Weakly distance-regular digraphs of valency three, I

Yuefeng Yang
Sch. Math. Sci. & Lab. Math. Com. Sys.,
Beijing Normal University,
Beijing, 100875, China
yangyf@mail.bnu.edu.cn

Benjian Lv
Sch. Math. Sci. & Lab. Math. Com. Sys.,
Beijing Normal University,
Beijing, 100875, China
bjlv@bnu.edu.cn

Kaishun Wang
Sch. Math. Sci. & Lab. Math. Com. Sys.,
Beijing Normal University,
Beijing, 100875, China
wangks@bnu.edu.cn

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Abstract

Suzuki (2004) classified thin weakly distance-regular digraphs and proposed the project to classify weakly distance-regular digraphs of valency 3. The case of girth 2 was classified by the third author (2004) under the assumption of the commutativity. In this paper, we continue this project and classify these digraphs with girth more than 2 and two types of arcs.

Keywords: Weakly distance-regular digraph; Cayley digraph

1 Introduction

A *digraph* Γ is a pair (X, A) where X is a finite set of *vertices* and $A \subseteq X^2$ is a set of *arcs*. Throughout this paper we use the term 'digraph' to mean a finite directed graph with no loops. We write $V\Gamma$ for X and $A\Gamma$ for A. A *path* of length r from u to v is a finite sequence of vertices $(u = w_0, w_1, \ldots, w_r = v)$ such that $(w_{t-1}, w_t) \in A\Gamma$ for $t = 1, 2, \ldots, r$. A digraph is said to be *strongly connected* if, for any two distinct vertices x and y, there is a path from x to y. The length of a shortest path from x to y is called the *distance* from x to y in Γ, denoted by $\partial_\Gamma(x, y)$. The *diameter* of Γ is the maximum value of the distance function in Γ. Let $\widetilde{\partial}_\Gamma(x, y) = (\partial_\Gamma(x, y), \partial_\Gamma(y, x))$ and $\widetilde{\partial}(\Gamma) = \{\widetilde{\partial}_\Gamma(x, y) \mid x, y \in V\Gamma\}$. If no confusion occurs, we write $\partial(x, y)$ (resp. $\widetilde{\partial}(x, y)$) instead of $\partial_\Gamma(x, y)$ (resp. $\widetilde{\partial}_\Gamma(x, y)$). An arc (u, v) of Γ is of *type* $(1, r)$ if $\partial(v, u) = r$. A path $(w_0, w_1, \ldots, w_{r-1})$ is said to be a *circuit* of length r if $\partial(w_{r-1}, w_0) = 1$. A circuit is *undirected* if each of its arcs is of type $(1,1)$. The *girth* of Γ is the length of a shortest circuit.

Let $\Gamma = (X, A)$ and $\Gamma' = (X', A')$ be two digraphs. Γ and Γ' are *isomorphic* if there is a bijection σ from X to X' such that $(x, y) \in A$ if and only if $(\sigma(x), \sigma(y)) \in A'$. In this case, σ is called an *isomorphism* from Γ to Γ'. An isomorphism from Γ to itself is called an *automorphism* of Γ. The set of all automorphisms of Γ forms a group which is called the *automorphism group* of Γ and denoted by $\mathrm{Aut}(\Gamma)$. A digraph Γ is *vertex transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on $V\Gamma$.

Lam [5] introduced a concept of distance-transitive digraphs. A strongly connected digraph Γ is said to be *distance-transitive* if, for any vertices x, y, x' and y' of Γ satisfying $\partial(x, y) = \partial(x', y')$, there exists an automorphism σ of Γ such that $x' = \sigma(x)$ and $y' = \sigma(y)$. Damerell [4] generalized this concept to that of distance-regular digraphs. He showed that the girth g of a distance-regular digraph of diameter d is either 2, d or $d + 1$, and the one with $d = g$ is a coclique extension of a distance-regular digraph with $d = g - 1$. Bannai, Cameron and Kahn [2] proved that a distance-transitive digraph of odd girth is a Paley tournament or a directed cycle. Leonard and Nomura [6] proved that except directed cycles all distance-regular digraphs with $d = g - 1$ have girth $g \leqslant 8$. In order to find 'better' classes of digraphs with unbounded diameter, Damerell [4] also proposed a more natural definition of distance-transitivity, i.e., weakly distance-transitivity. In [8], Wang and Suzuki introduced weakly distance-regular digraphs as a generalization of distance-regular digraphs and weakly distance-transitive digraphs.

A strongly connected digraph Γ is said to be *weakly distance-transitive* if, for any vertices x, y, x' and y' satisfying $\widetilde{\partial}(x, y) = \widetilde{\partial}(x', y')$, there exists an automorphism σ of Γ such that $x' = \sigma(x)$ and $y' = \sigma(y)$. A strongly connected digraph Γ is said to be *weakly distance-regular* if, for all $\widetilde{h}$, $\widetilde{i}$, $\widetilde{j} \in \widetilde{\partial}(\Gamma)$ and $\widetilde{\partial}(x, y) = \widetilde{h}$, the number $p^{\widetilde{h}}_{\widetilde{i},\widetilde{j}} := |P_{\widetilde{i},\widetilde{j}}(x, y)|$ depends only on $\widetilde{h}$, $\widetilde{i}$, $\widetilde{j}$, where

$$P_{\widetilde{i},\widetilde{j}}(x, y) = \{z \in V\Gamma \mid \widetilde{\partial}(x, z) = \widetilde{i} \text{ and } \widetilde{\partial}(z, y) = \widetilde{j}\}.$$

The nonnegative integers $p^{\widetilde{h}}_{\widetilde{i},\widetilde{j}}$ are called the *intersection numbers*. We say that Γ is *commutative* (resp. *thin*) if $p^{\widetilde{h}}_{\widetilde{i},\widetilde{j}} = p^{\widetilde{h}}_{\widetilde{j},\widetilde{i}}$ (resp. $p^{\widetilde{h}}_{\widetilde{i},\widetilde{j}} \leqslant 1$) for all $\widetilde{i}$, $\widetilde{j}$, $\widetilde{h} \in \widetilde{\partial}(\Gamma)$. Note that a weakly distance-transitive digraph is weakly distance-regular.

Let G be a finite group and S a subset of G not containing the identity. The *Cayley digraph* $\Gamma = \mathrm{Cay}(G, S)$ is a digraph with the vertex set G and the arc set $\{(x, sx) \mid x \in G,\ s \in S\}$.

In [8], Wang and Suzuki determined all commutative 2-valent weakly distance-regular digraphs. In [7], Suzuki determined all thin weakly distance-regular digraphs and proved the nonexistence of noncommutative weakly distance-regular digraphs of valency 2. Moreover, he proposed the project to classify weakly distance-regular digraphs of valency 3. In [9], Wang classified all commutative weakly distance-regular digraphs of valency 3 and girth 2. In this paper, we continue this project, and obtain the following result.

Theorem 1. *Let Γ be a weakly distance-regular digraph of valency 3 and girth more than 2. If Γ has two types of arcs, then Γ is isomorphic to one of the following digraphs:*

(i) $\mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_g, \{(0,1),(2,1),(1,0)\})$, *where* $g = 3$ *or* $g \geqslant 5$.

(ii) $\Gamma_{q,2mq,1}$, $\Gamma_{q,mq+2,q}$ *or* $\Gamma_{q,2mq-2q+2t,q+1-t}$ *in Construction 3, where* $q \geqslant 3$, $m \geqslant 1$ *and* $2 \leqslant t \leqslant q-1$.

This paper is organized as follows. In Section 2, we construct two families of weakly distance-regular digraphs of valency 3. In Section 3, we discuss some properties for circuits of weakly distance-regular digraphs. In Section 4, we prove our main theorem.

2 Constructions

In this section, we construct two families of weakly distance-regular digraphs of valency 3. For any element x in a residue class ring, we assume that $\hat{x}$ denotes the minimum nonnegative integer in x. Denote $\beta(w) = (1 + (-1)^{w+1})/2$ for any integer w.

Proposition 2. *Let* $g \geqslant 3$. *Then* $\Gamma_g := \mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_g, \{(1,0),(0,1),(2,1)\})$ *is a weakly distance-regular digraph if and only if* $g \neq 4$.

Proof. For any vertex (a,b) distinct with $(0,0)$, we have

$$\widetilde{\partial}((0,0),(a,b)) = \begin{cases} (\hat{a}, 4-\hat{a}), & \text{if } b = 0, \\ (\hat{b}+\beta(\hat{a}), g-\hat{b}+\beta(\hat{a})), & \text{if } b \neq 0. \end{cases}$$

Suppose $g \neq 4$. We will show that Γ_g is weakly distance-transitive. Let (a,b) and (x,y) be any two vertices satisfying $\widetilde{\partial}((0,0),(a,b)) = \widetilde{\partial}((0,0),(x,y))$. It suffices to verify that there exists an automorphism σ of Γ_g such that $\sigma(0,0) = (0,0)$ and $\sigma(a,b) = (x,y)$. If $(a,b) = (x,y)$, then the identity permutation is a desired automorphism. Now suppose $(a,b) \neq (x,y)$. Then $b \neq 0$, $y \neq 0$ and $(\hat{b}+\beta(\hat{a}), g-\hat{b}+\beta(\hat{a})) = (\hat{y}+\beta(\hat{x}), g-\hat{y}+\beta(\hat{x}))$. It follows that $b = y$ and $a - x = 2$. Let σ be the permutation on $V\Gamma_g$ such that

$$\sigma(u,v) = \begin{cases} (u,v), & \text{if } v \neq b, \\ (u+2,v), & \text{if } v = b. \end{cases}$$

Routinely, σ is a desired automorphism.

In Γ_4, $\widetilde{\partial}((0,0),(0,2)) = \widetilde{\partial}((0,0),(2,0)) = (2,2)$. But $P_{(1,3),(3,3)}((0,0),(0,2)) = \{(1,0)\}$ and $P_{(1,3),(3,3)}((0,0),(2,0)) = \emptyset$. Hence, Γ_4 is not a weakly distance-regular digraph. ☐

Construction 3. Let q, s, k be integers with $q > 2$, $s > 2$ and $\max\{1, q-s+2\} \leqslant k \leqslant q$. Write $s = 2mq + p$ with $m \geqslant 0$ and $0 \leqslant p < 2q$. Let $\Gamma_{q,s,k}$ be the digraph with the vertex set $\mathbb{Z}_q \times \mathbb{Z}_s$ whose arc set consists of $((a,b),(a+1,b))$, $((a,c),(a,c+1))$, $((a,d),(a+1,d-1))$, $((a,-1),(a-k+1,0))$ and $((a,0),(a+k,-1))$, where $a \in \mathbb{Z}_q$, $b,c,d \in \mathbb{Z}_s$, $\hat{c} \neq s-1$ and $d \neq 0$. See Figure 1.

In the following, we will prove that $\Gamma_{q,s,k}$ is a weakly distance-regular digraph if and only if one of the following holds:

C1: $p = 0$ and $k = 1$.

C2: $p = q+2$ or $p = 2$, and $k = q$.

C3: $4 \leqslant p \leqslant 2q-2$, p is even and $k = q+1-p/2$.

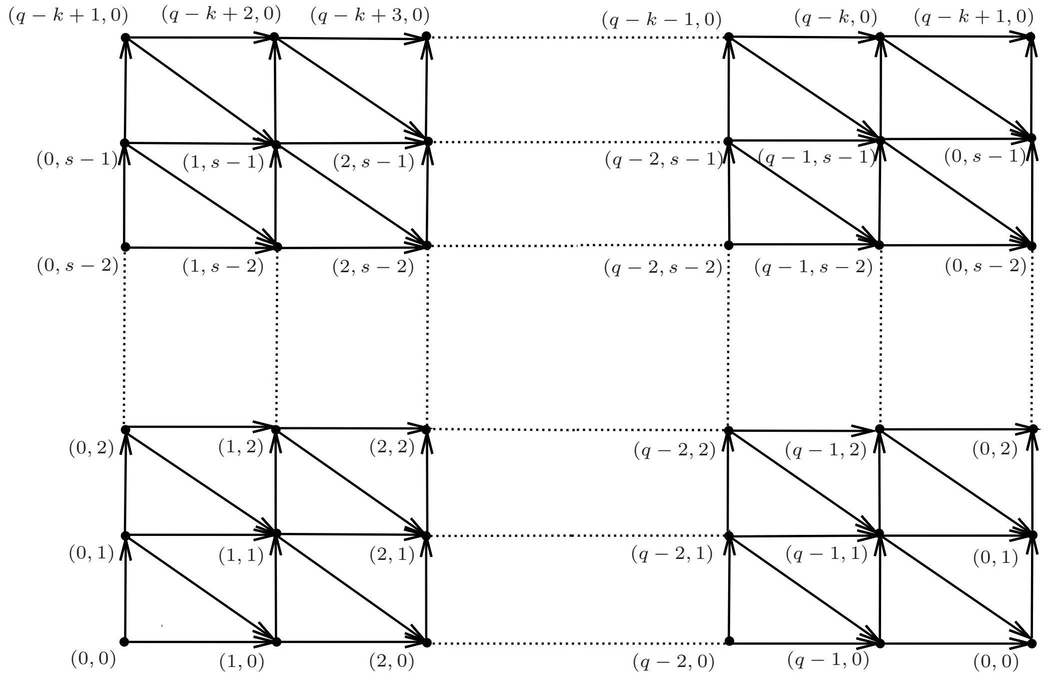

Figure 1: The digraph $\Gamma_{q,s,k}$.

Lemma 4. *$\Gamma_{q,s,k}$ is a vertex transitive digraph.*

Proof. Pick any vertex (a,b). It suffices to show that there exists an automorphism σ of $\Gamma_{q,s,k}$ such that $\sigma(0,0)=(a,b)$. Let σ be the permutation on $V\Gamma_{q,s,k}$ such that

$$\sigma(x,y)=\begin{cases}(x+a,y+b), & \text{if } \hat{y}\in\{0,1,2,\dots,s-1-\hat{b}\},\\ (x+a-k+1,y+b), & \text{otherwise.}\end{cases}$$

Routinely, σ is a desired automorphism. $\square$

For any two integers i and j, we write $i\equiv j$ instead of $i\equiv j \pmod q$. For any vertex (a,b) of $\Gamma_{q,s,k}$, let $f(a,b)$, $g(a,b)$ and $h(a)$ be nonnegative integers less than q such that

$$f(a,b)\equiv \hat{a}+\hat{b}-k-p+1,\ g(a,b)\equiv q-\hat{a}-\hat{b} \text{ and } h(a)\equiv k-\hat{a}-1. \tag{1}$$

By the structure of $\Gamma_{q,s,k}$, we have

$$\widetilde{\partial}((0,0),(a,b))=(\min\{\hat{a}+\hat{b},s-\hat{b}+f(a,b)\},\min\{\hat{b}+g(a,b),s-\hat{b}+h(a)\}). \tag{2}$$

Lemma 5. *Let* C1, C2 *or* C3 *hold. In $\Gamma_{q,s,k}$, $\partial((0,0),(a,b))=\hat{a}+\hat{b}$ if and only if $\partial((a,b),(0,0))=\hat{b}+g(a,b)$.*

Proof. Let $M=s-2\hat{b}-\hat{a}+f(a,b)$, $N=s-2\hat{b}+h(a)-g(a,b)$ and $\hat{b}=n'q+r'$ with $0\leqslant r'<q$. By (2), we only need to prove $M\geqslant 0$ if and only if $N\geqslant 0$. From (1), note that $f(a,b)+g(a,b)$ equals to $k-1$ or $q+k-1$, and $h(a)$ equals to $k-\hat{a}-1$ or $q+k-\hat{a}-1$.

Case 1. $f(a,b)+g(a,b)=k-1$ and $h(a)=k-\hat{a}-1$, or $f(a,b)+g(a,b)=q+k-1$ and $h(a)=q+k-\hat{a}-1$.

In this case, it is routine to check $M=N$, as desired.

Case 2. $f(a,b)+g(a,b)=k-1$ and $h(a)=q+k-\hat{a}-1$.

In this case, only C1 or C3 holds by $k<\hat{a}+1$.

Assume that C1 holds. Then $f(a,b)=g(a,b)=0$ and $h(a)=q-\hat{a}$, which imply that $\hat{a}+r'=0$ or q. Since $h(a)<q$, one gets $\hat{a}\neq 0$. Hence, $\hat{a}+r'=q$. Then $M=2(m-n')q-q-r'\geqslant 0$ if and only if $N=2(m-n')q-r'\geqslant 0$.

Assume that C3 holds. Then $f(a,b)+g(a,b)=q-p/2$ and $h(a)=2q-p/2-\hat{a}$. By (1), note that $g(a,b)=q-\hat{a}-r'$ or $2q-\hat{a}-r'$, and $p/2+\hat{a}>q$. Suppose $g(a,b)=q-\hat{a}-r'$. Then $f(a,b)=\hat{a}+r'-p/2$. From $\hat{a}+r'\leqslant q$, we have $0<p/2-r'<q$, which implies $M=2(m-n')q+p/2-r'\geqslant 0$ if and only if $N=2(m-n')q+q+p/2-r'\geqslant 0$. Suppose $g(a,b)=2q-\hat{a}-r'$. Since $f(a,b)=\hat{a}-q+r'-p/2$, we have $0\leqslant r'-p/2<q$, which implies $M=2(m-n')q-q+p/2-r'\geqslant 0$ if and only if $N=2(m-n')q+p/2-r'\geqslant 0$.

Case 3. $f(a,b)+g(a,b)=q+k-1$ and $h(a)=k-\hat{a}-1$.

In this case, only C1 or C3 holds by $f(a,b)+g(a,b)\leqslant 2(q-1)$.

Assume that C1 holds. Then $h(a)=\hat{a}=0$ and $r'\neq 0$, which imply that $f(a,b)=r'$ and $g(a,b)=q-r'$. Then $M=2(m-n')q-r'\geqslant 0$ if and only if $N=2(m-n')q-q-r'\geqslant 0$.

Assume that C3 holds. Then $f(a,b)+g(a,b)=2q-p/2$ and $h(a)=q-p/2-\hat{a}$. By (1), note that $g(a,b)=q-\hat{a}-r'$ or $2q-\hat{a}-r'$, and $p/2+\hat{a}\leqslant q$. Suppose $g(a,b)=q-\hat{a}-r'$. Then $f(a,b)=q+\hat{a}+r'-p/2$ and $0\leqslant q+r'-p/2<q$, which imply $M=2(m-n'+1)q-q-r'+p/2\geqslant 0$ if and only if $N=2(m-n')q+p/2-r'\geqslant 0$. Suppose $g(a,b)=2q-\hat{a}-r'$. Since $f(a,b)=\hat{a}+r'-p/2$, we have $p/2-r'\leqslant\hat{a}<q$, which implies $M=2(m-n')q+p/2-r'\geqslant 0$ if and only if $N=2(m-n')q-q+p/2-r'\geqslant 0$.

Thus, desired result follows. □

Lemma 6. *If* C1, C2 *or* C3 *holds, then* $\Gamma_{q,s,k}$ *is a weakly distance-regular digraph.*

Proof. We will prove that $\Gamma_{q,s,k}$ is weakly distance-transitive. Let (a,b) and (x,y) be two vertices satisfying $\widetilde{\partial}((0,0),(a,b))=\widetilde{\partial}((0,0),(x,y))$. It suffices to find $\sigma\in\mathrm{Aut}(\Gamma_{q,s,k})$ such that $\sigma(0,0)=(0,0)$ and $\sigma(a,b)=(x,y)$. By (2), we divide the proof into two cases.

Case 1. $\partial((0,0),(a,b))=\hat{a}+\hat{b}$.

Suppose $\partial((0,0),(x,y))=\hat{x}+\hat{y}$. Then $g(a,b)=g(x,y)$. By Lemma 5, we have $\hat{b}+g(a,b)=\hat{y}+g(x,y)$. This implies that $a=x$ and $b=y$. Hence, the identity permutation is a desired automorphism.

Suppose $\partial((0,0),(x,y))=s-\hat{y}+f(x,y)$. Then $\hat{a}+\hat{b}=s-\hat{y}+f(x,y)$, which implies $\hat{x}\equiv\hat{a}+\hat{b}+k-1$ by $s\equiv p$. Hence, $\hat{x}=f(a,b)$ and $g(a,b)=h(x)$. From Lemma 5, we have $\hat{b}+g(a,b)=s-\hat{y}+h(x)$. This implies $\hat{y}=s-\hat{b}$. Let σ be the permutation on $V\Gamma_{q,s,k}$ such that

$$\sigma(u,v)=\begin{cases}(u,v), & \text{if } v=0,\\ (f(u,v),-v), & \text{if } v\neq 0.\end{cases}$$

Routinely, σ is a desired automorphism.

Case 2. $\partial((0,0),(a,b)) = s - \hat{b} + f(a,b)$.

Suppose $\partial((0,0),(x,y)) = s - \hat{y} + f(x,y)$. Then $\hat{y} - \hat{b} = f(x,y) - f(a,b)$. We have $\hat{y} - \hat{b} \equiv \hat{x} + \hat{y} - \hat{a} - \hat{b}$. This implies $x = a$. By Lemma 5, one gets $s - \hat{b} + h(a) = s - \hat{y} + h(x)$, which implies that $y = b$. Hence, the identity permutation is a desired automorphism.

Suppose $\partial((0,0),(x,y)) = \hat{x} + \hat{y}$. It is similar to Case 1 and the desired result holds. □

By Lemma 4, for vertices (a,b) and (x,y) of $\Gamma_{q,s,k}$, we have

$$\widetilde{\partial}((a,b),(x,y)) = \begin{cases} \widetilde{\partial}((0,0),(x-a,y-b)), & \text{if } \hat{y} \in \{\hat{b}, \hat{b}+1, \ldots, s-1\}, \\ \widetilde{\partial}((0,0),(x-a+k-1,y-b)), & \text{otherwise.} \end{cases}$$

Lemma 7. *If* $\Gamma_{q,s,k}$ *is weakly distance-regular, then* C1, C2 *or* C3 *holds.*

Proof. Suppose for the contrary that C1, C2 and C3 do not hold. Let $e = (0,0)$, $z = (0,1)$, $w = (k, s-1)$ be the vertices of $\Gamma_{q,s,k}$, and $\alpha(v) = (3 + (-1)^v)/4$ for $v \in \mathbb{Z}$. By (2), note that $\widetilde{\partial}(e,z) = \widetilde{\partial}(e,w) = (1,q)$. To prove this lemma, we would pick proper $x, y \in V\Gamma_{q,s,k}$ such that $\widetilde{\partial}(e,x) = \widetilde{\partial}(e,y)$, and

$$P_{(1,q),\widetilde{\partial}(z,x)}(e,y) = \emptyset \quad \text{or} \quad P_{(1,q),\widetilde{\partial}(w,x)}(e,y) = \emptyset, \tag{3}$$

which contrary to $z \in P_{(1,q),\widetilde{\partial}(z,x)}(e,x)$ or $w \in P_{(1,q),\widetilde{\partial}(w,x)}(e,x)$.

Case 1. $k \neq q$ and $2k + p \geqslant 2(q - \alpha(p) + 2)$.

Let $x = (k, \alpha(s) - 1 + s/2)$ and $y = (k, \alpha(s) + s/2)$. In this case, $\widetilde{\partial}(e,x) = \widetilde{\partial}(e,y) = \widetilde{\partial}(z,y) = ((m+1)q + 1, (m+2)q - k)$ and $\widetilde{\partial}(w,x) = ((m+1)q, -\alpha(s) + s/2)$. Since $\partial(y,w) = -\alpha(s) + s/2 - 1$, we have (3) holds.

Case 2. $k \neq q$, $p \geqslant q$ and $2k + p \leqslant 2(q - \alpha(p) + 1)$.

Let $x = (0, \alpha(s) + s/2)$ and $y = (q + \alpha(p) + 1 - k - p/2, (m-1)q + p + k - 1)$. In this case, $\widetilde{\partial}(e,x) = \widetilde{\partial}(e,y) = (\alpha(s) + s/2, k - \alpha(s) - 1 + s/2)$ and $\widetilde{\partial}(z,x) = (\alpha(s) - 1 + s/2, k - \alpha(s) + s/2)$. Since $\widetilde{\partial}(z,y) = (\alpha(s) - 1 + s/2, k - \alpha(s) - 1 + s/2)$ and $\partial(w,y) = \alpha(s) + s/2$, we have (3) holds.

Case 3. $k \neq q$, $p < q$ and $2(\alpha(p) + 1) < 2k + p \leqslant 2(q - \alpha(p) + 1)$.

Let $x = (0, \alpha(s) + s/2)$ and $y = (q + \alpha(p) + 1 - k - p/2, (m-1)q + p + k - 1)$. In this case, $\widetilde{\partial}(e,x) = \widetilde{\partial}(e,y) = (\alpha(s) + s/2, k - \alpha(s) - 1 + s/2)$ and $\partial(w,y) = \alpha(s) + s/2$. If $p = 2(1 - \alpha(p))$, then $\widetilde{\partial}(z,x) = (\alpha(s) - 1 + s/2, \alpha(s) - 1 + s/2)$ and $\widetilde{\partial}(z,y) = (\alpha(s) - 1 + s/2, (m-1)q + p + k - 2)$; if $p \neq 2(1 - \alpha(p))$, then $\widetilde{\partial}(z,x) = (\alpha(s) - 1 + s/2, k - \alpha(s) + s/2)$ and $\widetilde{\partial}(z,y) = (\alpha(s) - 1 + s/2, k - \alpha(s) - 1 + s/2)$. Hence, (3) holds.

Case 4. $k \neq q$, $p < q$ and $2k + p \leqslant 2(\alpha(p) + 1)$.

Let $x = (k, \alpha(s) - 1 + s/2)$ and $y = (k, \alpha(s) + s/2)$. In this case, $\widetilde{\partial}(e,x) = \widetilde{\partial}(e,y) = \widetilde{\partial}(z,y) = (mq + 1, q - \alpha(s) - 1 + s/2)$ and $\widetilde{\partial}(w,x) = (mq, -\alpha(s) + s/2)$. Since $\partial(y,w) = -\alpha(s) - 1 + s/2$, we have (3) holds.

Case 5. $k = q$ and $p \geqslant q + 3$.

Let $x = (q-1, (m-1)q + p)$ and $y = (0, (m+1)q)$. In this case, $\widetilde{\partial}(e,x) = \widetilde{\partial}(e,y) = ((m+1)q, (m+1)q)$ and $\partial(z,x) = mq + p - 2$. Since $\partial(z,y) = (m+1)q - 1$ and $\partial(w,y) = (m+1)q$, we have (3) holds.

Case 6. $k = q$ and $3 \leqslant p \leqslant q+1$.

Let $x = (q-2, mq+2)$ and $y = (q-2, mq+1)$. In this case, $\widetilde{\partial}(e,x) = \widetilde{\partial}(e,y) = \widetilde{\partial}(z,x) = ((m+1)q-1, mq+2)$. Since $\widetilde{\partial}(z,y) = ((m+1)q-2, mq+2)$ and $\partial(y,w) = mq+3$, we have (3) holds.

Case 7. $k = q$ and $p \leqslant 1$.

Let $x = (q-1, mq+p)$ and $y = (0, mq)$. In this case, $\widetilde{\partial}(e,x) = \widetilde{\partial}(e,y) = (mq, mq)$ and $\partial(z,x) = (m+1)q+p-2$. Since $\partial(z,y) = mq-1$ and $\partial(w,y) = mq$, we have (3) holds.

Therefore, the desired result holds. □

Combining Lemmas 6 and 7, we obtain the following theorem.

Theorem 8. *$\Gamma_{q,s,k}$ is weakly distance-regular, if and only if* C1, C2 *or* C3 *holds.*

Finally, we shall show that every weakly distance-regular digraph $\Gamma_{q,s,k}$ is a Cayley digraph.

Proposition 9. *Let $d = \frac{p}{2\gcd(q,p)}$, $l = \max\{w \mid 2^w$ divides $\gcd(q,p)\}$, $h = \frac{s}{2^l}$, $i = 2\{d\}$ and u be an integer such that $2^i q$ divides $(up - \gcd(q,p))$, where $\{d\}$ denotes the fractional part of d and $\gcd(q,p)$ denotes the greatest common divisor of q and p.*

(i) *If* C1 *holds, then $\Gamma_{q,s,k}$ is isomorphic to* $\mathrm{Cay}(\mathbb{Z}_q \times \mathbb{Z}_{2mq}, \{(1,0),(0,1),(1,2mq-1)\})$, *$m \geqslant 1$ and $q \geqslant 3$.*

(ii) *If* C2 *holds, then $\Gamma_{q,s,k}$ is isomorphic to* $\mathrm{Cay}(\mathbb{Z}_{(mq+2)q}, \{1, mq+2, mq+1\})$, *$m \geqslant 1$ and $q \geqslant 3$.*

(iii) *If* C3 *holds, then $\Gamma_{q,s,k}$ is isomorphic to* $\mathrm{Cay}(\mathbb{Z}_{2^i q} \times \mathbb{Z}_{2^{-i}(2mq+p)}, \{(2^i ud, 1), (2^i - 2^i ud, ih-1), (2^i, ih)\})$, *where $q \geqslant 3$, $m \geqslant 0$, $4 \leqslant p \leqslant 2q-2$ and p is even.*

Proof. If C1 holds, then (i) is obvious. If C2 holds, then the mapping σ from $\Gamma_{q,s,k}$ to the digraph in (ii) satisfying $\sigma(a,b) = \hat{a}(mq+2) + \hat{b}$ is an isomorphism.

Now suppose C3 holds. Let σ be the mapping from $\Gamma_{q,s,k}$ to the digraph in (iii) such that $\sigma(a,b) = (2^i\hat{a} + 2^i ud\hat{b}, ih\hat{a} + \hat{b})$. Note that σ is well defined.

We will show that σ is injective. It is clear for $i = 0$. If $i = 1$, by $2|p$, then $l \geqslant 1$. Assume that $\sigma(x_1,y_1) = \sigma(x_2,y_2)$ for $(x_1,y_1),(x_2,y_2) \in V\Gamma_{q,s,k}$. Let $x = 2ud(\widehat{y_2} - \widehat{y_1}) - 2(\widehat{x_1} - \widehat{x_2})$ and $y = (\widehat{y_2} - \widehat{y_1}) - h(\widehat{x_1} - \widehat{x_2})$. Since $\sigma(x_1,y_1) = \sigma(x_2,y_2)$, we have $2q|x$ and $(mq+p/2)|y$, which imply $2^{l-1}h|y$ by $s = 2mq+p$. Hence, $h|(\widehat{y_2} - \widehat{y_1})$.

We claim $2^j|(\widehat{y_2} - \widehat{y_1})$ for $1 \leqslant j \leqslant l$. By $2q|(up - \gcd(q,p))$, we get $(2q/\gcd(q,p))|(2ud - 1)$, which implies $2ud$ is odd. Since $2q|x$, one obtains $2|(\widehat{y_2} - \widehat{y_1})$. Suppose $2^j|(\widehat{y_2} - \widehat{y_1})$ for some $j < l$. From $2^j|(mq+p/2)$, we have $2^j|y$ and $2^j|(\widehat{x_1} - \widehat{x_2})$, which imply $2^{j+1}|(\widehat{y_2} - \widehat{y_1})$ by $2^{j+1}|x$. So our claim is valid.

By $\gcd(2^l, h) = 1$, we obtain $(2mq+p)|(\widehat{y_2} - \widehat{y_1})$. Thus, $y_1 = y_2$ and $x_1 = x_2$. Therefore σ is a bijection. One can verify that $((x_1,y_1),(x_2,y_2))$ is an arc if and only if $(\sigma(x_1,y_1), \sigma(x_2,y_2))$ is an arc. Hence, σ is an isomorphism. □

3 Circuits

In this section, we will discuss some properties for circuits of weakly distance-regular digraphs.

Let Γ be a digraph. Let $R = \{\Gamma_{\widetilde{i}} \mid \widetilde{i} \in \widetilde{\partial}(\Gamma)\}$, where $\Gamma_{\widetilde{i}} = \{(x, y) \in V\Gamma \times V\Gamma \mid \widetilde{\partial}(x, y) = \widetilde{i}\}$. If Γ is weakly distance-regular, then $(V\Gamma, R)$ is an association scheme. For more information about association schemes, see [3, 11]. For two nonempty subsets E, $F \subseteq R$, define

$$EF := \{\Gamma_{\widetilde{h}} \mid \sum_{\Gamma_{\widetilde{i}} \in E} \sum_{\Gamma_{\widetilde{j}} \in F} p^{\widetilde{h}}_{\widetilde{i},\widetilde{j}} \neq 0\},$$

and write $\Gamma_{\widetilde{i}}\Gamma_{\widetilde{j}}$ instead of $\{\Gamma_{\widetilde{i}}\}\{\Gamma_{\widetilde{j}}\}$. For each nonempty subset F of R, define $\langle F \rangle$ to be the minimal equivalence relation containing F. Let

$$V\Gamma/F := \{F(x) \mid x \in V\Gamma\} \quad \text{and} \quad \Gamma^F_{\widetilde{i}} := \{(F(x), F(y)) \mid y \in F\Gamma_{\widetilde{i}}F(x)\},$$

where $F(x) := \{y \in V\Gamma \mid (x, y) \in \cup_{f \in F} f\}$. The digraph $(V\Gamma/F, \cup_{(1,s)\in\widetilde{\partial}(\Gamma)}\Gamma^F_{1,s})$ is said to be the *quotient digraph* of Γ over F, denoted by Γ/F. The size of $\Gamma_{\widetilde{i}}(x) := \{y \in V\Gamma \mid \widetilde{\partial}(x, y) = \widetilde{i}\}$ depends only on $\widetilde{i}$, denoted by $k_{\widetilde{i}}$. For any $(a, b) \in \widetilde{\partial}(\Gamma)$, we usually write $k_{a,b}$ (resp. $\Gamma_{a,b}$) instead of $k_{(a,b)}$ (resp. $\Gamma_{(a,b)}$).

Now we shall introduce some basic results which are used frequently in this paper.

Lemma 10. *Let Γ be a weakly distance-regular digraph. For each $\widetilde{i} := (a, b) \in \widetilde{\partial}(\Gamma)$, define $\widetilde{i}^* = (b, a)$.*

(i) $k_{\widetilde{h}}\, p^{\widetilde{h}}_{\widetilde{i},\widetilde{j}} = k_{\widetilde{i}}\, p^{\widetilde{i}}_{\widetilde{h},\widetilde{j}^*} = k_{\widetilde{j}}\, p^{\widetilde{j}}_{\widetilde{i}^*,\widetilde{h}}$.

(ii) $k_{\widetilde{i}}\, k_{\widetilde{j}} = \sum_{\widetilde{h}\in\widetilde{\partial}(\Gamma)} k_{\widetilde{h}}\, p^{\widetilde{h}}_{\widetilde{i},\widetilde{j}}$.

(iii) $|\Gamma_{\widetilde{i}}\, \Gamma_{\widetilde{j}}| \leqslant \gcd(k_{\widetilde{i}}, k_{\widetilde{j}})$.

Proof. See Proposition 2.2 in [3, pp. 55-56] and Proposition 5.1 in [1]. □

In the remaining of this paper, we assume that Γ is a weakly distance-regular digraph of valency 3 satisfying $k_{1,q-1} = 1$ and $k_{1,g-1} = 2$, where $q, g \geqslant 3$ and $q \neq g$. Let $A_{i,j}$ denote a binary matrix with rows and columns indexed by $V\Gamma$ such that $(A_{i,j})_{x,y} = 1$ if and only if $\widetilde{\partial}(x, y) = (i, j)$.

Lemma 11. *The following hold:*

$$A_{1,q-1}A_{1,g-1} = A_{1,g-1}A_{1,q-1}, \tag{4}$$

$$A_{1,g-1}A_{g-1,1} = A_{g-1,1}A_{1,g-1}. \tag{5}$$

Proof. By Lemma 10 (iii), we may assume that

$$A_{1,g-1}A_{1,q-1} = A_{i,j} \quad \text{and} \quad A_{1,q-1}A_{1,g-1} = A_{s,t}, \quad i, s \in \{1, 2\}.$$

We claim that $i = s = 2$. Suppose $i = 1$. Then $j = g-1$ because of $k_{1,q-1} = 1$. By Lemma 10 (i), we get $p^{(1,q-1)}_{(g-1,1),(1,g-1)} = 2p^{(1,g-1)}_{(1,g-1),(1,q-1)} = 2$. By Lemma 10 (iii), $A_{g-1,1}A_{1,g-1} = 2I + 2A_{1,q-1}$, contrary to the fact that $A_{g-1,1}A_{1,g-1}$ is a symmetric matrix. Hence, $i = 2$. Similarly, $s = 2$ and our claim is valid.

Pick a path (x_0, x_1, x_2) with $\widetilde{\partial}(x_0, x_1) = (1, g-1)$ and $\widetilde{\partial}(x_1, x_2) = (1, q-1)$. Then $\partial(x_2, x_0) = j$. We may choose a path $(x_2, x_3, \dots, x_{j+1}, x_0)$. Since Γ has just two types of arcs, there exists an $i \in \{1, 2, \dots, j+1\}$ such that $\widetilde{\partial}(x_i, x_{i+1}) = (1, q-1)$ and $\widetilde{\partial}(x_{i+1}, x_{i+2}) = (1, g-1)$, where $x_{j+2} = x_0$ and $x_{j+3} = x_1$. Since $\widetilde{\partial}(x_i, x_{i+2}) = (2, t)$, one has $t \leqslant j$. Similarly, $j \leqslant t$. Hence, $j = t$ and (4) holds.

In view of Lemma 10 (iii), we have

$$A_{1,g-1}A_{g-1,1} = 2I + p^{(s,s)}_{(1,g-1),(g-1,1)}\ A_{s,s}, \quad s \geqslant 2, \tag{6}$$

$$A_{g-1,1}A_{1,g-1} = 2I + p^{(t,t)}_{(g-1,1),(1,g-1)}\ A_{t,t}, \quad t \geqslant 2. \tag{7}$$

By Lemma 10 (ii), we have $k_{s,s}p^{(s,s)}_{(1,g-1),(g-1,1)} = k_{t,t}p^{(t,t)}_{(g-1,1),(1,g-1)} = 2$, which implies that $p^{(s,s)}_{(1,g-1),(g-1,1)}, p^{(t,t)}_{(g-1,1),(1,g-1)} \in \{1, 2\}$. Let x_0 and x_s be two vertices satisfying $\widetilde{\partial}(x_0, x_s) = (s, s)$. Suppose $p^{(s,s)}_{(1,g-1),(g-1,1)} = 2$. Pick two distinct vertices $x, y \in P_{(1,g-1),(g-1,1)}(x_0, x_s)$. By $x_0 \in P_{(g-1,1),(1,g-1)}(x, y)$ and (7), $\widetilde{\partial}(x, y) = (t, t)$. It follows that $p^{(t,t)}_{(g-1,1),(1,g-1)} = 2$. Similarly, if $p^{(t,t)}_{(g-1,1),(1,g-1)} = 2$, then $p^{(s,s)}_{(1,g-1),(g-1,1)} = 2$ by (6). Hence, $p^{(s,s)}_{(1,g-1),(g-1,1)} = p^{(t,t)}_{(g-1,1),(1,g-1)}$. In order to show (5), we shall prove $s = t$. Pick $x \in P_{(1,g-1),(g-1,1)}(x_0, x_s)$ and a path $P := (x_0,\ x_1, \dots, x_s)$.

Case 1. P contains an arc of type $(1, g-1)$.

By (4), without loss of generality, we may assume that $\widetilde{\partial}(x_0, x_1) = (1, g-1)$. Pick $y \in \Gamma_{1,g-1}(x_s) \setminus \{x\}$. In view of (7), if $x \neq x_1$, from $x_0 \in P_{(g-1,1),(1,g-1)}(x_1, x)$, then $\partial(x_1, x) = t \leqslant s$; if $x = x_1$, from $x_0 \in P_{(g-1,1),(1,g-1)}(x, y)$, then $\partial(x, y) = t \leqslant s$.

Case 2. P only contains arcs of type $(1, q-1)$.

In this case, $A^s_{1,q-1} \neq I$. By (4), there exists a path $(x_0, y_1, y_2, \dots, y_s, x)$ containing the unique arc (x_0, y_1) of type $(1, g-1)$. If $x = y_1$, by Lemma 10 (iii), we have $A^s_{1,q-1} = I$, a contradiction. Therefore, $x \neq y_1$. By $x_0 \in P_{(g-1,1),(1,g-1)}(y_1, x)$ and (7), one has $\partial(y_1, x) = t \leqslant s$.

Similarly, $t \geqslant s$, which implies $s = t$, as desired. □

In the following, let $F = \langle \Gamma_{1,g-1} \rangle$ and fix $x \in V\Gamma$. Then Γ/F is isomorphic to a circuit C_m of length m. Let Δ be a digraph with the vertex set $F(x)$ such that (y, z) is an arc of Δ if (y, z) is an arc of type $(1, g-1)$ in Γ.

Lemma 12. *Suppose that every circuit of length g contains arcs of the same type in Γ. Then $\Delta_{t,g-t}(x_0) = \Gamma_{t,g-t}(x_0)$ for each $x_0 \in F(x)$ and $t \in \{1, 2, \dots, g-1\}$.*

Proof. Note that every arc of type $(1, g-1)$ is contained in a circuit of length g with all arcs of type $(1, g-1)$. It follows that, for any such circuit $(x_0, x_1, \dots, x_{g-1})$, we have

$\widetilde{\partial}_\Gamma(x_0,x_i)=(i,g-i)$, where $1\leqslant i\leqslant g-1$. Then every arc of Δ is contained in a circuit of length g in Δ.

For any $x_t\in\Gamma_{t,g-t}(x_0)$, there exists a circuit $C_g:=(x_0,x_1,\ldots,x_t,\ldots,x_{g-1})$ in Γ. Hence, C_g only contains the arcs of same type. Suppose that each arc of C_g is of type $(1,q-1)$. Then, $q<g$ and every circuit of length q in Γ only contains arcs of type $(1,q-1)$. It follows that $A^q_{1,q-1}=I$. Since $x_0\neq x_l$ for $1\leqslant l\leqslant g-1$, $k_{1,q-1}=1$ implies that g is the minimum positive integer such that $A^g_{1,q-1}=I$, a contradiction. Consequently, each arc of C_g is of type $(1,g-1)$. Therefore, $(x_0,x_t)\in\Delta_{t,g-t}$; and so $\Gamma_{t,g-t}(x_0)\subseteq\Delta_{t,g-t}(x_0)$. Conversely, pick any $x_t\in\Delta_{t,g-t}(x_0)$. Then, in Γ, there exists a circuit $(x_0,x_1,\ldots,x_t,\ldots,x_{g-1})$ each of whose arcs is of type $(1,g-1)$. Hence, $(x_0,x_t)\in\Gamma_{t,g-t}$; and so $\Delta_{t,g-t}(x_0)\subseteq\Gamma_{t,g-t}(x_0)$. Thus, the desired result holds. □

Lemma 13. *If $F(x)=V\Gamma$, then there exists a circuit of length g containing different types of arcs.*

Proof. Suppose for the contrary that every circuit of length g contains the same type of arcs. By the Lemma 12, $\Gamma_{t,g-t}=\Delta_{t,g-t}$ for any $1\leqslant t\leqslant g-1$. By (5), the proof of Proposition 4.3 in [8] implies that Δ is isomorphic to $\Gamma_1:=\mathrm{Cay}(\mathbb{Z}_{2g},\{1,g+1\})$ or $\Gamma_2:=\mathrm{Cay}(\mathbb{Z}_g\times\mathbb{Z}_g,\{(0,1),(1,0)\})$.

Case 1. $\Delta\simeq\Gamma_1$.

Choose $y\in\mathbb{Z}_{2g}\setminus\{0,1,g+1\}$ and $t\in\mathbb{Z}_{2g}$ such that $\widetilde{\partial}_\Gamma(0,y)=(1,q-1)$, $\hat{t}\equiv\hat{y}\pmod g$ and $\hat{t}\in\{0,2,3,\ldots,g-1\}$. Since $(y+1,y+2,\ldots,y-t+g-1,0,y)$ is a path of length $g-\hat{t}$, $\partial_\Gamma(y+1,y)=g-1\leqslant g-\hat{t}$. It follows that $t=0$, and so $\hat{y}=g$. Therefore, $\widetilde{\partial}_\Gamma(0,g)=(1,q-1)$. Similarly, $\widetilde{\partial}_\Gamma(g,0)=(1,q-1)$. Hence, $q=2$, a contradiction.

Case 2. $\Delta\simeq\Gamma_2$.

Pick $(i,j)\in\Gamma_{1,q-1}(0,0)$. Since $\widetilde{\partial}_\Delta((0,0),(0,j))=(\hat{j},g-\hat{j})$, by Lemma 12, we have $\widetilde{\partial}_\Gamma((0,0),(0,j))=(\hat{j},g-\hat{j})$. It follows that $i\neq0$. By Lemma 10 (i), one gets $p^{(1,q-1)}_{(\hat{\imath},g-\hat{\imath}),(\hat{j},g-\hat{j})}=k_{\hat{\imath},g-\hat{\imath}}\,p^{(\hat{\imath},g-\hat{\imath})}_{(1,q-1),(g-\hat{j},\hat{j})}$. Since $(i,j)\in P_{(1,q-1),(g-\hat{j},\hat{j})}((0,0),(i,0))$ in Γ, $p^{(\hat{\imath},g-\hat{\imath})}_{(1,q-1),(g-\hat{j},\hat{j})}=1$, which implies that $p^{(1,q-1)}_{(\hat{\imath},g-\hat{\imath}),(\hat{j},g-\hat{j})}=k_{\hat{\imath},g-\hat{\imath}}$.

Let $((a,b),(a',b'))$ be an arc of type $(1,q-1)$. Then $P_{(\hat{\imath},g-\hat{\imath}),(\hat{j},g-\hat{j})}((a,b),(a',b'))=\Gamma_{\hat{\imath},g-\hat{\imath}}(a,b)$. Since $(a+i,b),(a,b+i)\in\Delta_{\hat{\imath},g-\hat{\imath}}(a,b)$, by Lemma 12, $(a',b')\in\Gamma_{\hat{j},g-\hat{j}}(a+i,b)\cap\Gamma_{\hat{j},g-\hat{j}}(a,b+i)$. By Lemma 12 again,

$$(a',b')\in\{(a+i+j,b),(a+i,b+j)\}\cap\{(a+j,b+i),(a,b+i+j)\}.$$

Since $i\neq0$, we have $(a',b')=(a+i,b+j)=(a+j,b+i)$, which implies that $i=j$. Thus, $\Gamma\simeq\mathrm{Cay}(\mathbb{Z}_g\times\mathbb{Z}_g,\{(1,0),(0,1),(i,i)\})$. Since $g\neq q$, one gets $i\neq1$. Let $g=n\hat{i}+r$ with $0\leqslant r\leqslant\hat{\imath}-1$. If $r\neq0$, then $\widetilde{\partial}_\Gamma((0,0),(1,1))=\widetilde{\partial}_\Gamma((0,0),(i,i+1))=(2,n+2r-2)$; if $r=0$, then $\widetilde{\partial}_\Gamma((0,0),(1,1))=\widetilde{\partial}_\Gamma((0,0),(i,i+1))=(2,n+2\hat{\imath}-3)$. But we have $(1,0)\in P_{(1,g-1),(1,g-1)}((0,0),(1,1))$ and $P_{(1,g-1),(1,g-1)}((0,0),(i,i{+}1))=\emptyset$ in Γ, a contradiction. □

Lemma 14. *Every circuit of length q in Γ only contains the arcs of the same type. In particular,*

$$A^2_{1,q-1}=A_{2,q-2}. \tag{8}$$

Proof. If $F(x) = V\Gamma$, then $q < g$ by Lemma 13 and the desired result follows. Suppose $F(x) \neq V\Gamma$. Assume the contrary, namely, there exists a circuit $(x_0, x_1, \ldots, x_{q-1})$ containing arcs of different types. Since $\Gamma/F \simeq C_m$ with $m \geqslant 2$, there exist at least two arcs of type $(1, q-1)$ in this circuit. By (4), we may assume that $\widetilde{\partial}(x_0, x_1) = \widetilde{\partial}(x_1, x_2) = (1, q-1)$ and $\widetilde{\partial}(x_{q-1}, x_0) = (1, g-1)$. By the claim in Lemma 11, $\widetilde{\partial}(x_{q-1}, x_1) = (2, q-2)$. Since $k_{1,q-1} = 1$, by Lemma 10 (ii), one has $k_{\widetilde{\partial}(x_0,x_2)} = 1$. Therefore, $\widetilde{\partial}(x_0, x_2) = (2, q-2)$. But $P_{(1,q-1),(1,q-1)}(x_0, x_2) = \{x_1\}$ and $P_{(1,q-1),(1,q-1)}(x_{q-1}, x_1) = \emptyset$, a contradiction. Lemma 10 (iii) implies (8). □

Lemma 15. *For any circuit $(x_0, x_1, \ldots, x_{l-1})$ with $\widetilde{\partial}(x_{l-1}, x_0) = (1, g-1)$, there exists $i \in \{0, 1, \ldots, l-2\}$ such that $\widetilde{\partial}(x_i, x_{i+1}) = (1, g-1)$.*

Proof. Suppose for the contradiction that $\widetilde{\partial}(x_i, x_{i+1}) = (1, q-1)$ for any $i = 0, 1, \ldots, l-2$. By Lemma 10 (iii), we have $A_{g-1,1} = A_{1,q-1}^{l-1}$. Then $A_{g-1,1}$ is a permutation matrix, a contradiction. □

Lemma 16. *$F(x) \neq V\Gamma$ if and only if every circuit of length g in Γ only contains the arcs of the same type.*

Proof. Suppose $F(x) \neq V\Gamma$. Assume the contrary, namely, $(x_0, x_1, \ldots, x_{g-1})$ is a circuit containing arcs of different types such that $\widetilde{\partial}(x_0, x_1) = (1, g-1)$. By (4) and Lemma 15, we may assume that $\widetilde{\partial}(x_1, x_2) = (1, q-1)$ and $\widetilde{\partial}(x_{g-1}, x_0) = (1, g-1)$. By the claim in Lemma 11, $\widetilde{\partial}(x_0, x_2) = (2, g-2)$. If $\partial(x_{g-1}, x_1) = 1$, from $F(x) \neq V\Gamma$, then $\widetilde{\partial}(x_{g-1}, x_1) = (1, g-1)$, which implies $(x_1, x_2, \ldots, x_{g-1})$ is a circuit of length $g-1$ containing an arc of type $(1, g-1)$, a contradiction. Hence, $\widetilde{\partial}(x_{g-1}, x_1) = (2, g-2)$. The fact that $x_2 \notin F(x_0)$ implies that $P_{(1,g-1),(1,g-1)}(x_0, x_2) = \emptyset$, contradicting to $x_0 \in P_{(1,g-1),(1,g-1)}(x_{g-1}, x_1)$.

The converse is true by Lemma 13. □

4 The proof of Theorem 1

In this section, we assume that $F = \langle \Gamma_{1,g-1} \rangle$ and x is a fixed vertex of Γ.

Lemma 17. *If $F(x) \neq V\Gamma$, then $\Gamma/F \simeq C_2$.*

Proof. Suppose for the contradiction that $\Gamma/F \simeq C_m$ with $m \geqslant 3$. Choose a path (x_0, x_1, x_2, x_3) such that $\widetilde{\partial}(x_0, x_1) = \widetilde{\partial}(x_1, x_2) = (1, q-1)$ and $\widetilde{\partial}(x_2, x_3) = (1, g-1)$. Since $\partial(F(x_0), F(x_2)) = 2$, $k_{1,q-1} = 1$ implies that $\widetilde{\partial}(x_0, x_3) = (3, l)$ for some l. Then there exists a shortest path $(x_3, x_4, y_2, \ldots, x_{l+2}, x_0)$. By Lemma 15 and (4), we may assume that $\widetilde{\partial}(x_3, x_4) = (1, g-1)$. Since $\partial(F(x_1), F(x_4)) = 1$ and $k_{1,q-1} = 1$, we have $\partial(x_1, x_4) = 2$ or 3. If $\partial(x_1, x_4) = 2$, by $F(x) \neq V\Gamma$, then $\widetilde{\partial}(x_2, x_4) = (1, g-1)$, which implies $g = 2$ by $x_2 \in P_{(g-1,1),(1,g-1)}(x_3, x_4)$ and (7), a contradiction. Hence, $\widetilde{\partial}(x_1, x_4) = (3, t)$ for some $t \leqslant l$. From $m \geqslant 3$ and (4), there exists a path $(x_4, y_1, y_2, \ldots, y_{t-2}, x_0, x_1)$. Then $(x_3, x_4, y_1, y_2, \ldots, y_{t-2}, x_0)$ is a path of length t; and so $l \leqslant t$. Hence, $l = t$. By (8), $x_2 \in P_{(2,q-2),(1,g-1)}(x_0, x_3)$. Then there exists $y \in P_{(2,q-2),(1,g-1)}(x_1, x_4)$. From $k_{1,q-1} = 1$, $\widetilde{\partial}(x_2, y) = (1, q-1)$, which implies $\Gamma_{1,q-1} \in F$, a contradiction. □

Proposition 18. *If $F(x) \neq V\Gamma$, then Γ is isomorphic to one of the digraphs in Theorem 1* (i).

Proof. By Lemma 17, $V\Gamma$ has a partition $F(x)\dot{\cup}F(x')$. Let Δ and Δ' be the subdigraphs of Γ induced on $F(x)$ and $F(x')$, respectively. By (4) and $k_{1,q-1} = 1$, $\sigma : F(x) \to F(x')$, $y \mapsto y'$ is an isomorphism mapping from Δ to Δ', where $y' \in \Gamma_{1,q-1}(y)$. By Lemmas 12 and 16, $\Gamma_{r,g-r}(y) = \Delta_{r,g-r}(y)$ for each $y \in F(x)$ and $r \in \{1, 2, \ldots, g-1\}$. By (5), the proof of Proposition 4.3 in [8] implies that Δ is isomorphic to $\Gamma_1 := \mathrm{Cay}(\mathbb{Z}_g \times \mathbb{Z}_g, \{(1, 0), (0, 1)\})$ or $\Gamma_2 := \mathrm{Cay}(\mathbb{Z}_{2g}, \{1, g + 1\})$. Suppose that τ_i is an isomorphism from Γ_i to Δ if Γ_i is isomorphic to Δ.

We claim that $\Delta \simeq \Gamma_2$. Suppose for the contrary that $\Delta \simeq \Gamma_1$. Write $\tau_1(a, b) = (a, b, 0)$ and $\sigma(a, b, 0) = (a, b, 1)$ for each $(a, b) \in \mathbb{Z}_g \times \mathbb{Z}_g$. Let $((0, 0, 1), (c, d, 0))$ be an arc of type $(1, q - 1)$. By (8), $\widetilde{\partial}_\Gamma((0, 0, 0), (c, d, 0)) = (2, q - 2)$. Lemma 12 implies that $c \neq 0$ and $d \neq 0$. By Lemma 12 again, we have $(c, d, 0) \in P_{(2,q-2),(g-\hat{d},\hat{d})}((0, 0, 0), (c, 0, 0))$ and $\widetilde{\partial}_\Gamma((0, 0, 0), (c, 0, 0)) = \widetilde{\partial}_\Gamma((0, 0, 0), (0, c, 0))$. By $k_{2,q-2} = 1$, we have $(0, c, 0) \in \Gamma_{g-\hat{d},\hat{d}}(c, d, 0)$. Then $(0, c, 0) \in \{(c, 0, 0), (c - d, d, 0)\}$ by Lemma 12. Hence, $c = d$.

Suppose $\hat{c} = g - 1$. Since $((0, 0, 1), (-1, -1, 0), (0, -1, 0), (0, 0, 0))$ is a shortest path, $q = 4$, contrary to Lemma 14. Suppose $\hat{c} \neq g - 1$. Then $\widetilde{\partial}_\Gamma((0, 0, 0), (c, c + 1, 0)) = (3, l)$ for some l. Pick a path $((c, c + 1, 0), x_1, x_2, \ldots, x_{l-1}, (0, 0, 0))$. By Lemma 15 and (4), we may assume that $\widetilde{\partial}_\Gamma((c, c + 1, 0), x_1) = (1, g - 1)$. By (7), we have $\widetilde{\partial}_\Gamma((0, 0, 1), x_1) = (3, t)$ for some $t \leqslant l$. Since $F(x) \neq V\Gamma$, $k_{1,q-1} = 1$ implies that there exists a path $(x_1, y_1, y_2, \ldots, y_{t-2}, (0, 0, 0), (0, 0, 1))$. Then $((c, c + 1, 0), x_1, y_1, y_2, \ldots, y_{t-2}, (0, 0, 0))$ is a path of length t; and so $l \leqslant t$. Hence $l = t$. By (8) and $x_1 \in V\Delta$, one has $(c, c, 0) \in P_{(2,q-2),(1,g-1)}((0, 0, 0), (c, c+1, 0))$ and $P_{(2,q-2),(1,g-1)}((0, 0, 1), x_1) = \emptyset$ in Γ, a contradiction. Therefore, our claim is valid.

Write $\tau_2(a) = (a, 0)$ and $\sigma(a, 0) = (a, 1)$ for each $a \in \mathbb{Z}_{2g}$. Let $((a, 1), (a + k_a, 0))$ be an arc of type $(1, q-1)$. Then $k_a \neq 0$. By (8), $\widetilde{\partial}_\Gamma((a, 0), (a+k_a, 0)) = (2, q-2)$. By Lemma 12, $\widetilde{\partial}_\Delta((a, 0), (a+k_a, 0)) \neq (t, g-t)$ for any $t \in \{1, 2, \ldots, g-1\}$. Since $\bigcup_{1\leqslant t\leqslant g-1} \Delta_{t,g-t}(a, 0) = V\Delta \setminus \{(a, 0), (a + g, 0)\}$, one has $\widehat{k_a} = g$. Then, $\Gamma \simeq \mathrm{Cay}(\mathbb{Z}_4 \times \mathbb{Z}_g, \{(0, 1), (1, 0), (2, 1)\})$ and the result holds by Proposition 2. □

Lemma 19. *If $F(x) = V\Gamma$, then $p^{(1,q-1)}_{(1,g-1),(1,g-1)} = 2$.*

Proof. By Lemma 13, there exists a circuit of length g with different types of arcs. Let $C := (x_0, x_1, \ldots, x_{g-1})$ be such a circuit with the minimum number of arcs of type $(1, g-1)$. Suppose C contains t arcs of types $(1, g-1)$. Lemma 15 implies that $t \geqslant 2$. By (4), we may assume that $\widetilde{\partial}(x_i, x_{i+1}) = (1, g - 1)$ for $0 \leqslant i \leqslant t$. We claim that $\widetilde{\partial}(x_0, x_2) = (1, q - 1)$. Suppose not. By the claim in Lemma 11 and (7), we have $\widetilde{\partial}(x_{g-1}, x_1) = \widetilde{\partial}(x_0, x_2) = (2, g - 2)$. Since $x_0 \in P_{(1,q-1),(1,g-1)}(x_{g-1}, x_1)$, there exists $x'_1 \in P_{(1,q-1),(1,g-1)}(x_0, x_2)$. The circuit $C' := (x_0, x'_1, x_2, \ldots, x_{g-1})$ contains just $t-1$ arcs of type $(1, g-1)$, a contradiction. Thus, our claim is valid. It follows that $p^{(1,g-1)}_{(1,q-1),(g-1,1)} = 1$. By Lemma 10 (i), the desired result holds. □

Let $H = \langle\Gamma_{1,q-1}\rangle$ and $H(x_{0,0}), H(x_{0,1}), \ldots, H(x_{0,s-1})$ be all pairwise distinct vertices of Γ/H. Since $q < g$, the subdigraph induced on each $H(x_{0,j})$ is a circuit of length q with arcs of type $(1, q-1)$, say $(x_{0,j}, x_{1,j}, \ldots, x_{q-1,j})$. It follows that $s \geqslant 2$.

Proposition 20. *If $F(x) = V\Gamma$, then Γ is isomorphic to one of the digraphs in Theorem 1* (ii).

Proof. Suppose $\partial(H(x_{0,0}), H(x_{0,1})) = 1$. By (4), we may assume that $\widetilde{\partial}(x_{0,0}, x_{0,1}) = (1, g-1)$. By Lemma 19, one has $\widetilde{\partial}(x_{0,1}, x_{1,0}) = (1, g-1)$, which implies $\partial(H(x_{0,1}), H(x_{0,0})) = 1$. Since $F(x) = V\Gamma$, Γ/H is a connected undirected graph. By $k_{1,g-1} = 2$, Γ/H is an undirected circuit of length s. Suppose $s = 2$. Pick $y \in \Gamma_{1,g-1}(x_{0,1}) \setminus \{x_{1,0}\}$. Then $y = x_{i,0}$ for some $i \geqslant 2$, and $(x_{0,1}, y, x_{i+1,0}, \ldots, x_{q-1,0}, x_{0,0})$ is a path of length $q - i + 1$ from $x_{0,1}$ to $x_{0,0}$, contrary to the fact $\partial(x_{0,1}, x_{0,0}) = g - 1$. Hence, $s \geqslant 3$.

Let $(H(x_{0,0}), H(x_{0,1}), \ldots, H(x_{0,s-1}))$ be an undirected circuit. By (4), we may assume that $(x_{0,0}, x_{0,1}, \ldots, x_{0,s-1})$ is a path with arcs of type $(1, g-1)$. By Lemma 19, $(x_{0,j}, x_{0,j+1}, x_{1,j}, x_{1,j+1}, x_{2,j}, \ldots, x_{q-1,j}, x_{q-1,j+1})$ is a circuit with arcs of type $(1, g-1)$ for $j = 0, 1, \ldots, s-2$. Hence, there exists $k \in \{1, 2, \ldots, q\}$ such that $\widetilde{\partial}(x_{0,s-1}, x_{q-k+1,0}) = (1, g-1)$, where the first subscription of x are taken modulo q. By Lemma 19 again, we obtain $\widetilde{\partial}(x_{i,s-1}, x_{i-k+1,0}) = \widetilde{\partial}(x_{i-k+1,0}, x_{i+1,s-1}) = (1, g-1)$ for each i. Since

$$(x_{0,0}, x_{0,1}, \ldots, x_{0,s-1}, x_{q-k+1,0}, x_{q-k+2,0}, \ldots, x_{q-1,0})$$

is a circuit of length $s+k-1$ with different types of arcs, by Lemma 14, we get $s+k-1 > q$. From Theorem 8, the desired result follows. $\square$

Combining Propositions 18 and 20, we complete the proof of Theorem 1.

In the forthcoming paper [10], we shall classify cubic commutative weakly distance-regular digraphs with one type of arcs.

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