Expansions of a chord diagram and alternating permutations

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Abstract

A chord diagram is a set of chords of a circle such that no pair of chords has a common endvertex. A chord diagram E with n chords is called an n-crossing if all chords of E are mutually crossing. A chord diagram E is called nonintersecting if E contains no 2-crossing. For a chord diagram E having a 2-crossing $S = \{x_1x_3, x_2x_4\}$, the expansion of E with respect to S is to replace E with $E_1 = (E \setminus S) \cup \{x_2x_3, x_4x_1\}$ or $E_2 = (E \setminus S) \cup \{x_1x_2, x_3x_4\}$. It is shown that there is a one-to-one correspondence between the multiset of all nonintersecting chord diagrams generated from an n-crossing with a finite sequence of expansions and the set of alternating permutations of order $n + 1$.

Keywords: chord diagram; alternating permutation; Entringer number; Euler number; Ptolemy's theorem

1 Introduction

Let us consider a set of chords of a circle. A set of chords is called a *chord diagram*, if they have no common endvertex. If a chord diagram consists of a set of n mutually crossing chords, it is called an *n-crossing*. A 2-crossing is simply called a crossing as well. If a chord diagram contains no crossing, it is called *nonintersecting*.

Let V be a set of $2n$ vertices on a circle, and let E be a chord diagram of order n, where each chord has endvertices of V. We denote the family of all such chord diagrams by $\mathcal{CD}(V)$. Let $x_1, x_2, x_3, x_4 \in V$ be placed on a circle in clockwise order. Let $E \in \mathcal{CD}(V)$. For a crossing $S = \{x_1x_3, x_2x_4\} \subset E$, let $S_1 = \{x_2x_3, x_4x_1\}$, and $S_2 = \{x_1x_2, x_3x_4\}$. The *expansion* of E with respect to S is defined as a replacement of E with $E_1 = (E \setminus S) \cup S_1$ or $E_2 = (E \setminus S) \cup S_2$ (see Figure 1). In this procedure, E is called the *predecessor* of E_1

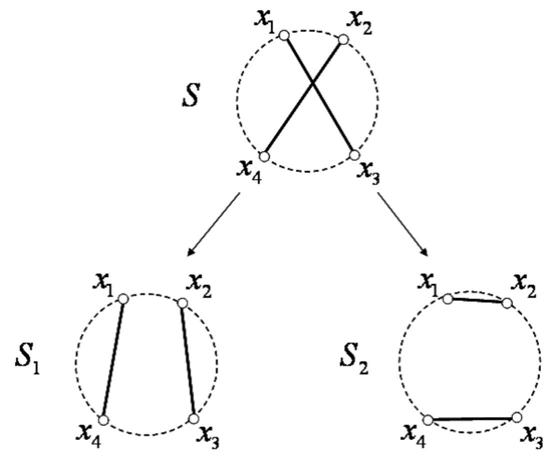

Figure 1: The expansion of a chord diagram.

and E_2, and E_1 and E_2 are called the *successors* of E. A chord of a chord diagram is called *isolated*, if it intersects no other chord.

For $E \in \mathcal{CD}(V)$, let us denote the number of 2-crossings of E by $c(E)$. Let E' be a successor of E such that $E' = (E \setminus S) \cup S'$, where S is an original 2-crossing and S' is a pair of additional chords.

We claim that $c(E') < c(E)$. Indeed, for $e \in E \cap E'$, let t (resp. t') be the number of chords of S (resp. S') intersecting e.

It is not difficult to see that if $t \leqslant 1$ then we have $t' = t$, and if $t = 2$ then we have $t' = 2$ or $t' = 0$. Hence, we have $t' \leqslant t$. Since S is a crossing of E which is removed in E', we have $c(E') < c(E)$.

Lemma 1. Let $E \in \mathcal{CD}(V)$ be a chord diagram. Then beginning from E, the resulting mutiset of nonintersecting chord diagrams generated by a maximal set of expansions is uniquely determined.

Proof. We proceed by induction on the number of crossings c of a chord diagram E.

If $c = 0$ or 1, there is nothing to prove. Let $c \geqslant 2$ and let $c(E) = c$. By inductive hypothesis, for a chord diagram E' with $c(E') \leqslant c-1$, we define $\mathcal{NCD}(E')$ as the resulting multisets of nonintersecting chord diagrams generated by E'. Moreover, for a set of chord diagrams $\mathcal{E}$ such that $E' \in \mathcal{E}$ with $c(E') \leqslant c-1$, let us denote $\mathcal{NCD}(\mathcal{E}) = \cup_{E' \in \mathcal{E}} \mathcal{NCD}(E')$.

Let S_1 and S_2 be two 2-crossings of E, and let E_{i1} and E_{i2} be two successors of E by an expansion with respect to S_i for $i = 1, 2$. Let $\mathcal{E}_i = \{E_{i1}, E_{i2}\}$ for $i = 1, 2$. What we want to show is that $\mathcal{NCD}(\mathcal{E}_1) = \mathcal{NCD}(\mathcal{E}_2)$.

<u>Case 1.</u> $S_1 \cap S_2 = \emptyset$.

For E_{11} and E_{12}, by an expansion with respect to S_2, we have a set $\mathcal{E}'$ of four chord diagrams. Then we have $\mathcal{NCD}(\mathcal{E}_1) = \mathcal{NCD}(\mathcal{E}')$. In the same way, for E_{21} and E_{22}, by an expansion with respect to S_1, we have $\mathcal{E}'$, and we have $\mathcal{NCD}(\mathcal{E}_2) = \mathcal{NCD}(\mathcal{E}')$. Hence, we have $\mathcal{NCD}(\mathcal{E}_1) = \mathcal{NCD}(\mathcal{E}_2)$.

Case 2. $S_1 \cap S_2 \neq \emptyset$.

We may assume $S_1 = \{e_0, e_1\}$ and $S_2 = \{e_0, e_2\}$, where $e_i = x_i y_i$ for $0 \leqslant i \leqslant 2$. Let $V_0 = \{x_0, x_1, x_2, y_0, y_1, y_2\}$ and let $E' = E \setminus \{e_0, e_1, e_2\}$. Beginning from $\mathcal{E}_i$ with $i = 1, 2$, let us consider expansions with respect to a crossing induced by V_0.

Case 2.1. e_1 and e_2 are not crossing.

We may assume x_0, x_1, x_2, y_0, y_2, y_1 are placed on a circle in clockwise order. By iterating possible expansions, not depending on the order of the expansions, we always have a set of four chord diagrams $\mathcal{E}' = \{\ E' \cup \{x_0x_1, x_2y_0, y_2y_1\},\ E' \cup \{x_0x_1, x_2y_1, y_0y_2\},\ E' \cup \{x_0y_1, x_1y_2, x_2y_0\},\ E' \cup \{x_0y_1, x_1x_2, y_0y_2\}\ \}$.

Case 2.2. e_1 and e_2 are crossing.

We may assume x_0, x_1, x_2, y_0, y_1, y_2 are placed on a circle in clockwise order. By iterating possible expansions, not depending on the order of the expansions, we always have a set of five chord diagrams $\mathcal{E}' = \{\ E' \cup \{x_0x_1, x_2y_0, y_1y_2\},\ E' \cup \{x_0y_2, x_1x_2, y_0y_1\},\ E' \cup \{x_0x_1, x_2y_2, y_0y_1\},\ E' \cup \{x_0y_0, x_1x_2, y_1y_2\},\ E' \cup \{x_0y_2, x_1y_1, x_2y_0\}\ \}$.

In any case, we have $\mathcal{NCD}(\mathcal{E}_i) = \mathcal{NCD}(\mathcal{E}')$ for $i = 1, 2$, as required. □

Let us denote the multiset of nonintersecting chord diagrams generated by $E \in \mathcal{CD}(V)$ by $\mathcal{NCD}(E)$. For $E \in \mathcal{CD}(V)$, let us define $f(E)$ as the cardinality of $\mathcal{NCD}(E)$ as a multiset.

Example 2. Let C_n be an n-crossing. Then we have $f(C_2) = 2$, $f(C_3) = 5$ and $f(C_4) = 16$. (See Figure 2.)

A background of expansions of a chord diagram is Ptolemy's theorem and its generalization. For two points x, y on a circle, let $\overline{xy}$ be the length of a chord xy. Ptolemy's theorem states that if $E = \{x_1x_3, x_2x_4\}$ itself is a 2-crossing, then we have $\overline{x_1x_3} \cdot \overline{x_2x_4} = \overline{x_2x_3} \cdot \overline{x_4x_1} + \overline{x_1x_2} \cdot \overline{x_3x_4}$. In other words, we have

$$\prod_{e \in E} \overline{e} \ = \ \prod_{e \in E_1} \overline{e} + \prod_{e \in E_2} \overline{e}, \tag{1}$$

where E_1 and E_2 are two successors of E. In general, for a given $E \in \mathcal{CD}(V)$, by iterating expansions with applications of Ptolemy's theorem, we have

$$\prod_{e \in E} \overline{e} \ = \sum_{E' \in \mathcal{NCD}(E)} \prod_{e \in E'} \overline{e}. \tag{2}$$

If E is a 3-crossing, the equation (2) is known as Fuhrmann's Theorem ([2]).

2 Main Results

For two nonnegative integers k and n with $k \leqslant n$, we define $A(n, k)$ as a chord diagram of order $n + 1$, in which there is an n-crossing E_0 with an extra chord e such that e crosses

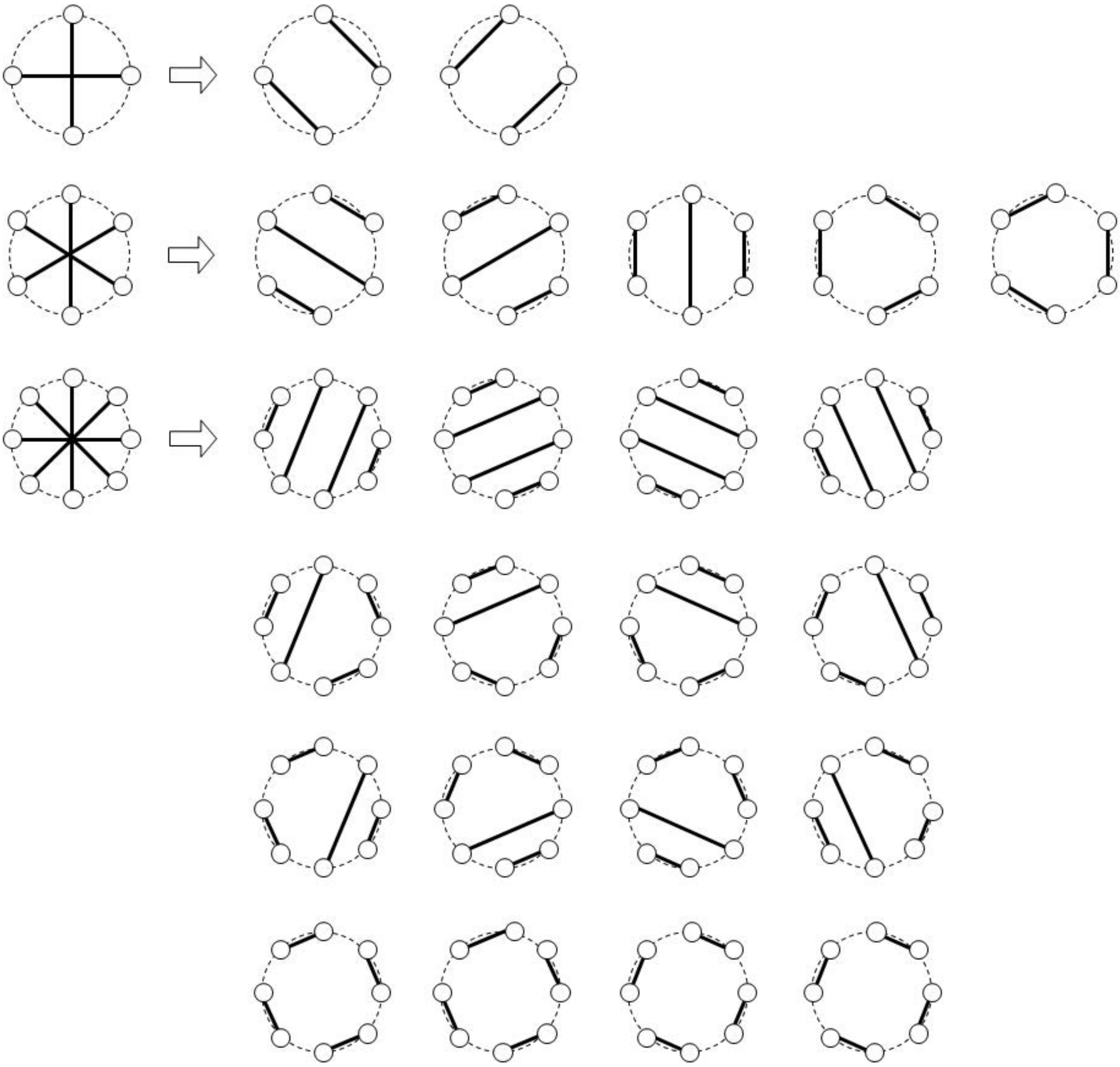

Figure 2: Multisets of nonintersecting chord diagrams generated by a 2-crossing(upper), a 3-crossing(middle) and a 4-crossing (lower).

exactly k chords of E_0. Note that $A(n-1, n-1)$ is simply an n-crossing, and that $A(n, 0)$ is a union of an n-crossing and an isolated chord. Hence, we have $f(A(n-1, n-1)) = f(A(n, 0))$. The values of $f(A(n, k))$ for small nonnegative integers n and k are shown in Table 1.

Table 1: $f(A(n, k))$ for $0 \leqslant k \leqslant n \leqslant 6$.

$n \setminus k$	0	1	2	3	4	5	6
0	1						
1	1	2					
2	2	4	5				
3	5	10	14	16			
4	16	32	46	56	61		
5	61	122	178	224	256	272	
6	272	544	800	1024	1202	1324	1385

A permutation σ of $[n] = \{1, 2, \ldots, n\}$ is called an *alternating permutation* if $(\sigma(i) - \sigma(i-1))(\sigma(i+1) - \sigma(i)) < 0$ for $2 \leqslant i \leqslant n-1$ (see [9] for an excellent survey of alternating permutations). An alternating permutation σ is called an *up-down permutation* (resp. *down-up permutation*) if $\sigma(1) < \sigma(2)$ (resp. $\sigma(1) > \sigma(2)$). Let $\mathcal{UDP}(n, k)$ denote the set of up-down permutations of $[n]$ with the first term at most k. Similarly, let $\mathcal{DUP}(n, k)$ denote the set of down-up permutations of $[n]$ with the first term at least $n-k+1$. Note that by definition, there is a natural bijection from $\mathcal{UDP}(n, k)$ to $\mathcal{DUP}(n, k)$.

The main result of the paper is the following theorem.

Theorem 3. For $0 \leqslant k \leqslant n$, there is a bijection from $\mathcal{NCD}(A(n, k))$ to $\mathcal{UDP}(n+2, k+1)$.

For $0 \leqslant k \leqslant n$, Entringer number $E_{n,k}$ is defined as the number of down-up permutations of $[n+1]$ with the first term $k+1$ [1], which equals the cardinality of $\mathcal{UDP}(n, k)$. Since for $n \geqslant 1$, $E_{n+1,1}$ equals Euler number E_n, the number of down-up permutations of $[n]$, we have the following Corollary.

Corollary 4. For $0 \leqslant k \leqslant n$, we have $f(A(n, k)) = E_{n+2,k+1}$. In particular, we have $f(A(n, 0)) = E_{n+1}$.

Several combinatorial interpretations for Entringer numbers are known ([4, 5, 6, 7, 8]). The generating function for Entringer number is treated in [3] as an exercise, Exer. 6.75. According to [3], it follows that

$$\sum_{n \geqslant 0} \sum_{k \geqslant 0} E_{n+k,k} \frac{x^n}{n!} \frac{y^k}{k!} = \frac{\cos x + \sin y}{\cos(x+y)}.$$

By Corollary 4, we have

$$\begin{aligned} \sum_{n \geqslant 0} \sum_{k \geqslant 0} f(A(n+k, k)) \frac{x^n}{n!} \frac{y^k}{k!} &= \frac{\partial^2}{\partial x \partial y} \left(\sum_{n \geqslant 0} \sum_{k \geqslant 0} E_{n+k,k} \frac{x^n}{n!} \frac{y^k}{k!} \right) \\ &= \frac{\cos x + \sin y}{\cos(x+y)(1 - \sin(x+y))}. \end{aligned}$$

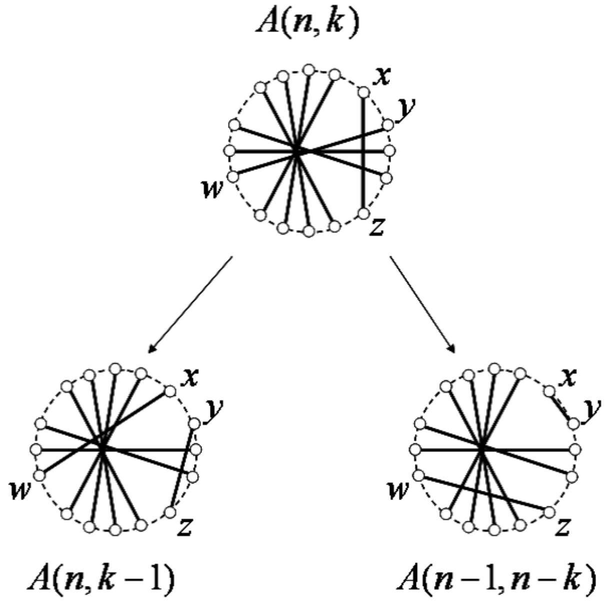

Figure 3: Two successors of $A(n,k)$, where $n = 7$ and $k = 3$.

3 Proof of Theorem 3

For two chord diagrams $F_i \in \mathcal{CD}(V_i)$ for $i = 1, 2$, let $F_i' \in \mathcal{CD}(V_i')$ be a chord diagram such that F_i' consists of the set of all nonisolated chords of F_i. Suppose that $|F_1'| = |F_2'|$. Let $V_i' = \{v_{i,0}', v_{i,1}', \ldots, v_{i,2n'-1}\}$, and the vertices are placed on a circle in clockwise order for each $i = 1, 2$. Suppose that $v_{1,\alpha}' v_{1,\beta}' \in F_1'$ holds if and only if $v_{2,\alpha}' v_{2,\beta}' \in F_2'$ holds. Then we say that F_1' and F_2' are isomorphic, and furthermore we say that F_1 and F_2 are isomorphic as well.

In order to prove Theorem 3, we will recursively construct a bijection from $\mathcal{NCD}(A(n,k))$ to $\mathcal{UDP}(n+2, k+1)$ for $0 \leqslant k \leqslant n$.

Firstly, we will show a recurrence for $\mathcal{NCD}(A(n,k))$, which is a key ingredient for the proof of Theorem 3.

Lemma 5. For $1 \leqslant k \leqslant n$, we have a bijection between $\mathcal{NCD}(A(n,k))$ and $\mathcal{NCD}(A(n,k-1)) \cup \mathcal{NCD}(A(n-1,n-k))$. In particular, we have $f(A(n,k)) = f(A(n,k-1)) + f(A(n-1,n-k))$.

Proof. Let E be a chord diagram isomorphic to $A(n,k)$. We may assume E contains an n-crossing E_0 and an extra edge $e = xz$ such that e crosses exactly k edges of E_0.

Let $f = yw$ be an edge of E_0 such that (1) x, y, z, w are placed on a circle in clockwise order and (2) there is no endvertex of E_0 between x and y. (See Figure 3.)

Put $S = \{xz, yw\}$. Let us expand E with respect to S. We have two successors E_1, E_2 of E, where $E_1 = (E \setminus S) \cup \{yz, wx\}$ and $E_2 = (E \setminus S) \cup \{xy, zw\}$. Then E_1 is isomorphic to $A(n,k-1)$ and E_2 is isomorphic to $A(n-1,n-k)$. Hence, we have a bijection between $\mathcal{NCD}(A(n,k))$ and $\mathcal{NCD}(A(n,k-1)) \cup \mathcal{NCD}(A(n-1,n-k))$. □

For the sake of completeness, we recall the well-known recurrence relation for $\mathcal{UDP}(n,k)$.

Lemma 6. For $1 \leqslant k \leqslant n$, we have a bijection between $\mathcal{UDP}(n+2,k+1)$ and $\mathcal{UDP}(n+2,k) \cup \mathcal{UDP}(n+1,n-k+1)$.

Proof. By the definition, $\mathcal{UDP}(n+2,k+1)$ is a set of up-down permutations of $[n+2]$ with the first term at most $k+1$. $\mathcal{UDP}(n+2,k+1)$ is partitioned into $\mathcal{UDP}(n+2,k)$ and $\mathcal{T} = \mathcal{UDP}(n+2,k+1) \setminus \mathcal{UDP}(n+2,k)$, where $\mathcal{T}$ is a set of up-down permutations of $[n+2]$ with the first term $k+1$.

For $\sigma \in \mathcal{T}$, let us remove the first term of σ. The resulting permutation σ' is a down-up permutation of $[n+2] \setminus \{k+1\}$ with the first term at least $k+2$. Hence, there is a natural bijection from $\mathcal{T}$ to $\mathcal{DUP}(n+1,n-k+1)$, which has a one-to-one correspondence to $\mathcal{UDP}(n+1,n-k+1)$. $\square$

Now, we return to the proof of Theorem 3.

For $n=0$ and $k=0$, a set of a single chord of $\mathcal{NCD}(A(0,0))$ clearly corresponds to a single permutation 12 of $\mathcal{UDP}(2,1)$.

Let $n \geqslant 1$ and $k \geqslant 0$. By the inductive hypothesis, we have a bijection from $\mathcal{NCD}(A(n',k'))$ to $\mathcal{UDP}(n'+2,k'+1)$ for $n' < n$ or $n' = n$ and $k' < k$.

For $k=0$, $A(n,0)$ is isomorphic to $A(n-1,n-1)$. Hence, there is a bijection from $\mathcal{NCD}(A(n,0))$ to $\mathcal{NCD}(A(n-1,n-1))$. On the other hand, let $\sigma \in \mathcal{UDP}(n+2,1)$. By removing the first term of σ, we have a down-up permutation σ' of $[n+2] \setminus \{1\}$. Hence, there is a natural bijection from $\mathcal{UDP}(n+2,1)$ to $\mathcal{DUP}(n+1,n)$, which has a one-to-one correspondence to $\mathcal{UDP}(n+1,n)$. Therefore, we have a bijection from $\mathcal{NCD}(A(n,0))$ to $\mathcal{UDP}(n+2,1)$.

Let $k \geqslant 1$. In this case, by Lemma 5 and Lemma 6, we can recursively construct a bijection from $\mathcal{NCD}(A(n,k))$ to $\mathcal{UDP}(n+2,k+1)$.

This completes the proof. $\square$

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