s is not firable no matter how many chips it has, it makes sense to define a configuration to be an element in $\mathbb{N}^{V \setminus \{s\}}$. When a chip goes into the sink, it vanishes. Two configurations are considered to be the same if they have the same number of chips on every vertex except for the sink. Note that in this section we consider only one fixed sink, but in subsequent sections we will consider the CFG relative to different choices of sink, and therefore we will need some more notations. Let us not be overburdened yet, a configuration on G with sink s is an element in $\mathbb{N}^{V \setminus \{s\}}$.

Definition. A stable configuration c is *recurrent* if and only if for any configuration a there is a configuration b such that $c = \overline{a + b}$.

There are several equivalent definitions of *recurrent* configurations. For an undirected graph G (regarded as a digraph) with a particular vertex s, which is called a *sink*, let H be the graph which is obtained from G by removing all out-going arcs of s. Then recurrent configurations of H can be defined by the firing rule of the Dollar game on G as follows. A configuration is recurrent if it remains unchanged under firing the sink and stabilizing the resulting configuration [2, 10]. For a graph G with global sink, recurrent configurations can be defined naturally by using the notion of recurrent states of the Markov chain which is defined as follows. The state space is the set of all stable configurations of G. When at a state c, the next state is obtained by adding one chip to c at some vertex (distinct from the sink) chosen uniformly at random, and then stabilizing the resulting configuration. The recurrent configurations can be defined to be the recurrent states of this Markov chain [9, 10, 18]. The one we present in this paper says that c is recurrent if and only if it can be reached from any other configuration a by adding some chips (according to b) and then stabilize the resulting configuration.

Dhar proved that the set of recurrent configurations has an elegant algebraic structure [9]. Fix a linear order $v_1 \prec v_2 \prec \cdots \prec v_{n-1} \prec v_n$ on the vertices, where $n = |V|$. We suppose that $v_n = s$. Now a configuration of G can be represented as a vector in $\mathbb{Z}^{n-1}$. The Laplacian matrix Δ of G is an $n \times n$ matrix which is defined by

$$\Delta_{ij} = \begin{cases} -\deg_G(v_i, v_j,) & \text{if } i \neq j \\ \deg_G^+(v_i) & \text{if } i = j \end{cases}.$$

Note that the Laplacian matrix of a general graph (not necessarily having a global sink) is defined by the same formula as above. Let Λ be the matrix which is obtained from Δ by deleting the last row and column of Δ. Let Λ_i denote the ith row of Λ. Firing the index i in a configuration c corresponds to adding the vector $-\Lambda_i$ to c. We define a binary relation $\sim$ over $\mathbb{Z}^{n-1}$ by $b \sim c$ if and only if there exist $a_1, a_2, \cdots, a_{n-1} \in \mathbb{Z}$ such that $b - c = \sum_{1 \leqslant i \leqslant n-1} a_i \Lambda_i$, *i.e.* b and c are linked by a (possibly impossible to perform) sequence of firings. The following result states the nice algebraic structure of the set of all recurrent configurations of G with sink s.

Lemma 3. *[16] The set of all recurrent configurations of G is an Abelian group under the operation $(a, b) \mapsto \overline{a + b}$. This group is isomorphic to $\mathbb{Z}^{n-1}/\langle \Lambda_1, \Lambda_2, \cdots, \Lambda_{n-1} \rangle$. Moreover, each equivalence class of $\mathbb{Z}^{n-1}/\sim$ contains exactly one recurrent configuration, and the number of recurrent configurations is equal to the number of equivalence classes.*

The group in Lemma 3 is called the *Sandpile group* of G. For two functions $f, g : X \to \mathbb{R}$ we write $f \leqslant g$ if $f(x) \leqslant g(x)$ for every $x \in X$. The following simple properties can be derived easily from the definition of recurrent configuration.

Lemma 4. *The following holds:*

1. *Let c be a configuration such that $c(v) \geqslant \deg_G^+(v) - 1$ for every $v \neq s$. Then $\overline{c}$ is recurrent.*

2. *Let c and d be two configurations (elements of $\mathbb{N}^{V \setminus \{s\}}$) such that $c \leqslant d$. Then*

$$\sum_{v \neq s} c(v) - \sum_{v \neq s} \overline{c}(v) \leqslant \sum_{v \neq s} d(v) - \sum_{v \neq s} \overline{d}(v).$$

Moreover, if $\overline{c}$ is recurrent then $\overline{d}$ is also recurrent.

Proof.

1. For any configuration a let $b = c - \overline{a}$. Clearly, b is a configuration. We have $\overline{a + b} = \overline{a + c - \overline{a}} = \overline{\overline{a} + c - \overline{a}} = \overline{c}$, therefore $\overline{c}$ is recurrent.

2. Let $\mathfrak{f} = (v_1, v_2, \cdots, v_k)$ be a firing sequence of c such that $c \xrightarrow{\mathfrak{f}} \overline{c}$. Since $\sum_{v \neq s} c(v) - \sum_{v \neq s} \overline{c}(v)$ is the number of chips lost into the sink, we have

$$\sum_{v \neq s} c(v) - \sum_{v \neq s} \overline{c}(v) = \sum_{1 \leqslant i \leqslant k} \deg_G(v_i, s).$$

Since $c(v) \leqslant d(v)$ for any $v \neq s$, $\mathfrak{f}$ is also a firing sequence of d. Therefore there is a firing sequence $\mathfrak{g} = (v_1, v_2, \cdots, v_k, v_{k+1}, v_{k+2}, \cdots, v_l)$ of d such that $d \xrightarrow{\mathfrak{g}} \overline{d}$. For the same reason we have

$$\sum_{v \neq s} d(v) - \sum_{v \neq s} \overline{d}(v) = \sum_{1 \leqslant i \leqslant l} \deg_G(v_i, s).$$

The first claim follows.

Let a be an arbitrary configuration. Since $\overline{c}$ is recurrent, there is a configuration b such that $\overline{a+b} = \overline{c}$. Let $e = b + d - c$. We have

$$\overline{a+e} = \overline{a+b+d-c} = \overline{\overline{a+b}+d-c} = \overline{\overline{c}+d-c} = \overline{c+d-c} = \overline{d},$$

thus $\overline{d}$ is recurrent. □

2.3 Recurrent configurations of an Eulerian digraph with a sink

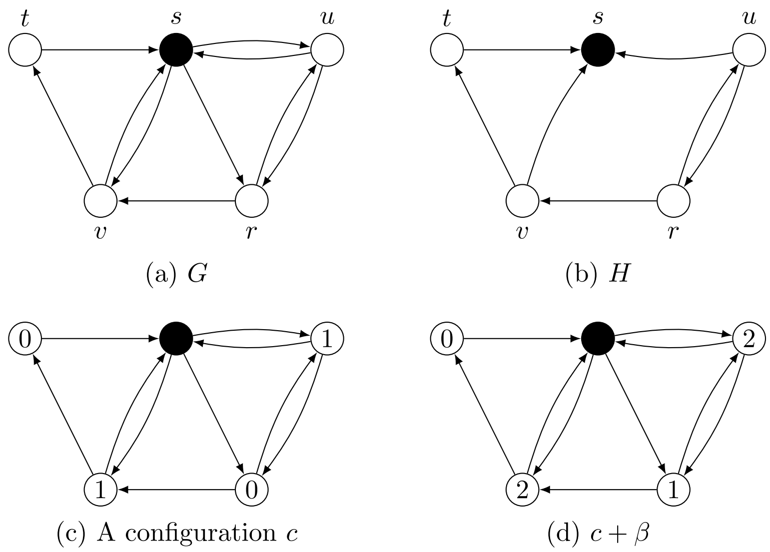
In this subsection we work with an Eulerian graph G and present properties that recurrent configurations have in that case. As in the previous subsection, the definition of recurrent configuration is based on the convergence of the game. Therefore a global sink plays an important role in the definition. The digraph G is strongly connected, therefore it has no global sink and the game may play forever from some initial configurations. To overcome this issue, we distinguish a particular vertex s of G that plays the role of the sink. By removing all outgoing arcs of s from G we obtain the digraph H that has a global sink s. The Chip-firing game on G with sink s is the ordinary Chip-firing game that is defined on H, and recurrent configurations are defined as presented in the previous subsection, on H. Figures 1a and 1b present an example of G and H. It is a good way to think of the Chip-firing game on an Eulerian digraph with a sink as the ordinary Chip-firing game on G with a fixed vertex that never fires in the game no matter how many chips it has. In this section we will work with G, s and H.

A configuration of the Chip-firing game on G with sink s is a map in $\mathbb{N}^{V\setminus\{s\}}$. To verify the recurrence of a configuration c, we have to test the condition that for any configuration a there is a configuration b such that $\overline{a+b} = c$. This is a tiresome task. However, in the case of Eulerian digraphs we have the following useful criterion.

Lemma 5. *[9, 16] A configuration c is recurrent if and only if $\overline{c+\beta} = c$, where β is the configuration defined by $\beta(v) = \deg_G(s, v)$ for every $v \neq s$. Moreover, if c is recurrent then each vertex distinct from s occurs exactly once in any firing sequence $\mathfrak{f}$ from $c+\beta$ to $\overline{c+\beta}$.*

Figure 1c presents a configuration c. The configuration $c + \beta$ is presented in Figure 1d, adding β corresponds to firing the sink. To verify the recurrence of c, one computes $\overline{c+\beta}$. Starting with $c+\beta$ we fire consecutively the vertices v_1, v_3, v_2, v_4 in this order and get exactly the configuration c, therefore c is recurrent. This procedure is called *Burning algorithm.*

For two integers p, q we denote by $[p..q]$ the set $\{x \in \mathbb{Z} : p \leqslant x \leqslant q\}$. The following lemma will be important later.



(a) G (b) H

(c) A configuration c (d) $c+\beta$

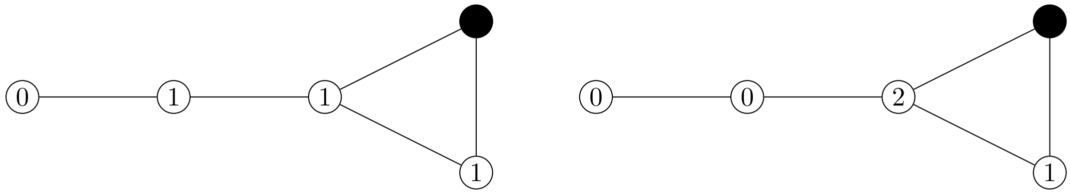
Figure 1: Burning algorithm

Lemma 6. *Let c and d be two stable configurations such that c and d are in the same equivalence class. If c is recurrent then $\sum\limits_{v\neq s} d(v) \leqslant \sum\limits_{v\neq s} c(v)$.*

Proof. Let β be the configuration that is defined as in Lemma 5 and a be an arbitrary stable configuration. We claim that for any firing sequence $\mathfrak{f} = (v_1, v_2, \cdots, v_k)$ of $a+\beta$ such that $a+\beta \xrightarrow{\mathfrak{f}} \overline{a+\beta}$, each vertex of G occurs at most once in $\mathfrak{f}$. For a contradiction we assume otherwise. This assumption implies that there is a first repetition, *i.e.*, there is $p \in [1..k]$ such that $v_1, v_2, \cdots, v_{p-1}$ are pairwise-distinct and $v_p = v_q$ for some $q \in [1..p-1]$. We denote b the configuration obtained from $a+\beta$ after the vertices $v_1, v_2, \cdots, v_{p-1}$ have been fired. We will now show that v_p is not firable in b, a contradiction. Let r be the number of chips v_p has received from its in-neighbors when the vertices $v_1, v_2, \cdots, v_{p-1}$ have been fired. Since adding β corresponds to firing the sink and v_p has been fired exactly once during the process of firing vertices in the sequence $(v_1, v_2, \ldots, v_{p-1})$, it follows that $b(v_p) = a(v_p) + \deg_G(s, v_p) + r - \deg^+_G(v_p)$. Since $v_1, v_2, \cdots, v_{p-1}$ are pairwise-distinct and different from the sink, we have $r + \deg_G(s, v_p) \leqslant \deg^-_G(v_p)$. The digraph G is Eulerian, therefore $\deg^-_G(v_p) = \deg^+_G(v_p)$ and from the previous equality we have $b(v_p) \leqslant a(v_p)$, but a is stable so vertex v_p is not firable in configuration b, which is absurd.

Since each of the in-neighbors of s is fired at most once in any firing sequence $\mathfrak{f}$ of $a+\beta$ such that $a+\beta \xrightarrow{\mathfrak{f}} \overline{a+\beta}$, it follows that during the stabilization process of $a+\beta$ the number of chips lost in the sink is not greater than $\deg^-_G(s) = \deg^+_G(s)$. This implies that

$$\sum_{v\neq s}(a+\beta)(v) - \sum_{v\neq s}\overline{a+\beta}(v) \leqslant \deg^+_G(s).$$



(a) A recurrent configuration (b) A non-recurrent configuration

Figure 2: Two stable configurations from the same equivalence class on an undirected graph

Since

$$\sum_{v\neq s}(a+\beta)(v) = \sum_{v\neq s} a(v) + \sum_{v\neq s}\beta(v) = \sum_{v\neq s} a(v) + \deg^+_G(s)$$

it follows that

$$\sum_{v\neq s} a(v) \leqslant \sum_{v\neq s}\overline{a+\beta}(v).$$

Repeating the application of this inequality n times we have

$$\sum_{v\neq s} a(v) \leqslant \sum_{v\neq s}\overline{a+n\beta}(v),$$

where $n\beta$ is the configuration given by $(n\beta)(v) = n\,\beta(v)$ for any $v \neq s$. This reasoning can be applied to d and we have

$$\sum_{v\neq s} d(v) \leqslant \sum_{v\neq s}\overline{d+n\beta}(v) \text{ for any } n \in \mathbb{N}.$$

Since for any vertex $v \neq s$ and any w being an out-neighbor of s there is a path in H from w to v, with a sufficiently large n there is an appropriate firing sequence $\mathfrak{g}$ of $d+n\beta$ and a configuration e such that $d+n\beta \xrightarrow{\mathfrak{g}} e$ and $e(v) \geqslant \deg^+_G(v) - 1$ for any $v \neq s$. When stabilizing e, it follows from Lemma 1 (convergence), Lemma 4 (recurrence) and Lemma 3 (uniqueness of recurrent configuration in an equivalent class) that it leads to c, that is, $c = \overline{e} = \overline{d+n\beta}$. This finishes the proof. □

Question. Does the claim of Lemma 6 hold for a general digraph with a global sink?

Note that if this statement is true, then it is tight. Figure 2 presents an example, on an undirected graph, of a recurrent configuration and a non-recurrent configuration belonging to the same equivalence class, such that they contain the same total number of chips. As a consequence, the recurrent configuration is not necessarily the unique configuration of maximum total number of chips over stable configurations of its equivalence class.

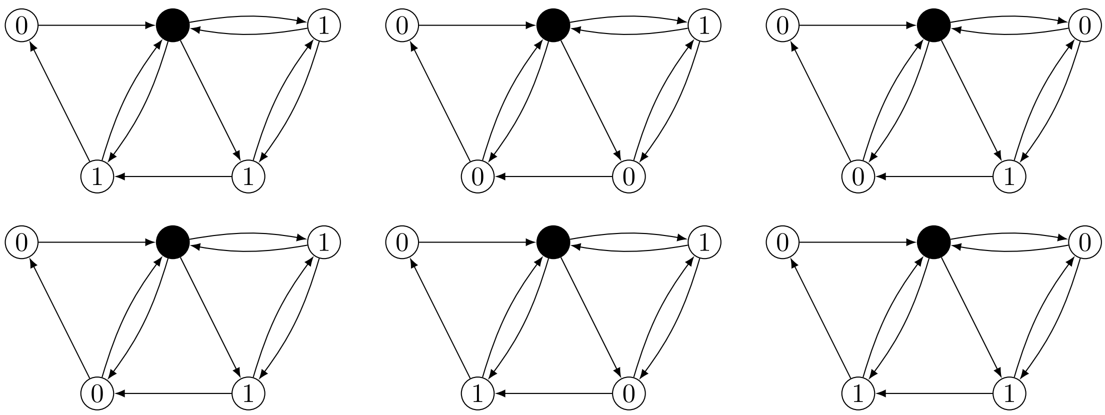


Figure 3: Recurrent configurations with respect to sink s

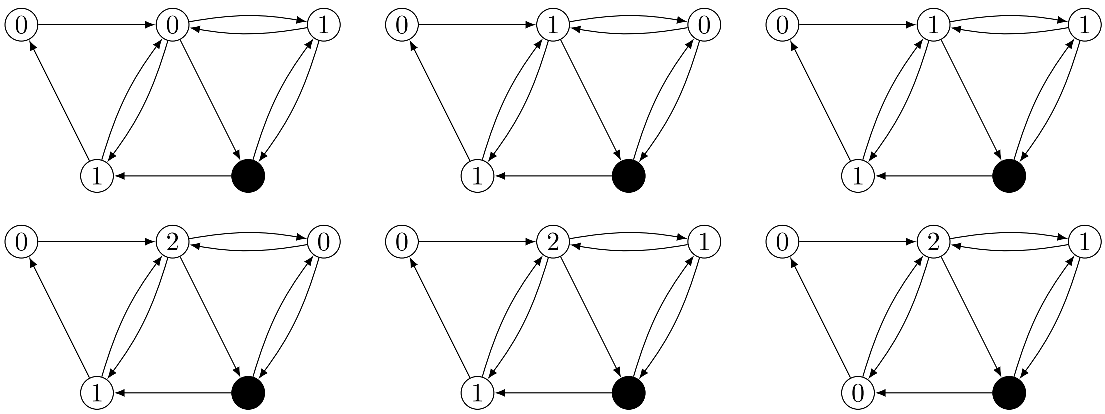


Figure 4: Recurrent configurations with respect to sink r

3 Sink-independence of generating function of recurrent configurations of an Eulerian digraph

Key observation. Let us give an important observation that motivates the study presented in this paper. We consider the Chip-firing game on the digraph drawn on Figure 1a. In this game the vertex s is chosen to be sink. All the recurrent configurations are presented in Figure 3. For each recurrent configuration we compute the sum of chips on the vertices different from the sink. We get the sorted sequence of numbers $(1, 1, 2, 2, 2, 3)$. If r is chosen to be the sink of the game, all the recurrent configurations are given in Figure 4, and the sum of chips on vertices different from the sink gives the sorted sequence $(2, 2, 3, 3, 3, 4)$. The two sequences are the same up to adding a constant sequence. This property also holds with other choices of sink, therefore, up to a constant, this sequence is characteristic of the support graph itself. This interesting property is the main discovery exploited in this paper, and allows to generalize the construction presented in [1] and proved in [19] of an analogue for the Tutte polynomial to the class of Eulerian digraphs. It is stated in the following theorem.

Theorem 1. *Let G be an Eulerian digraph and s a vertex of G. For each recurrent configuration with respect to sink s, let* $\texttt{sum}_G(c)$ *denote* $\deg_G^+(s)+\sum_{v\neq s} c(v)$. *Let $c_1, c_2, \ldots, c_p$ be an enumeration of recurrent configurations with sink s. Then the sequence* $(\texttt{sum}_G(c_i))_{1\leqslant i\leqslant p}$ *is independent of the choice of s up to a permutation of the entries.*

It makes sense to call an Eulerian digraph G *undirected* if for any two vertices v, w of G we have $\deg_G(v,w) = \deg_G(w,v)$. The result of Merino López [19] implies that Theorem 1 is true for undirected graphs. The following known result is thus a particular case of Theorem 1, for the class of undirected graphs.

Theorem 2. *[19] Let $\mathcal{C}$ be the set of all recurrent configurations with respect to some sink s. If G is an undirected graph (defined as a digraph) then*

$$T_G(1,y) = \sum_{c\in\mathcal{C}} y^{level(c)},$$

where $T_G(x,y)$ is the Tutte polynomial of G and for any $c \in \mathcal{C}$,

$$level(c) = -\frac{|A|}{2} + \deg_G^+(s) + \sum_{v\neq s} c(v).$$

In the rest of this section we work with an Eulerian digraph $G = (V, A)$. For simplicity we remove G in some notations such as $\deg_G^+(v)$, $\deg_G^-(v)$, $\texttt{sum}_G(c)$, $\deg_G(v,w)$. In order to prove Theorem 1, we consider the following natural approach. Let r and s be two arbitrary distinct vertices of G. Let $U = V\backslash\{r\}$ and $W = V\backslash\{s\}$. We denote by $\mathcal{R}$ and $\mathcal{S}$ the sets of all recurrent configurations with respect to sink r and s, respectively. We are going to construct a bijection θ from $\mathcal{R}$ to $\mathcal{S}$ such that $\texttt{sum}(c) = \texttt{sum}(\theta(c))$ for every $c \in \mathcal{R}$. Clearly, this bijection will imply Theorem 1. We recall that it follows from [16] that $|\mathcal{R}| = |\mathcal{S}|$.

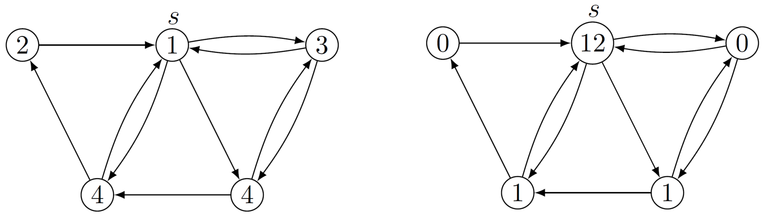
For each configuration $c \in \mathbb{N}^V$ we denote by $\overline{c}^r$ (resp. $\overline{c}^s$) the *stabilization* of c with respect to sink r (resp. sink s). Note that in the process of stabilizing c with respect to sink r, when a chip arrives at r it does not vanish but stays at r forever since r is never fired in the process. See Figure 5 for an illustration. **Warning:** let us underline that a *configuration* is now an element of $\mathbb{N}^V$.

Notation. For a function $f : X \to Y$ and a subset $Z \subseteq X$ we denote by $f|_Z$ the restriction of f to Z.

We denote by $\hat{\mathcal{R}}$ the set of configurations $c \in \mathbb{N}^V$ such that $c|_U \in \mathcal{R}$ and $c(r) \geqslant \deg^+(r)$, and by $\hat{\mathcal{S}}$ the set of configurations $c \in \mathbb{N}^V$ such that $c|_W \in \mathcal{S}$ and $c(s) \geqslant \deg^+(s)$. For each $c \in \mathcal{R}$ (resp. $\mathcal{S}$) let $\tilde{c}$ denote the configuration in $\hat{\mathcal{R}}$ (resp. $\hat{\mathcal{S}}$) such that $\tilde{c}|_U = c$ (resp. $\tilde{c}|_W = c$) and $\tilde{c}(r) = \deg^+(r)$ (resp. $\tilde{c}(s) = \deg^+(s)$).

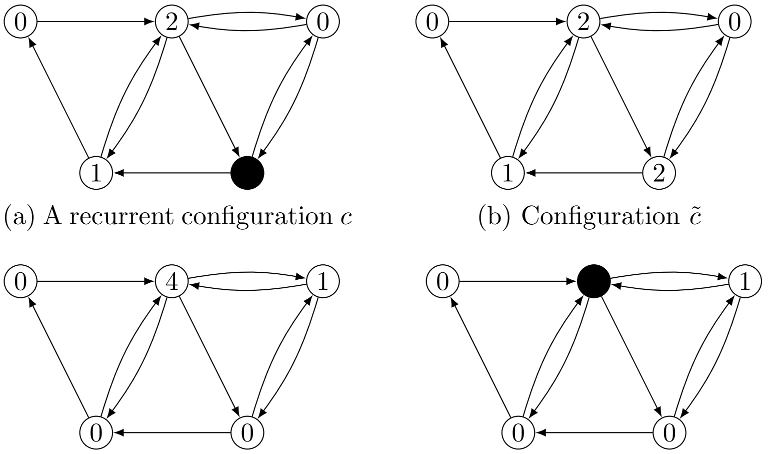
The following shows the first relation between $\hat{\mathcal{R}}$ and $\hat{\mathcal{S}}$ under the stabilization.

Lemma 7. *Let $c \in \mathcal{R}$, then $\overline{\tilde{c}}^s \in \hat{\mathcal{S}}$. Symmetrically, for each $c \in \mathcal{S}$ we have $\overline{\tilde{c}}^r \in \hat{\mathcal{R}}$.*



(a) A configuration c of G (b) $\overline{c}^s$

Figure 5: Chip-unvanished stabilization



(a) A recurrent configuration c (b) Configuration $\tilde{c}$

(c) $\overline{\tilde{c}}^s$ (d) Recurrent configuration $\overline{\tilde{c}}^s|_W$

Figure 6: Change of sink

Since the concept of this lemma is at the heart of the construction of the map θ, we give an illustration of the claim before going into the details of the proof. We consider the Eulerian digraph given in Figure 1. Figure 6a shows a recurrent configuration with respect to sink r. The configuration $\tilde{c}$ is given in Figure 6b. Figure 6c and Figure 6d show $\overline{\tilde{c}}^s$ and its restriction to W. Using the Burning algorithm, one easily checks that the configuration in Figure 6d is indeed recurrent with respect to sink s.

Proof of Lemma 7. We will once again use Lemma 5 (the Burning algorithm), which provides a firing sequence associated to a recurrent configuration of $\mathcal{R}$, that we will manipulate to built a firing sequence associated to a recurrent configuration of $\mathcal{S}$. In $\tilde{c}$, only r is firable, and after firing it, we will use the firing sequence leading back to c, provided by Lemma 5.

Let $\beta \in \mathbb{N}^U$ (resp. $\gamma \in \mathbb{N}^W$) be given by $\beta(v)$ (resp. $\gamma(v)$) is equal to $\deg(r, v)$ (resp. $\deg(s, v)$). Let a be such that $\tilde{c} \xrightarrow{r} a$. We have $a|_U = c + \beta$, and can therefore apply Lemma 5 (since c is recurrent), providing a firing sequence $\mathfrak{f} = (v_1, v_2, \cdots, v_{|V|})$ of $\tilde{c}$ such that $v_1 = r$ and each vertex of G occurs exactly once in this sequence. Let k be such that $s = v_k$ and b be the configuration reached after vertices $(v_1, v_2, \ldots, v_{k-1})$ have been fired. Vertex s is firable in b, thus $b(s) \geqslant \deg^+(s)$. Since the game is convergent, and this intermediate configuration b is reachable from $\tilde{c}$ without firing s, when we stabilize $\tilde{c}$ with respect to sink s we end up with at least as many chips in s as in b, and therefore $\overline{\tilde{c}}^s(s) \geqslant \deg^+(s)$.

Since $\tilde{c} \xrightarrow{f} \tilde{c}$, the sequence $\mathfrak{g} = (s, v_{k+1}, v_{k+2}, \cdots, v_{|V|}, v_1, v_2, \cdots, v_{k-1})$ is a firing sequence of b. We now consider the game with respect to sink s (that is, on G where the out-going arcs of s are removed), and the configuration $b|_W$. Let d be such that $b \xrightarrow{s} d$. We have $d|_W = b|_W + \gamma$, and the rest of the firing sequence implies that $b|_W + \gamma \xrightarrow{*} b|_W$, therefore $b|_W + n\gamma \xrightarrow{*} b|_W$ for any $n \in \mathbb{N}$. The recurrence of $\overline{b|_W}$ can be shown similarly as in the proof of Lemma 6 as follows. With n large enough there is $e \in \mathbb{N}^W$ such that $b|_W \xrightarrow{*} e$ and $e(v) \geqslant \deg^+(v)$ for any $v \in W$. By Lemma 4 we have $\overline{e} = \overline{b|_W}$ is recurrent. Since $\overline{\tilde{c}}^s(s) \geqslant \deg^+(s)$ and $\overline{b|_W} = \overline{\tilde{c}}^s|_W$, it follows that $\overline{\tilde{c}}^s \in \hat{\mathcal{S}}$. □

Lemma 7 naturally suggests a bijection from $\mathcal{R}$ to $\mathcal{S}$ that is defined by $c \mapsto \overline{\tilde{c}}^s|_W$. However, this does not give the intended bijection since it does not necessarily preserve the `sum` of chips, as shown on Figure 6. We will improve the above map by adding some extra chips to $\overline{c}$ at the vertex r so that the map preserves the `sum`. The required number of extra chips added to r follows from Lemma 10 below and is a non-constructive quantity. That is what we are going to present now. We need the following notation. We denote by $\mathbf{1}_r$ (resp. $\mathbf{1}_s$) the configuration on G which has one chip at r (resp. s) and none in other vertices. By this notation for any $i \in \mathbb{N}$ we have $i\,\mathbf{1}_r$ (resp. $i\,\mathbf{1}_s$) is the configuration which has i chips at r (resp. i chips at s) and none in other vertices.

Lemma 8. *For any $c \in \hat{\mathcal{R}}$ we have $\overline{c}^s \in \hat{\mathcal{S}}$. Symmetrically, for any $c \in \hat{\mathcal{S}}$ we have $\overline{c}^r \in \hat{\mathcal{R}}$.*

Proof. Clearly, we have $\widetilde{c|_U} \leqslant c$. It follows from Lemma 7 and the second item of Lemma 4 that $\overline{c}^s|_W \in \mathcal{S}$. By using the same arguments as in the proof of Lemma 7 we have $\overline{c}^s(s) \geqslant \deg^+(s)$. This finishes the proof. □

The lemma means that the correspondence $c \mapsto \overline{c}^s$ is a map from $\hat{\mathcal{R}}$ to $\hat{\mathcal{S}}$. The following implies the injectivity of this map.

Lemma 9. *For any $c \in \hat{\mathcal{R}}$ we have $\overline{(\overline{c}^s)}^r = c$.*

Proof. Let a denote $\overline{(\overline{c}^s)}^r$. It follows from Lemma 8 that both a and c are in $\hat{\mathcal{R}}$. Since G is Eulerian and a can be obtained from c by a sequence of firings, it follows that $a|_U$ and $c|_U$ are in the same equivalence class. Both are recurrent, hence from Lemma 3 they are equal. Since a and c contain the same total number of chips, and are equal on the vertices different from r, it follows that $a = c$. □

The aim is now to find, for every recurrent configuration $c \in \mathcal{R}$, the good i so as to get a bijection from $\mathcal{R}$ to $\mathcal{S}$ that preserves the `sum` of chips. We first concentrate on the `sum` conservation: if one wants to have

$$\mathtt{sum}(c) = \mathtt{sum}\left(\overline{\tilde{c} + i\,\mathbf{1}_r}^s|_W\right),$$

then the number i must be chosen so that $\overline{\tilde{c} + i\,\mathbf{1}_r}^s(s) = i + \deg^+(s)$, because the i extra chips are not counted in both `sums` in this case. The following shows that such an i always exists.

Lemma 10. *For every $c \in \mathcal{R}$ there exists i such that $\overline{\tilde{c} + i\,\mathbf{1}_r}^s(s) = i + \deg^+(s)$.*

Proof. For each $i \in \mathbb{N}$ let d_i denote the $\overline{\tilde{c} + i\,\mathbf{1}_r}^s$. Let the function $f : \mathbb{N} \to \mathbb{Z}$ be defined by $f(i) = d_i(s) - \deg^+(s) - i$. We are going to prove that there exists $j \in \mathbb{N}$ such that $f(j) = 0$. Since $\tilde{c} + i\,\mathbf{1}_r \leqslant \tilde{c} + (i+1)\,\mathbf{1}_r$, it follows from Lemma 4 (using the trick $\sum_{v\in W} c(v) = \sum_{v\in V} c(v) - c(s)$) that $d_i(s) - c(s) \leqslant d_{i+1}(s) - c(s)$, therefore $d_i(s) \leqslant d_{i+1}(s)$. As a consequence $f(i+1) - f(i) \geqslant -1$ for every i, that is, the function f decreases by at most one.

By Lemma 8 we have $d_0(s) \geqslant \deg^+(s)$, therefore $f(0) \geqslant 0$. Since $f(i+1) - f(i) \geqslant -1$ for any $i \in \mathbb{N}$, the proof is completed by showing that there is $j \in \mathbb{N}$ such that $f(j) \leqslant 0$. In particular, we are going to prove that $f(N-1) \leqslant 0$, where $N = |\mathcal{S}|$. Note that N is the order of the Sandpile group of G with respect to sink s.

$f(N-1) = d_{N-1}(s) - \deg^+(s) - (N-1)$. We are going to use Lemma 6, which states that the recurrent configuration has maximum total number of chips over stable configurations of its equivalence class, in order to upper bound $d_{N-1}(s)$ by $c(s) + N$, and the result follows since vertex s is stable in the recurrent configuration c (meaning that $c(s) \leqslant \deg^+(s) - 1$).

Let a denote $\tilde{c} + (N-1)\,\mathbf{1}_r$ and b denote $\tilde{c} - \mathbf{1}_r$. Note that $b \in \mathbb{N}^V$ since G is an Eulerian graph. We have $a = b + N\,\mathbf{1}_r$, thus the choice of N implies that a and b are in the same equivalence class with respect to sink s, and the first contains N more chips than the latter. Since $d_{N-1} = \overline{a}^s$ and the configuration $d_{N-1}|_W$ is recurrent, it follows from Lemma 6 that $\sum_{v\in W} b(v) \leqslant \sum_{v\in W} d_{N-1}(v)$. It remains to exploit the total number of chips difference between the two configurations: $N + \sum_{v\in V} b(v) = \sum_{v\in V} d_{N-1}(v)$. Replacing $\sum_{v\in W} x(v)$ by $\sum_{v\in V} x(v) - x(s)$ on both sides, the inequality given by Lemma 6 thus becomes $b(s) + N \geqslant d_{N-1}(s)$, and equivalently $c(s) + N \geqslant d_{N-1}(s)$. $\square$

We can now construct the intended bijection θ. For each $c \in \mathcal{R}$ (resp. $c \in \mathcal{S}$), let $\mathcal{I}(c)$ (resp. $\mathcal{J}(c)$) denote the smallest number $i \in \mathbb{N}$ such that $\overline{\tilde{c} + i\,\mathbf{1}_r}^s(s) = \deg^+(s) + i$ (resp. $\overline{\tilde{c} + i\,\mathbf{1}_s}^r(r) = \deg^+(r) + i$). The positive integers $\mathcal{I}(c)$ and $\mathcal{J}(c)$ are called the *swap numbers* of c from r to s and from s to r, respectively. By Lemma 10 we know that swap numbers are well-defined and unique.

$$\begin{array}{rccl} \theta: & \mathcal{R} & \to & \mathcal{S} \\ & c & \mapsto & d|_W, \text{where } d = \overline{\tilde{c} + \mathcal{I}(c)\,\mathbf{1}_r}^s. \end{array}$$

The map θ satisfies

$$\mathtt{sum}(c) = \mathtt{sum}(\theta(c)).$$

Since $|\mathcal{R}| = |\mathcal{S}|$ is finite, in order to prove Theorem 1 it remains to show that θ is injective. Let us first present some properties of swap numbers. A configuration $c \in \mathcal{R}$ is called *minimal* if there is no configuration $d \in \mathcal{R}$ such that $c \neq d$ and $d \leqslant c$, and *minimum* if $\sum_{v\in U} c(v)$ is minimum over all configurations in $\mathcal{R}$.

Proposition 1. *Let $c \in \mathcal{R}$. If c is minimum then $\mathcal{I}(c) = 0$.*

Proof. Let a denote $\overline{\tilde{c}}^s$. By definition of $\mathcal{I}(c)$, the aim is to prove that $a(s) = \deg^+(s)$. The proof relies intuitively on Lemma 7: it says that the lower bound for $a(s)$ is always reached for the configuration containing the minimum total number of chips. Let us assume it is false, therefore we can define $b \neq a$ such that $b(v) = a(v)$ for $v \in W$ and $b(s) = \deg^+(s)$, and we will get a contraction to the minimality of c.

By Lemma 8 we have $a(s) \geqslant \deg^+(s)$. By the assumption $b \neq a$ we have $a(s) > \deg^+(s)$. This implies that

$$\sum_{v \in V} b(v) < \sum_{v \in V} a(v) = \sum_{v \in V} \tilde{c}(v).$$

By Lemma 8 the configuration a is in $\hat{\mathcal{S}}$, and so is b. Applying again Lemma 8 to s and $\mathcal{S}$, we have $\overline{b}^r \in \hat{\mathcal{R}}$, and therefore $\overline{b}^r(r) \geqslant \deg^+(r) = \tilde{c}(r)$. Now, since

$$\sum_{v \in V} \overline{b}^r(v) = \sum_{v \in V} b(v) < \sum_{v \in V} \tilde{c}(v),$$

it follows that

$$\sum_{v \in U} \overline{b}^r(v) < \sum_{v \in U} \tilde{c}(v) = \sum_{v \in U} c(v),$$

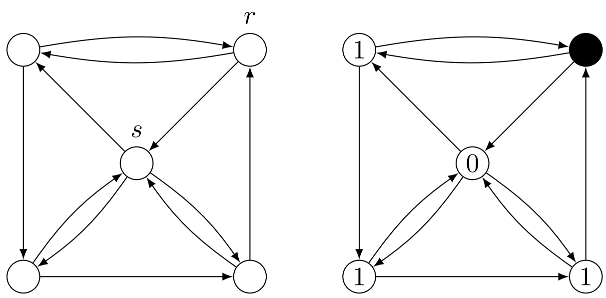
a contradiction to the minimality of c. □

Proposition 2. *Let $c, d \in \mathcal{R}$. If $c \leqslant d$ then $\mathcal{I}(c) \leqslant \mathcal{I}(d)$.*

Proof. Regarding Proposition 1, we would intuitively expect that the number $\mathcal{I}(x)$ increases monotonically with the total number of chips of x. We use the same construction as in the proof of Lemma 10.

Let k denote $\mathcal{I}(d)$. Let a and b denote $\overline{\tilde{c} + k\,\mathbf{1}_r}^s$ and $\overline{\tilde{d} + k\,\mathbf{1}_r}^s$, respectively. Since $\tilde{c} + k\,\mathbf{1}_r \leqslant \tilde{d} + k\,\mathbf{1}_r$, it follows from Lemma 4 that $a(s) - c(s) \leqslant b(s) - d(s)$. Since by hypothesis $c(s) \leqslant d(s)$, we have $a(s) \leqslant b(s)$, therefore $a(s) - \deg^+(s) - k \leqslant b(s) - \deg^+(s) - k = 0$. Let $f : \mathbb{N} \to \mathbb{Z}$ be given by $f(i) = \overline{(\tilde{c} + i\,\mathbf{1}_r)}^s(s) - \deg^+(s) - i$, from the previous inequality it verifies that $f(k) \leqslant 0$, and $\mathcal{I}(c)$ is by definition the smallest j such that $f(j) = 0$. By the same arguments as in the proof of Lemma 10, we have $f(0) \geqslant 0$ and $f(i+1) - f(i) \geqslant -1$ for any $i \in \mathbb{N}$. As a consequence, there is $j \in [0..k]$ such that $f(j) = 0$, therefore $\mathcal{I}(c) \leqslant k = \mathcal{I}(d)$. □

By Propositions 1 and 2, for a recurrent configuration $c \in \mathcal{R}$ the number $\mathcal{I}(c)$ increases monotonically as we add chips to c, starting from 0 when the configuration is minimum. When G is an undirected graph, every minimal recurrent configuration is minimum since every minimal recurrent configuration has the same number of chips, namely $\frac{|A|}{2} - \deg^+(s)$ [23]. Thus $\mathcal{I}(c) = 0$ for every minimal recurrent configuration c. One tends to think that a minimal configuration c should also have $\mathcal{I}(c) = 0$, but it may indeed be strictly positive as shown on Figure 7.



(a) An Eulerian graph

(b) A minimal recurrent configuration with respect to sink r

Figure 7: A minimal recurrent configuration c with $\mathcal{I}(c) = 1$

Question. Give an upper bound of $\mathcal{I}(c)$ when c is minimal.

There is a nice relation between swap numbers for c in $\mathcal{R}$ from r to s, and for $\theta(c)$ from s to r. The following proposition does most part of the work to prove Theorem 1, and the latter can be considered as a corollary of this result.

Proposition 3. *For all $c \in \mathcal{R}$, we have $\mathcal{I}(c) = \mathcal{J}(\theta(c))$.*

Proof. The proposition is proved with two inequalities.

- $\mathcal{J}(\theta(c)) \leqslant \mathcal{I}(c)$:

  Let l denote $\mathcal{I}(c)$ and let $a = \overline{\tilde{c} + l\,\mathbf{1}_r}^s$. We have $a|_W = \theta(c)$. First, by definition of l we have $a(s) = \deg^+(s) + l$, therefore $a = \widetilde{\theta(c)} + l\,\mathbf{1}_s$. Second, by Lemma 9 applied to $\tilde{c} + l\,\mathbf{1}_r$, we have

  $$\overline{a}^r = \overline{\overline{\tilde{c} + l\,\mathbf{1}_r}^s}^r = \tilde{c} + l\,\mathbf{1}_r.$$

  This implies that $\overline{\widetilde{a|_W} + l\,\mathbf{1}_s}^r(r) = \overline{a}^r(r) = \deg^+(r) + l$. Since $\mathcal{J}(a|_W)$ is the minimal number i such that $\overline{\widetilde{a|_W} + i\,\mathbf{1}_s}^r(r) = \deg^+(r) + i$, it follows that $\mathcal{J}(\theta(c)) = \mathcal{J}(a|_W) \leqslant l$.

- $\mathcal{J}(\theta(c)) \geqslant \mathcal{I}(c)$:

  This part of the lemma is more involved. For convenience, let us denote $\theta = \theta(c)$. Let p denote $\mathcal{J}(\theta)$. The previous inequality implies $p \leqslant l$. In order to get a contradiction, let us suppose that $p < l$. Let $b \in \mathcal{R}$ be such that $\overline{\tilde{\theta} + p\,\mathbf{1}_s}^r = \tilde{b} + p\,\mathbf{1}_r$ (the existence of b is due to Lemma 7 and the definition of p). The above inequality applied to θ implies that $b \neq c$.

  We have

  $$\overline{\tilde{c} + l\,\mathbf{1}_r}^s = \tilde{\theta} + l\,\mathbf{1}_s \quad \text{and} \quad \overline{\tilde{b} + p\,\mathbf{1}_r}^s = \tilde{\theta} + p\,\mathbf{1}_s \ \text{(By Lemma 9)}, \tag{1}$$

therefore the two configurations

$$\tilde{c} + l\,\mathbf{1}_r|_W \quad \text{and} \quad b + p\,\mathbf{1}_r|_W$$

are in the same equivalence class for the CFG with sink s. Removing p chips to r in both configurations does not affect the equivalence relation, hence with $k = l - p > 0$,

$$\tilde{c} + k\,\mathbf{1}_r|_W \quad \text{and} \quad \tilde{b}|_W$$

are also in the same equivalence class for the CFG with sink s, and from Lemma 8 and the uniqueness of the recurrent configuration in an equivalence class (Lemma 3),

$$\overline{\tilde{c} + k\,\mathbf{1}_r}^s|_W \;=\; \overline{\tilde{b}}^s|_W. \tag{2}$$

From equation (1) there are k more chips in $\tilde{c} + l\,\mathbf{1}_r$ than in $\tilde{b} + p\,\mathbf{1}_r$, thus it follows from the above equality (2) that

$$\overline{\tilde{c} + k\,\mathbf{1}_r}^s(s) \;=\; \overline{\tilde{b}}^s(s) + k. \tag{3}$$

We now consider the two configurations d and e defined by

$$\begin{array}{rl} & d = \tilde{c} + (k-1)\,\mathbf{1}_r \\ \iff & d + \mathbf{1}_r = \tilde{c} + k\,\mathbf{1}_r \end{array} \quad \text{and} \quad \begin{array}{rl} & e = \tilde{b} - \mathbf{1}_r \\ \iff & e + \mathbf{1}_r = \tilde{b} \end{array}.$$

It follows from equality (2) that

$$\overline{\overline{d}^s + \mathbf{1}_r}^s|_W \;=\; \overline{\overline{e}^s + \mathbf{1}_r}^s|_W. \tag{4}$$

As we will see, it is not possible that both:

- these two configurations are equal;
- enough chips go to s during these stabilization processes so that equation (3) is verified.

Let us present a reasoning contradicting equation (3).

We first work on the total chip content of $\overline{d}^s|_W$ and $\overline{e}^s|_W$. For the same reason as above, $d|_W$ and $e|_W$ belong to the same equivalence class for the CFG with sink s, and so do $\overline{d}^s|_W$ and $\overline{e}^s|_W$ because the firing process does not affect the equivalence relation. We have $\overline{d}^s|_W = \overline{\tilde{c} + (k-1)\,\mathbf{1}_r}^s|_W$ with $k - 1 \geqslant 0$, thus from Lemma 8 it is recurrent. Furthermore $\overline{e}^s|_W$ is stable, and since they belong to the same equivalence class, it follows from Lemma 6 that

$$\sum_{v \in W} \overline{d}^s(v) \quad \geqslant \quad \sum_{v \in W} \overline{e}^s(v). \tag{5}$$

Now we compare the number of chips going into the sink s. Let $\mathfrak{f} = (v_1, \ldots, v_m)$ and $\mathfrak{g} = (w_1, \ldots, w_n)$ be two firing sequences such that

$$\overline{d}^{s} + \mathbf{1}_r \xrightarrow{\mathfrak{f}} \overline{\overline{d}^{s} + \mathbf{1}_r}^{s} \quad \text{and} \quad \overline{e}^{s} + \mathbf{1}_r \xrightarrow{\mathfrak{g}} \overline{\overline{e}^{s} + \mathbf{1}_r}^{s}.$$

Obviously $s \notin \mathfrak{f}$ and $s \notin \mathfrak{g}$, and it follows from equations (4) and (5) that during the stabilization process, more chips go to s in $\mathfrak{f}$ than in $\mathfrak{g}$:

$$\sum_{1 \leqslant i \leqslant m} \deg(v_i, s) \quad \geqslant \quad \sum_{1 \leqslant i \leqslant n} \deg(w_i, s). \tag{6}$$

In order to get the intended contradiction with (3), let us have a close look at the chip content in both sinks s, using the fact that from the minimality of l,

$$\overline{\tilde{c} + (k-1)\,\mathbf{1}_r}^{s}(s) \quad > \quad \deg^{+}(s) + (k-1).$$

$$\begin{aligned}
\overline{\tilde{c} + k\,\mathbf{1}_r}^{s}(s) \quad &= \quad \overline{\overline{d}^{s} + \mathbf{1}_r}^{s}(s) \\
&= \quad \overline{d}^{s}(s) + \textstyle\sum_{1 \leqslant i \leqslant m} \deg(v_i, s) \\
&= \quad \overline{\tilde{c} + (k-1)\,\mathbf{1}_r}^{s}(s) + \textstyle\sum_{1 \leqslant i \leqslant m} \deg(v_i, s) \\
&> \quad \deg^{+}(s) + (k-1) + \textstyle\sum_{1 \leqslant i \leqslant m} \deg(v_i, s) \\
&\underset{\text{equation (6)}}{\geqslant} \quad \deg^{+}(s) + (k-1) + \textstyle\sum_{1 \leqslant i \leqslant n} \deg(w_i, s) \\
&\underset{\text{stability}}{\geqslant} \quad \deg^{+}(s) + (k-1) + \textstyle\sum_{1 \leqslant i \leqslant n} \deg(w_i, s) + \overline{e}^{s}(s) - \deg^{+}(s) + 1 \\
&= \quad k + \overline{\overline{e}^{s} + \mathbf{1}_r}^{s}(s) \\
&= \quad k + \overline{\tilde{b}}^{s}(s),
\end{aligned}$$

which contradicts equation (3). □

Theorem 1 is now easy to prove.

Proof of Theorem 1. Since $|\mathcal{R}| = |\mathcal{S}|$, it remains to prove that the map θ is injective. For a contradiction, suppose it is not, that is, there exist c and d belonging to $\mathcal{R}$ and such that

$$c \neq d \quad \text{and} \quad \theta(c) = \theta(d).$$

By Proposition 3 we have $\mathcal{I}(c) = \mathcal{I}(d)$. Let k denote $\mathcal{I}(c)$ and let e denote $\theta(c)$. It follows from Lemma 9 that

$$c \;=\; \overline{\overline{\tilde{c} + k\,\mathbf{1}_r}^{s}}^{r}\big|_U \;=\; \overline{\tilde{e} + k\,\mathbf{1}_s}^{r}\big|_U \;=\; \overline{\overline{\tilde{d} + k\,\mathbf{1}_r}^{s}}^{r}\big|_U \;=\; d,$$

a contradiction. □

4 Tutte-like properties of generating function of recurrent configurations

We present in this section a natural generalization of the partial Tutte polynomial $T_G(1, y)$ in one variable for the class of Eulerian graphs. In order to set up the most general setting, we introduce it for the class of Eulerian graphs with loops. Note that loops are not interesting regarding the Chip-firing game: a loop simply "freezes" a chip on one vertex, that is the reason why we did not consider them in previous sections. The Tutte-like polynomial we present is constructed from the generating function of the set of recurrent configurations with respect to an arbitrary sink, and its uniqueness is based on the sink-independence property exposed in Theorem 1.

Let us first present the extension of Theorem 1 to the class of Eulerian graphs with loops. We begin with the definition of the Chip-firing game for this class of graphs. Note that the out-degree of a vertex v is the number of arcs whose tail is v, therefore includes loops. Let $G = (V, A)$ be an Eulerian graph possibly having loops. A vertex v is *firable* in a configuration c if $c(v) \geqslant \deg^+(v)$ and $\deg^+(v) - \deg(v, v) \geqslant 1$, where $\deg(v, v)$ is the number of loops at v. Firing a firable vertex v means the process that decreases $c(v)$ by $\deg^+(v)$ and increases each $c(w)$ by $\deg(v, w)$ for all w, or equivalently decreases $c(v)$ by $\deg^+(v) - \deg(v, v)$ and increases each $c(w)$ with $w \neq v$ by $\deg(v, w)$. The Burning algorithm presented in Lemma 5 remains valid for Eulerian graphs with loops.

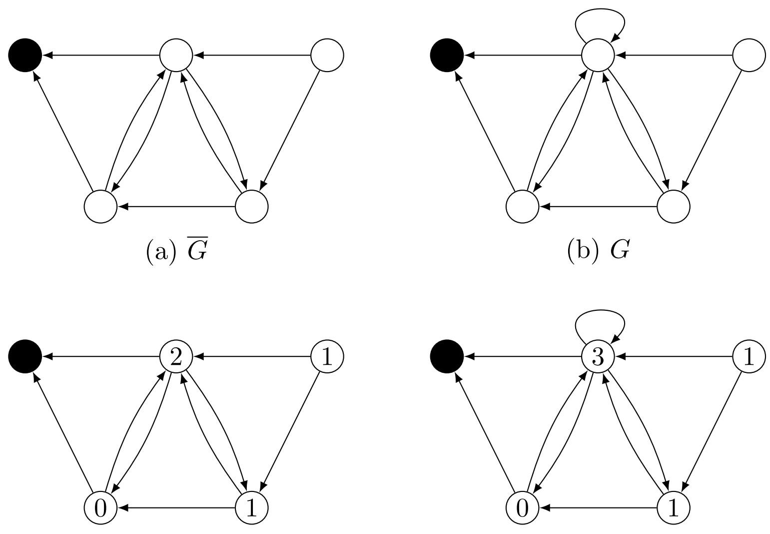
As pointed out above, the CFG on a digraph possibly having loops is very close to the CFG on the digraph where the loops are removed. For a digraph G we denote by $\overline{G}$ the digraph G in which all loops are removed, and denote by $L(G)$ the number of loops of G. Regarding undirected graphs, the influence of a loop is the same as a directed loop (it also "freezes" one chip). For two arcs a and b of G, a is *reverse* of b if $a^- = b^+$ and $a^+ = b^-$. An undirected graph with loops is converted to an Eulerian graph in the same way as an undirected graph without loops, however with the exception that each undirected loop e is replaced by exactly one directed loop that has the same endpoint as e.

Theorem 1 is generalized to the class of Eulerian graphs possibly having loops with the following lemma. From now on, we will always consider an arbitrary fixed sink denoted s, therefore a configuration means an element in $\mathbb{N}^{V\setminus\{s\}}$.

Lemma 11. *Let $G = (V, A)$ be an Eulerian graph with sink s. Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ be the sets of recurrent configurations of G and $\overline{G}$ with respect to sink s, respectively. For each configuration $c \in \overline{\mathcal{C}}$, let $\mu(c) : V\setminus\{s\} \to \mathbb{N}$ be given by $\mu(c)(v) = c(v) + \deg(v, v)$ for any $v \neq s$. Then μ is a bijection from $\overline{\mathcal{C}}$ to $\mathcal{C}$. Moreover, $\sum\limits_{v\neq s} c(v) - \sum\limits_{v\neq s}\mu(c)(v) = -\sum\limits_{v\neq s} \deg(v, v)$ for any $c \in \overline{\mathcal{C}}$.*

This lemma can be proved easily by using the definition of recurrent configuration with the observation that if a configuration c is recurrent with respect to G then $c(v) \geqslant \deg(v, v)$ for any $v \neq s$. An illustration of Lemma 11 is given in Figure 8.

In the rest of this section, we work with an Eulerian graph $G = (V, A)$ possibly having loops and an arbitrary but fixed vertex s of G that plays the role of sink for the game. We now introduce the partial Tutte polynomial generalization, which is defined as the



(a) $\overline{G}$ (b) G

(c) $c \in \overline{\mathcal{C}}$ (d) $\mu(c) \in \mathcal{C}$

Figure 8: Relation between $\mathcal{C}$ and $\overline{\mathcal{C}}$

generating function of the set of recurrent configurations. The generating function is based on the concept of *level* of recurrent configurations, which corresponds to the previously defined $\mathtt{sum}$ normalized according to the smallest level of a recurrent configuration. For an Eulerian graph G, let

$$\kappa(\overline{G}) \text{ denote the minimum of } \mathtt{sum}(c) = \deg^+(s) + \sum_{v \neq s} c(v)$$

over all recurrent configurations c of $\overline{G}$ with respect to sink s. Theorem 1 implies that $\kappa(\overline{G})$ is independent of the choice of s. It follows from [23] that the problem of finding $\kappa(\overline{G})$ is NP-hard for Eulerian graphs, and as a consequence the Tutte-like polynomial we present is also NP-hard to compute. In addition, when G is undirected (and defined as a digraph *i.e.*, each edge is represented by two reverse arcs), the number $\kappa(\overline{G})$ has an exact formula, namely $\kappa(\overline{G}) = \frac{|A(\overline{G})|}{2}$. For a recurrent configuration c of G with respect to sink s we define

$$level_G(c) = \mathtt{sum}(c) - \kappa(\overline{G}).$$

This is a generalization of the level that was defined in [1], because we recover the latter when G is undirected. Let $\mathcal{C}$ denote the set of all recurrent configurations of G with respect to sink s. The generating function of $\mathcal{C}$ is given by

$$\mathcal{T}_G(y) = \sum_{c \in \mathcal{C}} y^{level_G(c)},$$

and we claim that it is a natural generalization of the partial Tutte polynomial, for the class of Eulerian graphs. First, it follows from Theorem 1 that $\mathcal{T}_G(y)$ is independent of the choice of s, thus is characteristic of the support graph G itself. We are going to present in this section a number of properties of $\mathcal{T}_G(y)$ that can be considered as the generalizations of those of the Tutte polynomial in one variable, namely $T_G(1, y)$, that is defined on undirected graphs. The most interesting and new feature is that when G is an undirected graph we get back to the well-known Tutte polynomial. This fact is straightforward to notice.

- $\mathcal{T}_G(y) = T_G(1, y)$ if G is an undirected graph.
- $\mathcal{T}_G(1)$ counts the number of oriented spanning tree of G rooted at s [16]. It generalizes the evaluation $T_G(1, 1)$ that counts the number of spanning tree of an undirected graph.
- $\mathcal{T}_G(0)$ counts the number of maximum acyclic arc sets with exactly one sink s [23]. Therefore $\mathcal{T}_G(0)$ is a natural generalization of $T_G(1, 0)$ that counts the number of acyclic orientations with a fixed source of an undirected graph.

For an undirected graph G, $T_G(1, 2)$ counts the number of spanning connected subgraphs of G. It would be interesting to investigate a combinatorial interpretation for $\mathcal{T}_G(2)$. We propose the following question for future work.

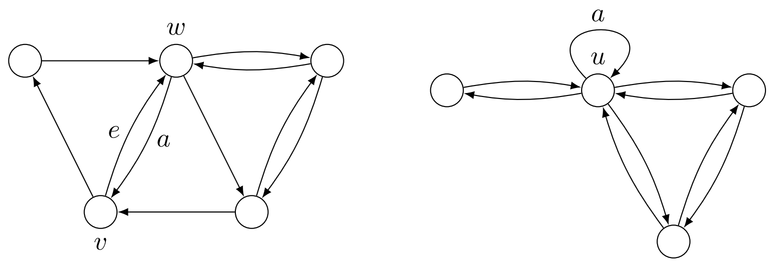
Question. What does $\mathcal{T}_G(2)$ count?

These evaluations set up a promising ground for further investigations, but it is definitely not trivial to find out the objects counted by evaluations of graph polynomials. We are now going to present the extension to $\mathcal{T}_G(y)$ of four known recursive formulas for the Tutte polynomial in the undirected case. We will need the two following simple lemmas.

For a subset B of A let $G\backslash B$ denote the graph $(V, A\backslash B)$. We write $G\backslash e$ for $G\backslash\{e\}$ if e is an arc of G. For an arc e of G with two endpoints v and w let G/e denote the digraph that is made from G by removing e from G, replacing v and w by a new single vertex u, and for each remaining arc a if the head (resp. tail) of a in G is v or w then the head (resp. tail) of a in G/e is u. This procedure is called *arc contraction*.

An analogue of arc contraction may also be defined for vertices. For a subset W of V, let G/W denote the digraph constructed from G by replacing all vertices in W by a new vertex w, and each arc $e \in A$ such that $e^- \in W$ (resp. $e^+ \in W$) in G by $e^- = w$ (resp. $e^+ = w$) in G/W.

In the case of undirected graphs Merino López gave a relation between the recurrent configurations of G and the recurrent configurations of the contracted graph G/e, where e is an edge of G [19]. In particular the author showed a bijection between the set of recurrent configurations which have maximum value at some fixed neighbor t of the sink and the set of recurrent configurations of the contracted graph $G/(s, t)$ with the sink being the new vertex from the edge contraction. The following lemma is a generalization of that relation for the class of Eulerian graphs. The proof idea is originally due to Merino López [19]. We recall that $\mathcal{C}$ is the set of recurrent configurations of the CFG on G with respect to sink s.



(a) G (b) G/e

Figure 9: Arc contraction

Lemma 12. *Let W be a non-empty subset of the set of out-neighbors of s,* i.e.*, for every $v \in W$ we have $v \neq s$ and $(s,v) \in A$. Let t be the new vertex in $G/W \cup \{s\}$ resulting from contracting the set of vertices $W \cup \{s\}$. For any $c \in \mathcal{C}$, if $c(v) \geqslant \deg^+(v) - \deg(s,v)$ for all $v \in W$, then $c|_{V\setminus(W\cup\{s\})}$ is a recurrent configuration of $G/W \cup \{s\}$ with respect to sink t. Conversely, if d is a recurrent configuration of $G/W \cup \{s\}$ with respect to sink t, then every configuration $c : V\setminus\{s\} \to \mathbb{N}$, satisfying $c(v) = d(v)$ for all $v \in V\setminus(W \cup \{s\})$ and $\deg^+(v) > c(v) \geqslant \deg^+(v) - \deg(s,v)$ for all $v \in W$, is in $\mathcal{C}$.*

Proof. This proof is straightforward. We use Lemma 5 (Burning algorithm) and the hypothesis that a configuration is recurrent, thus it admits a firing sequence, in order to construct a firing sequence for the considered configuration, which proves that it is recurrent (again by Lemma 5).

We denote the vertices in W by $w_1, w_2, \cdots, w_q$. Let the configuration $\beta : V\setminus\{s\} \to \mathbb{N}$ be given by $\beta(v) = \deg(s,v)$ for any $v \in V\setminus\{s\}$. The condition $c(w_i) \geqslant \deg^+(s) - \deg(s,w_i)$ for any i implies that w_i is firable in $c+\beta$ for any i. It follows from Lemma 1 and Lemma 5 that there is a firing sequence $\mathfrak{f} = (v_1, v_2, \cdots, v_k)$ of $c+\beta$ in G such that $c+\beta \xrightarrow{\mathfrak{f}} c$, $v_i \neq s$ for any i, each vertex of G distinct from s occurs exactly once in $\mathfrak{f}$, and $v_i = w_i$ for any $i \in [1..q]$. Let a be such that $c+\beta \xrightarrow{w_1,w_2,\cdots,w_p} a$. Let $\gamma : V\setminus(W \cup \{s\}) \to \mathbb{N}$ be given by $\gamma(v) = \deg(t,v)$ for any $v \in V\setminus(W \cup \{s\})$. Clearly, we have $a|_{V\setminus(W\cup\{s\})} = c|_{V\setminus(W\cup\{s\})} + \gamma$. Since $\mathfrak{g} = (v_{p+1}, v_{p+2}, \cdots, v_k)$ is a firing sequence of a, $\mathfrak{g}$ is also a firing sequence of $c|_{V\setminus(W\cup\{s\})} + \gamma$ in $G/W \cup \{s\}$. It follows from Lemma 5 that $c|_{V\setminus(W\cup\{s\})}$ is a recurrent configuration of $G/W \cup \{s\}$ with respect to sink t.

For the converse statement let $\mathfrak{h} = (u_1, u_2, \cdots, u_p)$ be a firing sequence of d such that $d + \gamma \xrightarrow{\mathfrak{h}} d$ in $G/W \cup \{s\}$, then $u_i \notin W \cup \{s\}$ for any i, and each vertex of G not in $W \cup \{s\}$ occurs exactly once in $\mathfrak{h}$. Let b be such that $c + \beta \xrightarrow{w_1,w_2,\cdots,w_q} b$ in G. Clearly, $b|_{V\setminus(W\cup\{s\})} = d + \gamma$, therefore $(w_1, w_2, \cdots, w_q, u_1, u_2, \cdots, u_p)$ is a firing sequence of $c + \beta$ in G. It follows that $c \in \mathcal{C}$. □

Lemma 13. *Let e and e' be two reverse arcs of G such that they are not loops and $e^- = s$. Let H denote $G\setminus\{e, e'\}$ and w denote e^+. If H is connected then $\{c \in \mathcal{C} : c(w) < \deg^+(w) - 1\}$ is the set of all recurrent configurations of H with respect to sink s.*

Proof. We prove a double inclusion, using again both directions of Lemma 5 (Burning algorithm).

Let $\beta : V\backslash\{s\} \to \mathbb{N}$ be given by $\beta(v) = \deg_G(s, v)$ for any $v \in V\backslash\{s\}$, and $\gamma : V\backslash\{s\} \to \mathbb{N}$ be given by $\gamma(v) = \deg_H(s, v)$ for any $v \in V\backslash\{s\}$. We have $\beta(v) = \gamma(v)$ for any $v \neq w$, and $\beta(w) - \gamma(w) = 1$. Let $c \in \mathcal{C}$ such that $c(w) < \deg^+(w) - 1$. We have to prove that c is also a recurrent configuration of H with respect to sink s. Let $\mathfrak{f} = (v_1, v_2, \cdots, v_k)$ be a firing sequence of c in G such that $v_i \neq s$ for any i, $c + \beta \xrightarrow{\mathfrak{f}} c$, and each vertex of G distinct from s occurs exactly once in $\mathfrak{f}$. We will show that $\mathfrak{f}$ is a firing sequence of $c + \gamma$ in H. Let j be such that $v_j = w$. Clearly, $(v_1, v_2, \cdots, v_{j-1})$ is a firing sequence of $c + \beta$ and $c + \gamma$ in G and H, respectively. Let a be such that $c + \beta \xrightarrow{v_1,v_2,\cdots,v_{j-1}} a$ in G and b be such that $c + \gamma \xrightarrow{v_1,v_2,\cdots,v_{j-1}} b$ in H. It follows from $\beta(w) - \gamma(w) = 1$ that $a(w) - b(w) = 1$. To prove that $\mathfrak{f}$ is a firing sequence of $c + \gamma$ in H it suffices to show that v_j is firable in b with respect to H. Since v_j is firable in a with respect to G, we have $a(w) \geqslant \deg^+(w)$, therefore $b(w) \geqslant \deg^+(w) - 1$. It follows that w is firable in b with respect to H. This implies that $\mathfrak{f}$ is also a firing sequence of $c + \gamma$ with respect to H. By Lemma 5, c is a recurrent configuration of H with respect to sink s.

For the converse, let d be a recurrent configuration of H with respect to sink s. Let $\mathfrak{g} = (u_1, u_2, \cdots, u_p)$ be a firing sequence of $d + \gamma$ in H such that $u_i \neq s$ for any i, $d + \gamma \xrightarrow{\mathfrak{g}} d$ in H, and each vertex of H distinct from s occurs exactly once in $\mathfrak{g}$. We have $d(w) \leqslant \deg^+(w) - 1 < \deg^+(w) - 1$. By similar arguments as above, $\mathfrak{g}$ is also a firing sequence of $d + \beta$ in G, therefore d is a recurrent configuration of G with respect to sink s. □

First, the Tutte polynomial on undirected graphs has the recursive formula $T_G(1, y) = y\,T_{G\backslash e}(1, y)$ if e is a loop. We have the following generalization.

Proposition 4. *If e is a loop then $\mathcal{T}_G(y) = y\,\mathcal{T}_{G\backslash e}(y)$.*

Proof. Let s denote e^-. Let $\mathcal{C}$ be the set of all recurrent configurations of G with sink s. Clearly, $\mathcal{C}$ is also the set of all recurrent configurations of $G\backslash e$ with sink s. Since $\deg^+(s) - \deg^+(s) = 1$, for any $c \in \mathcal{C}$ we have $level_G(c) - level_H(c) = 1$. This implies that $\mathcal{T}_G(y) = y\mathcal{T}_{G\backslash e}(y)$. □

Second, in order to generalize the recursive formula $T_G(1, y) = T_{G/e}(1, y)$ if e is a bridge, we recall the definition of strong bridge of a directed graph.

Definition. Let H be a directed graph. An arc a of H is called *strong bridge* if $H\backslash a$ has more strongly connected components than H.

The next lemma is used in the proofs of subsequent propositions. It aims at showing that there is a similarity between a strong bridge of an Eulerian graph and a bridge of an undirected graph.

Lemma 14. *Let a be a strong bridge of G. Then there is a subset X of V such that $\{a\} = \{e \in A : e^- \in X, e^+ \notin X\}$. Moreover, there is an arc b in G such that $\{b\} = \{e \in A : e^- \notin X, e^+ \in X\}$.*

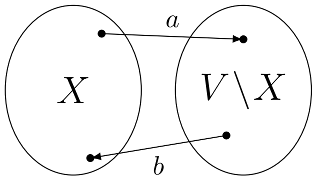


Figure 10: Strong bridge

Proof. Let X be the set of all vertices v of G such that there is a path in $G\backslash a$ from a^- to v. We claim that $a^+ \notin X$. For a contradiction we assume that $a^+ \in X$. This implies that there is a path P in $G\backslash a$ from a^- to a^+. Since G is strongly connected and $G\backslash e$ is not strongly connected, there exists two vertices v, w of G such that every path in G from v to w must contain a. Let Q be a path in G from v to w. We can assume that a occurs exactly once in Q. Let Q_1 be a subpath of Q from v to a^-, and Q_2 be a subpath of Q from a^+ to w. Then, (Q_1, P, Q_2) is a path in G from v to w that does not contain a, a contradiction.

It follows from the definition of X and the above claim that $\{a\} = \{e \in A : e^- \in X, e^+ \notin X\}$. Since G in an Eulerian graph, for every subset X of V there are as many arcs from X to $V\backslash X$ as from $V\backslash X$ to X. The second claim follows. $\square$

See Figure 10 for the illustration of Lemma 14. Note that the set X satisfying the condition of the lemma may not be unique. The following shows that the definition of strong bridge is a natural generalization of the notion of bridge on undirected graphs.

Proposition 5. *Suppose that G is an undirected graph (seen as a directed graph). An arc a is a strong bridge of G if and only if there is a reverse arc b of a in G and $G\backslash\{a, b\}$ is not connected.*

Proof. $\Rightarrow$ Let X be a subset of V that satisfies the condition in Lemma 14. Since G is undirected, there is a reverse arc b of a in G. Clearly, $b \in \{a \in E : a^- \notin X, a^+ \in X\}$. Lemma 14 implies that $\{b\} = \{a \in E : a^- \notin X, a^+ \in X\}$. It follows that $G\backslash\{a, b\}$ contains no arc from X to $V\backslash X$ and vice versa. Therefore $G\backslash\{a, b\}$ is not connected.

$\Leftarrow$ Since G is connected and $G\backslash\{a, b\}$ is not connected, a^- and a^+ are in different connected components of $G\backslash\{a, b\}$. Let X and Y be two connected components of $G\backslash\{a, b\}$ such that $a^- \in X$ and $a^+ \in Y$. Let $v \in X$ and $w \in Y$. Since there is no arc in $G\backslash a$ from a vertex in X to a vertex in Y, there is no path in $G\backslash a$ from v to w. This implies that $G\backslash a$ is not strongly connected. Therefore a is a strong bridge. $\square$

The second relation, extending the recursive formula on undirected graphs $T_G(1, y) = T_{G/e}(1, y)$ if e is a strong bridge, is split into the two following propositions, depending on whether the strong bridge has a reverse arc.

Proposition 6. *Let e be a strong bridge of G such that it does not have a reverse arc. Then $\mathcal{T}_G(y) = \mathcal{T}_{G/e}(y)$.*

Proof. We construct a bijection from the set of recurrent configurations of G/e to the set of recurrent configurations of G that preserves the level. We prove two intermediate claims, and the result follows.

Let s denote e^- and t denote e^+. Let $\mathcal{C}$ be the set of all recurrent configurations of G with respect to the sink s. We claim that for any $c \in \mathcal{C}$ we have $c(t) = \deg^+(t) - 1$. For a contradiction we assume that $c(t) < \deg^+(t) - 1$. Let X be a subset of V that satisfies the condition of Lemma 14. Let $\beta : V\backslash\{s\} \to \mathbb{N}$ be given by $\beta(v) = \deg(s, v)$ for any $v \in V\backslash\{s\}$. The choice of X straightforwardly implies that $\beta(t) = 1$, and $\beta(v) = 0$ for any $v \in V\backslash(X \cup \{t\})$. Let $\mathfrak{f} = (v_1, v_2, \cdots, v_k)$ be a firing sequence of $c + \beta$ such that $v_i \neq s$ for any i, $c + \beta \xrightarrow{\mathfrak{f}} c$, and each vertex v of G distinct from s occurs exactly once in the sequence. Since $c(t) < \deg^+(t) - 1$, there is no firable vertex of $c + \beta$ in $V\backslash X$. This implies that $v_1 \in X$. Let j be the smallest index such that $v_j \in X$ and $v_{j+1} \notin X$, and d be the configuration reach after the j first vertices have been fired, that is, such that $c \xrightarrow{v_1, v_2, \cdots, v_j} d$. Since v_{j+1} is not firable in $c + \beta$ and firable in d, there is at least one vertex $v_p \in \{v_1, v_2, \cdots, v_j\}$ that gives chips to v_{j+1} when it is fired. It follows that there is at least one arc a of G such that $a^- = v_p$ and $a^+ = v_{j+1}$. Clearly, $a \neq e$ and $a \in \{e \in A : e^- \in X, e^+ \notin X\}$, a contradiction to Lemma 14.

Let H denote G/e, let r denote the vertex of H resulting from replacing s and t in G/e, and let $\mathcal{D}$ denote the set of all recurrent configurations of H with the sink r. We claim that $\kappa(\overline{G}) = \kappa(\overline{H})$. By Lemma 11 we have $\kappa(\overline{G}) = \min\{\mathtt{sum}(c) - L(G) : c \in \mathcal{C}\}$ and $\kappa(\overline{H}) = \min\{\mathtt{sum}(c) - L(H) : c \in \mathcal{D}\}$, where $L(G)$ and $L(H)$ are the numbers of loops of G and H, respectively. It follows from the above claim and Lemma 12 that the map $\mu : \mathcal{D} \to \mathcal{C}$, defined by $\mu(c)(v) = c(v)$ if $v \neq t$, and $\mu(c)(t) = \deg^+(t) - 1$, is a bijection. Therefore

$$\min\left\{\sum_{v \neq s} c(v) : c \in \mathcal{C}\right\} - \min\left\{\sum_{v \neq s} c(v) : c \in \mathcal{D}\right\} = \deg^+(t) - 1.$$

Note that $\deg^+(r) = \deg^+(s) + \deg^+(t) - 1$. Finally, since e does not have a reverse arc, we have $L(G) = L(H)$, and the claim follows the fact that $\mathtt{sum}(c) = \deg^+(s) + \sum_{v \neq s} c(v)$.

We can conclude the proof: for any $c \in \mathcal{D}$ we have

$$\begin{aligned} level_G(\mu(c)) &= \deg^+(s) + \sum_{v \neq s} \mu(c)(v) - \kappa(\overline{G}) \\ &= \deg^+(s) + \sum_{v \neq r} c(v) + \deg^+(t) - 1 - \kappa(\overline{H}) \\ &= \deg^+(r) + \sum_{v \neq r} c(v) - \kappa(\overline{H}) \\ &= level_H(c). \end{aligned}$$

This implies $\mathcal{T}_G(y) = \mathcal{T}_H(y)$. □

Proposition 7. *Let a be a strong bridge of G such that it has a reverse arc b, and let H denote G/a. Then $\mathcal{T}_G(y) = \frac{1}{y}\mathcal{T}_H(y)$ and $\mathcal{T}_G(y) = \mathcal{T}_{H\backslash b}(y)$.*

As shown on Figure 10, deleting b in H corresponds to erasing the loop created by the contraction of a.

Proof. It follows from Lemma 14 that b is the unique reverse arc of a. Let t be the new vertex in G/a resulting from replacing the two endpoints of a. Let $\mathcal{C}$ and $\mathcal{D}$ be the sets of all recurrent configurations of G and H with respect to the sinks s and t, respectively. The following can be proved by similar arguments as used in the proof of Proposition 6, noting that b is a loop in G/a.

- $L(H) = L(G) + 1$.
- the map $\mu : \mathcal{D} \to \mathcal{C}$, defined by $\mu(c)(v) = c(v)$ if $v \neq a^+$, and $\mu(c)(a^+) = \deg^+(a^+) - 1$, is a bijection.
- $\kappa(\overline{H}) = \kappa(\overline{G}) - 1$.
- for any $c \in \mathcal{D}$ $level_H(c) = level_G(\mu(c)) + 1$.

The assertions above imply that $\mathcal{T}_G(y) = \frac{1}{y}\mathcal{T}_H(y)$. Since b is a loop in H, it follows from Proposition 4 that $\mathcal{T}_G(y) = \frac{1}{y}\mathcal{T}_H(y) = \frac{1}{y}\, y\, \mathcal{T}_{H\backslash b}(y) = \mathcal{T}_{H\backslash b}(y)$. □

Third, the recursive formula $T_G(1, y) = T_{G\backslash e}(1, y) + T_{G/e}(1, y)$ if e is neither a loop nor a strong bridge has the following generalization.

Proposition 8. *Let a be an arc of G such that a is neither a loop nor a strong bridge, and a has a reverse arc b. Then*

$$\mathcal{T}_G(y) = y^{1+\kappa(\overline{G\backslash\{a,b\}})-\kappa(\overline{G})}\, \mathcal{T}_{G\backslash\{a,b\}}(y) + y^{\kappa(\overline{H})-\kappa(\overline{G})}\, \mathcal{T}_H(y),$$

where H denotes G/a. Moreover, if G is undirected then

$$\mathcal{T}_G(y) = \mathcal{T}_{G\backslash\{a,b\}}(y) + y^{-\deg(a^-,a^+)+1}\, \mathcal{T}_{H\backslash b}(y).$$

In this formula, we reduce $\mathcal{T}_G(y)$ to the sum of the polynomial for G on which both a and its reverse arc b are removed (corresponding to the bridge deletion of the undirected case, see Proposition 7) and the polynomial for G on which a is contracted. The terms y^α are used for re-normalizing according to the definition of level.

Proof. We first give names to useful elements, and then prove both statements of the result one after the other. Let s and t denote a^- and a^+, respectively. Since a is neither a loop nor a strong bridge, $G\backslash\{a, b\}$ is connected. Let $\mathcal{C}_1$ be the set of all recurrent configurations c of G with sink s such that $c(t) = \deg^+(t) - 1$, and let $\mathcal{C}_2$ be the set of all recurrent configurations c of G with sink s such that $c(t) < \deg^+(t) - 1$. We have $\mathcal{C} = \mathcal{C}_1 \cup C_2$, and we will see that each element of this partition corresponds to one of the two terms of the sum. Let r denote the vertex of H resulting from replacing s and t in G. Let $\mathcal{D}$ be the set of all recurrent configurations of H with sink r.

First statement. We begin with $\sum_{c\in\mathcal{C}_1} z^{level_G(c)}$, corresponding to the second term of the sum. It follows from Lemma 12 that the map $\mu : \mathcal{D} \to \mathcal{C}_1$, defined by $\mu(c)(v) = c(v)$ if $v \neq t$, and $\mu(c)(t) = \deg^+(t) - 1$, is bijective. For any $c \in \mathcal{C}_1$ we have

$$level_G(c) = \deg^+(s) + \sum_{v\neq s} c(v) - \kappa(\overline{G})$$

$$= \deg^+(s) + \deg^+(t) - 1 + \sum_{v \notin \{s,t\}} c(v) - \kappa(\overline{H}) + \kappa(\overline{H}) - \kappa(\overline{G})$$
$$= \deg^+(r) + \sum_{v \notin \{s,t\}} c(v) - \kappa(\overline{H}) + \kappa(\overline{H}) - \kappa(\overline{G})$$
$$= level_H(\mu^{-1}(c)) + \kappa(\overline{H}) - \kappa(\overline{G}).$$

This implies that $\sum_{c \in \mathcal{C}_1} z^{level_G(c)} = y^{\kappa(\overline{H}) - \kappa(\overline{G})} \mathcal{T}_H(y)$, which is the second term of the sum.

Regarding $\sum_{c \in \mathcal{C}_2} z^{level_G(c)}$, it follows from Lemma 13 that $\mathcal{C}_2$ is the set of all recurrent configurations of $G \backslash \{a, b\}$ with sink s. For any $c \in \mathcal{C}_2$ we have

$$\begin{aligned} level_G(c) &= \deg^+(s) + \sum_{v \neq s} c(v) - \kappa(\overline{G}) \\ &= 1 + \deg^+(s) + \sum_{v \neq s} c(v) - \kappa(\overline{G \backslash \{a, b\}}) + \kappa(\overline{G \backslash \{a, b\}}) - \kappa(\overline{G}) \\ &= level_{G \backslash \{a,b\}}(c) + 1 + \kappa(\overline{G \backslash \{a, b\}}) - \kappa(\overline{G}). \end{aligned}$$

This implies that

$$\sum_{c \in \mathcal{C}_2} y^{level_G(c)} = y^{1 + \kappa(\overline{G \backslash \{a,b\}}) - \kappa(\overline{G})} \mathcal{T}_{G \backslash \{a,b\}}(y).$$

Since

$$\mathcal{T}_G(y) = \sum_{c \in \mathcal{C}_1} y^{level_G(c)} + \sum_{c \in \mathcal{C}_2} y^{level_G(c)},$$

the first statement follows.

Second statement. G is an undirected graph, so are $G \backslash \{a, b\}$ and H. Thus $1 + \kappa(\overline{G \backslash \{a, b\}}) - \kappa(\overline{G}) = 1 + \frac{|A(\overline{G \backslash \{a,b\}})|}{2} - \frac{|A(\overline{G})|}{2} = 0$. Since b is a loop in H, we have $y^{\kappa(\overline{H}) - \kappa(\overline{G})} \mathcal{T}_H(y) = y^{1 + \kappa(\overline{H}) - \kappa(\overline{G})} \mathcal{T}_{H \backslash b}(y)$. The second statement is completed by showing that $\kappa(\overline{H}) - \kappa(\overline{G}) = -\deg(s, t)$. We have

$$\begin{aligned} \kappa(\overline{H}) - \kappa(\overline{G}) &= \frac{|A(H)| - L(H)}{2} - \frac{|A(G)| - L(G)}{2} \\ &= \frac{|A(H)| - |A(G)|}{2} - \frac{L(H) - L(G)}{2} \\ &= -\frac{1}{2} - \frac{(2\deg(s,t) - 1)}{2} \\ &= -\deg(s, t). \end{aligned}$$

□

Let us present a recursive formula for the partial Tutte polynomial which in the case of undirected graphs is very likely to be known. However, we are unable to find its existence in the literature. If G is undirected, then it contains at least one arc that is a loop, or it satisfies the conditions of Proposition 6, Proposition 7 or Proposition 8. In every case $\mathcal{T}_G(y)$ can be defined by a recursive formula on smaller graphs. However, the digraph given in Figure 11 is an example of Eulerian graph that does not contain any such arc,

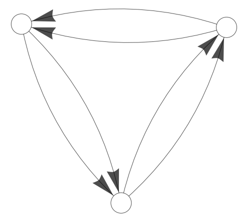


Figure 11: An Eulerian graph that does not satisfy any usual condition

therefore no recursive formula generalizing those of the classical Tutte polynomial can be applied. Neither of the recursive formulas in Proposition 4, Proposition 6, Proposition 7 and Propostion 8 is useful in this case. The following new recursive formula handles this case, in order to complete the recursive definitions of $\mathcal{T}_G(y)$ on the class of general Eulerian graphs. Note that its intuitive shape comes from the Möbius inversion formula that is stated as follows.

Möbius inversion formula. Let X be a non-empty finite set and $f : 2^X \to \mathbb{Z}$. We define $g : 2^X \to \mathbb{Z}$ by $g(A) = \sum_{A\subseteq Y} f(Y)$. Then for every $A \in 2^X$ we have

$$f(A) = \sum_{A\subseteq Y} (-1)^{|Y|-|A|} g(Y).$$

Proposition 9. *Let G be an Eulerian graph, s be a vertex of G, and N be the set of all out-neighbors of s. Then*

$$\mathcal{T}_G(y) = \sum_{\substack{W\subseteq N\\ W\neq\emptyset}} (-1)^{|W|+1} y^{\kappa(\overline{G/W\cup\{s\}})-\kappa(\overline{G})-\deg(s,W)} \frac{1}{(1-y)^{|W|}} \prod_{v\in W} \left(1 - y^{\deg(s,v)}\right) \mathcal{T}_{G/W\cup\{s\}}(y),$$

where $\deg(s,W)$ *denotes the number of arcs e of G such that $e^- = s$ and $e^+ \in W$.*

Note that the number of vertices of the digraph $G/W \cup \{s\}$ is strictly smaller than G. Moreover the digraph $G/W \cup \{s\}$ is likely to have more loops than G, hence we could apply Proposition 4 to remove the loops in $G/W \cup \{s\}$.

Proof. Let $\mathcal{C}$ be the set of all recurrent configurations of G with sink s. For each $c \in \mathcal{C}$, let $N_F(c)$ be the set of out-neighbors of s that, from the configuration c, become firable when s is fired, formally $N_F(c) = \{v \in N : c(v) \geqslant \deg^+(v) - \deg(s,v)\}$. We define

$$P_W(y) = \sum_{\substack{c\in\mathcal{C}\\ W\subseteq N_F(c)}} y^{level_G(c)},$$

so that $\mathcal{T}_G(y) = P_\emptyset(y)$. We will give thereafter a closed formula for $P_W(y)$, which is not interesting if $W = \emptyset$. In order to overcome this issue, let us express $P_\emptyset(y)$ in terms of $P_W(y)$ for $W \neq \emptyset$, using the Möbius inversion formula.

We define $Q_W(y) = \sum\limits_{\substack{c\in\mathcal{C}\\ W=N_F(c)}} y^{level_G(c)}$ so that $P_W(y) = \sum\limits_{W\subseteq S\subseteq N} Q_S(y)$. Moreover, from the Burning algorithm (Lemma 5) it follows that $\{c \in \mathcal{C} : N_F(c) = \emptyset\} = \emptyset$, therefore $Q_\emptyset(y) = 0$. Applying the Möbius inversion formula for the Boolean lattice 2^N we have

$$0 = Q_\emptyset(y) = \sum_{W\subseteq N} (-1)^{|W|} P_W(y),$$

which allows to express $P_\emptyset(y)$ in terms of the other components of the sum,

$$\mathcal{T}_G(y) = P_\emptyset(y) = \sum_{\substack{W\subseteq N\\ W\neq\emptyset}} (-1)^{|W|+1} P_W(y).$$

For the second part of the proof, we claim that

$$P_W(y) = y^{\kappa(\overline{H})-\kappa(\overline{G})-\deg(s,W)} \frac{1}{(1-y)^{|W|}} \prod_{v\in W} \left(1 - y^{\deg(s,v)}\right) \mathcal{T}_H(y),$$

where H denotes $G/W \cup \{s\}$.

The vertices in W are denoted by $w_1, w_2, \cdots, w_p$ for some p, and let $\mathcal{D}$ be the set of all recurrent configurations of H with sink t, where t is the new vertex in H resulting from replacing the vertices in $W \cup \{s\}$. It follows from Lemma 12 and the definition of level that

$$\begin{aligned} P_W(y) &= \sum_{\substack{c\in\mathcal{C}\\ W\subseteq N_F(c)}} y^{level_G(c)} = \sum_{c\in\mathcal{D}} \sum_{\substack{d\in\mathcal{C}\\ d|_{V\setminus(W\cup\{s\})}=c}} y^{level_G(d)} \\ &= \sum_{c\in\mathcal{D}} \left(y^{-\kappa(\overline{G})+\deg^+(s)} \left(\sum_{\substack{d\in\mathcal{C}\\ d|_{V\setminus(W\cup\{s\})}=c}} y^{\sum\limits_{v\in W} d(v)} \right) y^{\sum\limits_{v\notin(W\cup\{s\})} c(v)} \right). \end{aligned}$$

For each $i \in [1..p]$, let $I_i = \{\deg^+(w_i)-\deg(s,w_i), \deg^+(w_i)-\deg(s,w_i)+1, \cdots, \deg^+(w_i)-1\}$. It follows from Lemma 12 that the map $\mu : I_1 \times I_2 \times \cdots \times I_p \times \mathcal{D} \to \mathcal{C}$, defined by $\mu(i_1, i_2, \cdots, i_p, c)(v)$ is equal to $c(v)$ if $v \notin W$, and equal to i_j if $v = w_j$, is bijective, which means that, for a configuration $c \in \mathcal{D}$, the configurations on the graph G constructed from c by putting any number of chips in I_i to w_i produces the whole set $\mathcal{C}$. As a consequence,

$$\begin{aligned} P_W(y) &= \sum_{c\in\mathcal{D}} \left(y^{-\kappa(\overline{G})+\deg^+(s)} \left(\prod_{1\leqslant i\leqslant p} \sum_{j\in I_i} y^j \right) y^{\sum\limits_{v\notin(W\cup\{s\})} c(v)} \right) \\ &= y^{-\kappa(\overline{G})+\deg^+(s)} \left(\prod_{1\leqslant i\leqslant p} \sum_{j\in I_i} y^j \right) \sum_{c\in\mathcal{D}} y^{\sum\limits_{v\notin(W\cup\{s\})} c(v)} \\ &= y^{-\kappa(\overline{G})+\deg^+(s)} \prod_{w\in W} y^{\deg^+(w)-\deg(s,w)} \prod_{w\in W} \frac{\left(1-y^{\deg(s,w)}\right)}{1-y} \sum_{c\in\mathcal{D}} y^{\sum\limits_{v\notin(W\cup\{s\})} c(v)} \end{aligned}$$

$$= y^{-\kappa(\overline{G})-\deg(s,W)} y^{\sum\limits_{v\in W\cup\{s\}}\deg^+(v)} \frac{1}{(1-y)^{|W|}} \prod_{w\in W}\left(1-y^{\deg(s,w)}\right)\sum_{c\in\mathcal{D}} y^{\sum\limits_{v\notin W\cup\{s\}} c(v)}$$

$$= y^{-\kappa(\overline{G})-\deg(s,W)} y^{\deg^+(t)} \frac{1}{(1-y)^{|W|}} \prod_{w\in W}\left(1-y^{\deg(s,w)}\right)\sum_{c\in\mathcal{D}} y^{\sum\limits_{v\neq W\cup\{s\}} c(v)}$$

$$= y^{\kappa(\overline{H})-\kappa(\overline{G})-\deg(s,W)} \frac{1}{(1-y)^{|W|}} \prod_{w\in W}\left(1-y^{\deg(s,w)}\right)\sum_{c\in\mathcal{D}} y^{\deg^+(t)+\sum\limits_{v\notin W\cup\{s\}} c(v)-\kappa(\overline{H})}$$

$$= y^{\kappa(\overline{H})-\kappa(\overline{G})-\deg(s,W)} \frac{1}{(1-y)^{|W|}} \prod_{w\in W}\left(1-y^{\deg(s,w)}\right)\sum_{c\in\mathcal{D}} y^{level_H(c)}$$

$$= y^{\kappa(\overline{H})-\kappa(\overline{G})-\deg(s,W)} \frac{1}{(1-y)^{|W|}} \prod_{w\in W}\left(1-y^{\deg(s,w)}\right)\mathcal{T}_H(y),$$

which proves our claim and finishes the proof. □

5 Some open problems

In this paper we defined a natural analogue of the Tutte polynomial in one variable, for the class of general Eulerian graphs. From a sink-independence property of the generating function of the set of recurrent configurations of the Chip-firing game, it turns out that this polynomial $\mathcal{T}_G(y)$ is characteristic of the support graph itself, regardless of the chosen sink. Most interestingly, this polynomial is equal to the well-known Tutte polynomial $T_G(1, y)$ on undirected graphs. We presented evaluations of $\mathcal{T}_G(y)$ generalizing the evaluations of $T_G(1, y)$, and we hope that new objects counted by evaluations of $\mathcal{T}_G(y)$ will be discovered. Finally, we showed recursive formulas for this polynomial, which again account for natural generalization of those of the Tutte polynomial on undirected graphs. We ended up with a new recursive formula for $\mathcal{T}_G(y)$ in order to get a complete set of recursive formulas defining this polynomial.

It is now natural to ask whether there exists such a natural generalization of $T_G(1, y)$ to the class of connected digraphs. We believe there is such a generalization to the class of strongly connected digraphs by the following surprising conjecture.

Let $G = (V, E)$ be a strongly connected digraph and s be a vertex of G. We denote by $G\backslash s^+$ the digraph constructed from G by removing all out-going arcs of s. Clearly, $G\backslash s^+$ has a global sink s. Fix a linear order $v_1 \prec v_2 \prec \cdots \prec v_{n-1}$ on the set of all vertices of G distinct from s, where $n = |V|$. Let $r_1, r_2, \cdots, r_{n-1} \in \mathbb{Z}^{n-1}$ be given by $r_{i,j} = \deg(v_i, v_j)$ if $i \neq j$, and $r_{i,i} = \deg^+(v_i)$, and let $\beta = (\beta_1, \beta_2, \cdots, \beta_{n-1}) \in \mathbb{Z}^{n-1}$ be given by $\beta_i = \deg(s, v_i)$. We define an equivalence relation $\sim$ on the set $\mathcal{C}$ of all recurrent configurations of $G\backslash s^+$ by $c_1 \sim c_2$ if and only if $c_1 - c_2 \in \langle r_1, r_2, \cdots, r_{n-1}, \beta\rangle$, where $\langle r_1, r_2, \cdots, r_{n-1}, \beta\rangle$ is the subgroup of $(\mathbb{Z}^{n-1}, +)$ generated by $r_1, r_2, \cdots, r_{n-1}, \beta$. Note that if G is Eulerian then $\beta \in \langle r_1, r_2, \cdots, r_{n-1}\rangle$, therefore $\langle r_1, r_2, \cdots, r_{n-1}, \beta\rangle = \langle r_1, r_2, \cdots, r_{n-1}\rangle$. This implies that if G is Eulerian, each equivalence class contains exactly one recurrent configuration. For each $B \in \mathcal{C}/\sim$ let $\texttt{sum}(B)$ denote $\max\{\deg^+(s) + \sum\limits_{v\neq s} c(v) : c \in B\}$. By simulation experiment we make the following conjecture.

Conjecture 1. The sequence $(\mathtt{sum}(B))_{B\in\mathcal{C}/\sim}$ is independent of the choice of s, up to a permutation on the entries.

Note that the cardinality of $\mathcal{C}/\!\sim$ is independent of the choice of s since $|\mathcal{C}/\!\sim| = |(\{x \in \mathbb{Z}^n : \sum_{1\leqslant i\leqslant n} x_i = 0\}, +)/\langle \Delta_1, \Delta_2, \ldots, \Delta_n\rangle|$, where Δ is the Laplacian matrix of G and Δ_i denotes the ith row of Δ. For an Eulerian graph each equivalence class of $\sim$ contains exactly one recurrent configuration. Theorem 1 implies that the conjecture holds for the class of Eulerian graphs. If the conjecture holds in general, we have a generalization of $T_G(1, y)$ to the class of strongly connected digraphs.

It is also important to ask whether there is a generalization of the Tutte polynomial in two variables to the class of Eulerian graphs. The bijection presented in [8] gives a promising direction for this problem, that is, to look for its generalization to the class of Eulerian graphs. In addition, one has to generalize the concepts of internal and external activities to the class of Eulerian graphs. This task is hard, but the generalization of bridge presented in this paper may give insights to address the question.

This question could be addressed by looking for an alternative recursive formula for the Tutte polynomial in two variables on undirected graphs so that it works on Eulerian graphs, possibly for general digraphs. The new recursive formula in Proposition 9 could suggest such a formula since it uses only the vertex contraction in its recursive terms, and the notion of vertex contraction has a natural generalization to general digraphs. Moreover, the following conjecture makes us believe that such a generalization exists.

Conjecture 2. Let G be a connected undirected graph, s a vertex of G, and N the set of all neighbors of s (not including s). Then $T_G(x, y)$ is in the ideal generated by $\{T_{\overline{G/W\cup\{s\}}}(x, y) : \emptyset \subsetneq W \subseteq N\}$ in $\mathbb{Q}[x, y]$, where $\overline{H}$ denotes H in which all loops have been removed.

Equivalently, the conjecture means that there exist polynomials $P_W(x, y) \in \mathbb{Q}[x, y]$, with $\emptyset \subsetneq W \subseteq N$, such that

$$T_G(x, y) = \sum_{\emptyset\subsetneq W\subseteq N} P_W(x, y) T_{\overline{G/W\cup\{s\}}}(x, y).$$

Let us give an illustrative example for the explanation of this conjecture. The first graph in Figure 12 shows an undirected graph G with a chosen vertex s. The remaining graphs are the graphs which are obtained from G by contracting vertex sets $\{s\} \cup W$, where W is a non-empty subset of $\{u, v\}$, and then removing the resulting loops. The vertices x in the contracted graphs are the new vertices which are from the vertex contractions. The Tutte polynomials are shown below the corresponding graphs. By using a Gröbner basis we can verify that the first polynomial is in the ideal generated by the remaining polynomials.

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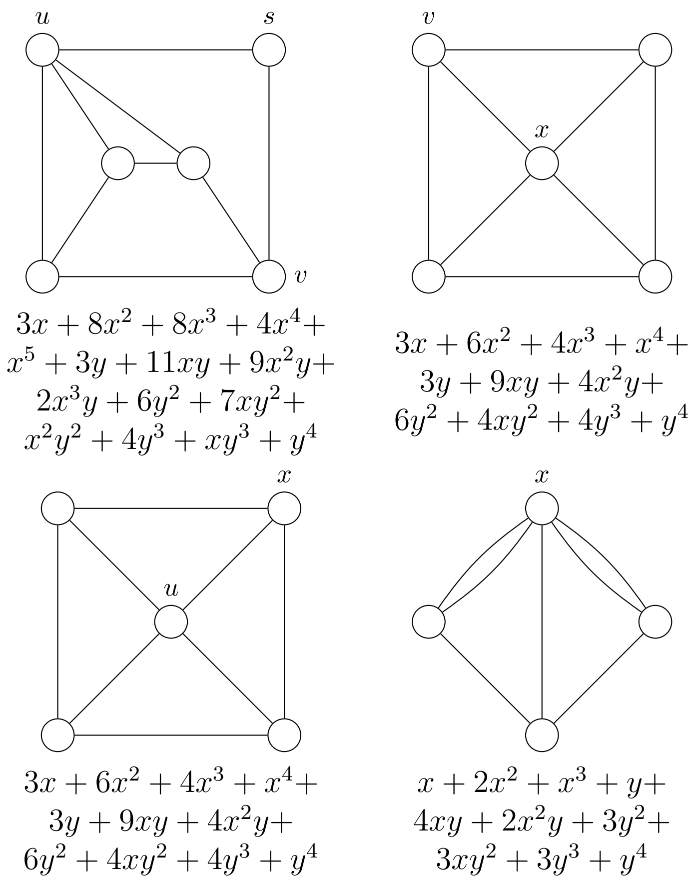

Figure 12: An undirected graph and its vertex contractions

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