Graphs with no $\bar{P}_7$-minor

Guoli Ding*
Department of Mathematics
Louisiana State University
Baton Rouge, Louisiana, USA
ding@math.lsu.edu

Chanun Lewchalermvongs
Department of Mathematics
Faculty of Science
Mahidol University
Bangkok, Thailand
chanun.lew@mahidol.ac.th

John Maharry
Department of Mathematics
The Ohio State University
Columbus, Ohio, USA
maharry.1@osu.edu

Submitted: Jul 12, 2015; Accepted: Apr 6, 2016; Published: Apr 15, 2016
Mathematics Subject Classifications: 05C83, 05C75

Abstract

Let $\bar{P}_7$ denote the complement of a path on seven vertices. We determine all 4-connected graphs that do not contain $\bar{P}_7$ as a minor.

Keywords: forbidden minor, 4-connected graph

1 Introduction

In this paper, a graph G is called *H-free*, where H is a graph, if no minor of G is isomorphic to H. Many important problems in graph theory can be formulated in terms of H-free graphs. For instance, the four-color theorem can be equivalently stated as: all K_5-free graphs are 4-colorable. To solve problems involving H-free graphs, it is often desirable to explicitly determine all H-free graphs. In this area, the two most famous open problems are to determine K_6-free and Petersen-free graphs. Notice that both graphs have fifteen edges.

For each 3-connected graph H with at most eleven edges, all H-free graphs have been completely determined. A survey of these results can be found in [3]. For 3-connected graphs with twelve edges, the characterization problem is solved for the cube [6], the octahedron [2, 7], and the Wagner graph V_8 [8]. In addition, 4-connected Oct^+-free

*Support in part by NSF grant DMS-1500699.

graphs are also determined [5], where Oct^+ is the unique 13-edge graph obtained from the octahedron by adding an edge. In this paper we consider $\bar{P}_7$-free graphs, where $\bar{P}_7$, a 15-edge graph, is the complement of a path on seven vertices. Our result makes $\bar{P}_7$ the largest graph H for which 4-connected H-free graphs are completely determined. In contrast, 6-connected K_6-free graphs are not determined (although there is a conjecture on these graphs) and nothing is known about 6-connected Petersen-free graphs.

To state our main result we need to define a few classes of graphs. For each integer $n \geqslant 3$, let DW_n denote a *double-wheel*, which is a graph on $n+2$ vertices obtained from a cycle C_n by adding two adjacent vertices and connecting them to all vertices on the cycle. Let $\mathcal{DW} = \{DW_n : n \geqslant 3\}$. For each integer $n \geqslant 5$, let C_n^2 be a graph obtained from a cycle C_n by joining all pairs of vertices of distance two on the cycle. Notice that $C_5^2 = DW_3 = K_5$, and C_n^2 is nonplanar when n is odd. Let $\mathcal{C}_0 = \{C_{2n}^2 : n \geqslant 3\}$, $\mathcal{C}_1 = \{C_{2n+1}^2 : n \geqslant 2\}$, and $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$. Let $\mathcal{K}$ consist of graphs that are 4-connected nonplanar minors of some $K_{4,n}$. In other words, these are 4-connected nonplanar graphs obtained from some $K_{4,n}$ ($n \geqslant 1$) by adding edges to the color class of size four. It is routine to check that $\mathcal{K}$ contains exactly one graph (K_5) on five vertices, two ($K_6 \backslash e$, DW_4) on six vertices, six ($K_{4,3}^1, K_{4,3}^2, K_{4,3}^3, K_{4,3}^4, K_{4,3}^5, K_{4,3}^6$ in Figure 4.2) on seven vertices, and eleven on n ($n \geqslant 8$) vertices. Given a graph G, the *line graph* of G, denoted by $L(G)$, is the graph with vertex set $E(G)$ and edge set $\{xy : x, y \in E(G)$ are adjacent in $G\}$. Our main result is the following.

Theorem 1.1. *A 4-connected graph G is $\bar{P}_7$-free if and only if either G is planar or G belongs to $\mathcal{DW} \cup \mathcal{C}_1 \cup \mathcal{K} \cup \{K_6,\ L(K_{3,3}), \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$, where $\Gamma_1, \ldots, \Gamma_5$ are the five graphs shown below.*

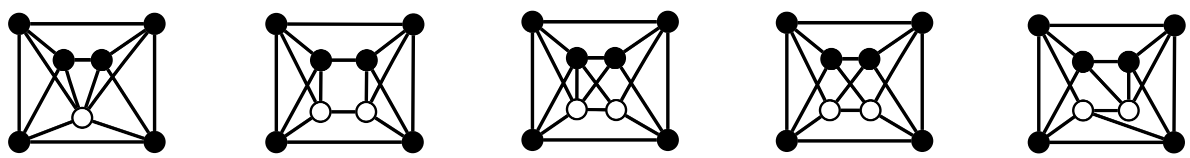

Figure 1.1: Graphs Γ_1, Γ_2, Γ_3, Γ_4, Γ_5

This theorem implies the following.

Corollary 1.2. *A 4-connected graph G is C_7^2-free if and only if either G is planar or G belongs to $\mathcal{DW} \cup \mathcal{K} \cup \{K_6,\ L(K_{3,3}),\ \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$.*

We remark that Theorem 1.1 is not a complete characterization of $\bar{P}_7$-free graphs, since we do not know those $\bar{P}_7$-free graphs that have a low connectivity. As observed in [2], $\bar{P}_7$-free graphs are precisely graphs constructed by 0-, 1-, 2-, and 3-sums starting from K_1, K_2, K_3, K_4, and internally 4-connected $\bar{P}_7$-free graphs. It follows that we need to determine all internally 4-connected $\bar{P}_7$-free graphs in order to obtain a complete characterization. Theorem 1.1 determines all 4-connected $\bar{P}_7$-free graphs, but it seems that there are still many internally 4-connected ones that are not 4-connected. For instance, consider graphs obtained from two disjoint copies of $K_{2,n}$ by adding a perfect matching in between. All such graphs are internally 4-connected and $\bar{P}_7$-free.

A closely related problem is to determine the extreme function for $\bar{P}_7$-free graphs. Such a function has been determined for many classes, including K_6-free graphs and Petersen-free graphs [4]. As a consequence of Theorem 1.1, every 4-connected graph with n vertices and $4n-9$ edges must contain a $\bar{P}_7$-minor (the extreme graphs are those in $\mathcal{K}$). This conclusion is no longer valid if the connectivity is weakened. For instance, for $n = 3k$, 3-summing $k-1$ copies of K_6 over the same triangle results in a 3-connected $\bar{P}_7$-free graph with n vertices and $4n-9$ edges. It seems reasonable to conjecture that every graph on n vertices and $4(n-2)$ edges must contain a $\bar{P}_7$-minor.

We close this section by providing an outline of the rest of the paper. In the next section we explain how our approach works. In particular, we introduce a chain theorem for 4-connected graphs, which says that all 4-connected graphs are "extensions" of certain basic graphs. Our proof of Theorem 1.1 will be divided into two parts. First, in Section 3, we determine $\bar{P}_7$-free extensions of every basic graph that is not K_5. Then, in Section 4, we determine $\bar{P}_7$-free extensions of K_5. Finally, we prove Theorem 1.1 and Corollary 1.2 in the end of Section 4.

2 Basic lemmas

Our main tool is a chain theorem for 4-connected graphs. To explain this result we need a few definitions. A cubic graph G is called *cyclically 4-connected* if G has four disjoint paths between any two disjoint cycles of G. It is not difficult to see that every cyclically 4-connected cubic graph is 3-connected (this was also observed in [9]). Let $\mathcal{L}$ denote the class of line graphs of cyclically 4-connected cubic graphs.

All graphs considered in this paper are simple. In particular, we use G/e to denote the graph obtained from G by first contracting e and then deleting all but one edge from each parallel family. When both ends of e have degree at least four, the inverse operation of this modified contraction is called *splitting* a vertex, which is formally defined as follows. Let v be a vertex of a graph G. Let $N_G(v)$ denote the set of vertices of G that are adjacent to v, which are also known as *neighbors* of v. Let $X, Y \subseteq N_G(v)$ such that $X \cup Y = N_G(v)$ and $|X|, |Y| \geqslant 3$. Let G' be obtained from $G \backslash v$ by adding two adjacent vertices x, y and then joining x to all vertices in X and y to all vertices in Y. We call G' a *split of* G. Now we can state the chain theorem [10] that we will use.

Theorem 2.1. *Every 4-connected graph can be obtained from a graph in $\mathcal{C} \cup \mathcal{L}$ by repeatedly splitting vertices.*

We also make the following observation.

Lemma 2.2. *If G' is obtained from a 4-connected graph G by splitting a vertex v, then G' is also 4-connected.*

Proof. Suppose, to the contrary, that G' has a vertex cut S of size at most three. Let x, y and X, Y be as in the definition of vertex split. Since $G = G'/xy$ is 4-connected, exactly one of x, y, say x, is in S. Then, for the same reason, y is an isolated vertex in $G'\backslash S$, which contradicts the assumption $|Y| \geqslant 3$. □

The above two results suggest an algorithm for generating all 4-connected graphs. We begin with graphs in $\mathcal{C} \cup \mathcal{L}$, which are known to be 4-connected. In the general step, we split each vertex of each constructed graph in all possible ways. Theorem 2.1 implies that graphs generated by this procedure include all 4-connected graphs, and Lemma 2.2 ensures that the generated graphs are precisely all 4-connected graphs. We will follow this algorithm to generate all 4-connected $\bar{P}_7$-free graphs.

When analyzing cubic graphs we will need the following version of Menger theorem, which can be found in Section 3.3 of [1].

Lemma 2.3. *Let G be a graph and let $\mathcal{P}$ be a set of k disjoint paths of G between disjoint $A, B \subseteq V(G)$. If G has a set $\mathcal{Q}$ of $k+1$ disjoint paths between A and B, then $\mathcal{Q}$ can be chosen so that each end of a path in $\mathcal{P}$ is also an end of a path in $\mathcal{Q}$.*

A graph G is a *subdivision* of a graph H if $G = H$ or G is obtained from a subdivision of H smaller than G by deleting an edge xy, and adding a new vertex z and two new edges zx, zy. The next is an easy lemma which was also observed in [7].

Lemma 2.4. *If a subdivision of H is a subgraph of G then $L(H)$ is a minor of $L(G)$.*

We also need the following result from [5].

Theorem 2.5. *If a nonplanar graph G is obtained from a 4-connected planar graph by splitting a vertex, then G contains $\bar{P}_7$ as a minor.*

3 Extensions of large graphs

Let $Ext(G)$ be the class of $\bar{P}_7$-free graphs that are either G or obtained from G by repeatedly splitting vertices. By Theorem 2.1, we need to determine $Ext(G)$ for every $G \in \mathcal{C} \cup \mathcal{L}$. In this section we consider extension of graphs in $(\mathcal{C} - \{K_5\}) \cup \mathcal{L}$, and we will consider $Ext(K_5)$ in the next section. As usual, a degree-three vertex will be called *cubic*.

We first consider planar graphs in $\mathcal{C} \cup \mathcal{L}$. The result follows from Theorem 2.5 and Lemma 2.2.

Lemma 3.1. *If $G \in \mathcal{C} \cup \mathcal{L}$ is planar then all graphs in $Ext(G)$ are planar.*

Next we consider nonplanar graphs in $\mathcal{L}$.

Lemma 3.2. *$L(K_{3,3})$ is C_7^2-free.*

Proof. Since $L(K_{3,3})$ is connected, if C_7^2 is a minor of $L(K_{3,3})$, the minor can be obtained by contracting two edges e, f and then deleting some edges. Let $e = xy$ and let xyz be the unique triangle containing e (see Figure 3.1). Notice that z is cubic in $L(K_{3,3})/e$, so f has to be incident with z. If f is not in the triangle xyz then $L(K_{3,3})/e/f$ has a cubic vertex, and hence cannot contain C_7^2. If f is in the triangle xyz then $L(K_{3,3})/e/f$ is isomorphic to Γ_1. To obtain C_7^2, we have to delete one edge from Γ_1. However, any edge deletion results in a cubic vertex, which implies that $L(K_{3,3})$ is C_7^2-free. □

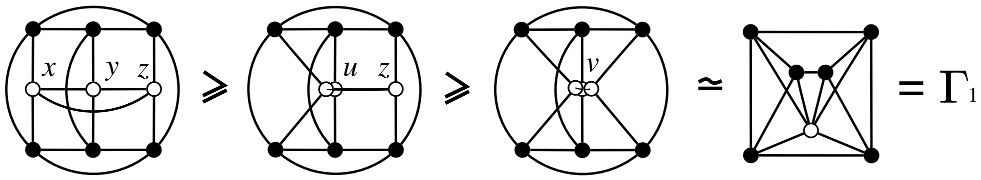

Figure 3.1: Contracting edges of $L(K_{3,3})$

Lemma 3.3. *If G is a cyclically 4-connected nonplanar cubic graph, then either $G = K_{3,3}$ or G contains a subdivision of V_8.*

Proof. This result follows from a characterization of V_8-free graphs [8]. However, instead of explaining the characterization, we provide a short direct proof of this lemma.

Since G is cubic and nonplanar, G contains a subgraph H that is a subdivision of $K_{3,3}$. Let $x_1, x_2, x_3, y_1, y_2, y_3$ be the cubic vertices of H and let P_{ij}, where $i, j \in \{1, 2, 3\}$, be the x_iy_j-path of H corresponding to edge x_iy_j of $K_{3,3}$. If $|V(H)| = 6$ then $G = K_{3,3}$ since G is connected. So we assume that P_{11} has interior vertices. Since G is 3-connected, $G\backslash\{x_1, y_1\}$ has a path Q between $P_{11}\backslash\{x_1, y_1\}$ and $H\backslash V(P_{11})$. If an end of Q is on P_{ij} for some $i, j \in \{2, 3\}$ then $H \cup Q$ is a subdivision of V_8. So we assume without loss of generality that Q has an end on P_{12}. Let A be the cycle contained in $P_{11} \cup P_{12} \cup Q$ and let B be the union of P_{ij} for $i = 2, 3$ and $j = 1, 2, 3$. By Lemma 2.3, since G is cyclically 4-connected, G has four disjoint paths Q_1, Q_2, Q_3, Q_4 between A, B and such that y_i $(i = 1, 2, 3)$ is an end of Q_i. Now it is easy to check that the union of A, B and Q_1, Q_2, Q_3, Q_4 contains a subdivision of V_8. □

Lemma 3.4. *If $G \in \mathcal{L}$ is nonplanar then $Ext(G) = \emptyset$, unless $G = L(K_{3,3})$, and in this case $Ext(G) = \{G\}$.*

Proof. We first observe that the line graph of any planar cubic graph is planar. So if $G \in \mathcal{L}$ is nonplanar and $G = L(H)$, then H is nonplanar. If H is not $K_{3,3}$, by Lemma 3.3, H contains a subdivision of V_8. Notice that $L(V_8)$ contains a $\bar{P}_7$-minor (see Figure 3.2), so we deduce from Lemma 2.4 that G contains a $\bar{P}_7$-minor, which proves $Ext(G) = \emptyset$.

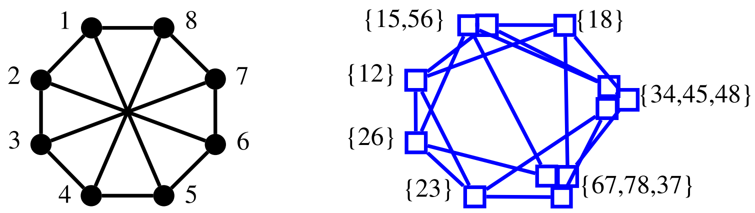

Figure 3.2: V_8 and a $\bar{P}_7$-minor of $L(V_8)$

It remains to consider the case $H = K_{3,3}$. Since $\bar{P}_7$ can be obtained from C_7^2 by joining two nonadjacent vertices, by Lemma 3.2, $L(K_{3,3})$ is $\bar{P}_7$-free. To complete the proof, we show that any split of $L(K_{3,3})$ contains a $\bar{P}_7$-minor. Clearly, we only need to consider the cases that both the two new vertices have degree four, because other splits contain these special splits. Up to symmetry, there are two such splits, and both of them contain a $\bar{P}_7$-minor, as illustrated in Figure 3.3. □

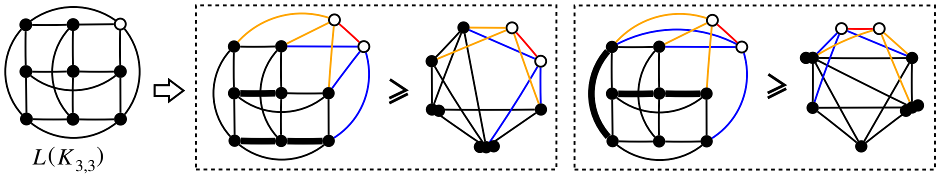

Figure 3.3: Two splits of $L(K_{3,3})$; they have a $\bar{P}_7$-minor by contracting the thick edges.

Remark. In Figure 3.3, the uncontracted edge is colored red and the other edges incident with the two new vertices are colored blue and orange, respectively. We will use this color scheme throughout the paper.

Finally we consider nonplanar graphs in $\mathcal{C} - \{K_5\}$. Note that these are exactly graphs in $\mathcal{C}_1 - \{K_5\}$ since graphs in $\mathcal{C}_0$ are planar.

Lemma 3.5. *For every integer $n \geqslant 3$, C^2_{2n+1} is $\bar{P}_7$-free.*

Proof. If a simple connected graph $G = (V, E)$ has an embedding in the projective plane, then by Euler formula, the embedding has $k = |E| - |V| + 1$ faces. If the size of the faces are $f_1, f_2, \ldots, f_k$, then $2|E| = f_1 + f_2 + \ldots + f_k \geqslant 3k = 3|E| - 3|V| + 3$, which implies $|E| \leqslant 3|V| - 3$.

For any graph G, let $G + u$ denote the graph obtained from G by adding a new vertex u and joining u to all vertices of G. Then $\bar{P}_7 + u$ is not projective since it has 8 vertices and $22 > 3|V| - 3$ edges. However, it is easy to see that C^2_{2n+1} can be embedded in the Möbius strip with all vertices on the boundary, hence $C^2_{2n+1} + u$ admits a projective embedding. As a result, $G + u$ is projective for every minor G of C^2_{2n+1} and thus $\bar{P}_7$ is not a minor of C^2_{2n+1}. □

Lemma 3.6. *For any $n \geqslant 3$, $Ext(C^2_{2n+1}) = \{C^2_{2n+1}\}$.*

Proof. By Lemma 3.5, we only need to show that every split of C^2_{2n+1} ($n \geqslant 3$) contains a $\bar{P}_7$-minor. We prove this by induction on n. First, every split of C^2_7 contains a $\bar{P}_7$-minor since it contains a split, that both new vertices have degree four, as shown in Figure 3.4.

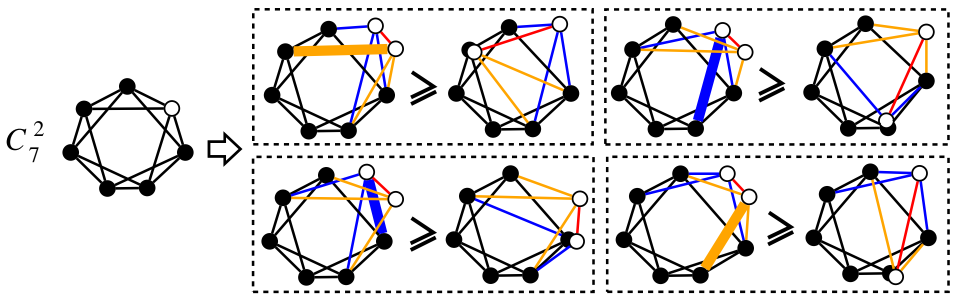

Figure 3.4: Four splits C^2_7; they have a $\bar{P}_7$-minor by contracting the thick edges.

Next, suppose $n > 3$. We claim that every split of C^2_{2n+1} contains a split of C^2_{2n-1} as a minor. Let $\{v_1, v_2, \ldots, v_{2n+1}\}$ be the vertex set of C^2_{2n+1} such that for all $1 \geqslant i \geqslant 2n+1$, $N(v_i) = \{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$, where the indices are taken modulo $2n+1$. Since vertices of C^2_{2n+1} are symmetric, we may choose to split v_4 and let G' be the resulted graph. Then we contract edges $v_{2n-1}v_{2n+1}$ and $v_{2n}v_1$ in G'. Let G'' be the resulted graph and let v'_{2n-1}, v'_1 be the new vertices obtained from contracting $v_{2n-1}v_{2n+1}$ and $v_{2n}v_1$, respectively. Then $N_{G''}(v'_{2n-1}) = \{v_{2n-3}, v_{2n-2}, v'_1, v_2\}$ and $N_{G''}(v'_1) = \{v_{2n-2}, v'_{2n-1}, v_2, v_3\}$. Since $n > 3$, $v_{2n-1}, v_{2n+1}, v_{2n}, v_1$ are not adjacent to v_4. So G'' is a split of C^2_{2n-1}, which proves the claim. Now the induction hypothesis implies that G' contains a $\bar{P}_7$-minor, which completes our induction and thus the lemma is proved. □

4 Extensions of the last graph

In this section we determine all graphs in $Ext(K_5)$.

Lemma 4.1. $Ext(K_5) = \mathcal{DW} \cup \mathcal{K} \cup \{K_6, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$

We divide the proof of Lemma 4.1 into a sequence of lemmas.

Lemma 4.2. *Every graph in* $\mathcal{DW}$ *is* C^2_7*-free.*

Proof. Each double-wheel has a set of at most two vertices whose deletion results in a graph of maximum degree at most two. This is a property preserved by all its minors. It is easy to check that C^2_7 does not have this property, so it is not a minor of any double-wheel. □

Lemma 4.3. *Every graph in* $\mathcal{K}$ *is* C^2_7*-free.*

Proof. Every $K_{4,n}$ has a set of at most four vertices that covers all edges of the graph. This is a property preserved by all its minors. It is easy to check that C^2_7 does not have this property, so it is not a minor of any $G \in \mathcal{K}$ since G is a minor of some $K_{4,n}$. □

Lemma 4.4. *All graphs in* $\{K_6, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$ *are* C^2_7*-free.*

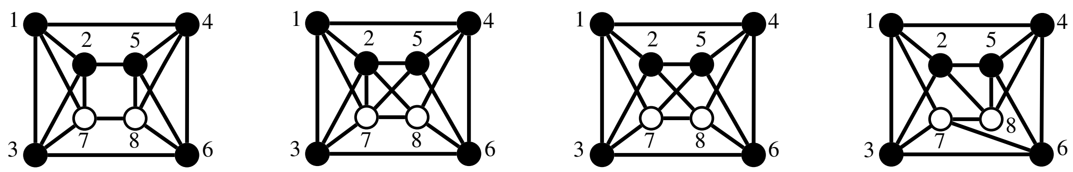

Figure 4.1: Graphs Γ_2, Γ_3, Γ_4, and Γ_5

Proof. It is clear that K_6 is C^2_7-free because $|V(K_6)| < |V(C^2_7)|$. Since $|V(\Gamma_1)| = |V(C^2_7)|$, $|E(\Gamma_1)| = |E(C^2_7)| + 1$, and $\Gamma_1 \backslash e$ has a cubic vertex for every edge e, it follows that Γ_1 is C^2_7-free. For $i = 2, 3, 4, 5$, notice that Γ_i has eight vertices, see Figure 4.1. If Γ_i contains a

$\bar{P}_7$-minor, we may assume that one of its edges is contracted. Some edges of Γ_i cannot be contracted since its contraction destroys the 4-connectivity of the graph. In Γ_2, we can contract only edge 14, 25, 36, or 78, and the resulted graph is isomorphic to Γ_1. In Γ_3, we can contract only edge 14, 25, 28, 36, 57, or 78, and the resulted graph is isomorphic to Γ_1 or $K^3_{4,3}$, where $K^3_{4,3}$ is a graph in $\mathcal{K}$ as shown in Figure 4.2. In Γ_4, we can contract only edge 14, 25, 28, 36, 57, or 78, and the resulted graph is isomorphic to Γ_1 or $K^4_{4,3}$, where $K^4_{4,3}$ is a graph in $\mathcal{K}$ as shown in Figure 4.2. In Γ_5, we can contract only edge 14 or 78, and the resulted graph is isomorphic to Γ_1. By Lemma 4.3, $\Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$ are C^2_7-free. □

In the next few lemmas we determine all extensions of K_5. The process is illustrated in Figure 4.2. We first determine all three splits of K_5. Then we determine all $\bar{P}_7$-free splits of each of these three. We will repeat this procedure and further determine all $\bar{P}_7$-free splits of these 7-vertex graphs. Finally, we show that any further split will create a $\bar{P}_7$-minor except for graphs in $\mathcal{DW} \cup \mathcal{K}$. In the following, we will denote graphs in $\mathcal{K}$ with seven or eight vertices by $K^j_{4,i}$ as shown in Figure 4.2.

Lemma 4.5. *The only splits of K_5 are K_6, $K_6 \backslash e$, and DW_4.*

Proof. When a vertex of K_5 is split, the degree sum of the two new vertices could be 8, 9, or 10, and these correspond to DW_4, $K_6 \backslash e$, and K_6, which proves the lemma. □

Lemma 4.6. *Every split of K_6 contains a $\bar{P}_7$-minor.*

Proof. To prove this lemma, we may assume that both the two new vertices have degree four. Up to symmetry, K_6 has only one such split, which contains $\bar{P}_7$ as a spanning subgraph (this is more clear if we consider the complements of the two graphs). □

The proofs of the last two lemmas are easy since the conclusions are simple. In proving the remainder lemmas we will see more cases. Typically, when we split a vertex we first consider the case when both the two new vertices have degree four. Then we view other splits as obtained from these minimal splits by adding edges. The following is a useful lemma for this approach. Let $G + e$ denote a graph obtained from a graph G by adding an edge e between two nonadjacent vertices.

Lemma 4.7. *(i) $DW_5 + e$ contains a $\bar{P}_7$-minor;*
(ii) $\Gamma_i + e$ contains a $\bar{P}_7$-minor, unless $i = 4$ and $\Gamma_4 + e$ is isomorphic to Γ_3;
(iii) $K^i_{4,3} + e$, where e is between hollow vertices, contains a $\bar{P}_7$-minor, unless $i = 4$;
(iv) $K^i_{4,4} + e$, where e is between hollow vertices, contains a $\bar{P}_7$-minor, unless $i = 11$.

Proof. Part (i). Notice that the complement of $DW_5 + e$ is P_5 together with two isolated vertices, which is a subgraph of P_7, so $DW_5 + e$ contains $\bar{P}_7$ as a subgraph.

Part (ii). Notice that the complement of $\Gamma_1 + e$ is P_6 together with one isolated vertex, which is a subgraph of P_7, so $\Gamma_1 + e$ contains $\bar{P}_7$ as a subgraph. For $i = 2, 3, 4, 5$, Γ_i can be obtained from Γ_1 by splitting the degree-5 vertex, as shown in Figure 4.1, where the two new vertices are 7 and 8. If e is incident to neither 7 nor 8, then $\Gamma_i + e$ contains a $(\Gamma_1 + e)$-minor by contracting 78. So in each Γ_i, we only need to consider the addition of those

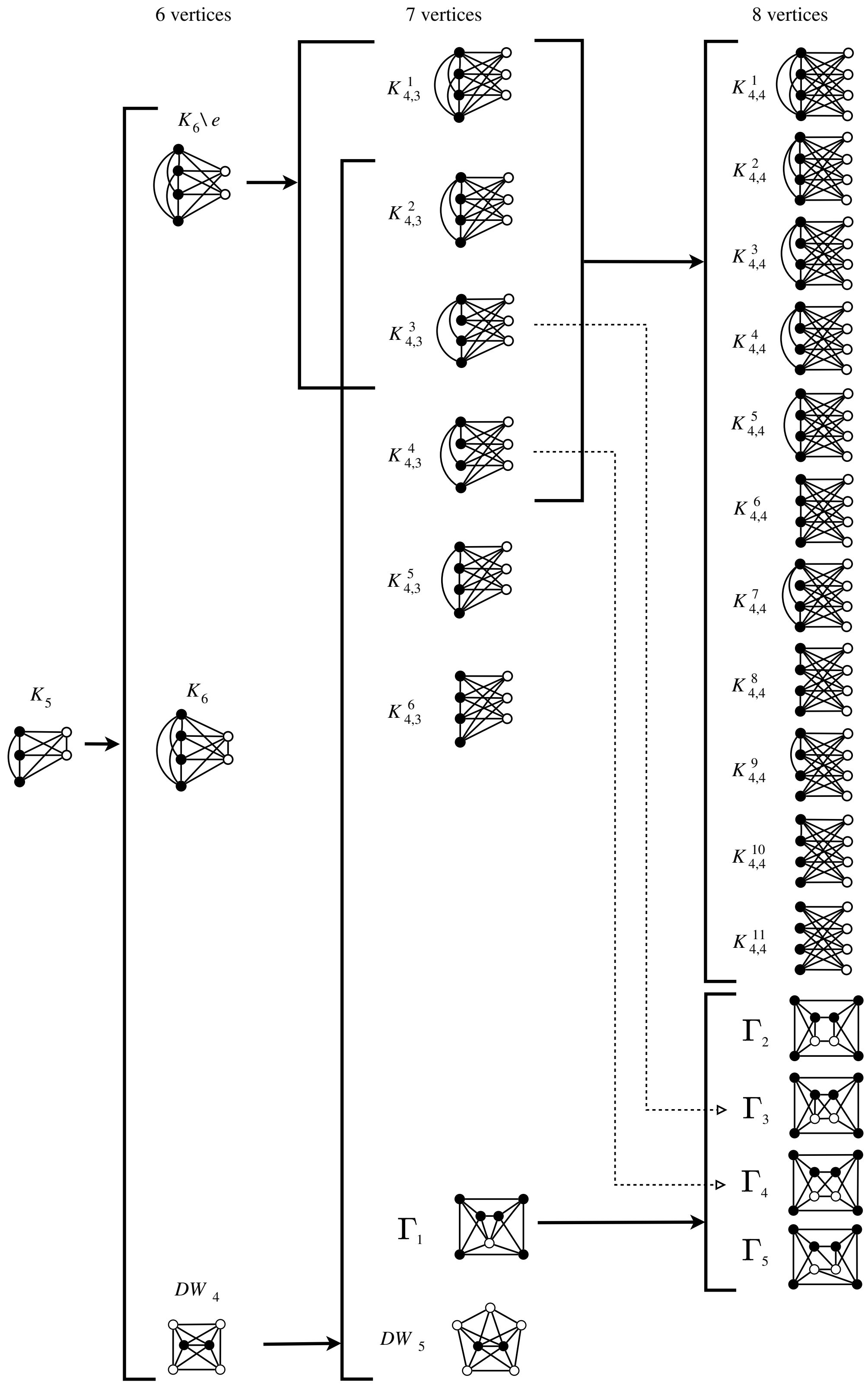

Figure 4.2: $\bar{P}_7$-free splits of K_5 on 6, 7, and 8 vertices

missing edges e that are incident with every edge f of Γ_i with Γ_i/f isomorphic to Γ_1. We use labels in Figure 4.1. In Γ_2, since Γ_2/f is isomorphic to Γ_1 for $f \in \{14, 25, 36, 78\}$, Γ_2+e contains a $\bar{P}_7$-minor. In Γ_3, since Γ_3/f is isomorphic to Γ_1 for $f \in \{25, 28, 57, 78\}$, we only consider adding $e = 58$. The resulted graph contains a $\bar{P}_7$-minor because it contains Γ_2+e as a spanning subgraph. In Γ_4, since Γ_4/f is isomorphic to Γ_1 for $f \in \{25, 28, 57, 78\}$, we only consider adding $e \in \{27, 58\}$ and the resulted graph is isomorphic to Γ_3. In Γ_5, since Γ_5/f is isomorphic to Γ_1 for $f \in \{14, 78\}$, we only consider adding $e \in \{18, 47\}$. Notice that these two additions are isomorphic. Suppose $e = 18$, then by contracting 23, the resulted graph is isomorphic to $\bar{P}_7$.

Part (iii). Since $K^4_{4,3} + e$ is isomorphic to $K^3_{4,3}$, by Lemma 4.3, $K^4_{4,3} + e$ is $\bar{P}_7$-free. For $i = 1, 2, 3, 5$, $K^i_{4,3}$ contains $K^6_{4,3}$ as a subgraph. We only need to consider $K^6_{4,3} + e$. Notice that the complement of $K^6_{4,3} + e$ consists of P_4 with one edge and one isolated vertex, which is a subgraph of P_7, so $K^6_{4,3} + e$ contains $\bar{P}_7$ as a subgraph.

Part (iv). Since $K^{11}_{4,4} + e$ is isomorphic to $K^{10}_{4,4}$, by Lemma 4.3, $K^{11}_{4,4} + e$ is $\bar{P}_7$-free. For $i = 1, \ldots, 9$, $K^i_{4,4}$ contains $K^{10}_{4,4}$ as a subgraph. We only need to consider $K^{10}_{4,4} + e$. Notice that there is an edge f in $K^{10}_{4,4} + e$ incident to two degree-4 vertices in $K^{10}_{4,4} + e$. Then $K^{10}_{4,4} + e$ contains a $(K^3_{4,3} + e)$-minor by contracting f. So $K^{10}_{4,4} + e$ contains a $\bar{P}_7$-minor. □

Lemma 4.8. *The only $\bar{P}_7$-free splits of $K_6\backslash e$ are $K^1_{4,3}$, $K^2_{4,3}$, and $K^3_{4,3}$.*

Proof. We first claim that splitting a degree-4 vertex of $K_6\backslash e$ must result in a $\bar{P}_7$-minor. To prove this we may assume that both the two new vertices have degree four. Up to symmetry, $K_6\backslash e$ has only one such split. The complement of the split is P_5 together with two isolated vertices, which is a subgraph of P_7, so the split contains $\bar{P}_7$ as a subgraph.

Next we consider splitting a degree-5 vertex of $K_6\backslash e$. Suppose both of the two new vertices, x^1, x^2, have degree four. Up to symmetry, there are exactly three such splits. They are denoted by G_1, G_2, G_3 and are shown in Figure 4.3. The first two splits, G_1 and G_2, contain $\bar{P}_7$ as a spanning subgraph. The third split G_3 is isomorphic to $K^3_{4,3}$, which is $\bar{P}_7$-free by Lemma 4.3.

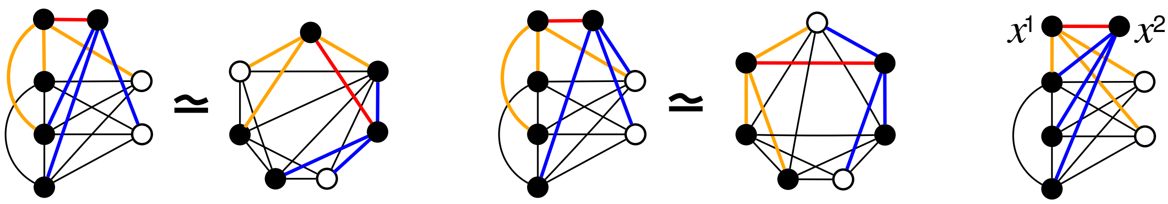

Figure 4.3: Three splits G_1, G_2, G_3 of $K_6\backslash e$.

Now suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by G, is obtained from G_1, G_2, or G_3 by adding edges. If G contains G_1 or G_2 then G contains a $\bar{P}_7$-minor. So we assume that G is obtained from G_3 by adding edges. If we only add edges incident with x^1, then G is isomorphic to $K^1_{4,3}$ or $K^2_{4,3}$, both of which are $\bar{P}_7$-free by Lemma 4.3. If we also add edges incident with x^2 then, by Lemma 4.7(iii), G contains a $\bar{P}_7$-minor because G contains $K^i_{4,3} + e$ $(i = 1, 2, 3)$ as a spanning subgraph. Hence, the only $\bar{P}_7$-free splits of $K_6\backslash e$ are $K^1_{4,3}$, $K^2_{4,3}$, and $K^3_{4,3}$. □

Proofs for lemmas 4.9, 4.11, and 4.12 are of the same flavor. We generate all possible splits and we identify the ones that are $\bar{P}_7$-free. Since these lemmas only talk about graphs with fewer than ten vertices, the conclusions can be verified by a computer. We reproduced the process (splitting vertices and testing for minors) using Mathematica and we found that our conclusions agree with results produced by computer. So readers who are comfortable with computer-assisted proofs can fast forward to Lemma 4.13 for a summary and then move on to the next lemma where we will deal with graphs of unbounded size.

Lemma 4.9. *The only $\bar{P}_7$-free splits of DW_4 are Γ_1, DW_5, and $K^i_{4,3}$ for $i = 2, 3, 4, 5, 6$.*

Proof. We first consider splitting a degree-4 vertex of DW_4. Suppose both of the two new vertices have degree four. Up to symmetry, there are exactly three such splits. They are denote by G_1, G_2, G_3 and are shown in Figure 4.4. The first two splits, G_1 and G_2, contain $\bar{P}_7$ as a spanning subgraph. The third split G_3 is isomorphic to DW_5, which is $\bar{P}_7$-free by Lemma 4.2. Now suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by G, is obtained from G_1, G_2, or G_3 by adding edges. If G contains G_1 or G_2 then G contains a $\bar{P}_7$-minor. So we assume that G is obtained from G_3 by adding edges. Then G contains a $\bar{P}_7$-minor by Lemma 4.7(i) since G contains $DW_5 + e$ as a spanning subgraph.

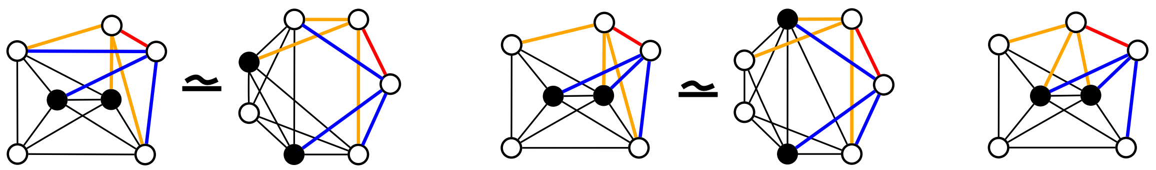

Figure 4.4: Three splits G_1, G_2, G_3 of DW_4.

Next we consider splitting a degree-5 vertex. Suppose both of the two new vertices, x^1, x^2, have degree four. Up to symmetry, there are exactly four such splits. They are denoted by H_1, H_2, H_3, H_4 and are shown in Figure 4.5. The first split H_1 contains $\bar{P}_7$ as a spanning subgraph. The other three are isomorphic to Γ_1, $K^4_{4,3}$, and $K^6_{4,3}$, respectively, which are $\bar{P}_7$-free by lemmas 4.3 and 4.4.

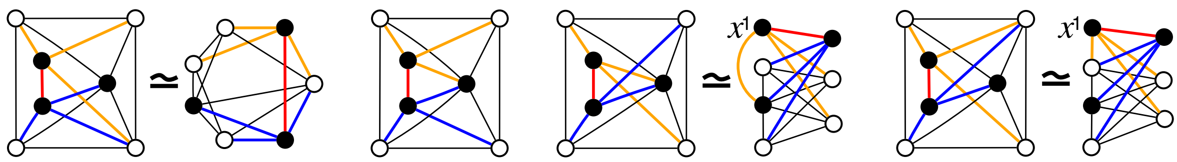

Figure 4.5: Another four splits H_1, H_2, H_3, H_4 of DW_4

Now suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by H, is obtained from H_1, H_2, H_3, or H_4 by adding edges. If H contains H_1 then H contains a $\bar{P}_7$-minor. If H is obtained from H_2 by adding edges then, by Lemma 4.7(ii), H contains a $\bar{P}_7$-minor since H contains $\Gamma_1 + e$. We assume that H is obtained from H_i ($i \in \{3,4\}$) by adding edges. If we only add edges incident with x^1 then H is isomorphic to $K^2_{4,3}$, $K^3_{4,3}$, or $K^5_{4,3}$, which are $\bar{P}_7$-free by Lemma 4.3. The conclusion is the same if $i = 3$ and we only add edges incident with x^2. Finally, if we add an edge incident with x^2, and, in case $i = 3$, we add at least one edge incident with x^1, then we deduce from Lemma 4.7(iii) that H contains a $\bar{P}_7$-minor. In summary, the only $\bar{P}_7$-free splits of DW_4 are Γ_1, and $K^i_{4,3}$ for $i = 2, \dots, 6$. □

Lemma 4.10. *Let x be a degree-4 vertex of a graph G, which does not have a 4-cycle with vertex set $N_G(x)$. Then every split G' of G at x contains a $G+e$-minor, for some e between two nonadjacent vertices of $N_G(x)$.*

Proof. We only need to consider the case that both the two new vertices x^1 and x^2 have degree four, because other splits contain these special splits. Then there are $y_1, y_2 \in N_G(x)$ such that $y_1 \in N_{G'}(x^1) - N_{G'}(x^2)$, and $y_2 \in N_{G'}(x^2) - N_{G'}(x^1)$. Since x has degree four in G, we let $N_{G'}(x^1) \cap N_{G'}(x^2) = \{u_1, u_2\}$. Since $y_1u_1y_2u_2$ is not a 4-cycle of G, we may assume by symmetry that y_1u_1 is not an edge in G. So G' contains a $(G + y_1u_1)$-minor by contracting x^1y_1. □

Lemma 4.11. *The only $\bar{P}_7$-free splits of Γ_1 are Γ_2, Γ_3, Γ_4, and Γ_5, and no split of Γ_i ($i = 2, 3, 4, 5$) is $\bar{P}_7$-free.*

Proof. We first claim that splitting a degree-4 vertex of Γ_1 must result in a $\bar{P}_7$-minor. Let x be a degree-4 vertex of Γ_1. Since Γ_1 does not have a 4-cycle with vertex set $N_{\Gamma_1}(x)$, by Lemma 4.10, every split G' of Γ_1 at x contains a $(\Gamma_1 + e)$-minor. By Lemma 4.7(ii), G' contains a $\bar{P}_7$-minor. Next we consider splitting the degree-6 vertex of Γ_1. Suppose both the two new vertices have degree four. Up to symmetry, there are only three such splits, which we denote by G_1, G_2, and G_3. These splits are isomorphic to Γ_2, Γ_4, and Γ_5, respectively, as shown in Figure 4.1, where the two new vertices are 7 and 8. By Lemma 4.4, these splits are $\bar{P}_7$-free. Now suppose at least one of the two new vertices has degree exceeding four. Then this split, denoted by G, is obtained from G_1, G_2, or G_3 by adding edges. If G is obtained from G_1 or G_3 by adding edges, then G contains $\Gamma_i + e$ for some $i = 2, 5$ as a spanning subgraph. By Lemma 4.7(ii), G contains a $\bar{P}_7$-minor. If G is obtained from $G_2 = \Gamma_4$ by adding edges then G contains $\Gamma_4 + e$ as a spanning subgraph. By Lemma 4.7(ii), either G is isomorphic to Γ_3 or G contains a $\bar{P}_7$-minor. So the only $\bar{P}_7$-free splits of Γ_1 are Γ_2, Γ_3, Γ_4, and Γ_5.

Next, we consider splits of Γ_i, $i = 2, 3, 4, 5$. For $i = 2, 5$, every vertex x in Γ_i has degree four and Γ_i does not have a 4-cycle with vertex set $N_{\Gamma_i}(x)$. By Lemma 4.10, every split G' of Γ_i at x contains a $(\Gamma_i + e)$-minor. Then by Lemma 4.7(ii), G' contains a $\bar{P}_7$-minor.

In Γ_3, for every degree-4 vertex x, Γ_3 does not have a 4-cycle with vertex set $N_{\Gamma_3}(x)$. By lemmas 4.7(ii) and 4.10, every split of Γ_3 at x contains a $\bar{P}_7$-minor. For splitting a degree-5 vertex in Γ_3, we claim that such splits contain a $\bar{P}_7$-minor. To prove this we may assume that both the two new vertices have degree four. Up to symmetry, Γ_3 has exactly six such splits, which contain a $\bar{P}_7$-minor, as shown in Figure 4.6.

In Γ_4, every vertex x in Γ_4 has degree four and Γ_4 does not have a 4-cycle with vertex set $N_{\Gamma_4}(x)$. By Lemma 4.10, every split G' of Γ_4 at x contains a $(\Gamma_4 + e)$-minor. By

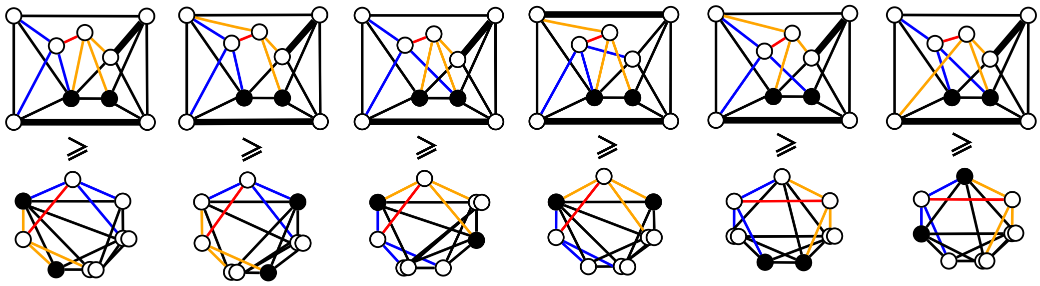

Figure 4.6: Six splits of Γ_3; they contain a $\bar{P}_7$-minor by contracting the thick edges.

Lemma 4.7(ii), G' contains a $\bar{P}_7$-minor unless $\Gamma_4 + e = \Gamma_3$. In this exception case, G' is also a split of Γ_3. Then our proof in the last paragraph shows that G' has a $\bar{P}_7$-minor. □

In Lemma 4.12 and Lemma 4.15, we will consider splits of graphs in $\mathcal{K}$. We will use the following terminology in both cases. For any graph K in $\mathcal{K}$, let X be a set of four vertices that cover all edges of K, and let $Y = V(K) - X$. Let G be a split of K, where a vertex x in X is split into x^1, x^2. Then there are two possibilities. If x^1 or x^2 is adjacent to no vertex in Y then we call G a *clean* split of K. It is clear that if G is clean then G belongs to $\mathcal{K}$. A non-clean split is called a *mixed* split. In other words, G is a mixed split if both x^1 and x^2 have a neighbor in Y.

Lemma 4.12. *Let G be a split of $H = K^i_{4,3}$, $i = 1, \ldots, 6$.*
(i) If G is obtained by splitting a vertex in Y then G contains a $\bar{P}_7$-minor.
(ii) If G is mixed then G contains a $\bar{P}_7$-minor, unless G is isomorphic to Γ_3 or Γ_4.

Proof. Suppose G is obtained by splitting some $y \in Y$ into y^1 and y^2. We may assume that y^1 and y^2 have degree four. Then G has a minor $\bar{P}_7$ because G contains a split as shown in Figure 4.7(a) as a subgraph. This proves (i).

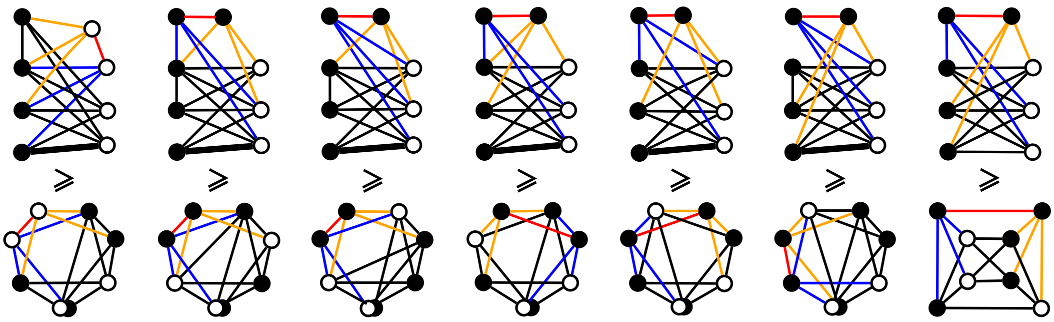

Figure 4.7: These splits of $K^i_{4,3}$ have a $\bar{P}_7$-minor by contracting the thick edges.

To prove (ii), we assume that G is mixed and is obtained by splitting $x \in X$ into x^1 and x^2. Let u, v, w denote the other three vertices of X. We first claim that if both x^1, x^2 have degree four then G contains a $\bar{P}_7$-minor, unless G contains Γ_4 as a spanning subgraph. We group the cases according to the degree of x. Suppose x has only one neighbor u in X.

Then we may assume $uv \in E(H)$ as $H[X]$ is connected. Upto symmetry, x can be split in two ways, as shown in Figure 4.7(b-c), and both of them contain a $\bar{P}_7$-minor. Next, suppose x has exactly two neighbors u, v in X. Upto symmetry, x can be split in three ways, as shown in Figure 4.7(d-f), where in (f) we assume without loss of generality that $wu \in E(H)$. In all three cases we find a $\bar{P}_7$-minor. Finally, suppose x has three neighbors in X. Since G is mixed, there is only one split which is shown in Figure 4.7(g). In this case G contains Γ_4 as a spanning subgraph. The claim is proved.

Now suppose at least one of x^1, x^2 has degree exceeding four. Then G must have a triangle x^1x^2z. It is easy to see that either $G\backslash zx^1$ or $G\backslash zx^2$ is a mixed split of H. Therefore, G is obtained from a mixed split G' by adding edges, where both of the two new vertices of G' have degree four. Now the result follows immediately from the above claim and Lemma 4.7(ii). □

Lemma 4.13. *Every $G \in Ext(K_5)$ satisfies at least one of the following.*
(i) $G \in \{K_6, DW_4, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$;
(ii) $G \in \mathcal{K}$ with $|V(G)| \leqslant 7$;
(iii) $G \in Ext(DW_5)$;
(iv) $G \in Ext(K)$ for some $K \in \mathcal{K}$ with $|V(K)| = 8$.

Proof. The result follows from Lemmas 4.5, 4.6, 4.8, 4.9, 4.11, and 4.12. □

In the next two lemmas we consider the two infinite families contained in $Ext(K_5)$.

Lemma 4.14. $Ext(DW_5) = \{DW_n : n \geqslant 5\}$.

Proof. Since C_7^2 is a subgraph of $\bar{P}_7$, by Lemma 4.2, every double-wheel is $\bar{P}_7$-free. Thus it is enough for us to show that, for each $n \geqslant 5$, the only $\bar{P}_7$-free split of DW_n is DW_{n+1}. Suppose DW_n is constructed from cycle $v_1v_2 \dots v_n$ and two adjacent vertices u_1, u_2. Let G be a split of DW_n.

Suppose G is obtained by splitting u_i. Let u_i^1 and u_i^2 be the two new vertices. In case $n \geqslant 6$ we assume without loss of generality that u_i^1 has degree exceeding four. Let v_j be a neighbor of u_i^1. If possible we choose j such that v_j is not a neighbor of u_i^2. Then $G' = G/v_jv_{j+1}$ is a split of DW_{n-1}, because $G'/u_i^1u_i^2 = DW_{n-1}$ and both u_i^1, u_i^2 have degree at least four in G'. By repeating this process we see that G contains a minor that is obtained from DW_5 by splitting a degree-6 vertex. Note that DW_5 has two splits of u_1 such that both of the two new vertices have degree four (see Figure 4.8). Since both splits contain a $\bar{P}_7$-minor, it follows that G contains a $\bar{P}_7$-minor.

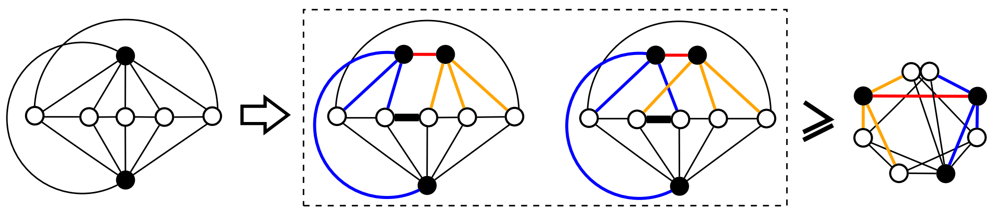

Figure 4.8: Two minimal splits of DW_5. Both contain a $\bar{P}_7$-minor.

Now suppose G is obtained by splitting v_i. If $G\backslash u_1u_2$ is nonplanar, by applying Theorem 2.5 to $G\backslash u_1u_2$ we deduce that G contains a $\bar{P}_7$-minor. So we assume that $G\backslash u_1u_2$ is planar. Then there are two cases as shown in Figure 4.9 with $i = 2$. So either G is isomorphic to DW_{n+1} or G contains a $\bar{P}_7$-minor. □

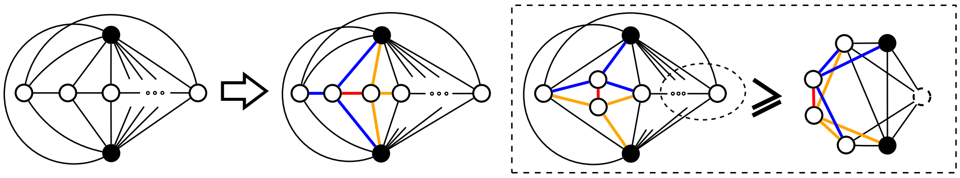

Figure 4.9: Two planar splits of DW_n: one is DW_{n+1} and the other contains a $\bar{P}_7$-minor.

Lemma 4.15. *If $K \in \mathcal{K}$ has at least eight vertices and G is a $\bar{P}_7$-free split of K, then G belongs to $\mathcal{K}$.*

Proof. Suppose the lemma is false. Then there exist $K \in \mathcal{K}$ with $|V(K)| \geqslant 8$ and a $\bar{P}_7$-free split G of K with $G \notin \mathcal{K}$. We choose such a K with $|V(K)|$ as small as possible. Recall that we denote by $X = \{x_1, x_2, x_3, x_4\}$ a set of four vertices that cover all edges of K and we let $Y = V(K) - X = \{y_1, y_2, \ldots, y_n\}$. If G is obtained by splitting some y_j, say, $j = 1$, we consider $G' = G/\{x_1y_4, x_1y_5, \ldots, x_1y_n\}$. Since $n \geqslant 4$, G' satisfies the assumption in Lemma 4.12(i), which implies that G' contains a $\bar{P}_7$-minor. This is a contradiction since G is $\bar{P}_7$-free. This contradiction shows that G is not obtained by splitting any vertex in Y, and thus G is obtained by splitting a vertex $x_i \in X$. Since G is not in $\mathcal{K}$, G must be a mixed split. Without loss of generality, let $i = 1$, let x_1^1 and x_1^2 be the two new vertices, and let y_j $(j = 1, 2)$ be a neighbor of x_1^j.

We first claim that in each G/x_iy_j $(i = 2, 3, 4$ and $j = 3, 4, \ldots, n)$, at lease one of x_1^1, x_1^2 has degree smaller than four. Suppose on the contrary that both x_1^1 and x_1^2 have degree at least four in G/x_iy_j. Since $(G/x_iy_j)/x_1^1x_1^2 = (G/x_1^1x_1^2)/x_iy_j = K/x_iy_j \in \mathcal{K}$, it follows that G/x_iy_j is a mixed split of K/x_iy_j. To proceed we consider two cases. First, assume $n = 4$. Then $K/x_iy_j = K_{4,3}^k$ for some $k = 1, 2, \ldots, 6$. By Lemma 4.12(ii) and our assumption that G is $\bar{P}_7$-free, it follows that G/x_iy_j is isomorphic to either Γ_3 or Γ_4. However, by Lemma 4.11, G contains a $\bar{P}_7$-minor, a contradiction. Next, assume $n \geqslant 5$. Observe that $G/x_iy_j \notin \mathcal{K}$, because if Z is a set of four vertices covering all edges of G/x_iy_j, then some vertex in $Y - \{y_j\}$ is not in Z, which implies $\{x_2, x_3, x_4\} \subseteq Z$, and thus a contradiction since we have to cover $x_1^1y_1$ and $x_1^2y_2$ with only one vertex. This observation shows that K/x_iy_j is a smaller counterexample, which contradicts the choice of G and K, and this contradiction completes the proof of our claim.

The above claim implies that for each edge x_iy_j $(i = 2, 3, 4$ and $j = 3, 4, \ldots, n)$, there exists $k \in \{1, 2\}$ such that $x_1^kx_iy_j$ is a triangle and x_1^k is a degree-4 vertex in G. In particular, each x_i $(i = 2, 3, 4)$ is adjacent to at least one of x_1^1, x_1^2. Consequently, $d_G(x_1^1) + d_G(x_1^2) \geqslant |Y| + (|X| - 1) + 2 \geqslant 9$, which implies that only one of x_1^1, x_1^2, say, x_1^1, has degree four. Now we conclude that $x_1^1x_iy_j$ is a triangle for all $i = 2, 3, 4$ and $j = 3, 4$, which contradicts $d_G(x_1^1) = 4$ and this contradiction proves the lemma. □

Proof of Lemma 4.1. The result follows from Lemmas 4.13, 4.14, and 4.15. □

Proof of Theorem 1.1. By Theorem 2.1, Lemmas 3.1, 3.4, and 3.6, we only need to determine $Ext(K_5)$. Then the result follows from Lemma 4.1. □

Proof of Corollary 1.2. Since C_{2n+1} ($n \geqslant 4$) contains a C^2_{2n-1}-minor, by Lemmas 3.2, 4.2, 4.3, and 4.4, the result follows from Theorem 1.1. □

Acknowledgements

The authors would like to thank the anonymous referees for their helpful comments and suggestions.

References

[1] Reinhard Diestel, *Graph Theory*, Third Edition, 2006.

[2] Guoli Ding, A characterization of graphs with no octahedron minor, *Journal of Graph Theory* **74**(2) (2013) 143-162.

[3] Guoli Ding and Cheng Liu, Excluding a small minor, *Discrete Applied Mathematics* **161**(3) (2013) 355-368.

[4] Kevin Hendrey and David Wood, The extremal function for Petersen minors, arXiv:1508.04541.

[5] John Maharry, An excluded minor theorem for the octahedron plus an edge, *Journal of Graph Theory* **57**(2) (2008) 124-130.

[6] John Maharry, A characterization of graphs with no cube minor, *Journal of Combinatorial Theory*, Series B **80**(2) (2000) 179-201.

[7] John Maharry, An excluded minor theorem for the octahedron, *Journal of Graph Theory* **31**(2) (1999) 95-100.

[8] John Maharry and Neil Robertson, The structure of graphs not topologically containing the Wagner graph. https://u.osu.edu/maharry.1/files/2016/03/WagnerFreeGraphsMaharryRobertson2015JCTB-1ilaiar.pdf

[9] William McCuaig, Edge Reductions in cyclically k-connected cubic graphs, *Journal of Combinatorial Theory*, Series B **56** (1) (1992) 16-44.

[10] Nicola Martinov, Uncontractable 4-connected graphs, *Journal of Graph Theory* **6**(3) (1982) 343-344.