Enumeration of Hybrid Domino-Lozenge Tilings II: Quasi-octagonal regions

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Abstract

We use the subgraph replacement method to prove a simple product formula for the tilings of an octagonal counterpart of Propp's quasi-hexagons (Problem 16 in *New Perspectives in Geometric Combinatorics*, Cambridge University Press, 1999), called quasi-octagon.

Keywords: perfect matchings; tilings; dual graphs; Aztec diamonds; Aztec rectangles; urban renewal; quasi-hexagons; quasi-octagons.

1 Introduction

We are interested in (lattice) regions on the square lattice $\mathbb{Z}^2$ with (southwest-to-northeast) *diagonals* drawn in. Here the diagonals of the square lattice are the lines $x = y + c$, for some integer c. In 1996, Douglas [3] proved a conjecture posed by Propp on the number of tilings of a certain family of regions on the square lattice with all *second diagonals* drawn-in (i.e. all the lines $x = y + 2c$, for some integer c). In particular, Douglas showed that the region of order n (shown in Figure 1.1) has $2^{2n(n+1)}$ tilings. In 1999, Propp listed 32 open problems in enumeration of perfect matchings and tilings in his survey paper [16]. Problem 16 on the list asks for a formula for the number of tilings of a certain quasi-hexagonal region on the square lattice with all *third diagonals* drawn in (i.e. all the lines $x = y + 3c$, for some integer c). The problem has been solved and generalized by the author (see [10]) for a certain quasi-hexagons in which the drawn-in diagonals are arbitrary. The method, subgraph replacement method, provided also a generalization of Douglas' result above. See [1], [9], [12], [14], [15], [17] for more applications of the subgraph replacement method.

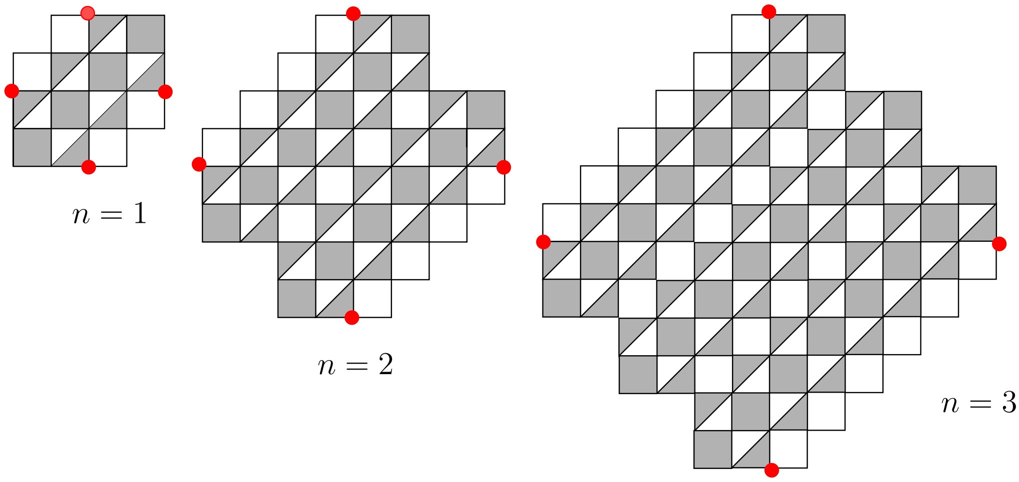


Figure 1.1: The Douglas' regions of order $n=1$, $n=2$ and $n=3$.

In this paper, we use the subgraph replacement method to enumerate tilings of a new family of regions that are inspired by Propp's quasi-hexagons (see [16], [10]). The new regions are defined in the next three paragraphs and illustrated by Figure 1.2.

Let $a, k, l, t; d_1, d_2, \dots, d_k; d'_1, \dots, d'_l; \overline{d}_1, \dots, \overline{d}_t$ be arbitrary positive integers. Denote by

$$\mathbf{d} := d_1 + d_2 + \ldots + d_k,$$

$$\mathbf{d}' := d'_1 + d'_2 + \ldots + d'_l,$$

and

$$\overline{\mathbf{d}} = \overline{d}_1 + \overline{d}_2 + \ldots + \overline{d}_t.$$

We say a diagonal α is *above* (resp., *below*) a diagonal β if α can be obtained by translating β upward (resp., downward).

We consider two distinguished drawn-in diagonals $\ell : x = y - \mathbf{d}$ and $\ell' : x = y - \mathbf{d} - \overline{\mathbf{d}}$ (ℓ and ℓ' are illustrated by the dashed lines in Figure 1.2). Draw in k diagonals $x = y$, $x = y - d_1, \dots, x = y - d_1 - d_2 - \ldots - d_{k-1}$ above ℓ; and l diagonals $x = y - \mathbf{d} - \overline{\mathbf{d}} - d'_1, x = y - \mathbf{d} - \overline{\mathbf{d}} - d'_1 - d'_2, \dots, x = y - \mathbf{d} - \overline{\mathbf{d}} - d'_1 - \ldots - d'_{t-1}$ below ℓ'. Draw in also $t-1$ additional diagonals $x = y - \mathbf{d} - \overline{d}_1, x = y - \mathbf{d} - \overline{d}_1 - \overline{d}_2, \dots, x = y - \mathbf{d} - \overline{d}_1 - \ldots - \overline{d}_{t-1}$ between ℓ and ℓ' (see Figure 1.2 for the case $k=2$, $l=2$, $t=3$, $d_1=5$, $d_2=3$, $d'_1=4$, $d'_2=3$, $\overline{d}_1=\overline{d}_2=\overline{d}_3=4$).

Next, we color the resulting dissection of the square lattice black and white, so that any two fundamental regions sharing an edge have opposite colors. Without loss of generality, we can assume that the triangles pointing toward ℓ and having bases on the top drawn-in diagonal are white. We draw a lattice path with unit steps south or east from the point $A = (0,0)$ so that at each step the black fundamental region is on the right. The lattice path meets ℓ at a lattice point B. It is easy to see that the x-coordinate of B is the sum $\sum_{i=1}^{k} \lfloor \frac{d_i+1}{2} \rfloor$. Continue from B to a vertex C on ℓ' in same fashion, with the difference that the black fundamental region is now on the left at each step. Finally, we go from C to a

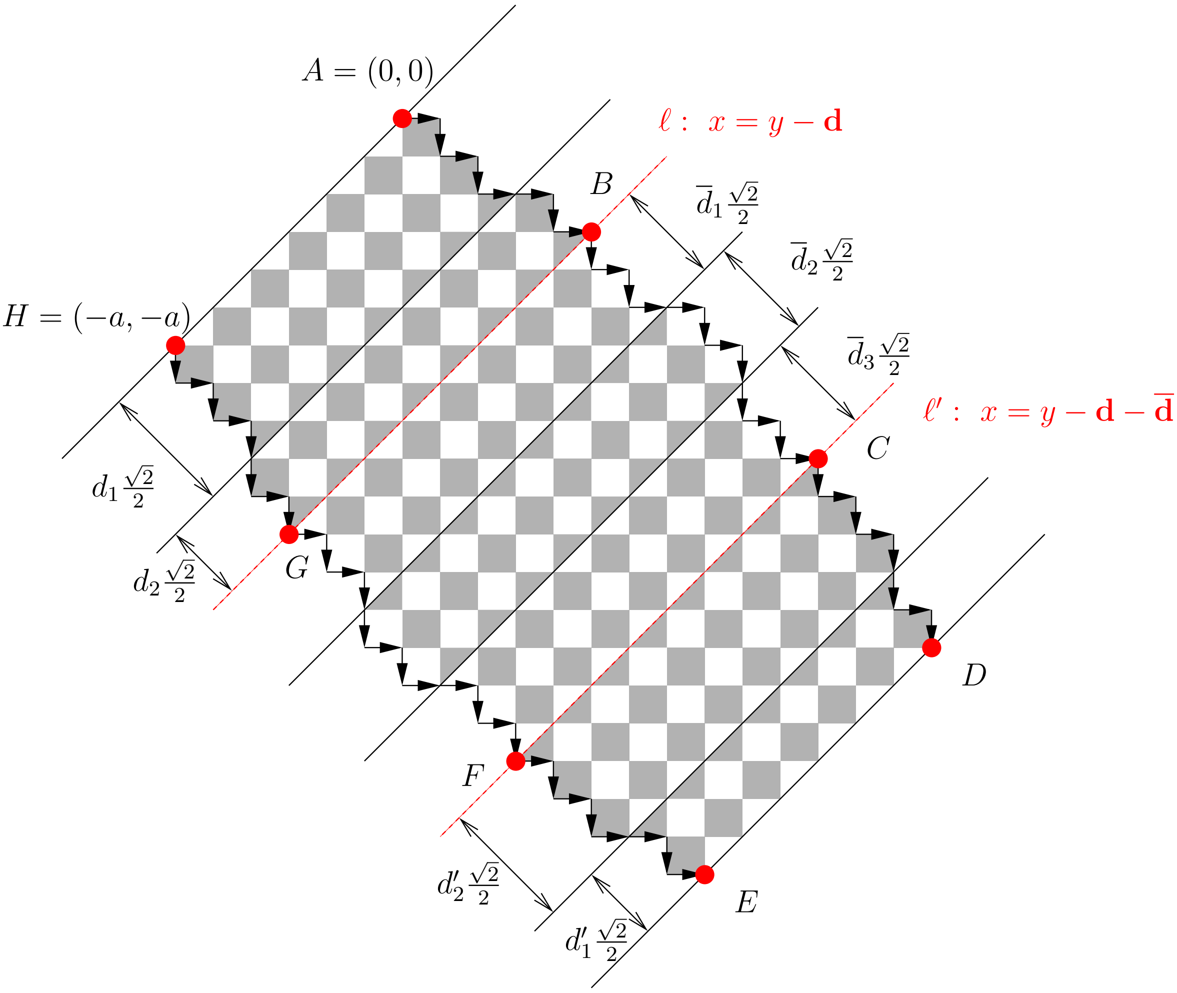


Figure 1.2: The quasi-octagon $\mathcal{O}_6(5,3;\ 4,4,4;\ 3,5)$.

vertex D on the bottom drawn-in diagonal (i.e. the line $x = y - \mathbf{d} - \overline{\mathbf{d}} - \mathbf{d}'$) in the same way with the black fundamental region is on the right at each step. The x-coordinates of C and D are $\sum_{i=1}^{k}\lfloor \frac{d_i+1}{2} \rfloor + \sum_{i=1}^{t}\lfloor \frac{\overline{d}_i+1}{2} \rfloor$ and $\sum_{i=1}^{k}\lfloor \frac{d_i+1}{2} \rfloor + \sum_{i=1}^{t}\lfloor \frac{\overline{d}_i+1}{2} \rfloor + \sum_{i=1}^{l}\lfloor \frac{d'_i+1}{2} \rfloor$, respectively. The described path from A to D is the northeastern boundary of the region.

The southwestern boundary is defined analogously, going from $H = (-a, -a)$ to a point $G \in \ell$, then to a point $F \in \ell'$, and to a point E on the bottom drawn-in diagonal. One can see that the southwestern boundary is obtained by reflecting the northeastern one about the perpendicular bisector of the segment AH. The segments AH and DE complete the boundary of the region, which we denote by $\mathcal{O}_a(d_1, \dots, d_k;\ \overline{d}_1, \dots, \overline{d}_t;\ d'_1, \dots, d'_l)$ (see Figure 1.2 for an example) and call a *quasi-octagon*.

Call the fundamental regions inside $\mathcal{O}_a(d_1, \dots, d_k;\ \overline{d}_1, \dots, \overline{d}_t;\ d'_1, \dots, d'_l)$ *cells*, and call them black or white according to the coloring described above. Note that there are two kinds of cells, square and triangular. The latter in turn come in two orientations: they may point towards ℓ' or away from ℓ'. A cell is called *regular* if it is a square cell or a triangular cell pointing away from ℓ'.

A *row of cells* consists of all the triangular cells of a given color with bases resting on a fixed lattice diagonal, or of all the square cells (of a given color) passed through by a fixed lattice diagonal.

Remark 1. Similar to the case of quasi-hexagons in [10] (see [10, Theorem 2.1(a)]), if the triangular cells running along the bottom drawn-in diagonal of a quasi-octagon are black, then we can not cover these cells by disjoint tiles, and the region has no tilings. *Therefore, from now on, we assume that the triangular cells running along the bottom diagonal are white.* This is equivalent to the the fact that the last step of the southwestern boundary is an east step.

The *upper, lower*, and *middle parts* of the region are defined to be the portions above ℓ, below ℓ', and between ℓ and ℓ' of the region. We define *the upper* and *lower heights* of our region to be the numbers of rows of black regular cells in the upper and lower parts. The *middle height* is the number of rows of *white* regular cells in the middle part. Denote by $h_1(\mathcal{O})$, $h_2(\mathcal{O})$ and $h_3(\mathcal{O})$ the upper, middle and lower heights of a quasi-octagon $\mathcal{O}$, respectively. Define also *the upper width* $w_1(\mathcal{O})$ and the *lower width* $w_2(\mathcal{O})$ of $\mathcal{O}$ to be the numbers of black triangles of the region with the bases resting on the segments BG and CF, respectively (i.e., $w_1(\mathcal{O}) = |BG|/\sqrt{2}$ and $w_2(\mathcal{O}) = |CF|/\sqrt{2}$, where $|BG|$ and $|CF|$ are the Euclidean lengths of the segments BG and CF). For example, the quasi-octagon in Figure 1.2 has the upper, middle and lower heights 5,6,5, respectively, and has the upper and lower widths equal to 8.

The main result of the paper concerns the number of tilings of quasi-octagons whose upper and lower widths are equal. In this case, the number of tilings is given by a simple product formula (unfortunately, in general, a quasi-octagon does not lead to a simple product formula).

The number of tilings of a quasi-octagon with equal widths is given by the theorem stated below.

Theorem 1.1. *Let* a, $d_1, \ldots, d_k$; $\overline{d}_1, \ldots, \overline{d}_t$; $d'_1, \ldots, d'_l$ *be positive integers for which the quasi-octagon* $\mathcal{O} := \mathcal{O}_a(d_1, \ldots, d_k;\ \overline{d}_1, \ldots, \overline{d}_t;\ d'_1, \ldots, d'_l)$ *has the upper, the middle, and the lower heights* h_1, h_2, h_3*, respectively, and has both upper and lower widths equal to* w*, with* $w > h_1, h_2, h_3$*.*

(*a*) *If* $h_1 + h_3 \neq w + h_2$*, then* $\mathcal{O}$ *has no tilings.*

(*b*) *If* $h_1 + h_3 = w + h_2$*, then the number of tilings of* $\mathcal{O}$ *is equal to*

$$2^{\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3-h_1(2w-h_1+1)/2-h_2(2w-h_2+1)/2-h_3(2w-h_3+1)/2}$$
$$\times\, 2^{\binom{h_1+2h_2+h_2}{2}-2h_2(w+h_2)-\binom{h_1+h_2}{2}-\binom{h_2+h_3}{2}}$$
$$\times\, \frac{\prod_{i=h_2+h_3+1}^{h_2+w}(i-1)!\prod_{i=h_1+h_2+1}^{h_2+w}(i-1)!\prod_{i=1}^{w-h_3}(i-1)!\prod_{i=1}^{w-h_1}(i-1)!}{\prod_{i=1}^{w-h_2}(h_2+i-1)!(w-i)!}, \tag{1.1}$$

where $\mathcal{C}_1$ *is the number of black regular cells in the upper part,* $\mathcal{C}_2$ *is the number of white regular cells in the middle part, and* $\mathcal{C}_3$ *is the number of black regular cells in the lower part of the region.*

We notice that if we consider the case when ℓ and ℓ' are superimposed in the definition of a quasi-octagon, we get a region with six vertices that is exactly a symmetric quasi-hexagon defined in [10]. We showed in [10] that the number of tilings of a symmetric quasi-hexagon is a power of 2 times the number of tilings of a hexagon on the triangular lattice (see Theorem 2.2 in [10]).

We will use a result of Krattenthaler [8] (about the number of perfect matchings of a certain family of Aztec rectangle graphs with holes) and several new subgraph replacement rules to prove Theorem 1.1 (see Section 3). As mentioned before, in general, the number of tilings of a quasi-octagon is *not* given by a simple product formula. However, we can prove a sum formula for the number of tilings of the region in the general case (see Theorem 3.9).

2 Preliminaries

This paper shares several preliminary results and definitions with its prequel [10]. The first result not reported in [10] is Lemma 2.5.

A *perfect matching* of a graph G is a collection of edges such that each vertex of G is adjacent to precisely one edge in the collection. A perfect matching is called a *dimer covering* in statistical mechanics, and also a *1-factor* in graph theory. The number of perfect matchings of G is denoted by $\mathrm{M}(G)$. More generally, if the edges of G carry weights, $\mathrm{M}(G)$ denotes the sum of the weights of all perfect matchings of G, where the weight of a perfect matching is the product of the weights on its constituent edges.

Given a periodic dissection of the plane, a *region* is a finite connected union of fundamental regions of that dissection. A *tile* is the union of any two fundamental regions sharing an edge. A *tiling* of the region R is a covering of R by tiles with no gaps or overlaps. The tilings of a region R can be naturally identified with the perfect matchings of its *dual graph* (i.e., the graph whose vertices are the fundamental regions of R, and

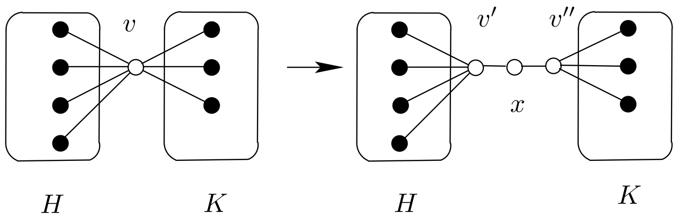


Figure 2.1: Vertex splitting.

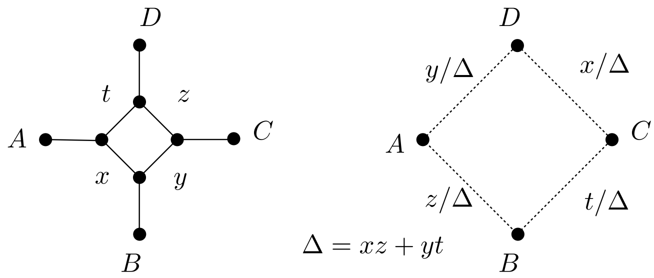


Figure 2.2: Urban renewal.

whose edges connect two fundamental regions precisely when they share an edge). In view of this, we denote by $\mathrm{M}(R)$ the number of tilings of R.

A *forced edge* of a graph G is an edge contained in every perfect matching of G. By removing a forced edge e, its endpoints, and all edges adjacent to these endpoints, we obtain a new graph G' that has the same number of perfect matchings as G. Indeed, each perfect matching μ of G has the form $\{e\} \cup \mu'$, where μ' is a perfect matching of G'; reversely, each perfect matching of G' is obtained by removing the edge e from a perfect matching of G.

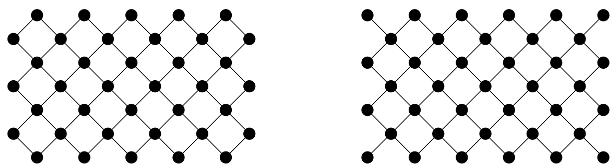
We present next three basic preliminary results stated below.

Lemma 2.1 (Vertex-Splitting Lemma, Lemma 2.2 in [1])**.** *Let G be a graph, v be a vertex of it, and denote the set of neighbors of v by $N(v)$. For any disjoint union $N(v) = H \cup K$, let G' be the graph obtained from $G \setminus v$ by including three new vertices v', v'' and x so that $N(v') = H \cup \{x\}$, $N(v'') = K \cup \{x\}$, and $N(x) = \{v', v''\}$ (see Figure 2.1). Then* $\mathrm{M}(G) = \mathrm{M}(G')$.

Lemma 2.2 (Star Lemma, Lemma 3.2 in [10])**.** *Let G be a weighted graph, and let v be a vertex of G. Let G' be the graph obtained from G by multiplying the weights of all edges that are adjacent to v by $t > 0$. Then $\mathrm{M}(G') = t\,\mathrm{M}(G)$.*

We note that the transformation in Lemma 2.2 is also called "*gauge transformation*" (see e.g. [7]).

The following result is a generalization due to Propp of the "urban renewal" trick first observed by Kuperberg (see e.g [17, pp. 284–285]).



(a) (b)

Figure 2.3: Two kinds of Aztec rectangles.

Lemma 2.3 (Spider Lemma)**.** *Let G be a weighted graph containing the subgraph K shown on the left in Figure 2.2 (the labels indicate weights, unlabeled edges have weight 1). Suppose in addition that the four inner black vertices in the subgraph K, different from A, B, C, D, have no neighbors outside K. Let G' be the graph obtained from G by replacing K by the graph $\overline{K}$ shown on right in Figure 2.3, where the dashed lines indicate new edges, weighted as shown. Then* $\mathrm{M}(G) = (xz + yt)\,\mathrm{M}(G')$.

The *Aztec rectangle (graph)* $AR_{m,n}$ is the graph whose vertex set is

$$\{(x, y) \mid 0 \leqslant x \leqslant 2n, 0 \leqslant y \leqslant 2m, x + y \text{ is odd}\},$$

and two vertices (x, y) and (x', y') are adjacent if and only if $|x - x'| = |y - y'| = 1$ (see Figure 2.3(a) for $AR_{3,4}$). The *odd Aztec rectangle* $OR_{m,n}$ is the graph whose vertices are the elements of the set

$$\{(x, y) \mid 0 \leqslant x \leqslant 2n, 0 \leqslant y \leqslant 2m, x + y \text{ is even}\},$$

and the vertices are also connected diagonally as in $AR_{m,n}$ (see Figure 2.3(b) for $OR_{3,4}$). If one removes the bottommost vertices in the graph $AR_{m,n}$, the resulting graph is denoted by $AR_{m-\frac{1}{2},n}$, and called a *baseless Aztec rectangle* (see the graph on the right in Figure 2.5 for an example with $m = 3$ and $n = 4$).

An *induced subgraph* of a graph G is a graph when vertex set is a subset U of the vertex set of G, and whose edge set consists of all edges of G with endpoints in U. The following lemma was proven in [10].

Lemma 2.4 (Graph Splitting Lemma, Lemma 3.6 in [10])**.** *Let $G = (V_1, V_2, E)$ be a bipartite graph with the two vertex classes V_1 and V_2. Let H be an induced subgraph of G.*

(a) Assume that H satisfies the following conditions.

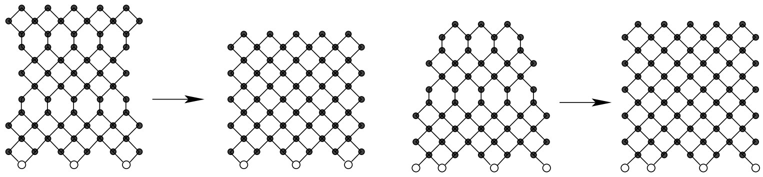
(i) The separating condition: there are no edges of G connecting a vertex in $V(H) \cap V_1$ and a vertex in $V(G - H)$,

(ii) The balancing condition: $|V(H) \cap V_1| = |V(H) \cap V_2|$.

Then

$$\mathrm{M}(G) = \mathrm{M}(H)\ \mathrm{M}(G - H). \tag{2.1}$$

(b) If H satisfies the separating condition, and but has $|V(H) \cap V_1| > |V(H) \cap V_2|$, then $\mathrm{M}(G) = 0$.



(a) (b) (c) (d)

Figure 2.4: Illustrating the transformations in Lemma 2.5.

Let $\mathcal{D} := D_a(d_1, \dots, d_k)$ be the portion of $\mathcal{O}_a(d_1, \dots, d_k;\ \overline{d}_1, \dots, \overline{d}_t;\ d'_1, \dots, d'_l)$ that is above the diagonal ℓ, we call it a *generalized Douglas region* (see [10], [11] and [13] for more details). The *height* and the *width* of $\mathcal{D}$ are defined to be the upper height and the upper width of the quasi-octagon, denoted by $h(\mathcal{D})$ and $w(\mathcal{D})$, respectively. We denote by $G_a(d_1, \dots, d_k)$ the dual graph of $D_a(d_1, \dots, d_k)$.

The *connected sum* $G\#G'$ of two disjoint graphs G and G' along the ordered sets of vertices $\{v_1, \dots, v_n\} \subset V(G)$ and $\{v'_1, \dots, v'_n\} \subset V(G')$ is the graph obtained from G and G' by identifying vertices v_i and v'_i, for $i = 1, \dots, n$. We have the following variant of Proposition 4.1 in [10].

Lemma 2.5 (Composite Transformations)**.** *Assume a, $d_1, d_2, \dots, d_k$ are positive integers for which $\mathcal{D} := D_a(d_1, \dots, d_k)$ is a generalized Douglas region having the height h and width w. Assume in addition that G is a graph, and that $\{v_1, v_2, \dots, v_{w-m}\}$ is an ordered subset of the vertex set of G, for some $0 \leqslant m < w$.*

(a) If the bottom row of cells in $\mathcal{D}$ is white, then

$$\mathrm{M}(\overline{G}_a(d_1, \dots, d_k)\#G) = 2^{\mathcal{C}-h(w+1)}\,\mathrm{M}(\overline{AR}_{h,w}\#G), \tag{2.2}$$

where $\overline{G}_a(d_1, \dots, d_k)$ and $\overline{AR}_{h,w}$ are the graphs obtained from $G_a(d_1, \dots, d_k)$ and $AR_{h,w}$ by removing the r_1-st, the r_2-nd, $\dots$, and the r_m-th vertices in their bottoms, respectively (if $m = 0$, then we do not remove any vertices from the bottom of the graphs); and where the connected sum acts on G along $\{v_1, v_2, \dots, v_{w-m}\}$ and on $\overline{G}_a(d_1, \dots, d_k)$ and $\overline{AR}_{h,w}$ along their bottom vertices ordered from left to rights. See the illustration of this transformation in Figures 2.4 (a) and (b).

(b) If the bottom row of cells in D is black, then

$$\mathrm{M}(\overline{G}_a(d_1, \dots, d_k)\#G) = 2^{\mathcal{C}-hw}\,\mathrm{M}(\overline{AR}_{h-\frac{1}{2},w-1}\#G), \tag{2.3}$$

where $\overline{G}_a(d_1, \dots, d_k)$ and $\overline{AR}_{h-\frac{1}{2},w-1}$ are the graphs obtained from $G_a(d_1, \dots, d_k)$ and $AR_{h-\frac{1}{2},w-1}$ by removing the r_1-st, the r_2-nd, $\dots$, and the r_m-th vertices in their bottoms, respectively; where the connected sum acts on G along the vertex set $\{v_1, v_2, \dots, v_{w-m}\}$, and on $\overline{G}_a(d_1, \dots, d_k)$ and $\overline{AR}_{h-\frac{1}{2},w-1}$ along their bottom vertices ordered from left to right (see Figures 2.4 (c) and (d)).

Proof. The lemma can be proven similarly to Proposition 4.1 in [10]. One sees that the graphs in our lemma and that in Proposition 4.1 in [10] are the same, except for m vertices

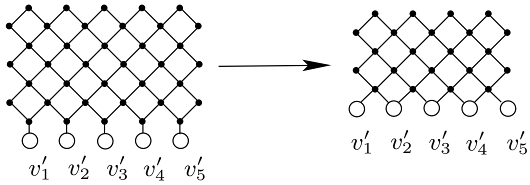


Figure 2.5: Illustrating the transformation in Lemma 2.6.

removed from their bottoms. However, all the transformations in the proof of Proposition 4.1 in [10] did not have any effects on the bottom vertices of the graphs. This allows us to apply the same transformations to the graphs in our lemma, and the equations (2.2) and (2.3) follow in the same way as the equations (4.1) and (4.2) in Proposition 4.1 in [10]. □

The following lemma is a special case of the Lemma 2.5.

Lemma 2.6 (Lemma 3.4 in [10]). *Let G be a graph and let $\{v_1, \dots, v_q\}$ be an ordered subset of its vertices. Then*

$$\mathrm{M}({}_{|}AR_{p,q}\#G) = 2^p\,\mathrm{M}(AR_{p-\frac{1}{2},q-1}\#G), \tag{2.4}$$

where ${}_{|}AR_{p,q}$ is the graph obtained from $AR_{p,q}$ by appending q vertical edges to its bottommost vertices; and where the connected sum acts on G along $\{v_1, \dots, v_q\}$, and on ${}_{|}AR_{p,q}$ and $AR_{p-\frac{1}{2},q}$ along their q bottommost vertices ordered from left to right. The transformation is illustrated in Figure 2.5.

The next result is due to Cohn, Larsen and Propp (see [2], Proposition 2.1; [5], Lemma 2). A (a, b)-*semihexagon* is the bottom half of a lozenge hexagon of side-lengths b, a, a, b, a, a (clockwise from top) on the triangular lattice.

Lemma 2.7. *Label the topmost vertices of the dual graph of (a, b)-semihexagon from left to right by $1, \dots, a + b$, and the number of perfect matchings of the graph obtained from it by removing the vertices with labels in the set $\{r_1, \dots, r_a\}$ is equal to*

$$\prod_{1\leqslant i<j\leqslant a} \frac{r_j - r_i}{j - i}, \tag{2.5}$$

where $1 \leqslant r_1 < \cdots < r_a \leqslant a + b$ are given integers.

We conclude this section by quoting the following result of Krattenthaler.

Lemma 2.8 (Krattenthaler [8], Theorem 14). *Let m, n, c, f be positive integers, and d be a nonnegative integer with $2n + 1 \leqslant 2m + d - 1 \geqslant n$ and $c + (2n - 2m - d + 1)f \leqslant n + 1$. Let G be a $(2m + d - 1) \times n$ Aztec rectangle, where all the vertices on the horizontal row that is by $d\sqrt{2}/2$ units below the central row, except for the c-th, $(c + f)$-th, …, and*

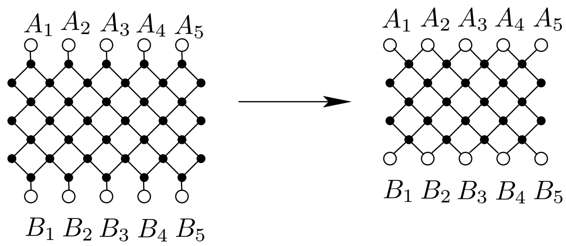


Figure 3.1: Illustrating the transformation in Lemma 3.1(a).

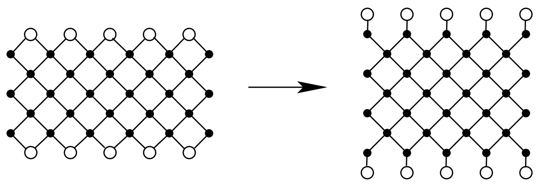


Figure 3.2: Illustrating the transformation in Lemma 3.1(b).

the $(c+(2n-2m-d+1)f)$-th vertex, have been removed. Then the number of perfect matchings of G equals

$$2^{\binom{2m+d}{2}+(n+1)(n-2m-d+1)}f^{m^2+(d-1)m+\binom{d}{2}+n(n-2m-d+1)}$$
$$\times\frac{\prod_{i=m+1}^{n+1}(i-1)!\prod_{i=m+d+1}^{n+1}(i-1)!\prod_{i=1}^{n-m+1}(i-1)!\prod_{i=1}^{n-m-d+1}(i-1)!}{\prod_{i=1}^{2n-2m-d+2}(c+f(i-1)-1)!(n+1-c-f(i-1))!},\tag{2.6}$$

where the product $\prod_{m+d+1}^{n+1}(i-1)!$ has to be interpreted as 1 if $n=m+d-1$, and as 0 if $n<m+d-1$, and similarly for the other products.

3 Proof of Theorem 1.1

Before proving Theorem 1.1, we present several new transformations stated below.

Lemma 3.1. *Let G be a graph and let $\{v_1,\dots,v_{2q}\}$ be an ordered subset of its vertices. Then*

(a)

$$\mathrm{M}\left({}^{|}_{|}AR_{p,q}\#G\right)=2^p\,\mathrm{M}(OR_{p,q-1}\#G),\tag{3.1}$$

where ${}^{|}_{|}AR_{p,q}$ is the graph obtained from $AR_{p,q}$ by appending q vertical edges to its top vertices, and q vertical edges to its bottom vertices; and where the connected sum acts on G along $\{v_1,v_2,\dots,v_{2q}\}$, and on ${}^{|}_{|}AR_{p,q}$ and $OR_{p,q-1}$ along their q top vertices ordered from left to right, then along their q bottom vertices ordered from left to right. See Figure 3.1 for the case $p=3$ and $q=5$.

(b)

$$\mathrm{M}(AR_{p,q}\#G)=2^p\,\mathrm{M}\left({}^{|}_{|}OR_{p,q-1}\#G\right),\tag{3.2}$$

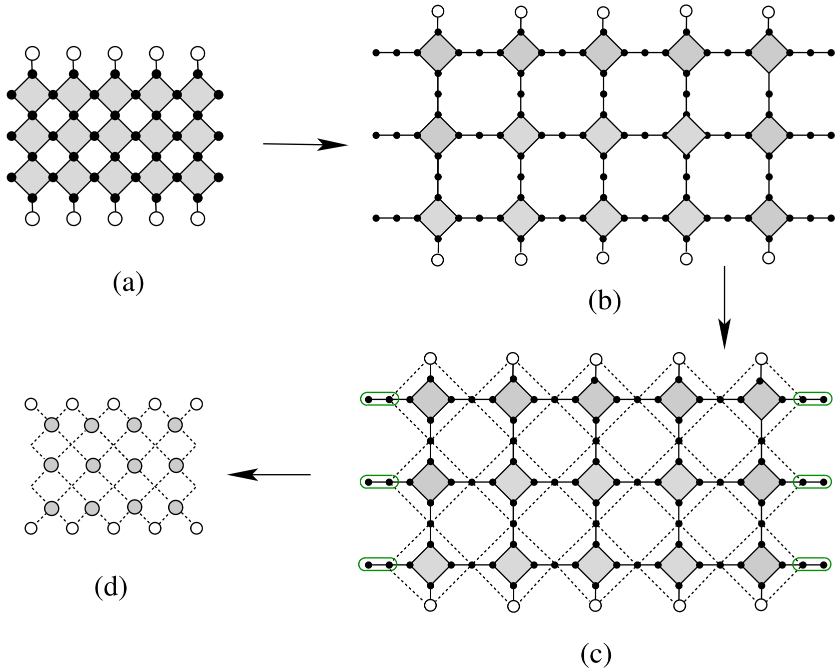


Figure 3.3: Illustrating the proof of Lemma 3.1(a).

where $^{|}_{|}OR_{p,q-1}$ *is the graph obtained from* $OR_{p,q-1}$ *by appending* q *vertical edges to its top vertices and* q *vertical edges to its bottom vertices; and where the connected sum acts on* G *along* $\{v_1, v_2, \dots, v_{2q}\}$*, and on* $AR_{p,q}$ *and* $^{|}_{|}OR_{p,q-1}$ *along their* q *top vertices ordered from left to right, then along their* q *bottom vertices ordered from left to right. See the illustration in Figure 3.2 for the case* $p = 3$ *and* $q = 5$*.*

Proof. We only prove part (a), and part (b) can be shown similarly.

Let G_1 be the graph obtained from $^{|}_{|}AR_{p,q}\#G$ by applying Graph-splitting Lemma 2.1 at all vertices that are not the end point of a vertical edge (see Figures 3.3 (a) and (b) for the case $p = 3$ and $q = 5$). Apply Spider lemma around all pq shaded diamonds in the graph G_1. Next, we remove $2q$ edges (as well as their endpoints) adjacent to a vertex of degree 1 from G_1, which are forced edges (illustrated in Figure 3.3 (c); the forced edges are the circled ones). The resulting graph is isomorphic to $OR^{1/2}_{p,q-1}\#G$, where $OR^{1/2}_{p,q-1}$ is the graph obtained from $OR_{p,q-1}$ by changing all weights of edges to 1/2 (see Figure 3.3(d); the dotted edges have weight 1/2). By Lemmas 2.1 and 2.3, we have

$$\mathrm{M}(^{|}_{|}AR_{p,q}\#G) = \mathrm{M}(G_1) = 2^{pq}\,\mathrm{M}(OR^{1/2}_{p,q-1}\#G). \tag{3.3}$$

Next, we apply Star Lemma 2.2 with weight factor $t = 2$ to all $p(q-1)$ shaded vertices of the graph $OR^{1/2}_{p,q-1}$ (see Figure 3.3(d)), the graph $OR^{1/2}_{p,q-1}\#G$ is turned into $OR_{p,q-1}\#G$. By the equality (3.3) and Spider Lemma 2.3, we get

$$\mathrm{M}(^{|}_{|}AR_{p,q}\#G) = 2^{pq}2^{-p(q-1)}\,\mathrm{M}(OR_{p,q-1}\#G), \tag{3.4}$$

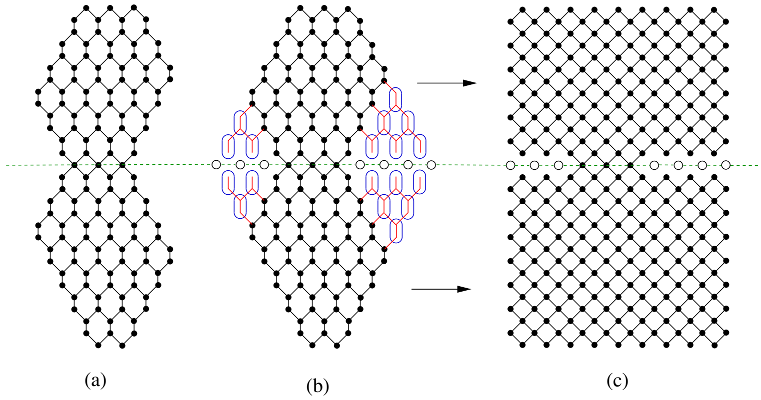


Figure 3.4: Illustrating the proof of Lemma 3.2.

which implies (3.1). □

Denote by $AR^d_{m,n}(A)$ the graph obtained from the Aztec rectangle $AR_{m,n}$ by removing all vertices at the positions in $A \subset \{1,\dots,n+1\}$ from the row that is by $d\sqrt{2}/2$ units below the central row (see Figure 3.4(c) for an example).

Lemma 3.2. *Let a, b, c, d, a', b' be positive integers, so that $a+b=a'+b'$, $c<\min(a,a')$, and $d<\min(a,a')$. Let $H^{a',b'}_{a,b,c,d} := H_1\#H_2$, where H_1 is the dual graph of a hexagon of sides b, $a-d$, d, $a+b-c-d$, c, $a-c$, and H_2 is the dual graph of a hexagon of sides $a+b-c-d$, d, $a'-d$, b', $a'-c$, c (in cyclic order, starting from the north side); and where the connected sum acts on H_1 along its $a+b-c-d$ bottom vertices ordered from left to right, and on H_2 along its $a+b-c-d$ top vertices ordered from left to right (see Figure 3.4(a) for the case $a=7$, $b=3$, $c=3$, $d=4$, $a'=8$, $b'=2$). Then*

$$\mathrm{M}(H^{a',b'}_{a,b,c,d}) = 2^{-a(a-1)/2-a'(a'-1)/2}\,\mathrm{M}(AR^{a-a'}_{a+a'-1,a+b-1}(A)), \tag{3.5}$$

where $A=\{1,\dots,c\}\cup\{a+b-d+1,\dots,a+b\}$.

Proof. Consider the graph $\widetilde{G} := S_1\#S_2$, where S_1 is the graph obtained from the dual graph of the (a,b)-semihexagon by removing the c leftmost and the d rightmost bottom vertices, and S_2 is the graph obtained from the dual graph of the (a',b')-semihexagon by removing the c leftmost and the d rightmost bottom vertices (see Figure 3.4(b)), and where the connected sum acts on S_1 and S_2 along their bottom vertices ordered from left to right. Since $H^{a',b'}_{a,b,c,d}$ is obtained from $\widetilde{G}$ by removing vertical forced edges and their endpoints (the circled edges indicate the forced edges in Figure 3.4(b)), $\mathrm{M}(H^{a',b'}_{a,b,c,d}) = \mathrm{M}(\widetilde{G})$.

Apply the transformation in Lemma 2.5(b) to S_1, we replace this graph by the graph $AR_{a-\frac{1}{2},a+b}$ with the c leftmost and the d rightmost bottom vertices removed. Apply this transformation one more time to S_2, then S_2 gets transformed into the graph $AR_{a'-\frac{1}{2},a'+b'}$ with the c leftmost and the d rightmost bottom vertices removed. This way, the graph $S_1\#S_2$ gets transformed precisely into the graph on the right hand side of (3.5) (see Figure 3.4(c)). Then the lemma follows from Lemmas 2.5 and 2.8. □

We introduce several new terminology and notations as follows.

We divide the family of quasi-octagons into four subfamilies, based on the color of the triangles running right above ℓ and the color of the triangles running right below ℓ'. In particular, *type-1 quasi-octagons* have black triangles running right above ℓ and right below ℓ'; *type-2 quasi-octagons* have those triangles white; *type-3 quasi-octagons* have the triangles right above ℓ black and the triangles right below ℓ' white; and *type-4 quasi-octagons* have white triangles right above ℓ, and black triangles right below ℓ'. To specify the type of a quasi-octagon, we denote by

$$\mathcal{O}_a^{(k)}(d_1,\dots,d_k;\ \overline{d}_1,\dots,\overline{d}_t;\ d'_1,\dots,d'_l)$$

the type-k quasi-octagon with corresponding parameters (i.e. we add the superscript k to the original denotation of the quasi-octagon). The dual graph of the region is denoted by

$$G_a^{(k)}(d_1,\dots,d_k;\ \overline{d}_1,\dots,\overline{d}_t;\ d'_1,\dots,d'_l).$$

The diagonals divide the middle part of a quasi-octagon into of t parts, called *middle layers*. We define the *height* of a middle layer to be the number of rows of white regular cells in the layer, the *width* of the layer to be the number of cells on each of those rows. Assume that the i-th middle layer has the height a_i and the width b_i, for $1 \leqslant i \leqslant t$; then the middle height of the quasi-octagon is $\sum_{i=1}^{t} a_i$. Moreover, one can see that $|b_i - b_{i+1}| = 1$, for any $i = 1,\dots,t-1$. A term b_i satisfying $b_i = b_{i-1} - 1 = b_{i+1} - 1$ (resp., $b_i = b_{i-1} + 1 = b_{i+1} + 1$) is called a *concave term* (resp., a *convex term*) of the sequence $\{b_j\}_{j=1}^{t}$, for $i = 2,\dots,t-1$.

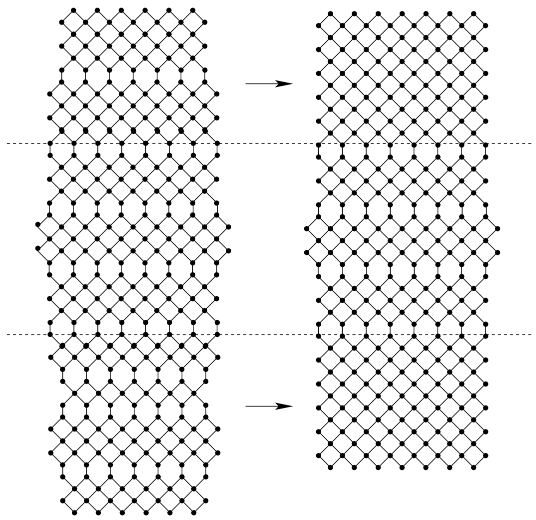
We are now ready to prove Theorem 1.1.

Outline of the proof:

There are 4 cases to distinguish, based on the type of the quasi-octagon.

- First, we prove in detail the case of type-1 quasi-octagons. The proof of this case is divided into 3 steps as follows:
  - Simplifying to the case when $k = l = 1$.
  - Simplifying further to the case when $k = l = t = 1$.
  - Proving the statement for $k = l = t = 1$.
- Second, we show that all other cases can be implied from the case of type-1 quasi-octagons.

Proof of Theorem 1.1. Without loss of generality, we can assume that $h_1 \geqslant h_3$ (otherwise, we reflect the region about ℓ and get a new quasi-octagon with the upper height larger than the lower height).



(a) (b)

Figure 3.5: Application of the transformation in Lemma 2.5 to the upper and the lower parts of the dual graph of a quasi-octagon.

Denote by $\mathcal{P}(c,f,m,d,n)$ the expression (2.6) in Lemma 2.8. The statement in part (b) of the theorem is equivalent to

$$\mathrm{M}(\mathcal{O}) = 2^{\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3-h_1(2w-h_1+1)/2-h_2(2w-h_2+1)/2-h_3(2w-h_3+1)/2} \times 2^{-\binom{h_1+h_2}{2}-\binom{h_2+h_3}{2}}\mathcal{P}(h_2+1,1,h_2+h_3,h_1-h_3,w+h_2-1). \tag{3.6}$$

We have four cases to distinguish, based on the type of the quasi-octagon.

Case 1. $\mathcal{O}$ is of type 1.

STEP 1. *Simplifying to the case when $k=l=1$.*

Let G be the dual graph of the type-1 quasi-octagon

$$\mathcal{O} := \mathcal{O}_a^{(1)}(d_1,\dots,d_k;\ \overline{d}_1,\dots,\overline{d}_t;\ d_1',\dots,d_l').$$

Apply the composite transformation in Lemma 2.5(b) with $m=0$, separately to the portions of G corresponding to the parts above ℓ and below ℓ' of the region $\mathcal{O}$ (which we call the *upper* and *lower parts* of the dual graph G; similarly, the *middle part* of G corresponds to the part between ℓ and ℓ' of $\mathcal{O}$). We replace the upper part of G by the graph $AR_{h_1-\frac{1}{2},w-1}$, and the lower part by the graph $AR_{h_2-\frac{1}{2},w-1}$ flipped over its base. This way, G gets transformed into the dual graph G' of the type-1 quasi-octagon

$$\overline{\mathcal{O}} = \mathcal{O}_{w-1}^{(1)}(2h_1-1;\ \overline{d}_1,\dots,\overline{d}_t;2h_3-1), \tag{3.7}$$

and by Lemma 2.5, we obtain

$$\mathrm{M}(\mathcal{O}) = 2^{\mathcal{C}_1-h_1w+\mathcal{C}_3-h_3w}\,\mathrm{M}(\overline{\mathcal{O}}) \tag{3.8}$$

(see Figure 3.5 for an example).

It is easy to check that $\mathcal{O}$ and $\overline{\mathcal{O}}$ have the same heights, the same widths, and the same middle part. This and the equality (3.8) imply that part (a) is true for $\mathcal{O}$ if and only if it is true for $\overline{\mathcal{O}}$. Suppose now that $h_1 + h_3 = h_2 + w$, the number of black regular cells in the upper part of $\overline{\mathcal{O}}$ is $\overline{\mathcal{C}}_1 = h_1 w$, and the number of black regular cells in the lower part of $\overline{\mathcal{O}}$ is $\overline{\mathcal{C}}_3 = h_3 w$ (note that we have $\mathcal{C}_2 = \overline{\mathcal{C}}_2$ since two regions have the same middle part). Therefore, by (3.8) again, (3.6) is equivalent to

$$\begin{aligned}\mathrm{M}(\overline{\mathcal{O}}) &= 2^{\overline{\mathcal{C}}_1+\overline{\mathcal{C}}_2+\overline{\mathcal{C}}_3-h_1(2w-h_1+1)/2-h_2(2w-h_2+1)/2-h_3(2w-h_3+1)/2}\\ &\quad\times 2^{-\binom{h_1+h_2}{2}-\binom{h_2+h_3}{2}}\mathcal{P}(h_2+1,1,h_2+h_3,h_1-h_3,w+h_2-1). \qquad (3.9)\end{aligned}$$

This means that the statement in part (b) is true for the region $\mathcal{O}$ if and only if it is true for the region $\overline{\mathcal{O}}$ that has only one layer in each of its upper part and lower parts. Therefore, without of loss generality, we can assume that our quasi-octagon $\mathcal{O}$ has $k = l = 1$.

STEP 2. Simplifying further to the case when $k = l = t = 1$.
Assume that the i-th middle layer of $\mathcal{O}$ has the height a_i and the width b_i, for $i = 1, 2, \dots, t$. Since $\mathcal{O}$ is a type-1 quasi-octagon, we have $b_1 = |BG| = w = |CF| = b_t$ and

$$\sum_{i=1}^{t-1}(b_i - b_{i+1}) = b_1 - b_t = 0. \qquad (3.10)$$

Since each term of the sum on the left-hand side of (3.10) is either 1 or -1, the numbers of 1's and -1's are equal. It implies that $t-1$ is even, or t is odd.

We now assume that $\mathcal{O}$ has $t \geqslant 3$ middle layers, i.e.

$$\mathcal{O} := \mathcal{O}^{(1)}_{w-1}(2h_1 - 1;\ \overline{d}_1, \dots, \overline{d}_t; 2h_3 - 1),$$

for some odd $t \geqslant 3$. Next, we show a way to construct a type-1 quasi-octagon $\mathcal{O}'$ having $t-2$ middle layers so that the statement of the theorem is true for $\mathcal{O}$ if and only if it is true for $\mathcal{O}'$.

We can find two consecutive terms having opposite signs in the sequence $\{(b_i - b_{i+1})\}_{i=1}^{t-1}$ (otherwise, the terms are all 1 or all -1; so the sum of them is different from 0, a contradiction to (3.10)). Assume that $(b_{j-1} - b_j)$ and $(b_j - b_{j+1})$ are such two terms, so b_j is a convex term or a concave term of the sequence $\{b_i\}_{i=1}^{t}$.

Suppose first that b_j is a convex term, i.e. $b_{j-1} = b_j - 1 = b_{j+1}$. Let B be graph obtained from the dual graph of the j-th middle layer by appending vertical edges to its topmost and bottommost vertices. In this case, B is isomorphic to the graph $|AR_{a_j,b_j}$ (see the graph between two dotted lines in Figure 3.6(a)). Apply the transformation in Lemma 3.1(1) to replace B by OR_{a_j,b_j-1}. This ways, the dual graph G of $\mathcal{O}$ is transformed into the dual graph G'' of the type-1 quasi-octagon

$$\mathcal{O}' := \mathcal{O}^{(1)}_{w-1}(2h_1 - 1; d_1, \dots, d_{j-2}, d_{j-1} + d_j + d_{j+1}, d_{j+2}, \dots, d_t; 2h_3 - 1) \qquad (3.11)$$

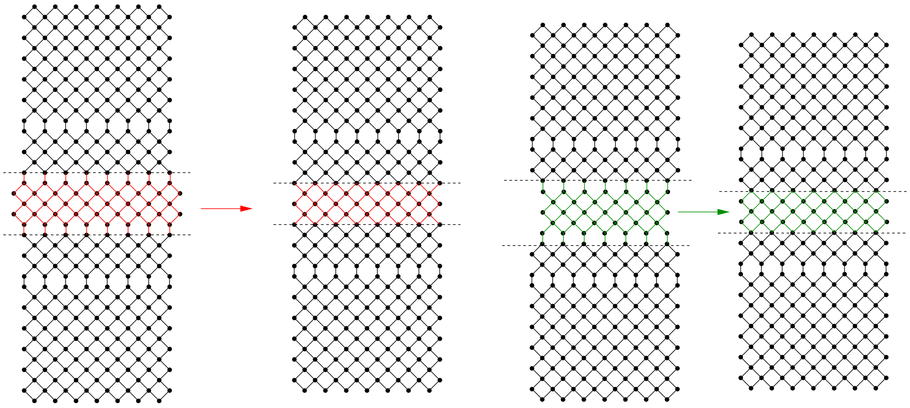


Figure 3.6: Removing convex and concave terms from the sequence the widths of the middle layers.

having $t-2$ middle layers (see Figures 3.6(a) and (b) for an example). By Lemma 3.1(a), we have

$$\mathrm{M}(\mathcal{O}) = 2^{a_j}\,\mathrm{M}(\mathcal{O}'). \tag{3.12}$$

Intuitively, we have just combined three middle layers (the $(j-1)$-th, the i-th and the $(j+1)$-th middle layers) of $\mathcal{O}$ into the $(j-1)$-th middle layer of $\mathcal{O}'$, and leave other parts of the region unchanged. The height of the $(j-1)$-th middle layer of $\mathcal{O}'$ is $a_{j-1}+a_j+a_{j+1}$, and the width of its is b_{j-1}. Thus

$$h_2 = \sum_{i=1}^{t} a_i = \sum_{i=1}^{j-2} a_i + (a_{j-1}+a_j+a_{j+1}) + \sum_{i=j+2}^{t-2} a_i = h_2'. \tag{3.13}$$

Moreover, the two regions $\mathcal{O}$ and $\mathcal{O}'$ have the same upper and lower parts, so $h_1 = h_1'$, $h_3 = h_3'$, $\mathcal{C}_1 = \mathcal{C}_1'$, $\mathcal{C}_3 = \mathcal{C}_3'$ and $w = w'$ (the primed symbols refer to $\mathcal{O}'$ and denote quantities corresponding to their unprimed counterparts of $\mathcal{O}$). This and the equality (3.12) imply that the statement in part (a) holds for $\mathcal{O}$ if and only if it holds for $\mathcal{O}'$.

Suppose that the condition $h_1 + h_3 = w + h_2$ in part (b) of the theorem holds. Note that the i-th middle layer of $\mathcal{O}$ has a_i rows of $b_i + 1$ white regular cells, for $i = 1, 2, \ldots, t$. Thus, the number of white regular cells in the middle part of $\mathcal{O}$ is given by

$$\mathcal{C}_2 = \sum_{i=1}^{t} a_i(b_i+1). \tag{3.14}$$

Similarly, the number of white regular cells in the middle part of $\mathcal{O}'$ is given by

$$\mathcal{C}_2' = \sum_{i=1}^{j-2} a_i(b_i+1) + (a_{j-1}+a_j+a_{j+1})(b_{j-1}+1) + \sum_{i=j+2}^{t} a_i(b_i+1).$$

Thus,

$$\begin{aligned}\mathcal{C}_2-\mathcal{C}_2' &= a_{j-1}(b_{j-1}+1)+a_{j-1}(b_j+1)+a_{j-1}(b_{j+1}+1)\\ &\quad-(a_{j-1}+a_j+a_{j+1})(b_{j-1}+1)\\ &= a_j, \end{aligned} \tag{3.15}$$

because we are assuming $b_{j-1}=b_j-1=b_{j+1}$. Therefore,

$$\begin{aligned}&2^{\mathcal{C}_1'+\mathcal{C}_2'+\mathcal{C}_3'-h_1'(2w'-h_1'+1)/2-h_2'(2w'-h_2'+1)/2-h_3'(2w'-h_3'+1)/2}\\ &\times 2^{\binom{h_1'+h_2'}{2}-\binom{h_2'+h_3'}{3}}\mathcal{P}(h_2'+1,1,h_1'+h_3',h_1'-h_3',w'+h_2'-1)\\ &=2^{\mathcal{C}_1+(\mathcal{C}_2-a_j)+\mathcal{C}_3-h_1(2w-h_1+1)/2-h_2(2w-h_2+1)/2-h_3(2w-h_3+1)/2}\\ &\times 2^{\binom{h_1+h_2}{2}-\binom{h_2+h_3}{3}}\mathcal{P}(h_2+1,1,h_1+h_3,h_1-h_3,w+h_2-1). \end{aligned} \tag{3.16}$$

By the equalities (3.12) and (3.16), the statement of part (b) holds for $\mathcal{O}$ if and only if it holds for $\mathcal{O}'$.

The case of concave b_j is perfectly analogous to the case treated above. The only difference is that we use the transformation in Lemma 3.1(b) (in reverse) instead of the transformation in Lemma 3.1(a) (see Figures 3.6(c) and (d) for an example). The resulting region is still the quasi-octagon $\mathcal{O}'$ defined as in (3.11); and by Lemma 3.1(b), we have now

$$\mathrm{M}(\mathcal{O})=2^{-a_j}\,\mathrm{M}(\mathcal{O}') \tag{3.17}$$

and

$$\mathcal{C}_2-\mathcal{C}_2'=-a_j. \tag{3.18}$$

Similar to the case of convex b_j, the statements in parts (a) and (b) hold for $\mathcal{O}$ if and only if they hold for $\mathcal{O}'$.

Keep applying this process if the resulting quasi-octagon still has more than one middle layer. Finally, we get a quasi-octagon $\widetilde{\mathcal{O}}$ with only one middle layer so that the statement of the theorem holds for $\mathcal{O}$ if and only if it holds for $\widetilde{\mathcal{O}}$. This means that, without loss of generality, we can assume that $t=1$.

STEP 3. *Proving the theorem for $k=l=t=1$.*

We have in this case $\mathcal{O}=\mathcal{O}^{(1)}_{w-1}(2h_1-1;\ 2h_2;\ 2h_3-1)$.

It is easy to see that the numbers of black cells and white cells must be the same if the region $\mathcal{O}$ admits tilings. One can check easily that the balancing condition between black and white cells in the region requires

$$h_1+h_3=w+h_2. \tag{3.19}$$

In particular, this implies the statement in part (a) of the theorem.

Assume that (3.19) holds, and let

$$\mathcal{O}'':=\mathcal{O}^{(1)}_{w-h_1}(\underbrace{1,\dots,1}_{h_1};\ 2h_2;\ \underbrace{1,\dots,1}_{h_3}). \tag{3.20}$$

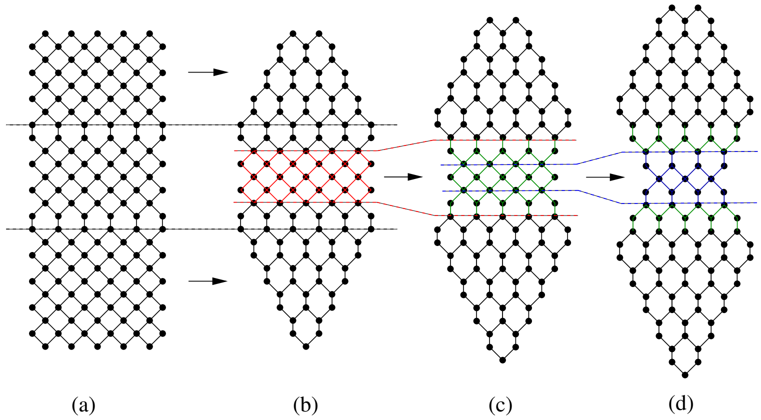


Figure 3.7: Transforming process for the middle layer in the case when $k = l = t = 1$.

Apply the equality (3.8) to the region $\mathcal{O}'$, we get

$$\mathrm{M}(\mathcal{O}'') = 2^{\mathcal{C}_1''-h_1''w''+\mathcal{C}_3''-h_3''w''}\,\mathrm{M}(\mathcal{O}^{(1)}_{w-1}(2h_1-1;\ 2h_2;\ 2h_3-1)) \tag{3.21}$$

$$= 2^{\mathcal{C}_1''-h_1''w''+\mathcal{C}_3''-h_3''w''}\,\mathrm{M}(\mathcal{O}), \tag{3.22}$$

where the double-primed symbols refer to the regions $\mathcal{O}''$ and denote quantities corresponding to their unprimed counterparts of $\mathcal{O}$. One readily sees that $h_1'' = h_1$, $w'' = w$, and $h_3'' = h_3$. Moreover, the numbers of black regular cells in the upper and lower parts of $\mathcal{O}''$ are given by

$$\mathcal{C}_1'' = \sum_{i=0}^{h_1-1}(w-i) = h_1w - \frac{h_1(h_1-1)}{2} \tag{3.23}$$

and

$$\mathcal{C}_3'' = \sum_{i=0}^{h_3-1}(w-i) = h_3w - \frac{h_3(h_3-1)}{2}. \tag{3.24}$$

Therefore, by (3.21), we have

$$\mathrm{M}(\mathcal{O}'') = 2^{-\frac{h_1(h_1-1)}{2}-\frac{h_3(h_3-1)}{2}}\,\mathrm{M}(\mathcal{O}) \tag{3.25}$$

(see Figures 3.7(a) and (b) for an example).

Let G'' be the dual graph of $\mathcal{O}''$. Consider the transforming process illustrated in Figures 3.7(b)–(d) as follows. Let B_1 be the subgraph consisting of all h_2-1 rows of $w-1$ diamonds in the middle part of G'', i.e. B_1 is isomorphic to $AR_{h_2-1,w-1}$ (see the graph between two inner dotted lines in Figure 3.7(b)). Apply the transformation in Lemma 3.1(b) to replace B_1 by the graph $|OR_{h_2-1,w-2}$ (see Figures 3.7(b) and (c)). Next, consider the subgraph B_2 of the resulting graph that consists of all h_2-2 rows of $w-2$ diamonds of the graph, so B_2 is isomorphic to $AR_{h_2-2,w-2}$ (see the graph between two inner dotted lines in Figure 3.7(c)). Apply the transformation in Lemma 3.1(b) again to transform B_2 into $|OR_{h_2-2,w-3}$. Keep applying the process until all rows of diamonds

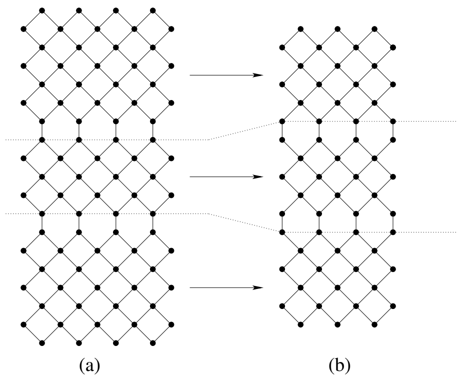


Figure 3.8: Obtaining the dual graph of a type-1 quasi-octagon form the dual graph of a type-2 quasi-octagon.

in the resulting graph have been eliminated (i.e. this process stops after $h_2 - 1$ steps). Denote by $\widetilde{G}$ the final graph of the process, by Lemma 3.1(b), we get

$$\mathrm{M}(\mathcal{O}'') = \mathrm{M}(G'') = 2^{\frac{h_2(h_2-1)}{2}}\,\mathrm{M}(\widetilde{G}). \tag{3.26}$$

By (3.25), we obtain

$$\mathrm{M}(\mathcal{O}) = 2^{\frac{h_1(h_1-1)}{2}+\frac{h_2(h_2-1)}{2}+\frac{h_3(h_3-1)}{2}} M(\widetilde{G}). \tag{3.27}$$

Moreover, $\widetilde{G}$ is exactly the graph $H^{h_2+h_3,w-h_3}_{h_1+h_2,w-h_1,h_2,h_2}$ in Lemma 3.2 (see Figure 3.7(d)). By Lemma 3.2, we have

$$\begin{aligned}\mathrm{M}(\widetilde{G}) =& 2^{-\frac{(h_1+h_2)(h_1+h_2-1)}{2}-\frac{(h_2+h_3)(h_2+h_3-1)}{2}}\\ &\times \mathrm{M}(AR^{|h_1-h_3|}_{h_1+2h_2+h_3-1,w+h_2}(A)),\end{aligned} \tag{3.28}$$

where $A = \{1,\dots,h_2\}\cup\{w+1,\dots,w+h_2\}$. Thus, (3.6) follows from the equality (3.28) and Lemma 2.8. This finishes our proof for type-1 quasi-octagons.

Case 2. $\mathcal{O}$ is of type 2.

Repeat the argument in Steps 1 and 2 of Case 1, we can assume that $k = l = t = 1$. We only need to prove the theorem for the type-2 quasi-octagon $\mathcal{O} := \mathcal{O}^{(2)}_w(2h_1;\ 2h_2;\ 2h_3)$. The dual graph G of $\mathcal{O}$ can be divided into three subgraphs ${}_{|}AR_{h_1,w}$, ${}^{|}AR_{h_2,w}$, and $AR_{h_3,w}$ (in order from top to bottom) as in Figure 3.8(a), for $w = 4$, $h_1 = 3$, $h_2 = 2$ and $h_3 = 4$. Apply the transformation in Lemma 2.6 separately to replace ${}_{|}AR_{h_1,w}$ by the graph $AR_{h_1-\frac{1}{2},w-1}$, and ${}^{|}AR_{h_3,w}$ by the graph $AR_{h_3-\frac{1}{2},w-1}$ flipped over a horizontal line. Next, apply Lemma 3.1(b) to transform $AR_{h_2,w}$ into ${}^{|}_{|}OR_{h_2,w-1}$. This way, the G gets transformed in to the dual graph of the type-1 quasi-octagon $\mathcal{O}^{(1)}_{w-1}(2h_1-1;\ 2h_2;\ 2h_3-1)$ (illustrated in Figure 3.8(b)); and by Lemmas 2.6 and 3.1, we obtain

$$\mathrm{M}(\mathcal{O}) = 2^{h_1+h_2+h_3}\,\mathrm{M}(\mathcal{O}^{(1)}_{w-1}(2h_1-1;\ 2h_2;\ 2h_3-1)). \tag{3.29}$$

Thus, both parts (a) and (b) of the theorem are reduced to the Case 1 treated above.

Case 3. $\mathcal{O}$ is of type 3.

Since $|BG| = |CF|$ and since $\mathcal{O}$ is of type 3, we have

$$-1 = (w-1) - w = b_1 - b_t = \sum_{i=1}^{t-1}(b_i - b_{i+1}).$$

Since $|b_i - b_{i+1}| = 1$, for $i = 1, 2, \ldots, t-1$, the number of 1 terms is one less than the number of -1 terms in the sequence $\{(b_i - b_{i+1})\}_{i=1}^{t-1}$. Thus, $t-1$ is odd, or t is *even* (as opposed to being odd in Case 1). By arguing similarly to Case 1, we can assume that $k = l = 1$ and $t = 2$.

The quasi-octagon $\mathcal{O}$ is now

$$\mathcal{O}^{(2)}_{w-1}(2h_1 - 1;\ 2x, 2y;\ 2h_3),$$

where x and y are two positive integers such that $x + y = h_2$ (see Figure 3.9(a) for an example with $h_1 = 5$, $h_2 = 6$, $x = 2$, $y = 2$, $w = 7$). Denote by G the dual graph of $\mathcal{O}$ as usual. Apply Vertex-splitting Lemma to all topmost vertices of the lower part of G, and divide the resulting graph by three horizontal dotted lines as in Figure 3.9(b). Apply the transformation in Lemma 2.6 to the bottom part, and the transformation in Lemma 3.1(a) to the second part from the top (sees Figures 3.9(b) and (c)). This way, G is transformed into the dual graph of the type-1 quasi-octagon

$$\mathcal{O}^{(1)}_{w-1}(2h_1 - 1;\ 2h_2;\ 2h_3 - 1)$$

(see Figure 3.9(c)); and, by Lemmas 2.6 and 3.1, we obtain

$$\mathrm{M}(\mathcal{O}) = 2^{h_2+h_3}\,\mathrm{M}(\mathcal{O}^{(1)}_{w-1}(2h_1 - 1;\ 2h_2;\ 2h_3 - 1)). \tag{3.30}$$

Again, we get the statements of the theorem from Case 1.

Case 4. $\mathcal{O}$ is of type-4.

The type-3 quasi-octagon

$$\mathcal{O}^* := \mathcal{O}^{(3)}_{|DE|}(d'_1, \ldots, d'_l;\ \overline{d}_t, \ldots, \overline{d}_1;\ d_1, \ldots, d_k)$$

is obtained from our type-4 quasi-octagon

$$\mathcal{O} := \mathcal{O}^{(4)}_{a}(d_1, \ldots, d_k;\ \overline{d}_1, \ldots, \overline{d}_t;\ d'_1, \ldots, d'_l)$$

by reflecting about ℓ. Thus, this case follows from Case 3. □

Theorem 1.1 requires $w > \max(h_1, h_2, h_3)$. The following theorem gives the number of tilings of a quasi-octagon when $w \leqslant \max(h_1, h_2, h_3)$.

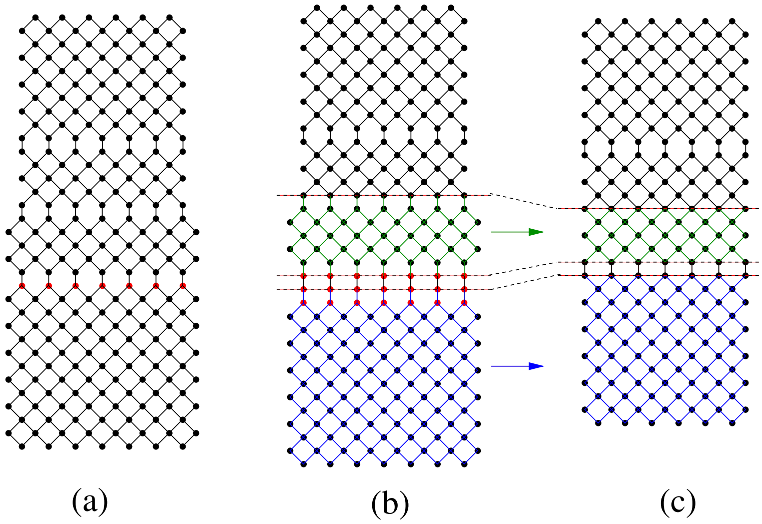


Figure 3.9: Illustrating Case 3 of the proof of Theorem 1.1.

Theorem 3.3. *Let a, $d_1, \dots, d_k$; $\overline{d}_1, \dots, \overline{d}_t$; $d'_1, \dots, d'_l$ be positive integers, for which the region $\mathcal{O} := \mathcal{O}_a(d_1, \dots, d_k;\ \overline{d}_1, \dots, \overline{d}_t;\ d'_1, \dots, d'_l)$ is a quasi-octagon satisfying the balancing condition (3.19), having the heights h_1, h_2, h_3, and having both widths equal to w. Let $\mathcal{C}_1$ be the numbers of black regular cells in the upper part, $\mathcal{C}_2$ be the number of white regular cells in the middle part, and $\mathcal{C}_3$ be the number of black regular cells in the lower part of the region.*

(a) If $w < \max(h_1, h_2, h_3)$, then $\operatorname{M}(\mathcal{O}) = 0$.

(b) If $h_2 = w$ (so, $h_1 = h_2 = h_3 = w$ by (3.19)), then

$$\operatorname{M}(\mathcal{O}) = 2^{\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3-h_1(2w-h_1+1)/2-h_2(2w-h_2+1)/2-h_3(2w-h_3+1)/2}. \tag{3.31}$$

(c) If $h_1 = w$ and $h_2 = h_3 < w$, then

$$\operatorname{M}(\mathcal{O}) = 2^{\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3-h_1(2w-h_1+1)/2-h_2(2w-h_2+1)/2-h_3(2w-h_3+1)/2}\operatorname{M}(H_{h_2,w-h_2,h_2}). \tag{3.32}$$

(d) The conclusion of part (2) is still true when $h_3 = w$ and $h_1 = h_2 < w$.

Proof. (a) By the same argument in Theorem 1.1, we can assume that $k = l = t = 1$ for type-1 and type-2 quasi-octagons, and $k = l = 1$ and $t = 2$ for type-3 and type-4 quasi-octagons. Then this part follows directly from Graph-splitting Lemma 2.1, part (b).

(b) Suppose that $\mathcal{O}$ is of type-1. By the arguments in the proof of Theorem 1.1, we can assume that $k = l = t = 1$ (then the general case can be obtained by induction on t). The quasi-octagon $\mathcal{O}$ is now $\mathcal{O}^{(1)}_{w-1}(2h_1 - 1; 2h_2;\ 2h_3 - 1)$. Let G be the dual graph of $\mathcal{O}$ as usual.

Define G_1 to be the graph obtained from the upper part of G by removing all bottom vertices, G_2 to be the graph obtained from the middle part of G by removing all top and bottom vertices, and G_3 to be the graph obtained from the lower part of G by removing all top vertices. Since $h_1 = h_2 = h_3 = w$, the graphs G_1, G_2 and G_3 are isomorphic to the dual graph of the Aztec diamond of order $w-1$, and satisfy the condition in part (a) of the Graph-splitting Lemma. Thus,

$$\begin{aligned} \mathrm{M}(G) &= \mathrm{M}(G_1)\,\mathrm{M}(G_2)\,\mathrm{M}(G_3)\,\mathrm{M}(G') & (3.33)\\ &= 2^{3w(w-1)/2}\,\mathrm{M}(G'), & (3.34)\end{aligned}$$

where G' is the graph obtained from G by removing G_1, G_2 and G_3. One readily sees that G' consists of $2q$ disjoint vertical edges, so $\mathrm{M}(G') = 1$. Moreover, we have $h_1 = h_2 = h_3 = w$ and $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3 = w^2$, then thus

$$\begin{aligned} \mathrm{M}(G) &= 2^{3w(w-1)/2} \\ &= 2^{\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3-h_1(2w-h_1+1)/2-h_2(2w-h_2+1)/2-h_3(2w-h_3+1)/2}, \end{aligned} \tag{3.35}$$

which implies (3.31) and finishes our proof for type-1 quasi-octagons.

Finally, the equalities (3.29) and (3.30) in the proof of Theorem 1.1 are still true in this case. Thus, the case when $\mathcal{O}$ is of type 2, 3 or 4 can also be reduced to the case treated above.

(c) Similar to part (b), we only need to consider the case when $\mathcal{O}$ is of type 1 and has $k = l = t = 1$. Since $h_1 = w$, the graph G_1 defined in part (b) is isomorphic to the dual graph of the Aztec diamond of order $w-1$ (so $\mathcal{C}_1 = w^2$) and satisfies the two conditions in Graph Splitting Lemma 2.4(a), thus

$$\mathrm{M}(G) = 2^{w(w-1)/2}\,\mathrm{M}(\overline{G}) = 2^{\mathcal{C}_1-h_1(2w-h_1+1)/2}\,\mathrm{M}(\overline{G}),$$

where $\overline{G}$ is the graph obtained from G by removing G_1. Remove all w vertical forced edges at the top of $\overline{G}$, we get precisely the dual graph of the symmetric quasi-hexagon $H_{w-1}(2h_2-1;\ 2h_3-1)$ (see [10]); and by Theorem 2.2 in [10], we have

$$M(\overline{G}) = 2^{\mathcal{C}_2-h_2(2w-h_2+1)/2+\mathcal{C}_3-h_3(2w-h_3+1)/2}\,\mathrm{M}(H_{h_2,w-h_2,h_2}).$$

This implies (3.32).

(d) Part (d) can be reduced to part (c) by considering the region $\mathcal{O}' := \mathcal{O}^{(1)}_{w-1}(2h_3-1; 2h_2;\ 2h_1-1)$ that is obtained by reflecting $\mathcal{O} = \mathcal{O}^{(1)}_{w-1}(2h_1-1; 2h_2;\ 2h_3-1)$ over ℓ'. $\square$

As mentioned before, we do *not* have a simple product formula for the number of tilings of a quasi-octagon when its widths are not equal, i.e. $w_1 = |BG| \neq |CF| = w_2$. However, we have a sum formula for the number of tilings in this case.

Let $S = \{s_1, s_2, \dots, s_k\}$ be a set of integers. We define the operation

$$\Delta(S) := \prod_{1\leqslant i<j\leqslant k} (s_j - s_i).$$

For any positive integer n, we denote by $[n]$ the set of the first n positive integers $\{1, 2, 3, \dots, n\}$. Let x be a number and A be a set of numbers, we define $x + A := \{y + x \mid y \in A\}$.

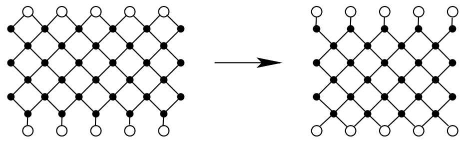


Figure 3.10: Illustration of the transformation in Lemma 3.5.

Theorem 3.4. *Let a, d_1, ..., d_k; $\overline{d}_1$, ..., $\overline{d}_t$; d'_1, ..., d'_l be positive integers, for which the region*

$$\mathcal{O} := \mathcal{O}_a(d_1, \ldots, d_k;\ \overline{d}_1, \ldots, \overline{d}_t;\ d'_1, \ldots, d'_l)$$

is a quasi-octagon having the upper, the middle, the lower heights h_1, h_2, h_3, respectively, and the upper and the lower widths w_1, w_2, respectively. Assume in addition that $w_1 > w_2$, $h_1 < w_1$, $h_2 < w_2$, and $h_3 < w_2$. Let $\mathcal{C}_1$ be the numbers of black regular cells in the upper part, $\mathcal{C}_2$ the number of white regular cells in the middle part, and $\mathcal{C}_3$ the number of black regular cells in the lower part of the region. Then

(a) If $h_1 + h_3 \neq w_1 + h_2$, then $\mathrm{M}(\mathcal{O}) = 0$.

(b) Assume that $h_1 + h_3 = w_1 + h_2$, then

$$\mathrm{M}(\mathcal{O}) = 2^{\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3-h_1(2w-h_1+1)/2-h_2(2w-h_2+1)/2-h_3(2w-h_3+1)/2} \times \sum_{(A,B)} \frac{\Delta([h_2+2w_1-w_2]\setminus(h_2+w_1-w_2+B))}{\Delta([h_1+h_2+w_1-w_2])}\frac{\Delta([w_2+h_2]\setminus(h_2+A))}{\Delta([h_2+h_3])}, \tag{3.36}$$

where the sum is taken over all pairs of disjoint sets A and B so that $A \cup B = [w_2 - h_2]$, $|A| = w_2 - h_3$ and $|B| = w_1 - h_1$.

The following transformation can be proven similarly to Lemma 3.1, and will be employed in the proof of Theorem 3.4.

Lemma 3.5. *Let G be a graph, and p, q two positive integers. Assume that $\{v_1, v_2, \ldots, v_{2q}\}$ be an ordered set of vertices of G. Then*

$$\mathrm{M}({}_{|}AR_{p,q}\#G) = 2^p\,\mathrm{M}({}^{|}OR_{p,q-1}\#G), \tag{3.37}$$

where ${}_{|}AR_{p,q}$ is defined as in Lemma 2.6, and ${}^{|}OR_{p,q-1}$ is the graph obtained from $OR_{p,q-1}$ by appending q vertical edges to its top vertices; and where the connected sum acts on G along $\{v_1, v_2, \ldots, v_{2q}\}$, and on ${}_{|}AR_{p,q}$ and ${}^{|}OR_{p,q-1}$ along their q top vertices ordered from left to right, then along their q bottom vertices ordered from left to right (see Figure 3.10 for the case $p = 2$ and $q = 5$).

Proof of Theorem 3.4. Similar to the proof of Theorem 1.1, we can assume that the quasi-octagon $\mathcal{O}$ is of type 1 and has $k = l = 1$.

Denote by K_j the dual graph of the j-th middle layer of the region. Assume that the j-th middle layer has the height and the width a_j and b_j, respectively.

We apply the process in the proof of Theorem 1.1 (Case 1, Step 3) using the transformations in Lemma 3.1 to eliminate all the concave and convex terms in the sequence

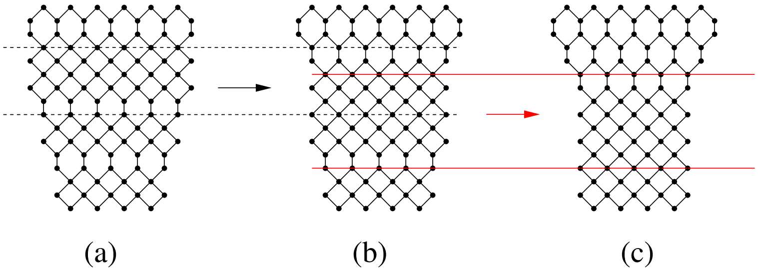


Figure 3.11: Transforming process for the middle parts of a quasi-octagon when $w_1 > w_2$.

of the widths of the middle layers $\{b_j\}_1^t$ (see Figure 3.6 for an example). It means that the sequence of the widths of the middle layers becomes a monotone sequence. Since $w_1 = b_1 > b_t = w_2$, we can assume that $b_1 > b_2 > \cdots > b_t$.

The dual graph K_j of the j-th middle layer is now isomorphic to the baseless Aztec rectangle $AR_{a_j-\frac{1}{2},b_j}$ reflected about its base. We consider a process using the transformation in Lemma 3.5 to the middle part of G as follows.

Assume that there exists a middle layer other than the last one which has positive height. Assume that the j_0-th middle layer is the first such layer. Consider the graph Q_1 obtained from K_{j_0} by removing all its top vertices and appending vertical edges to its bottom vertices. Apply the transformation in Lemma 3.5 to replace Q_1 by the graph ${}^{|}OR_{a_{j_0},b_{j_0}-1}$. This transformed G into the dual graph of new quasi-octagon $\mathcal{O}'$, which has the same upper and lower parts as $\mathcal{O}$, and the sequence of sizes of the middle layers

$$\big((0,b_1),(0,b_2),\dots,(0,b_{j_0}),(a_{j_0}+a_{j_0+1},b_{j_0+1}),(a_{j_0+2},b_{j_0+2}),\dots,(a_t,b_t)\big).$$

This step is illustrated in Figures 3.11(a) and (b), the subgraph between two dotted lines in Figure 3.11(a) is replaced by the one between these two lines in Figure 3.11(b). Apply again the transformation in Lemma 3.5 to the graph Q_2 obtained from the dual graph of the (j_0+1)-th middle layer $\mathcal{O}'$ by removing the top vertices and appending b_{j_0+1} vertical edges to bottom (see Figures 3.11(b) and (c)), and so on. The procedure stops when the heights of all middle layers in the resulting region, except for the last one, are equal to 0. Denote by $\mathcal{O}^*$ the final quasi-octagon, so $\mathcal{O}^*$ has the sequence of sizes of the middle layers

$$\big((0,b_1),(0,b_2),\dots,(0,b_{t-1}),(h_2,b_t)\big).$$

One readily sees that the above process preserves the heights, the widths, and the lower and upper parts of the quasi-octagon. Moreover, by Lemma 3.5 and the equality (3.14) in the proof of Theorem 1.1, we get

$$\begin{aligned}\operatorname{M}(\mathcal{O})/\operatorname{M}(\mathcal{O}^*) &= 2^{a_{j_0}+(a_{j_0}+a_{j_0+1})+\dots+(a_{j_0}+\dots+a_{i-1})}\\ &= 2^{\mathcal{C}_2}/2^{\mathcal{C}_2^*}\\ &= \mathcal{Q}(h_1,h_2,h_3,w_1,w_2,\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)/\mathcal{Q}(h_1^*,h_2^*,h_3^*,w_1^*,w_2^*,\mathcal{C}_1^*,\mathcal{C}_2^*,\mathcal{C}_3^*), \qquad (3.38)\end{aligned}$$

where the star symbols refer to the region $\mathcal{O}^*$ and denote quantities corresponding to their non-starred counterparts of $\mathcal{O}$, and where $\mathcal{Q}(h_1,h_2,h_3,w_1,w_2,\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3)$ denotes the

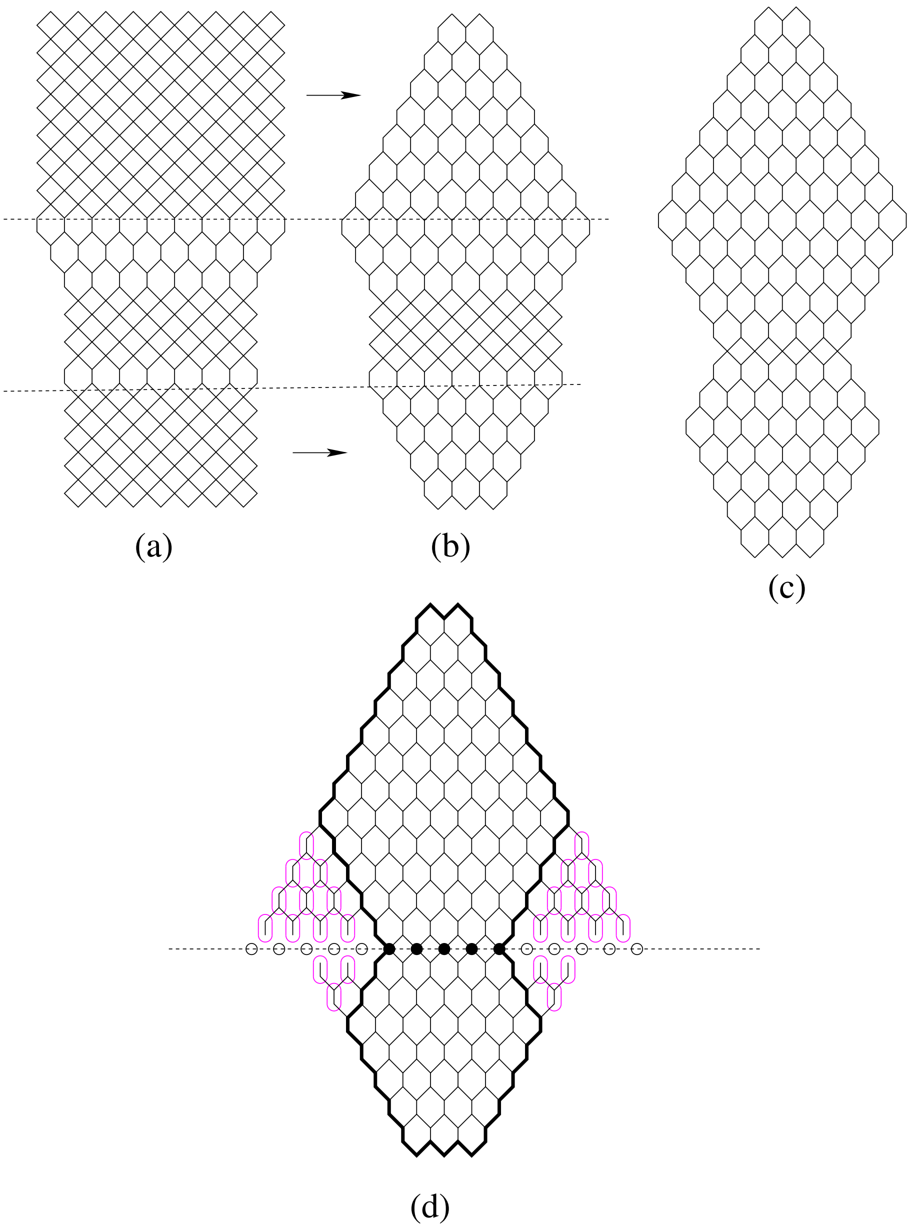


Figure 3.12: Illustrating the proof of Theorem 3.4.

expression on the right-hand side of (3.36). This implies that the statement of the theorem is true for $\mathcal{O}$ if and only its is true for $\mathcal{O}^*$. It means that we can assume that $a_j = 0$, for all $1 \leqslant j < t$.

Next, we prove the theorem for the case when $k = l = 1$ and $a_j = 0$, for $1 \leqslant j < t$ (illustrated in Figure 3.12).

Similar to the proof of Theorem 1.1 (Case 1, Step 3), we apply the transformation in Lemma 2.5(2) (in reverse) to transform the upper and lower parts of the dual graph G of $\mathcal{O}$ into the dual graphs of two semi-hexagons (see Figures 3.12(a) and (b)), and the transformation in Lemma 3.1(2) to transform the dual graph K_t of the last middle layer into a butterfly-shaped graph (see Figures 3.12(b) and (c)). This way, G gets transformed into the graph $\overline{G} = H_1 \# H_2$, where H_1 is the dual graph of a hexagon of sides $w_1 - h_1, h_1, h_2 + w_1 - w_2, w_2 - h_2, h_2 + w_1 - w_2, h_1$ and H_2 is the dual graph of a hexagon of sides $w_2 - h_2, h_2, h_3, w_2 - h_3, h_3, h_2$ (in cyclic order starting by the north side) on the triangular lattice (see Figures 3.12(a) and (c)). By Lemmas 2.5 and 3.1, we get

$$\begin{aligned} \mathrm{M}(\mathcal{O}) &= 2^{h_1(h_1-1)/2+h_2(h_2-1)/2+h_3(h_3-1)/2}\,\mathrm{M}(\overline{G}) \\ &= 2^{\mathcal{C}_1+\mathcal{C}_2+\mathcal{C}_3-h_1(2w-h_1+1)/2-h_2(2w-h_2+1)/2-h_3(2w-h_3+1)/2}\,\mathrm{M}(\overline{G}). \end{aligned} \tag{3.39}$$

The graph $\overline{G}$ is in turn obtained from $\widetilde{G} := S_1 \# S_2$ by removing the vertical forced edges and their endpoints (the forced edges are illustrated by the circled ones in Figure 3.12(d)), where S_1 is the dual graph of the $(h_1 + h_2 + w_1 - w_2, w_1 - h_1)$-semihexagon with the $h_2 + w_1 - w_2$ leftmost and the $h_2 + w_1 - w_2$ rightmost bottom vertices removed, and S_2 is the dual graph of $(h_2 + h_3, w_2 - h_3)$-semihexagon with the h_2 leftmost and the h_2 rightmost bottom vertices removed; and where the connected sum acts on S_1 and S_2 along their bottommost vertices ordered from left to right (see Figure 3.12(d)). Since removing forced edges and their endpoints does not change the number of perfect matchings of a graph, $\mathrm{M}(\overline{G}) = \mathrm{M}(\widetilde{G})$.

There are $w_2 - h_2$ vertices belonging to both S_1 and S_2; and we partition the set of perfect matchings of $\widetilde{G}$ into $2^{w_2-h_2}$ classes corresponding to all the possible choices for each of these vertices to be matched upward or downward. Each class is then the set of perfect matchings of a disjoint union of two graphs, being of the kind in Lemma 2.7. Part (a) follows from the requirement that the graph $\widetilde{G}$ has the numbers of vertices in two vertex classes equal, while part (b) follows from Lemma 2.7. □

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