On Isomorphisms of Vertex-transitive Graphs

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Abstract

The isomorphism problem of Cayley graphs has been well studied in the literature, such as characterizations of CI (DCI)-graphs and CI (DCI)-groups. In this paper, we generalize these to vertex-transitive graphs and establish parallel results. Some interesting vertex-transitive graphs are given, including a first example of connected symmetric non-Cayley non-GI-graph. Also, we initiate the study for GI and DGI-groups, defined analogously to the concept of CI and DCI-groups.

Keywords: coset graph; GI-graphs; isomorphisms; vertex-transitive graphs

1 Introduction

Throughout this paper, by (di)graph we mean finite digraph without loops or multi-edges, and all groups are assumed to be finite. Deciding whether two graphs are isomorphic is fundamental for the study of graphs, especially for determining isomorphism classes of graphs. A graph is said to be G-vertex-transitive if the subgroup G of its full automorphism group acts transitively on the vertex set. One would expect to determine the isomorphisms between two G-vertex-transitive graphs by the information of the

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group G. For Cayley graphs, such an approach was initiated by a conjecture of Ádám in 1967 [1], and has been extensively studied over the past decades, see for example [2, 4, 8, 11, 13, 21, 22, 23, 24] and more references listed in the survey [15]. Since a large number of vertex-transitive graphs are not Cayley graphs, it is natural to extend the study from Cayley graphs to vertex-transitive graphs. The isomorphism problem for metacirculants (not necessarily Cayley graphs) has been considered by Dobson [9].

To be precise, we need the concept of coset graphs. Let $\Gamma = (V, E)$ be a G-vertex-transitive graph, α be a vertex of Γ and S be the set of elements of G which maps α to its (out) neighbors. Then Γ is uniquely determined by the triple (G, G_α, S) in the following sense: writing $H = G_\alpha$ and identifying the vertex set V with the set $[G{:}H]$ of right cosets of H in G, the action of G on V is equivalent to the action of G on $[G{:}H]$ by right multiplication. In particular, if α is identified with $H \in [G{:}H]$ then the neighborhood $\Gamma(\alpha)$ consists of Hg with $g \in S$, and moreover, $Hx \sim Hy$ if and only if $yx^{-1} \in HSH$. This defines a *coset graph* representation of Γ, denoted by $\mathsf{Cos}(G, H, HSH)$. Note that H is core-free in G (that is, H does not contain any nontrivial normal subgroup of G) since G is a transitive permutation group on V, and $S \subseteq G \setminus H$ since Γ has no loops. Clearly, for any automorphism $\tau \in \mathrm{Aut}(G)$ we have $\mathsf{Cos}(G, H, HSH) \cong \mathsf{Cos}(G, H^\tau, H^\tau S^\tau H^\tau)$.

Definition 1. The G-vertex-transitive graph $\Gamma = \mathsf{Cos}(G, H, HSH)$ is called a *GI-graph* ('GI' stands for 'Group automorphism inducing Isomorphism') of G if for any graph $\Sigma = \mathsf{Cos}(G, H, HTH)$ with $T \subseteq G \setminus H$ and $\Gamma \cong \Sigma$, there exists $\tau \in \mathrm{Aut}(G)$ such that $H^\tau = H$ and $HS^\tau H = HTH$. A group G is called a *DGI-group* ('D' emphasizes that our graph may be Directed) if each G-vertex-transitive graph is a GI-graph of G. A group G is called a *GI-group* if each undirected G-vertex-transitive graph is a GI-graph of G.

Note that $\mathsf{Cos}(G, 1, S)$ is a Cayley graph of G, and the GI-graphs of G with $H = 1$ are exactly the so called *CI-graphs* of G. If each Cayley graph of G is a CI-graph of G, then G is called a *DCI-group*. If each undirected Cayley graph of G is a CI-graph of G, then G is called a *CI-group*. Clearly, a DGI-group is necessarily a GI-group, and a DGI-group (GI-group) is necessarily a DCI-group (CI-group). A small list of candidates for DCI and CI-groups has been obtained, through the effort of many mathematicians, see [15, Theorem 8.7] and [17, Corollary 1.5]. However, determining which groups in the list are indeed DCI or CI-groups is not easy and largely open. As being DGI-groups (GI-groups) is more restrictive than being DCI-groups (CI-groups), the explicit list of DGI-groups (GI-groups) would be smaller than that of DCI-groups (CI-groups). Thus we propose the problem:

Problem 2. Classify the finite DGI-groups (GI-groups).

In the literature, a crucial step to solve a conjecture of Babai and Frankel [5] stating that CI-groups are solvable was to determine whether there exists a non-CI-Cayley graph of A_5. After 20 years since Babai-Frankel conjecture was posed, a non-CI-Cayley graph of A_5 of valency 29 was constructed by Li [14], thus completing the proof of the conjecture. Although some other non-CI-Cayley graphs of A_5 was later constructed in [6, 26], Li's graph is the only known connected symmetric non-CI-graph of A_5 yet. Here a graph Γ is called G-*symmetric* for some $G \leqslant \mathrm{Aut}(\Gamma)$ if G acts transitively on the arc set of Γ, and Γ is simply called *symmetric* if Γ is $\mathrm{Aut}(\Gamma)$-symmetric. In general, constructing connected symmetric non-GI-graphs is not easy. Due to the significance of non-CI-Cayley graphs of A_5, one would ask:

Problem 3. Does there exist a connected symmetric non-GI-graph of A_5 other than Li's?

The layout of this paper is as follows. After this introduction, we give the criterion for GI-graph in Section 2, which enables us to construct GI and non-GI-graphs, respectively, in Section 3. In particular, we prove the theorem below by Example 11.

Theorem 4. *There exists a connected symmetric non-Cayley non-GI-graph of order* 40 *and valency* 12.

Then in Section 4 we establish some results on Problem 2. The final section is devoted to Problem 3, where it is shown that a connected A_5-symmetric graph is necessarily GI if its full automorphism group is almost simple or vertex-primitive.

2 Criterion for GI-graph

As mentioned in the introduction, G-vertex-transitive graphs can be represented as coset graphs of G: for a core-free subgroup H of G and a subset $S \subseteq G \setminus H$, define $\Gamma = \mathsf{Cos}(G, H, HSH)$ to be the graph with vertex set $V := [G{:}H]$ such that $Hx \sim Hy$ if and only if $yx^{-1} \in HSH$. For any $g \in G$, the right multiplication of g on the cosets in $[G{:}H]$ gives an element of $\mathrm{Sym}(V)$, denoted by $\hat{g}$. Moreover, denote $\hat{G} = \{\hat{g} \mid g \in G\}$. (The reader should be aware that this also depends on the subgroup H although the ˆ symbol does not indicate.) We list here some basic facts concerning coset graphs.

Lemma 5. *Let* $\Gamma = \mathsf{Cos}(G, H, HSH)$*.*

(a) Γ *is undirected if and only if* $HSH = HS^{-1}H$*, where* $S^{-1} := \{s^{-1} \mid s \in S\}$*.*

(b) G *acts faithfully and transitively on the vertex set* $[G{:}H]$ *by right multiplication, so* $\hat{G}$ *is a subgroup of* $\mathrm{Aut}(\Gamma)$ *isomorphic to* G*.*

(c) Γ *is connected if and only if* $\langle H, S\rangle = G$*.*

(d) Γ *is* G*-symmetric if and only if* $HSH = HgH$ *for some* $g \in G$*. In this case, the valency of* Γ *is equal to* $|H|/|H^g \cap H|$*.*

Let X and Y be permutation groups on Ω and Δ, respectively. We say that X is *permutation isomorphic* to Y if there exist a bijection $\sigma : \Omega \to \Delta$ and a group isomorphism $\varphi : X \to Y$ such that $(\alpha^x)^\sigma = (\alpha^\sigma)^{\varphi(x)}$ for any $\alpha \in \Omega$ and $x \in X$. The following folklore theorem is an extension of the criterion for a Cayley graph to be a CI-graph [3, 4] to those vertex-transitive graphs. The proof goes along the same lines as that of the CI-graph criterion, so we omit it.

Theorem 6. *A G-vertex-transitive graph Γ is a GI-graph of G if and only if subgroups of* $\mathrm{Aut}(\Gamma)$ *which are permutation isomorphic to $\hat{G}$ are all conjugate in* $\mathrm{Aut}(\Gamma)$.

Based on Theorem 6, we establish a sufficient condition on GI-graphs as follows.

Theorem 7. *Suppose that G is a finite group of odd order, p is the smallest prime divisor of $|G|$, Γ is a G-vertex-transitive graph and A is the full automorphism group of Γ. For any vertex α of Γ, if $\gcd(|G|, |A_\alpha|) = 1$, then Γ is a GI-graph of G. In particular, if Γ is connected of valency less than p, then Γ is a GI-graph of G.*

Proof. Since G is transitive on the vertices of Γ we have $A = GA_\alpha$. Assume that $\gcd(|G|, |A_\alpha|) = 1$. Then G is a Hall π-subgroup of A, where π is the set of the prime divisors of $|G|$. Note that π is a set of odd primes as $|G|$ is odd. Then for any $\sigma \in \mathrm{Sym}(V)$ with $G^\sigma \leqslant A$, one deduces from [12, Theorem A] that G and G^σ are conjugate in A as they are Hall π-subgroups of A. Hence according to Theorem 6, Γ is a GI-graph of G.

Now assume that Γ is connected of valency less than p. It suffices to prove that $\gcd(|G|, |A_\alpha|) = 1$. Suppose for a contradiction that there exists a prime number r dividing $\gcd(|G|, |A_\alpha|)$ and that R is a Sylow r-subgroup of A_α. Since Γ is connected, there exist a neighbor β of α and an element $x \in R$ such that $\beta^x \neq \beta$. It follows that the orbit of β under $\langle x \rangle$ has length at least r, contrary to our assumption that the valency of Γ is less than $p \leqslant r$. □

Below is a necessary condition for GI-graphs.

Theorem 8. *If $\mathsf{Cos}(G, H, HSH)$ is a GI-graph of a group G, then for any embedding $\varphi : \langle H, S\rangle \to G$ such that $H^\varphi = H$, there exists $\tau \in \mathrm{Aut}(G)$ such that $H^\tau = H$ and $\langle H, S\rangle^\tau = \langle H, S^\varphi\rangle$.*

Proof. Note that $\mathsf{Cos}(\langle H, S\rangle, H, HSH)$ and $\mathsf{Cos}(\langle H, S^\varphi\rangle, H, HS^\varphi H)$ are connected components of $\mathsf{Cos}(G, H, HSH)$ and $\mathsf{Cos}(G, H, HS^\varphi H)$, respectively. Then

$$\mathsf{Cos}(G, H, HSH) \cong \mathsf{Cos}(G, H, HS^\varphi H)$$

if and only if $\mathsf{Cos}(\langle H, S\rangle, H, HSH) \cong \mathsf{Cos}(\langle H, S^\varphi\rangle, H, HS^\varphi H)$. As φ induces an graph isomorphism from $\mathsf{Cos}(\langle H, S\rangle, H, HSH)$ to $\mathsf{Cos}(\langle H, S^\varphi\rangle, H, HS^\varphi H)$, we thus have an isomorphism $\mathsf{Cos}(G, H, HSH) \cong \mathsf{Cos}(G, H, HS^\varphi H)$. Since $\mathsf{Cos}(G, H, HSH)$ is a GI-graph of G, there exists $\tau \in \mathrm{Aut}(G)$ such that $H^\tau = H$ and $HS^\tau H = HS^\varphi H$. Consequently,

$$\langle H, S\rangle^\tau = \langle H, HSH\rangle^\tau = \langle H, HS^\tau H\rangle = \langle H, HS^\varphi H\rangle = \langle H, S^\varphi\rangle,$$

which completes the proof. □

3 Examples

First of all, the complete graphs and their complements are GI-graphs. We regard them as *trivial* GI-graphs. An observation of [16] says that every finite group of order greater than two has non-trivial CI-graphs. Thus we know that every finite group of order greater than two has non-trivial GI-graphs. Given a finite group G of odd order, recall that as Theorem 7 asserts, every G-vertex-transitive graph Γ of valency less than the smallest prime divisor of $|G|$ is a GI-graph of G. This provides us with more examples of GI-graphs.

A 2-*arc* of a graph Γ is a triple (α, β, γ) of pairwise distinct vertices of Γ such that $\alpha \sim \beta$ and $\beta \sim \gamma$. A graph is said to be $(G, 2)$-*arc-transitive* for some $G \leqslant \mathrm{Aut}(\Gamma)$ if G acts transitively on the set of 2-arcs. Recall that the *socle* of a group G is the product of all its minimal normal subgroups, denoted by $\mathrm{Soc}(G)$. We call a group *almost simple* if its socle is nonabelian simple. It is readily seen that the almost simple groups with a given socle T are precisely those groups G satisfying $T \leqslant G \leqslant \mathrm{Aut}(T)$, whence G/T is a solvable group by the well-known Schreier conjecture. The next example follows from [10, Theorem 1.3] and the criteria in Theorem 6.

Example 9. Let G be an almost simple group with socle $\mathrm{Sz}(2^{2n+1})$ or $G = \mathrm{Ree}(3^{2n+1})$. Then every connected undirected $(G, 2)$-arc transitive graph is a GI-graph of G.

Utilizing Theorem 8, we are able to construct some disconnected non-GI-graphs.

Example 10. Let m and n be integers such that $m \geqslant 2$ and $n \geqslant 2m + 6$. Take $G = \mathrm{A}_n$, $a = (5, 6)(7, 8, \ldots, 2m + 5, 2m + 6) \in G$, $b = (1, 2)(3, 4)a \in G$, $H = \langle a^2 \rangle = \langle b^2 \rangle$, $S = \{a, a^3, \ldots, a^{2m-3}, a^{2m-1}\}$ and $\varphi : a^i \mapsto b^i$ for any $i \in \mathbb{Z}$. Then φ is an embedding of $\langle H, S \rangle$ into G such that $H^\varphi = H$ and $\langle H, S^\varphi \rangle = \langle b \rangle$. Apparently, there does not exist $\tau \in \mathrm{Aut}(G)$ such that $\langle H, S \rangle^\tau = \langle a \rangle^\tau = \langle b \rangle$. Hence by Theorem 8, the coset graph $\mathsf{Cos}(G, H, HSH)$ is non-GI.

We close this section with the construction of a connected symmetric non-Cayley non-GI-graph, which proves Theorem 4.

Example 11. Let $X = \mathrm{PSL}_4(3)$ acting naturally on the set Ω of one-dimensional subspaces of $\mathbb{F}_3^4$, a four-dimensional vector space over $\mathbb{F}_3$. Take $\alpha \in \Omega$, and $G = \mathrm{P\Sigma U}_4(2) = \mathrm{PSp}_4(3){:}\mathrm{C}_2$ to be a maximal subgroup of X. There exists an involution $g \in G$ such that $\langle G_\alpha, g \rangle = G$ and $|G_\alpha|/|G_\alpha^g \cap G_\alpha| = 12$. Let $\Gamma = \mathsf{Cos}(G, G_\alpha, G_\alpha g G_\alpha)$. Then Γ is a connected G-symmetric and G-vertex-primitive graph of order $|\Omega| = 40$ and valency 12. Moreover, G has two conjugacy classes of subgroups isomorphic to S_6, fused in X, and the groups in both conjugacy classes are transitive on Ω. Take P to be a group in one of these two conjugacy classes, and Q be a group in the other. Since P and Q are conjugate in X, they are permutation isomorphic. We claim that Γ is a non-Cayley non-GI-graph of P.

In fact, the conclusion that Γ is not a Cayley graph is obvious as $|P| \neq 40$. Denote $Y = \mathrm{Aut}(\Gamma)$. In light of Theorem 6, it suffices to show that P and Q are not conjugate in Y. If $\mathrm{Soc}(Y) = \mathrm{Soc}(G) = \mathrm{PSU}_4(2)$, then $Y = G$ since $G \leqslant Y$ and $G = \mathrm{Aut}(\mathrm{PSU}_4(2))$, which indicates that P and Q are not conjugate in Y, as desired. Assume next that $\mathrm{Soc}(G) \neq \mathrm{Soc}(Y)$. Then there exists a subgroup H of Y such that $\mathrm{Soc}(G) \neq \mathrm{Soc}(H)$ and G is maximal in H. By [18], either G is maximal in $\mathrm{A}_{40}G$, or H is almost simple with socle

$\mathrm{PSL}_4(3)$. For the latter, $H_\alpha = \mathrm{C}_3^3{:}\mathrm{PSL}_3(3)$ or $\mathrm{C}_3^3{:}(\mathrm{PSL}_3(3) \times \mathrm{C}_2)$ since H is primitive on 40 points, but then H_α does not have a subgroup of index 12, violating the requirement that Γ is H-symmetric as Γ is G-symmetric. Therefore, G is maximal in $\mathrm{A}_{40}G$, and hence $Y \cap \mathrm{A}_{40}G = G$ or $\mathrm{A}_{40}G$. Because Γ is not a complete graph, we have $Y \not\geqslant \mathrm{A}_{40}$. It follows that $Y \cap \mathrm{A}_{40}G = G$. If $G \not\leqslant \mathrm{A}_{40}$, then $\mathrm{A}_{40}G = \mathrm{S}_{40}$ and thus $Y = Y \cap \mathrm{S}_{40} = G$, contrary to our assumption that $\mathrm{Soc}(G) \neq \mathrm{Soc}(Y)$. Consequently, $G \leqslant \mathrm{A}_{40}$, and so G has index two in Y. Since $\mathrm{Soc}(G)$ is a minimal normal subgroup of Y and $\mathrm{Soc}(G) \neq \mathrm{Soc}(Y)$, we conclude that Y has a minimal normal subgroup other than $\mathrm{Soc}(G)$, say N. Viewing that $N \not\leqslant G$, we have $N = \mathrm{C}_2$ and $Y = G \times N$. Hence P and Q are not conjugate in Y, proving our claim.

4 GI-groups

A group G is said to be *Hamiltonian* if every subgroup of G is normal. It is obvious that abelian groups are all Hamiltonian, but the converse is not true (for instance, the quaternion group Q_8 is Hamiltonian but not abelian).

Lemma 12. *Let G be a Hamiltonian group. Then G is DGI (GI) if and only if G is DCI (CI).*

Proof. For any coset graph $\mathsf{Cos}(G, H, HSH)$ of G, the condition that H is core-free in G forces $H = 1$ since G is Hamiltonian. This means that each coset graph of G is a Cayley graph of G. Hence the concepts of DGI (GI) and DCI (CI) coincide. □

Lemma 12 immediately shows up some DGI-groups (GI-groups) from the list of DCI-groups (CI-groups). For example, since the groups C_k, C_{2k} and C_{4k}, where k is odd square-free, are Hamiltonian and DCI [19, 20] simultaneously, we know that they are DGI-groups.

Theorem 13. D_{2p} *is a DGI-group for any odd prime p.*

Proof. Let $G = \mathrm{D}_{2p}$, N be the Sylow p-subgroup of G, and $\Gamma = \mathsf{Cos}(G, H, HSH)$ be a coset graph of G with vertex set $V = [G{:}H]$, where H is a core-free subgroup of G and $S \subseteq G \setminus H$. If $H = 1$, then Γ is a DGI-graph of G by [4]. Hence we assume that $H \neq 1$. As H is core-free in G, we conclude that $H = \mathrm{C}_2$ and $|V| = |G|/|H| = p$. Let X be a subgroup of $\mathrm{Aut}(\Gamma)$ such that $X = \varphi^{-1}\hat{G}\varphi$ for some $\varphi \in \mathrm{Sym}(V)$, and Y be a Sylow p-subgroup of X. Then $Y = \mathrm{C}_p$, and by the Sylow theorem, there exists $\tau \in \mathrm{Aut}(\Gamma)$ such that $Y = \tau^{-1}\hat{N}\tau$. It derives from $X = \varphi^{-1}\hat{G}\varphi$ that $Y = \varphi^{-1}\hat{N}\varphi$. Thereby we obtain $\varphi^{-1}\hat{N}\varphi = \tau^{-1}\hat{N}\tau$, or equivalently, $\varphi\tau^{-1} \in \mathbf{N}_{\mathrm{Sym}(V)}(\hat{N})$. Note that $\mathbf{N}_{\mathrm{Sym}(V)}(\hat{N}) \leqslant \mathbf{N}_{\mathrm{Sym}(V)}(\hat{G})$. This leads to $\varphi\tau^{-1} \in \mathbf{N}_{\mathrm{Sym}(V)}(\hat{G})$ and thus

$$X = \varphi^{-1}\hat{G}\varphi = \tau^{-1}(\varphi\tau^{-1})^{-1}\hat{G}(\varphi\tau^{-1})\tau = \tau^{-1}\hat{G}\tau.$$

Now appealing Theorem 6 we know that Γ is a DGI-graph of G, which proves the lemma. □

We close this section with a theorem stating that being DGI-groups (GI-groups) is inherited by subgroups.

Theorem 14. *If G is a DGI-group (GI-group), then any subgroup H of G is a DGI-group (GI-group).*

Proof. Suppose that G is a DGI-group (GI-group). Let $\Gamma = \mathsf{Cos}(H, K, KSK)$ and $\Sigma = \mathsf{Cos}(H, K, KTK)$ be two isomorphic (undirected) coset graphs of H, where K is a core-free subgroup of H and S, T are subsets of $H \setminus K$. Clearly, K is also core-free in G. Without loss of generality we assume that S and T are both unions of double cosets of K.

First assume that $\langle K, S\rangle = H$. Then Γ is connected, and so is Σ since $\Gamma \cong \Sigma$. Noticing that $\mathsf{Cos}(G, K, KSK)$ and $\mathsf{Cos}(G, K, KTK)$ are $|G|/|H|$ copies of Γ and Σ, respectively, we have $\mathsf{Cos}(G, K, KSK) \cong \mathsf{Cos}(G, K, KTK)$. Then as G is a DGI-group (GI-group), there exists $\tau \in \mathrm{Aut}(G)$ such that $K^\tau = K$ and $KS^\tau K = KTK$. It follows that

$$H^\tau = \langle K, S\rangle^\tau = \langle K, KSK\rangle^\tau = \langle K^\tau, KS^\tau K\rangle = \langle K, KTK\rangle = \langle K, T\rangle = H.$$

This shows that τ induces an automorphism of H.

Next assume that $\langle K, S\rangle \neq H$. Then $|K \cup S| \leqslant |H|/2$, and so

$$|K \cup (H \setminus S)| = |K| + |H \setminus (K \cup S)| > |H \setminus (K \cup S)| \geqslant |H|/2.$$

Let $\overline{S} = (H \setminus S) \setminus K$ and $\overline{T} = (H \setminus T) \setminus K$. Then $\langle K, \overline{S}\rangle = \langle K, H \setminus S\rangle = H$, which means that the complement graph $\overline{\Gamma}$ of Γ is connected and so is the complement graph $\overline{\Sigma}$ of Σ. From $\Gamma \cong \Sigma$ we deduce $\mathsf{Cos}(H, K, K\overline{S}K) = \overline{\Gamma} \cong \overline{\Sigma} = \mathsf{Cos}(H, K, K\overline{T}K)$. Hence $\mathsf{Cos}(G, K, K\overline{S}K) \cong \mathsf{Cos}(G, K, K\overline{T}K)$, and there exists $\tau \in \mathrm{Aut}(G)$ such that $K^\tau = K$ and $K\overline{S}^\tau K = K\overline{T}K$ since G is a DGI-group (GI-group). As a consequence,

$$H^\tau = \langle K, \overline{S}\rangle^\tau = \langle K, K\overline{S}K\rangle^\tau = \langle K^\tau, K\overline{S}^\tau K\rangle = \langle K, K\overline{T}K\rangle = \langle K, \overline{T}\rangle = H,$$

showing that τ induces an automorphism of H. Moreover,

$$KS^\tau K = (H \setminus K) \setminus (K\overline{S}^\tau K) = (H \setminus K) \setminus (K\overline{T}K) = KTK.$$

Thereby we conclude that there always exists $\tau \in \mathrm{Aut}(G)$ such that $K^\tau = K$ and $KS^\tau K = KTK$. This implies that H is a DGI-group (GI-group). □

5 GI-properties of connected A_5-symmetric graphs

For a group G, the expression $G = HK$ with proper subgroups H and K of G is called a *factorization* of G. The lemma below can be read off from [25].

Lemma 15. *If $T = GK$ is a factorization of a simple group T with $G = \mathrm{A}_5$, then either $(T, K) = (\mathrm{A}_n, \mathrm{A}_{n-1})$ with $n \in \{10, 12, 15, 20, 30, 60\}$ or (T, K) lies in* Table 1.

The following two theorems are the main results of this section.

Table 1:

row	T	K
1	A_6	A_4, S_4, $\mathrm{C}_3^2{:}\mathrm{C}_4$, A_5
2	A_7	$\mathrm{PSL}_2(7)$
3	A_8	$\mathrm{AGL}_3(2)$
4	$\mathrm{PSL}_2(11)$	C_{11}, $\mathrm{C}_{11}{:}\mathrm{C}_5$
5	$\mathrm{PSL}_2(19)$	$\mathrm{C}_{19}{:}\mathrm{C}_9$
6	$\mathrm{PSL}_2(29)$	$\mathrm{C}_{29}{:}\mathrm{C}_7$, $\mathrm{C}_{29}{:}\mathrm{C}_{14}$
7	$\mathrm{PSL}_2(59)$	$\mathrm{C}_{59}{:}\mathrm{C}_{29}$
8	M_{12}	M_{11}

Theorem 16. *Let $G = \mathrm{A}_5$ and Γ be a connected symmetric coset graph of G. If $\mathrm{Aut}(\Gamma)$ is almost simple, then Γ is a GI-graph of G.*

Proof. Suppose on the contrary that Γ is not a GI-graph of G. By Theorem 6, $\mathrm{Aut}(\Gamma) \neq \mathrm{A}_5$ or S_5. Let α be a vertex of Γ, $X = \mathrm{Aut}(\Gamma)$ and T be the socle of X. Then $T \neq \mathrm{A}_5$, $X = \hat{G}X_\alpha$ and $\hat{G} \cap T$ is a normal subgroup of $\hat{G}$. It follows that $\hat{G} \cap T = 1$ or $\hat{G}$ since $\hat{G} \cong G$ is simple. If $\hat{G} \cap T = 1$, then $\mathrm{A}_5 = \hat{G} \cong \hat{G}T/T \leqslant X/T$, contrary to Schreier conjecture. Hence $\hat{G} \cap T = \hat{G}$, or equivalently, $\hat{G} \leqslant T$. Thereby we have the factorization $T = \hat{G}T_\alpha$, which is classified in Lemma 15. If T acts 2-transitively on $[T{:}T_\alpha]$, then Γ is the complete graph on n vertices and $X = \mathrm{S}_n$, which implies that Γ is a GI-graph of G by Theorem 6, contrary to our assumption. Consequently, T does not act 2-transitively on $[T{:}T_\alpha]$, and so we deduce from Lemma 15 that one of the following three cases appears:

(i) $T = \mathrm{A}_6$ and $T_\alpha = \mathrm{A}_4$ or S_4;

(ii) $T = \mathrm{PSL}_2(11)$ and $T_\alpha = \mathrm{C}_{11}$;

(iii) $T = \mathrm{PSL}_2(29)$ and $T_\alpha = \mathrm{C}_{29}{:}\mathrm{C}_7$.

First suppose that case (i) appears. As X should have at least two conjugacy classes of subgroups isomorphic to A_5 by Theorem 6, the only possibilities for X are A_6 and S_6. If $X = \mathrm{A}_6$, then $\mathrm{A}_4 \leqslant X_\alpha \leqslant \mathrm{S}_4$ and hence X has only one conjugacy class of vertex-transitive subgroups isomorphic to A_5, which leads to a contradiction that Γ is a GI-graph of G by Theorem 6. If $X = \mathrm{S}_6$, then $\mathrm{A}_4 \leqslant X_\alpha \leqslant \mathrm{S}_4 \times \mathrm{S}_2$ and hence X has at most one conjugacy class of vertex-transitive subgroups isomorphic to A_5, again a contradiction.

Next suppose that case (ii) appears. As X should have at least two conjugacy classes of subgroups isomorphic to A_5 by Theorem 6, it derives that $X = \mathrm{PSL}_2(11)$ and so $X_\alpha = \mathrm{C}_{11}$. Since Γ is symmetric, there exists $g \in X \setminus X_\alpha$ such that $\Gamma \cong \mathsf{Cos}(X, X_\alpha, X_\alpha g X_\alpha)$. Let $Y = \mathrm{PGL}_2(11) > X$. One can take an involution $t \in \mathbf{N}_Y(X_\alpha)$ such that $X_\alpha g^t X_\alpha = X_\alpha g X_\alpha$. Let $H = \langle X_\alpha, t\rangle = X_\alpha\langle t\rangle$, and note $t \notin X$. Due to $X_\alpha g^t X_\alpha = X_\alpha g X_\alpha$ we have $tgt \in X_\alpha g X_\alpha$. For any $h_1, h_2 \in H$, if $h_1 g h_2 \in X$, then either $h_1, h_2 \in X_\alpha$ or $h_1, h_2 \notin X_\alpha$. Further, if $h_1, h_2 \notin X_\alpha$, then $h_1 t, th_2 \in X_\alpha$ and so $h_1 g h_2 = (h_1 t)tgt(th_2) \in X_\alpha g X_\alpha$. This shows that $(HgH) \cap X = X_\alpha g X_\alpha$. Then the map

$$X_\alpha x \mapsto Hx \quad \text{for } x \in X$$

is a graph isomorphism from $\mathsf{Cos}(X, X_\alpha, X_\alpha g X_\alpha)$ to $\mathsf{Cos}(Y, H, HgH)$. However, this implies that $Y = \mathrm{PGL}_2(11)$ is a group of automorphisms of $\Gamma \cong \mathsf{Cos}(Y, H, HgH)$, contrary to the condition that $\mathrm{Aut}(\Gamma) = X = \mathrm{PSL}_2(11)$.

Finally suppose that case (iii) appears. As X should have at least two conjugacy classes of subgroups isomorphic to A_5 by Theorem 6, it derives that $X = \mathrm{PSL}_2(29)$ and so $X_\alpha = \mathrm{C}_{29}{:}\mathrm{C}_7$. Thus Γ has order $|X|/|X_\alpha| = 60$. Take β to be a neighbor of α in Γ. Since Γ is X-symmetric, $|X_\alpha|/|X_{\alpha\beta}|$ equals the valency of Γ, which is less than 60. Hence $X_{\alpha\beta} = \mathrm{C}_{29}$ or C_7. If $X_{\alpha\beta} = \mathrm{C}_{29}$, then $X_{\alpha\beta}$ fixes each neighbor of α since $X_{\alpha\beta} \lhd X_\alpha$. This will cause a contradiction that $X_{\alpha\beta} = 1$ due to the connectivity of Γ. Consequently, $X_{\alpha\beta} = \mathrm{C}_7$ and Γ is of valency $|X_\alpha|/|X_{\alpha\beta}| = 29$. Note that X has a maximal subgroup $K = \mathrm{C}_{29}{:}\mathrm{C}_{14}$ containing X_α such that X acts 2-transitively on $[X{:}K]$. We deduce that X has an imprimitive block system $\mathcal{B} = \{V_1, V_2, \dots, V_{30}\}$ on the vertex set of Γ, where $|V_1| = \cdots = |V_{30}| = 2$, and the quotient graph of Γ with respect to the partition $\mathcal{B}$ is complete. Moreover, denoted by V_k the block in $\mathcal{B}$ such that $\alpha \in V_k$, the action of X_α on $\mathcal{B} \setminus \{V_k\}$ is transitive. Therefore, distinct neighbors of α lie in distinct blocks in $\mathcal{B}$, and so the induced graph $\Gamma[V_i \cup V_j]$ is a perfect matching for any two blocks V_i, V_j in $\mathcal{B}$. Now we see that interchanging the two vertices in each V_i is an automorphism of Γ. Then the kernel of X acting on $\mathcal{B}$ is non-trivial, contrary to the fact that $X = \mathrm{PSL}_2(29)$ is simple. □

Theorem 17. *Let $G = \mathrm{A}_5$ and Γ be a connected symmetric coset graph of G. If $\mathrm{Aut}(\Gamma)$ is vertex-primitive, then Γ is a GI-graph of G.*

Proof. Suppose on the contrary that Γ is not a GI-graph of G. A subgroup of G has order 1, 2, 3, 4, 5, 6, 10 or 12, whence the order of Γ is 60, 30, 20, 15, 12, 10, 6 or 5. In view of Theorem 16 we may assume that $X := \mathrm{Aut}(\Gamma)$ is not almost simple. Further, Theorem 6 requires X to have at least two conjugacy classes of transitive subgroups isomorphic to A_5. Then by [7, Appendix B], $X = \mathrm{Hol}(G)$ or $\mathrm{Soc}(\mathrm{Hol}(G))$, where the symbol Hol denotes the holomorph of a group. Let $N = \mathrm{Soc}(\mathrm{Hol}(G)) = G \times G$ and D be the full diagonal subgroup of N. Then the vertex set of Γ can be viewed as $[N{:}D]$, with the action of N by right multiplication. Moreover, let t be the permutation

$$D(g_1, g_2) \mapsto D(g_2, g_1) \quad \text{for } (g_1, g_2) \in N$$

on $[N{:}D]$, $\alpha = D \in [N{:}D]$, $H = \langle X_\alpha, t\rangle$ and $Y = \langle X, t\rangle$. Clearly, t is an involution in $Y \setminus X$. Since Γ is symmetric, there exists $g \in X \setminus X_\alpha$ such that $\Gamma \cong \mathsf{Cos}(X, X_\alpha, X_\alpha g X_\alpha)$.

First suppose that $g \in \mathrm{Soc}(\mathrm{Hol}(G))$. Then $g = (g_1, g_2)$ acts on $[N{:}D]$ by right multiplication for some $g_1, g_2 \in G$. Take $h_1 \in G$ such that $(g_1 g_2^{-1})^{h_1} = (g_1 g_2^{-1})^{-1}$ and write $h_2 = g_1^{-1} h_1^{-1} g_2$. For $i = 1, 2$ set x_i to be the right multiplication of (h_i, h_i) on $[N{:}D]$. It is routine to verify that $tgt = x_1 g x_2 \in X_\alpha g X_\alpha$. Hence $(HgH) \cap X = X_\alpha g X_\alpha$, and thus the map

$$X_\alpha x \mapsto Hx \quad \text{for } x \in X$$

is a graph isomorphism from $\mathsf{Cos}(X, X_\alpha, X_\alpha g X_\alpha)$ to $\mathsf{Cos}(Y, H, HgH)$. However, this implies that Y is a group of automorphisms of $\Gamma \cong \mathsf{Cos}(Y, H, HgH)$, contrary to the condition that $\mathrm{Aut}(\Gamma) = X < Y$.

Next suppose that $g \in \mathrm{Hol}(G) \setminus \mathrm{Soc}(\mathrm{Hol}(G))$. Then there exists an involution $\tau \in X_\alpha \setminus \mathrm{Soc}(\mathrm{Hol}(G))$ such that $t\tau = \tau t$ and $g\tau^{-1} \in \mathrm{Soc}(\mathrm{Hol}(G))$. In the previous paragraph we see that $t(g\tau^{-1})t \in X_\alpha g\tau^{-1}X_\alpha$. Hence

$$tgt = tg\tau^{-1}\tau t = t(g\tau^{-1})t\tau \in X_\alpha g\tau^{-1}X_\alpha\tau = X_\alpha gX_\alpha.$$

Consequently, $(HgH) \cap X = X_\alpha gX_\alpha$, and so the map

$$X_\alpha x \mapsto Hx \quad \text{for } x \in X$$

is a graph isomorphism from $\mathsf{Cos}(X, X_\alpha, X_\alpha gX_\alpha)$ to $\mathsf{Cos}(Y, H, HgH)$. However, this implies that Y is a group of automorphisms of $\Gamma \cong \mathsf{Cos}(Y, H, HgH)$, contrary to the condition that $\mathrm{Aut}(\Gamma) = X < Y$. □

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