Neighborhood Complexes of Some Exponential Graphs

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Abstract

In this article, we consider the bipartite graphs $K_2 \times K_n$. We first show that the connectedness of $\mathcal{N}(K_{n+1}^{K_n}) = 0$. Further, we show that $\text{Hom}(K_2 \times K_n, K_m)$ is homotopic to S^{m-2}, if $2 \leqslant m < n$.

Keywords: Hom complexes, Exponential graphs, Discrete Morse theory.

1 Introduction

Determining the chromatic number of a graph is a classical problem in graph theory and finds applications in several fields. The Kneser conjecture posed in 1955 and solved by Lovaśz [14] in 1978, dealt with the problem of computing the chromatic number of a certain class of graphs, now called the *Kneser graphs*. To prove this conjecture, Lovaśz first constructed the neighborhood complex $\mathcal{N}(G)$ of a graph G, which is a simplicial complex and then related the connectivity of this complex to the chromatic number of G.

In [1], Lovaśz introduced the notion of a prodsimplicial complex called the *Hom complex*, denoted by $\text{Hom}(G, H)$ for graphs G and H, which generalized the notion of a neighborhood complex. In particular, $\text{Hom}(K_2, G)$ (where K_2 denotes a complete graph with 2 vertices) and $\mathcal{N}(G)$ are homotopy equivalent. The idea was to be able to estimate the chromatic number of an arbitrary graph G by understanding the connectivity of the Hom complex from some standard graph into G. Taking H to be the complete graph K_n makes each of the complexes $\text{Hom}(G, K_n)$ highly connected. In [1] Babson and Kozlov made the following conjecture.

Conjecture 1. For a graph G with maximal degree d, $\text{Hom}(G, K_n)$ is at least $(n-d-2)$-connected.

In [4], Čukić and Kozlov presented a proof for the above conjecture. They further showed that in the case when G is an odd cycle, $\text{Hom}(G, K_n)$ is $(n-4)$-connected for all $n \geqslant 3$. From [12], it is seen that for any even cycle C_{2m}, $\text{Hom}(C_{2m}, K_n)$ is $(n-4)$-connected for all $n \geqslant 3$.

It is natural to ask whether it is possible to classify the class of graphs G for which the Hom complexes $\text{Hom}(G, K_n)$ are exactly $(n-d-2)$-connected. In this article, we consider the bipartite graphs $K_2 \times K_n$, which are $n-1$ regular graphs. Since $\text{Hom}(K_2 \times K_n, K_m) \simeq \text{Hom}(K_2, K_m^{K_n})$, it is sufficient to determine the connectedness of $\text{Hom}(K_2, K_m^{K_n})$ which is the same as the connectedness of $\mathcal{N}(K_m^{K_n})$. The main results of this article are

Theorem 2. *Let $n \geqslant 4$ and $p = \frac{n!(n-1)n}{2}$. Then*

$$H_k(\mathcal{N}(K_{n+1}^{K_n}); \mathbb{Z}_2) = \begin{cases} Z_2 & \text{if} \quad k = 0, 1 \text{ or } n-1 \\ Z_2^{p-n!+1} & \text{if} \quad k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3. *$conn(\mathcal{N}(K_{n+1}^{K_n})) = 0$ for all $n \geqslant 2$.*

Corollary 4. *Let $n \geqslant 2$ and $2 \leqslant m \leqslant n+1$. Then*

$$conn(Hom(K_2 \times K_n, K_m)) = \begin{cases} 0 & \text{if} \quad m = n+1 \\ -1 & \text{if} \quad m = n \\ m-3 & \text{otherwise.} \end{cases}$$

We make the following conjecture.

Conjecture 5. The lower bounds given in [4] are exact for all bipartite graphs of the type $K_2 \times K_n$.

2 Preliminaries

A graph G is a pair $(V(G), E(G))$, where $V(G)$ is the set of vertices of G and $E(G) \subset V(G) \times V(G)$ denotes the set of edges. If $(x, y) \in E(G)$, it is also denoted by $x \sim y$ and x is said to be adjacent to y. The *degree* of a vertex v is defined as $\deg(v) = |\{y \in V(G) \mid x \sim y\}|$.

- A *graph homomorphism* from G to H is a function $\phi : V(G) \to V(H)$ such that,

$$(v, w) \in E(G) \implies (\phi(v), \phi(w)) \in E(H).$$

- A *finite abstract simplicial complex* X is a collection of finite sets where $\tau \in X$ and $\sigma \subset \tau$, implies $\sigma \in X$. The elements of X are called the *simplices* of X. If $\sigma \in X$ and $|\sigma| = k+1$, then σ is said to be k *dimensional*. A $k-1$ dimensional subset of a k simplex σ is called a *facet* of σ.

- A *prodsimplicial complex* is a polyhedral complex each of whose cells is a direct product of simplices ([13]).

- Let v be a vertex of a graph G. The *neighborhood of* v is defined as $N(v) = \{w \in V(G) \mid (v,w) \in E(G)\}$. If $A \subset V(G)$, the neighborhood A is defined as $N(A) = \{x \in V(G) \mid (x,a) \in E(G) \; \forall \; a \in A\}$.

- The *neighborhood complex* $\mathcal{N}(G)$ of a graph G is the abstract simplicial complex whose elements are $N(A)$, for all subsets A of $V(G)$.

- Let G be a graph and $N(u) \subset N(v)$ for $u, v \in V(G)$. The graph $G \setminus \{u\}$ is called a *fold* of G. Here, $V(G \setminus \{u\}) = V(G) \setminus \{u\}$ and the edges in the subgraph $G \setminus \{u\}$ are all those edges of G which do not contain u.

- Let X be a simplicial complex and $\tau, \sigma \in X$ such that $\sigma \subsetneq \tau$ and τ is the only maximal simplex in X that contains σ. A *simplicial collapse* of X is the simplicial complex Y obtained from X by removing all those simplices γ of X such that $\sigma \subseteq \gamma \subseteq \tau$. σ is called a *free face* of τ and (σ, τ) is called a *collapsible pair* and is denoted by $X \searrow Y$.

- For any two graphs G and H, $\mathrm{Hom}(G,H)$ is the polyhedral complex whose cells are indexed by all functions $\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}$, such that if $(v,w) \in E(G)$ then $\eta(v) \times \eta(w) \subset E(H)$.

  Elements of $\mathrm{Hom}(G,H)$ are called cells and are denoted by $(\eta(v_1), \ldots \eta(v_k))$, where $V(G) = \{v_1, \ldots, v_k\}$. A cell $(A_1, \ldots, A_k)$ is called a *face* of $B = (B_1, \ldots B_k)$, if $A_i \subset B_i \; \forall \, 1 \leqslant i \leqslant k$. The Hom complex is often referred to as a topological space. Here, we are referring to the geometric realisation of the order complex of the poset. The simplicial complex whose simplices are the chains of the poset P is called the order complex of P.

- A topological space X is said to be n *connected* if $\pi_*(X) = 0$ for all $* \leqslant n$.
  By convention, $\pi_0(X) = 0$ means X is connected. The connectivity of a topological space X is denoted by $\mathrm{conn}(X)$, *i.e.*, $\mathrm{conn}(X)$ is the largest integer m such that X is m-connected. If X is a non empty disconnected space, it is said to be (-1)-connected and if it is empty, it is said to be $-\infty$ connected.

We now review some of the constructions related to the existence of an *internal hom* which is related to the categorical product. Details can be found in [8, 9, 15].

- The *categorical product* of two graphs G and H, denoted by $G \times H$ is the graph where $V(G \times H) = V(G) \times V(H)$ and $(g,h) \sim (g',h')$ in $G \times H$ if $g \sim g'$ and $h \sim h'$ in G and H respectively.

- If G and H are two graphs, then the *exponential graph* H^G is defined to be the graph where $V(H^G)$ contains all the set maps from $V(G)$ to $V(H)$. Any two vertices f and f' in $V(H^G)$ are said to be adjacent, if $v \sim v'$ in G implies that $f(v) \sim f'(v')$ in H.

Using tools from poset topology ([3]), it can be shown that given a poset P and a poset map $c : P \to P$ such that $c \circ c = c$ and $c(x) \geqslant x, \forall\, x \in P$, there is a strong deformation retract induced by $c : P \to c(P)$ on the relevant spaces. Here, c is called the *closure map*.

From [5, Proposition 3.5] we have a relationship between the exponential graph and the categorical product in the Hom-complex.

Proposition 6. *Let G, H and K be graphs. Then $Hom(G \times H, K)$ can be included in $Hom(G, K^H)$ so that $Hom(G \times H, K)$ is the image of the closure map on $Hom(G, K^H)$. In particular, there is a strong deformation retract $|Hom(G \times H, K)| \hookrightarrow |Hom(G, K^H)|$.*

From [1, Proposition 5.1] we have the following result which allows us to replace a graph by a subgraph in the Hom complex.

Proposition 7. *Let G and H be graphs such that u, v are distinct vertices of G and $N(u) \subset N(v)$. The inclusion $i : G \setminus \{u\} \hookrightarrow G$ respectively, the homomorphism $\phi : G \to G \setminus \{u\}$ which maps v to u and fixes all the other vertices, induces the homotopy equivalence $i_H : Hom(G, H) \to Hom(G \setminus \{u\}, H)$, respectively $\phi_H : Hom(G \setminus \{u\}, H) \to Hom(G, H)$.*

3 Tools from Discrete Morse Theory

We introduce some tools from Discrete Morse Theory which have been used in this article. R. Forman in [6] introduced what has now become a standard tool in Topological Combinatorics, Discrete Morse Theory. The principal idea of Discrete Morse Theory (simplicial) is to pair simplices in a complex in such a way that they can be cancelled by elementary collapses. This will reduce the original complex to a homotopy equivalent complex, which is not necessarily simplicial, but which has fewer cells. More details of discrete Morse theory can be found in [10] and [13].

Definition 8. A *partial matching* in a poset P is a subset $\mathcal{M}$ of $P \times P$ such that

- $(a, b) \in \mathcal{M}$ implies $b \succ a$, *i.e.* $a < b$ *and* $\nexists\, c$ *such that* $a < c < b$.
- Each element in P belongs to at most one element in $\mathcal{M}$.

In other words, if $\mathcal{M}$ is a *partial matching* on a poset P then there exists $A \subset P$ and an injective map $f : A \to P \setminus A$ such that $x \prec f(x)$ for all $x \in A$.

Definition 9. An *acyclic matching* is a partial matching $\mathcal{M}$ on the Poset P such that there does not exist a cycle

$$x_1 \prec f(x_1) \succ x_2 \prec f(x_2) \succ x_3 \cdots \prec f(x_t) \succ x_1, t \geqslant 2.$$

Given an acyclic partial matching on P, those elements of P which do not belong to the matching are said to be *critical* . To obtain the desired homotopy equivalence, the following result is used.

Theorem 10. *(Main theorem of Discrete Morse Theory)[6] Let X be a simplicial complex and let $\mathcal{A}$ be an acyclic matching such that the empty set is not critical. Then, X is homotopy equivalent to a cell complex which has a d -dimensional cell for each d -dimensional critical face of X together with an additional 0-cell.*

4 Main Result

To prove the Theorems 2 and 3, we first construct an acyclic matching on the face poset of $\mathcal{N}(K_{n+1}^{K_n})$ after which we construct the Morse Complex corresponding to this acyclic matching and use this complex to compute the homology groups.

In this article $n \geqslant 3$ and $[n]$ denotes the set $\{1, 2, \ldots, n\}$. Any vertex in the exponential graph $K_{n+1}^{K_n}$ is a set map $f : K_n \to K_{n+1}$.

Lemma 11. *The graph $K_m^{K_n}$ can be folded onto the graph G, where the vertices $f \in V(G)$ have images of cardinality either 1 or n.*

Proof. Consider the vertex f such that $1 < |\mathrm{Im}\, f| < n$. Since f is not injective there exist distinct $i, j \in [n]$ such that $f(i) = f(j) = \alpha$. Consider $\tilde{f} \in V(K_m^{K_n})$ such that $\tilde{f}([n]) = \alpha$. By the definition of the exponential graph, any neighbor h of f will not have α in its image and therefore h will be a neighbor of $\tilde{f}$ thereby showing that $N(f) \subset N(\tilde{f})$. $K_m^{K_n}$ can be folded to the subgraph $K_m^{K_n} \setminus \{f\}$. Repeating the argument for all noninjective, non constant maps from $[n]$ to $[m]$, $K_m^{K_n}$ can be folded to the graph G whose vertices are either constant or injective maps from $[n]$ to $[m]$. □

From Proposition 7, we observe that $\mathcal{N}(K_{n+1}^{K_n}) \simeq \mathcal{N}(G)$. Hence, it is sufficient to study the homotopy type of $\mathcal{N}(G)$.

Henceforth, if $f \in V(G)$ and $f([n]) = \{x\}$, f shall be denoted by $\langle x \rangle$. In the other cases the string $a_1a_2 \ldots a_n$ will denote the vertex f where $a_i = f(i)$, $1 \leqslant i \leqslant n$. Hence, if the notation $a_1a_2 \ldots a_n$ is used, it is understood that for $1 \leqslant i < j \leqslant n$, $a_i \neq a_j$

Let $f = a_1a_2 \ldots a_n \in V(G)$ and $x \notin \mathrm{Im}\, f$. Define A_i^f to be the set

$$\{a_1, a_2, \ldots, \hat{a}_i, \ldots, a_n\} = \{a_1, \ldots, a_n\} \setminus \{a_i\}.$$

The map f_k is defined on $[n]$ by

$$f_k(i) = \begin{cases} f(i), & \text{if } k \neq i. \\ x, & \text{if } k = i. \end{cases}$$

We first consider the maximal simplices of $\mathcal{N}(G)$.

Lemma 12. *Let $f \in V(G)$. Then*

(i) $f = a_1a_2 \ldots a_n,\, x \notin Im\, f \Rightarrow N(f) = \{f, \langle x\rangle, f_1, f_2, \ldots, f_n\}$.

(ii) $f = \langle x\rangle \ \Rightarrow N(f) = \{\langle y\rangle |\, y \neq x\} \cup \{g \in V(G) | x \notin Im\, g\}$.

Proof.

(i) Since $a_i \neq a_j$, $\forall\ i \neq j$, $f \sim f$. If $g = \langle x\rangle$, then $g(i) \neq f(j)$ for $i \neq j$ and thus $\langle x\rangle \in N(f)$. For any $l \in [n]$, $f_l(i) \neq f(j)$ for $i \neq j$ which implies $f_l \in N(f)$. Thus $\{f, \langle x\rangle, f_1, f_2, \ldots, f_n\} \subset N(f)$. Conversely if $\tilde{f} \in N(f)$, then $\tilde{f}(i) \in \{a_i, x\}$. Since $|\text{Im}\, \tilde{f}| = 1$ or n, if $\tilde{f} \neq \langle x\rangle$ then $\tilde{f}$ has to be f or f_l for some $l \in [n]$.

(ii) Let $f = \langle x\rangle$. Clearly, $\langle y\rangle \sim \langle x\rangle$ for all $y \neq x$. If $g \in V(G)$ and $x \notin \text{Im}\, g$, then $g \sim \langle x\rangle$. Conversely if $\tilde{f} \in N(f)$, then x cannot belong to the image of $\tilde{f}$. Since $|\text{Im}\, \tilde{f}|$ has to be either 1 or n from Lemma 11, the proof follows. □

We now determine the free faces in $\mathcal{N}(G)$.

Lemma 13. *Let $f \in V(G)$. Then*

(i) $f = a_1a_2 \ldots a_n \Rightarrow (\{f_s, f_t\}, N(f))$ *is a collapsible pair* $\forall\ 1 \leqslant s < t \leqslant n$.

(ii) $f = \langle y\rangle$, $g \neq \tilde{g} \in V(G)$ *non constant neighbors of f implies that* $(\{g, \tilde{g}\}, N(f))$ *is a collapsible pair.*

Proof. (i) From Lemma 12, $N(f) = \{f, \langle x\rangle, f_1, f_2, \ldots, f_n\}$, where $x \notin \text{Im} f$. All the maximal simplices of $\mathcal{N}(G)$ are of the form $N(g)$, where $g \in V(G)$. Suppose there exists $\tilde{f} \in V(G)$ such that $\{f_s, f_t\} \subset N(\tilde{f})$, for some $1 \leqslant s < t \leqslant n$. Since $A_s^{f_s} = [n+1] \setminus \{a_s, x\}$ and $A_i^{f_s} = [n+1] \setminus \{a_i, a_s\}$ if $i \neq s$, then for each $i \in [n]$ at least one of the sets $A_i^{f_s}$ or $A_i^{f_t}$ contains x. Therefore $A_i^{f_s} \cup A_i^{f_t} = [n+1] \setminus \{a_i\}\, \forall\, 1 \leqslant i \leqslant n$. Since $\tilde{f}$ is a neighbor of both f_s and f_t, $\tilde{f}(i) \neq f_s(j), f_t(j)\, \forall\, j \neq i$, which implies that $\tilde{f}(i) = a_i = f(i)$. Hence $\{f_s, f_t\}$ is free in $N(f)$.

(ii) Since $g, \tilde{g}$ are neighbors of f and $i \sim j$ in K_n $\forall\ i \neq j$, $f(j) \neq g(i), \tilde{g}(i)$ implies that $y \notin \text{Im}\, g, \text{Im}\, \tilde{g}$, which shows that $\text{Im}\, g = \text{Im}\, \tilde{g}$. Let $h \in V(G)$ such that $g, \tilde{g} \in N(h)$. Since $g, \tilde{g}$ are distinct and injective there exist $s \neq t \in [n]$ such that $g(s) \neq \tilde{g}(s)$ and $g(t) \neq \tilde{g}(t)$. $A_s^g = [n+1] \setminus \{y, g(s)\}$ and $A_s^{\tilde{g}} = [n+1] \setminus \{y, \tilde{g}(s)\}$. Since $g(s) \neq \tilde{g}(s)$, $A_s^g \cup A_s^{\tilde{g}} = [n+1] \setminus \{y\}$. h is a neighbor of g and $\tilde{g}$ and $i \sim s$ in K_n $\forall\ i \neq s$, implies $h(s) \neq g(i), \tilde{g}(i)$. In particular, $h(s) \notin A_s^g \cup A_s^{\tilde{g}}$ and therefore $h(s) = y$ (similarly $h(t) = y$). Therefore $h(i) = y\ \forall\, i \in [n]$ and is equal to f. □

Let $M(X)$ be the set of maximal simplices in the simplicial complex X.

Lemma 14. *In a simplicial complex X, let $\sigma = \{x_1, \ldots, x_t, y_1, \ldots, y_k\}$, $t \geqslant 2$ be a maximal simplex such that $\{x_i, x_j\}$ is a free face of σ for $1 \leqslant i < j \leqslant t$. X collapses to the subcomplex Y where $M' = M(X) \setminus \{\sigma\}$ and $M(Y) = M' \cup \{\{x_i, y_1, \ldots, y_k\} | 1 \leqslant i \leqslant t\}$.*

Proof. We first consider collapses with the faces $\{x_1, x_j\}$, $2 \leqslant j \leqslant t$.

Claim 15. *$X \searrow X'$ with $M(X') = M' \cup \{x_1, y_1, \dots, y_k\} \cup \{\sigma \setminus \{x_1\}\}$.*

Since $\{x_1, x_2\}$ is a free face of σ, $X \searrow X_{12}$ with $M(X_{12}) = M' \cup \{\sigma \setminus \{x_1\}\} \cup \{\sigma \setminus \{x_2\}\}$. In X_{12}, $\{x_1, x_3\}$ is a free face of $\sigma \setminus \{x_2\}$ and hence $X \searrow X_{13}$ with $M(X_{13}) = M' \cup \{\sigma \setminus \{x_1\}\} \cup \sigma \setminus \{x_2, x_3\}$. Inductively, we assume that $M(X_{1l}) = M' \cup \{\sigma \setminus \{x_1\}\} \cup \sigma \setminus \{x_2, \dots x_l\}$. In X_{1l}, $\{x_1, x_{l+1}\}$ is a free face of $\sigma \setminus \{x_2, \dots x_l\}$. Hence $X_{1l} \searrow X_{1l+1}$ with $M(X_{1l+1}) = M' \cup \{\sigma \setminus \{x_1\}\} \cup \sigma \setminus \{x_2, \dots x_{l+1}\}$. This proves the claim.

For $2 \leqslant i \leqslant t-1$, considering the pairs $\{x_i, x_j\}$, $i+1 \leqslant j \leqslant t$ and using Claim 15, the lemma follows. ☐

The Lemmas 13 and 14 show that $\mathcal{N}(G)$ collapses to a subcomplex Δ_1 with $M(\Delta_1) = M_1 \cup M_2$, where

$$M_1 = \{\{f, f_s, \langle x \rangle\} \mid f \in V(G),\ x \notin \text{Im } f \text{ and } s \in [n]\} \textit{ and}$$
$$M_2 = \{\{\langle y_1 \rangle, \langle y_2 \rangle, \dots, \langle y_n \rangle, g\} \mid \text{Im } g = \{y_1, y_2, \dots, y_n\}\}.$$

For any simplex $\sigma_g = \{\langle y_1 \rangle, \langle y_2 \rangle, \dots, \langle y_n \rangle, g\}$ in M_2, if $1 \in \text{Im}\, g$, let $y_1 = 1$ and if $1 \notin \text{Im}\, g$, let $y_1 = 2$. In σ_g for all $1 \leqslant i < j \leqslant n$, $\{\langle y_i \rangle, \langle y_j \rangle, g\}$ are free faces. Considering the faces $\{\langle y_2 \rangle, \langle y_j \rangle, g\}$, $3 \leqslant j \leqslant n$, we get the following result.

Claim 16. *$\Delta_1 \searrow \Delta'$ and $M(\Delta') = M_1 \cup \{M_2 \setminus \{\sigma_g\}\} \cup \{\sigma_g \setminus \{g\}\} \cup \{\sigma_g \setminus \{\langle y_2 \rangle\} \cup \{\langle y_1 \rangle, \langle y_2 \rangle, g\}$.*

Proof. Let $Y = M_1 \cup \{M_2 \setminus \{\sigma_g\}\} \cup \{\sigma_g \setminus \{g\}\}$. Since $\{\langle y_2 \rangle, \langle y_3 \rangle, g\}$ is a free face of σ_g, $\Delta_1 \searrow \Delta_{1,3}$ where $M(\Delta_{1,3}) = Y \cup \{\sigma_g \setminus \{\langle y_2 \rangle\}\} \cup \{\sigma_g \setminus \{\langle y_3 \rangle\}\}$. In $\Delta_{1,3}$, $\{\langle y_2 \rangle, \langle y_4 \rangle, g\}$ is a free face of $\{\sigma_g \setminus \{\langle y_3 \rangle\}\}$ and so $\Delta_{1,3} \searrow \Delta_{1,4}$ where $M(\Delta_{1,4}) = Y \cup \{\sigma_g \setminus \{\langle y_2 \rangle\}\} \cup \{\sigma_g \setminus \{\langle y_3 \rangle\ \langle y_4 \rangle\}\}$. Inductively, assume that $\Delta_1 \searrow \Delta_{1,n-1}$ where $M(\Delta_{1,n-1}) = Y \cup \{\sigma_g \setminus \{\langle y_2 \rangle\}\} \cup \{\sigma_g \setminus \{\langle y_3 \rangle, \langle y_4 \rangle, \dots, \langle y_{n-1} \rangle\}\}$. In $\Delta_{1,n-1}$, $\alpha = \{\langle y_2 \rangle, \langle y_n \rangle, g\}$ is a free face of $\sigma_g \setminus \{\langle y_3 \rangle, \langle y_4 \rangle, \dots, \langle y_{n-1} \rangle\}$. By a simplicial collapse, we get the complex Δ' where $M(\Delta') = Y \cup \{\sigma_g \setminus \{\langle y_2 \rangle\}\} \cup \{\langle y_1 \rangle, \langle y_2 \rangle, g\}$. ☐

For $3 \leqslant i \leqslant n-1$, using Claim 16 for the simplices $\{\langle y_i \rangle, \langle y_j \rangle, g\}$, $i+1 \leqslant j \leqslant n$, we get $\Delta_1 \searrow Z$ where $M(Z) = Y \cup \{\{\langle y_1 \rangle, \langle y_i \rangle, g\} | \, i \in \{2, 3, 4, \dots, n\}\}$. Repeating the above argument for the remaining elements of M_2, $\mathcal{N}(G)$ collapses to the subcomplex Δ, $M(\Delta) = M_1 \cup A_1 \cup A_2 \cup A_3$, where

- $A_1 = \{\{\langle 1 \rangle, \langle y \rangle, g\} \mid 1, y \in \text{Im } g\}$,
- $A_2 = \{\{\langle 2 \rangle, \langle y \rangle, g\} \mid 2, y \in \text{Im } g \text{ and } 1 \notin \text{Im } g\}$ and
- $A_3 = \{\{\langle y_1 \rangle, \langle y_2 \rangle, \dots, \langle y_n \rangle\} \mid y_1, y_2, \dots, y_n \in [n+1]\}$.

Since $\mathcal{N}(K_{n+1}^{K_n})$ is homotopy equivalent to $\mathcal{N}(G)$ which collapses to Δ, we have $\mathcal{N}(K_{n+1}^{K_n}) \simeq \Delta$. We now construct an acyclic matching on the face poset of Δ to compute the Morse Complex corresponding to this matching.

Let $(P, \subset)$ denote the face poset of Δ. Define $S_1 \subset P$ to be the set $\{\sigma \in P \mid \langle 1\rangle \notin \sigma,\ \sigma \cup \langle 1\rangle \in \Delta\}$ and the map $\mu_1 : S_1 \to P \setminus S_1$ by $\mu_1(\sigma) = \sigma \cup \langle 1\rangle$. μ_1 is injective and $(S_1, \mu_1(S_1))$ is a partial matching on P. Let $S' = P \setminus (S_1 \cup \mu_1(S_1))$.

Lemma 17. *Any 1-cell σ of S' will be of one of the following types.*

(I) $\{f, \langle x\rangle\}$, $1 \notin Im\, f$, $x \neq 1$.

(II) $\{f, \langle x\rangle\}$, $x \notin Im\, f$, $x \neq 1$.

(III) $\{f, f_i\}$, $a_k = 1$, $i \neq k$.

Proof. Let τ be a maximal simplex and $\sigma \subsetneq \tau$. From the above discussion, τ has to be an element in one of the sets M_1, A_1, A_2 or A_3.

$\tau \notin A_1$ since $\sigma \notin \mu_1(S_1) \cup S_1$. If $\tau \in A_3$, then σ has to be of the form $\{\langle x\rangle, \langle y\rangle\}$ for some $x, y \neq 1$. There exists $z \neq 1, x, y$ such that $\sigma \cup \langle 1\rangle \in N(\langle z\rangle)$, which implies $\sigma \in S_1$, a contradiction. Hence $\tau \notin A_3$.

If $\tau \in M_1$, then $\tau = \{f, f_i, \langle x\rangle\}$, where $f = a_1a_2 \dots a_n$ and $x \notin \mathrm{Im}\, f$. Clearly $x \neq 1$, as if $x = 1$, then $\sigma \in S_1 \cup \mu(S_1)$, a contradiction. Let $a_k = 1$. Since $\sigma \subset \tau$, $\sigma = \{f, \langle x\rangle\}$, $\{f_i, \langle x\rangle\}$ or $\{f, f_i\}$.

(i) $\sigma = \{f_i, \langle x\rangle\}$.
If $i \neq k$, then $a_i \neq 1, x \Rightarrow \sigma \cup \langle 1\rangle \in N(\langle a_i\rangle) \Rightarrow \sigma \in S_1$, a contradiction.
If $i = k$, then $A_k^{f_k} = [n+1] \setminus \{1, x\} \Rightarrow N(f_k, \langle x\rangle, \langle 1\rangle) = \emptyset$ and thus σ is of the type (I).

(ii) $\sigma = \{f, \langle x\rangle\}$.
Since $x \neq 1$, and $A_k^f \cup \{x\} \cup \{1\} = [n+1]$, $\sigma \notin S_1 \cup \mu(S_1)$. Hence $\sigma \in S'$ and is of the type (II).

(iii) $\sigma = \{f, f_i\}$.
$a_k = 1 \Rightarrow 1 \notin \mathrm{Im} f_k \Rightarrow \langle 1\rangle \in N(f_k) \Rightarrow \{f, f_k, \langle 1\rangle\} \in N(f_k)$. Since $\sigma \in S'$, σ cannot be $\{f, f_k\}$ and hence $i \neq k$. $A_k^{f_i} = [n+1] \setminus \{1, a_i\}$ and $A_k^{f_k} = [n+1] \setminus \{1, x\} \Rightarrow A_k^f \cup A_k^{f_i} \cup \{1\} = [n+1] \Rightarrow N(f, f_i, \langle 2\rangle) = \emptyset$. Here $\sigma = \{f, f_i\} \in S'$ is of the type (III).

Finally, consider the case when $\tau \in A_2$. There exists $f = a_1 \dots a_n$, $a_i \neq 1\ \forall\ i \in [n]$, such that $\tau = \{\langle 2\rangle, \langle y\rangle, f\}$, where $y \neq 1$.

For any $z \in [n+1] \setminus \{1, 2, y\}$, $N(\langle 1\rangle)$ contains $\langle 1\rangle, \langle 2\rangle$ and $\langle y\rangle$ which implies $\{\langle 2\rangle, \langle y\rangle\} \in S_1$. Since $\sigma \notin S_1$, σ is either $\{f, \langle 2\rangle\}$ or $\{f, \langle y\rangle\}$.

$A_i^f = [n+1] \setminus \{1, 2\}$, where $a_i = 2$ which implies $N(f, \langle 2\rangle, \langle 1\rangle) = \emptyset$ implying that $\{f, \langle 2\rangle\} \notin S_1$. Hence $\{f, \langle 2\rangle\} \in S'$ is of the form (I).

$A_j^f = [n+1] \setminus \{1, y\}$, where $a_j = y$ which implies $N(f, \langle y\rangle, \langle 1\rangle) = \emptyset$ thereby showing $\{f, \langle y\rangle\} \in S'$ and σ is of the type (I). ☐

Let S_2 be the set of all the 1-cells in S' except those of the type $\{f, \langle a_i \rangle\}$, $a_i \neq 1$, $1 \leqslant i \leqslant n$. We now define the map $\mu_2 : S_2 \longrightarrow P \setminus S_2$, as

(i) $\mu_2(\{f, f_i\}) = \begin{cases} \{f, f_i, \langle x \rangle\}, & \text{if } x > a_i \\ \{f, f_i, \langle a_i \rangle\}, & \text{if } x < a_i, \end{cases}$

where $a_k = 1$, $\operatorname{Im} f = [n+1] \setminus \{x\}$ and $i \neq k$.

(ii) $\mu_2(\{f, \langle x \rangle\}) = \{f, f_k, \langle x \rangle\}$ where $a_k = 1$, $\operatorname{Im} f = [n+1] \setminus \{x\}$.

(iii) $\mu_2(\{f, \langle y \rangle\}) = \{f, \langle y \rangle, \langle 2 \rangle\}$ where $y \neq 1, 2$ and $1 \notin \operatorname{Im} f$.

Claim 18. *μ_2 is injective.*

From the definition of μ_2, for any $\sigma \in S_2$, dim($\mu_2(\sigma)$) = 2 and $\sigma \subset \mu_2(\sigma)$. Therefore $\mu_2(\sigma) \succ \sigma$, for each σ. Let $\mu_2(\sigma_1) = \mu_2(\sigma_2) = \tau$ for some $\sigma_1, \sigma_2 \in S_2$. There are three possibilities for $\tau \in Im\, \mu_2$.

1. *$\tau = \{f, f_i, \langle x \rangle\}$, $x \notin Im\, f$, $a_k = 1$, $i \neq k$, $x > a_i$.*

   $\{f_i, \langle x \rangle\} \in S_1$ (since $\{f_i, \langle x \rangle, \langle 1 \rangle\} \in N(\langle y_i \rangle))$ and $\{f, \langle x \rangle\} \in S_2$ imply that both $\mu(\{f_i, \langle x \rangle\})$ and $\mu(\{f, \langle x \rangle\}$ are not equal to τ. Hence $\sigma_1 = \sigma_2 = \{f, f_i\}$.

   If $x < a_i$, then $\tau = \{f, f_i, \langle a_i \rangle\} = \{f_i, (f_i)_i, \langle a_i \rangle\}$ and the same argument as the one above holds.

2. *$\tau = \{f, f_k, \langle x \rangle\}$, $x \notin Im\, f$, $a_k = 1$.*

   $1 \notin Im\, f_k \Rightarrow \{f, f_k, \langle 1 \rangle\} \in N(f_k) \Rightarrow \{f, f_k\} \in S_1 \Rightarrow \sigma_1, \sigma_2 \neq \{f, f_k\}$.

   If $x \neq 2$, then $\mu_2(\{f_k, \langle x \rangle\}) = \{f_k, \langle x \rangle, \langle 2 \rangle\} \neq \tau$ and when $x = 2$, then $\{f_k, \langle 2 \rangle\} \notin S_2$. Hence both σ_1 and σ_2 have to be $\{f, \langle x \rangle\}$.

3. *$\tau = \{f, \langle 2 \rangle, \langle y \rangle\}$, $y \in Im\, f$, $y \neq 1$, $1 \notin Im\, f$.*

   Since $\{f, \langle 2 \rangle\} \notin S_2$ and $\{\langle 2 \rangle, \langle y \rangle\} \in S_1$, $\sigma_1 = \sigma_2 = \{f, \langle y \rangle\}$.

From Claim 18, $\mu_2 : S_2 \longrightarrow P \setminus S_2$ is a partial matching.

Since $S_2 \cap \mu_1(S_1), \mu_1(S_1) \cap \mu_2(S_2) = \emptyset$, the map $\mu : S_1 \cup S_2 \longrightarrow P$ defined by μ_1 on S_1 and μ_2 on S_2 is a partial matching on P. This map is well defined since $S_1 \cap S_2 = \emptyset$.

Lemma 19. *μ is an acyclic matching.*

Proof. Let $C = P \setminus \{S_1, S_2, \mu_1(S_1), \mu_2(S_2)\}$. Suppose there exists a sequence of cells $\sigma_1, \sigma_2, \ldots, \sigma_t \in P \setminus C$ such that $\mu(\sigma_1) \succ \sigma_2, \mu(\sigma_2) \succ \sigma_3, \ldots, \mu(\sigma_t) \succ \sigma_1$. If $\sigma_i \in S_1$, then $\mu(\sigma_i) = \sigma_i \cup \langle 1 \rangle \succ \sigma_{i+1(mod\ t)}$ which implies $\langle 1 \rangle \in \sigma_{i+1(mod\ t)}$, which is not possible by the construction of S_1 and S_2. Hence σ_i has to be in S_2, for each i, $1 \leqslant i \leqslant t$. From Lemma 17, σ has the following three forms.

1. $\sigma_i = \{f, \langle y \rangle\}$, $1 \notin \operatorname{Im} f$, $y \neq 1$.

   Since $\{f, \langle 2 \rangle\} \notin S_1 \cup S_2 \cup \mu(S_1) \cup \mu(S_2)$, $y \neq 2$. Further, $\mu(\sigma_i) = \{f, \langle y \rangle, \langle 2 \rangle\}$ shows that σ_{i+1} has to be $\{\langle y \rangle, \langle 2 \rangle\}$ or $\{f, \langle 2 \rangle\}$, both of which are impossible. Hence, σ_i is not of this form.

2. $\sigma_i = \{f, \langle x\rangle\}$, $x \notin \operatorname{Im} f$, $x \neq 1$, $a_k = 1$.

   $\mu(\sigma_i) = \{f, f_k, \langle x\rangle\}$ implies that σ_{i+1} has to be either $\{f_k, \langle x\rangle\}$ or $\{f, f_k\}$. Since $1 \notin \operatorname{Im} f_k$ and $x \neq 1$, $\sigma_{i+1} \neq \{f_k, \langle x\rangle\}$ (from the above case). Further, $\{f, f_k, \langle 1\rangle\} \in N(f_k)$ implies that $\{f, f_k\}$ is an element of S_1 and therefore this case too is not possible.

3. $\sigma_i = \{f, f_i\}$, $a_k = 1$, $\operatorname{Im} f = [n+1] \setminus \{x\}$, $i \neq k$.

$$\mu(\sigma_i) = \mu_2(\sigma_i) = \begin{cases} \{f, f_i, \langle x\rangle\}, & \text{if } x > a_i \\ \{f, f_i, \langle a_i\rangle\}, & \text{if } x < a_i. \end{cases}$$

   If $x > a_i$, case (2) shows that $\sigma_{i+1} \neq \{f, \langle x\rangle\}$. Since $a_i \neq 1, x$, and $a_i \notin \operatorname{Im} f_i$, $\{f_i, \langle 1\rangle, \langle x\rangle\} \in N(\langle a_i\rangle)$ which implies that $\{f_i, \langle x\rangle\} \in S_1$. A similar argument shows that the case $x < a_i$ is also not possible.

Our assumption that the above sequence $\sigma_1, \sigma_2, \ldots, \sigma_t$ exists is wrong. Therefore (S, μ) where $S = S_1 \cup S_2$, is an acyclic matching. □

Every element of C is a critical cell corresponding to this matching. We now describe the structure of the elements of C.

For any 0-cell $\langle x\rangle \neq \langle 1\rangle$ in Δ, if $y \neq 1, x$ then $\{\langle x\rangle, \langle 1\rangle\} \in N(\langle y\rangle)$, thereby implying that $\langle x\rangle \in S_1$. If f is a 0-cell with $|\operatorname{Im} f| = n$ then either $f, \langle 1\rangle \in N(< [n+1] \setminus \operatorname{Im} f >)$ or $\langle 1\rangle, f \in N(f)$ accordingly as $1 \in \operatorname{Im} f$ or not. In both cases $f \in S_1$ and therefore $\langle 1\rangle$ is the only critical 0-cell.

Any 2-cell σ of Δ belongs to $M_1 \cup A_1 \cup A_2 \cup A_3$. Each element of A_1 belongs to $\mu_1(S_1)$. If $\sigma \in A_2$, then $\sigma = \{f, \langle 2\rangle, \langle y\rangle\}$ with $\operatorname{Im} f = [n+1] \setminus \{1\}$, $y \neq 1, 2$. Since $\mu_2(\{f, \langle y\rangle\}) = \sigma$, $\sigma \notin C$. Therefore, if σ has to be a critical 2-cell, then σ has to belong to either M_1 or A_3.

If $\sigma \in M_1$, then $\sigma = \{f, f_i, \langle x\rangle\}$ and $x \notin \operatorname{Im} f$. Clearly, $x \neq 1$. Let $a_k = 1$. $\mu(\{f, \langle x\rangle\}) = \{f, f_k, \langle x\rangle\}$, $i \neq k$. If $x > a_i$, then $\mu(\{f, f_i\}) = \sigma$. If $x < a_i$ then $\{f_i, \langle 1\rangle, \langle x\rangle\} \in N(\langle a_i\rangle)$ which implies that $\sigma \notin \mu(S)$. Further, $\sigma \notin S_1, S_2$ and therefore $\sigma \in M_1$ will be a critical 2-cell if and only if $\sigma = \{f, f_i, \langle x\rangle\}$, where $x \notin \operatorname{Im} f$, $a_k = 1$, $i \neq k$ and $x < a_i$.

Finally from A_3, there exists exactly one critical cell $\{\langle y_1\rangle, \langle y_2\rangle, \ldots, \langle y_n\rangle\}$ of dimension $n-1$, where $y_1, y_2, \ldots, y_n \in [n+1] \setminus \{1\}$ (since any proper subset of $\{\langle y_1\rangle, \langle y_2\rangle, \ldots, \langle y_n\rangle\}$ belongs to S_1).

Therefore the $\langle z\rangle$ set of critical cells $C = \{\langle 1\rangle\} \cup C_1 \cup C_2 \cup C_3$, where

$$\begin{aligned} C_1 &= \{\{f, \langle 2\rangle\} \mid \operatorname{Im} f = [n+1] \setminus \{1\}\}, \\ C_2 &= \{\{f, f_i, \langle y\rangle\} \mid \operatorname{Im} f = [n+1] \setminus \{x\}, a_k = 1, i \neq k \text{ and } x < a_i\}, \\ C_3 &= \{\{\langle y_1\rangle, \langle y_2\rangle, \ldots, \langle y_n\rangle\} \mid [n+1] \setminus \{1\} = \{y_1, y_2, \ldots, y_n\}\}. \end{aligned}$$

Hence the critical cells are of dimension $0, 1, 2$ and $n-1$.
Clearly $|C_1| = n!$ and $|C_3| = 1$. For each fixed $x \neq 1$, let $r = |\{s \in [n+1] \,|\, s > x\}|$ and $Q = \{f \in V(G) \,|\, \operatorname{Im} f = [n+1] \setminus \{x\}\}$. The cardinality of Q is $n!$. For $f \in Q$,

$\{f, f_i, \langle x\rangle\} \in C_2$ if and only if $x < f(i)$. Hence $|\{\tau = \{f, f_i, \langle x\rangle\} \mid \{x\} = [n+1]\setminus \mathrm{Im}\, f, \tau \in C_2\}| = r(n!)$. Therefore $|C_2| = n! \sum_{r=1}^{n-1} r = \frac{n!(n-1)n}{2}$.

We now describe the Morse Complex $\mathcal{M} = (\mathcal{M}_i, \partial)$ corresponding to this acyclic matching on the poset P. If c_i denotes the number of critical i cells of C, then the free abelian group generated by these critical cells is denoted by $\mathcal{M}_i$. Our objective now is to first compute the $\mathbb{Z}_2$ homology groups of the Morse complex $\mathcal{M}$. We use the following version of Theorem 10, from which we explicitly compute the boundary maps in the Morse Complex $\mathcal{M}$.

Proposition 20. *(Theorem 11.13 [13]) Let X be a simplicial complex and μ be an acyclic matching on the face poset of $X \setminus \emptyset$. Let c_i denote the number of critical i cells of X. Then*

(a) X is homotopy equivalent to X_c, where X_c is a CW complex with c_i cells in dimension i.

(b) there is a natural indexing of cells of X_c with the critical cells of X such that for any two cells τ and σ of X_c satisfying dim τ = dim $\sigma + 1$, the incidence number $[\tau : \sigma]$ is given by

$$[\tau : \sigma] = \sum_c w(c).$$

The sum is taken over all (alternating) paths c connecting τ with σ i.e., over all sequences $c = \{\tau, x_1, \mu(x_1), \ldots, x_t, \mu(x_t), \sigma\}$ such that $\tau \succ x_1$, $\mu(x_t) \succ \sigma$, and $\mu(x_i) \succ x_{i+1}$ for $i = 1, \ldots, t-1$. The quantity $w(c)$ associated to this alternating path is defined by

$$w(c) := (-1)^t [\tau : \sigma][\mu(x_t) : \sigma] \prod_{i=1}^{t} [\mu(x_i) : x_i] \prod_{i=1}^{t-1} [\mu(x_i) : x_{i+1}]$$

where all the incidence numbers are taken in the complex X.

We now determine all the possible alternating paths between any two critical cells.

Lemma 21. *Let $\gamma \in \Delta$ be a k-simplex, $k > 0$, such that $\langle 1\rangle \in \gamma$. Then γ does not belong to any alternating path connecting two critical cells.*

Proof. Given two critical cells τ and σ, let $c = \{\tau, x_1, \mu(x_1), \ldots, x_t, \mu(x_t), \sigma\}$ be an alternating path and let $\gamma \in c$. Since $\langle 1\rangle \in \gamma$, $\gamma \in \mu_1(S_1)$, and therefore $\gamma \neq \tau, \sigma$. For some $i \in [t-1]$, there exists $x_i \in c$ such that $\gamma = \mu(x_i)$. Since $[\mu(x_i) : x_i] = \pm 1$, x_i has to be $\gamma \setminus \langle 1\rangle$. Any facet of γ different from x_i must contain $\langle 1\rangle$. But since $x_{i+1} < \mu(x_i)$, x_{i+1} has to be a facet of $\mu(x_i)$ and therefore must belong to S, which is impossible, as $\langle 1\rangle \in x_{i+1}$ implies $x_{i+1} \in \mu(S)$ and $\mu(S) \cap S = \emptyset$. Hence $\gamma \notin c$. $\square$

Lemma 22. *Let $\tau = \{f, f_i, \langle x\rangle\}$ be a critical 2-cell with $i \neq k$ and $a_k = 1$. There exists exactly one alternating path from τ to each of exactly 2 critical 1-cells $\alpha = \{f_k, \langle 2\rangle\}$ and $\beta = \{(f_i)_k, \langle 2\rangle\}$.*

Proof. Let $\tau = \{f, f_i, \langle x\rangle\} \in C_2$ be a critical 2-cell. For any alternating path c from τ to a critical 1 cell, $\tau \succ x_1$, i.e. x_1 is a facet of τ. We have three choices for x_1.

1. $x_1 = \{f, \langle x\rangle\}$.

   Since x_2 has to be a facet of $\mu(x_1) = \{f, f_k, \langle x\rangle\}$, it is either $\{f, f_k\}$ or $\{f_k, \langle x\rangle\}$. In the former case, $\mu(\{f, f_k\}) = \{f, f_k, \langle 1\rangle\}$ (since $\{f, f_k, \langle 1\rangle\} \in N(f_k)$) which contradicts Lemma 21.

   If $x = 2$, then $x_2 = \{f_k, \langle 2\rangle\}$ is a critical 1- cell and the alternating path is $\{\tau, x_1 = \{f, \langle 2\rangle\}, \{f, f_k, \langle 2\rangle\}, \{f_k, \langle 2\rangle\}\}$.

   If $x > 2$, then $\mu(x_2) = \{f_k, \langle x\rangle, \langle 2\rangle\}$ and x_3 has to be the critical 1-cell $\{f_k, \langle 2\rangle\}$ (since $\{\langle x\rangle, \langle 2\rangle, \langle 1\rangle\} \in N(\langle y\rangle)$, $y \neq 1, 2, x$). The alternating path is

   $$\{\tau, \{f, \langle x\rangle\}, \{f, \langle x\rangle, f_k\}, \{f_k, \langle x\rangle\}, \{f_k, \langle x\rangle, \langle 2\rangle\}, \{f_k, \langle 2\rangle\}\}.$$

2. $x_1 = \{f, f_i\}$.

   $\mu(x_1) = \{f, f_i, \langle a_i\rangle\}$ forces x_2 to be $\{f_i, \langle a_i\rangle\}$ (as $\{f, \langle a_i\rangle, \langle 1\rangle\} \in N(\langle x\rangle) \Rightarrow x_2 \neq \{f, \langle a_i\rangle\}$). $\mu(x_2) = \{f_i, \langle a_i\rangle, (f_i)_k\}$ implies that $x_3 = \{f_i, (f_i)_k\}$ or $\{(f_i)_k, , \langle a_i\rangle\}$. But, $\{f_i, (f_i)_k, \langle 1\rangle\} \in N((f_i)_k)$ shows that $x_3 = \{(f_i)_k, \langle a_i\rangle\}$. Since $x < a_i$ and $x \neq 1$, a_i is not 2 and thus $\mu(x_3) = \{(f_i)_k, \langle a_i\rangle, \langle 2\rangle\}$. Since $\{\langle a_i\rangle, \langle 2\rangle, \langle 1\rangle\}$ is a simplex in Δ, x_4 has to be the critical cell $\{(f_i)_k, \langle 2\rangle\}$. The alternating path is

   $$\{\tau, \{f, f_i\}, \{f, f_i, \langle a_i\rangle\}, \{f_i, \langle a_i\rangle\}, \{f_i, \langle a_i\rangle, (f_i)_k\},$$
   $$\{(f_i)_k, \langle a_i\rangle\}, \{(f_i)_k, \langle a_i\rangle, \langle 2\rangle\}, \{(f_i)_k, \langle 2\rangle\}\}.$$

3. $x_1 = \{f_i, \langle x\rangle\}$.
   Since $a_i \notin \operatorname{Im} f_i$ and $a_i \neq 1$, $\{f_i, \langle x\rangle, \langle 1\rangle\} \in N(\langle a_i\rangle)$. Thus x_1 can not be an element of c.

Hence, for each critical 2-cell $\tau = \{f, f_i, \langle x\rangle\} \in C_2$, there exist unique alternating paths from τ to exactly 2 critical 1-cells. □

Consider $\tau = \{\langle y_1\rangle, \langle y_2\rangle, \ldots, \langle y_n\rangle\} \in C_3$. There exists no alternating path from τ to any critical cell because each facet of τ belongs to S_1.

If $\alpha = \{f, \langle 2\rangle\}$ is a critical 1-cell, then $1 \notin \operatorname{Im} f$. Since $n \geqslant 3$, there exists $i \neq k$ such that $a_k \neq 2$ and $a_i = n + 1$. Since $a_k < a_i$, the 2-cell $\{f_k, (f_k)_i, \langle f(k)\rangle\}$ is a critical cell. From Lemma 22, there exists an alternating path between these two cells, showing that there exists at least one alternating path to each critical 1-cell.

Let $W_n = \{a_1a_2\ldots a_n \in V(G) \,|\{a_1, a_2, \ldots, a_n\} = [n+1] \setminus \{1\}\}$, where $n \in \mathbb{N}$, $n \geqslant 2$. Define a relation $\sim$ on W_n by, $a \sim b \iff \exists\, i, j \in [n], i \neq j$ such that $a_i = b_j$, $a_j = b_i$ and $a_k = b_k$ for all $k \neq i, j$. The cardinality of W_n is easily seen to be $n!$.

Lemma 23. *The $n!$ elements of W_n, $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_{n!}$ can be ordered in such a way that $\alpha_i \sim \alpha_{i+1}$, for $1 \leqslant i \leqslant n! - 1$.*

Proof. If $n = 2$, then $W_n = \{23, 32\}$ and $23 \sim 32$. Let us assume that $n \geqslant 3$. The proof is by induction on n.

Let $W_{n,i} = \{f \in W_n \mid a_1 = i\}$. Clearly $W_n = \bigcup_{i=2}^{n+1} W_{n,i}$, where each $W_{n,i}$ is in bijective correspondence with W_{n-1}. By the inductive hypothesis, assume that W_{n-1} has the required ordering $\alpha_1 \sim \alpha_2 \sim \cdots \sim \alpha_{(n-1)!}$, where $\alpha_1 = a_1 a_2 \ldots a_{n-1}$, $a_i \in \{2, 3, \ldots n\}$. For a fixed first element $iw_2w_3 \ldots w_n \in W_{n,i}$, the map $\phi_i : [n+1] \setminus \{1, i\} \to \{2, 3, \ldots n\}$ defined by $\phi_i(w_j) = a_{j-1}$ is bijective. Using the ordering in W_{n-1} and the map ϕ_i, we get an ordering in $W_{n,i}$. Beginning with $23 \ldots n+1 = 2w_2 \ldots w_n \in W_{n,2}$ and using the map ϕ_2 we order $W_{n,2}$. Let $2w'_2w'_3 \ldots w'_n$ be the last element of this ordering and $w'_j = 3$. Then, $2w'_2w'_3 \ldots w_n \sim 3w'_2 \ldots w'_{j-1}2w'_{j+1}w'_n$. Using the map ϕ_3 in the above method, we get an ordering for $W_{n,3}$. Repeating this argument for $4 \leqslant i \leqslant n+1$, we have the required ordering in W_n. ☐

Since every critical 1-cell contains $\langle 2 \rangle$, henceforth a critical 1-cell $\{f, \langle 2 \rangle\}$ shall be denoted by f.

Remark 24. There exist alternating paths from a critical 2-cell τ to α and β if and only if $\alpha \sim \beta$.

The set of critical 1-cells $C_1 = \{\alpha_i = \{f_i, \langle 2 \rangle\} | f_i \in W_n\}$ is in bijective correspondence with W_n. From Lemma 23, we have an ordering $\alpha_1 \sim \alpha_2 \sim \ldots \sim \alpha_{n!}$ of the elements of C_1. Let $C_2 = \{\tau_1, \tau_2, \ldots, \tau_{\frac{n!(n-1)n}{2}}\}$ and $A = [a_{ij}]$ be a matrix of order $|C_1| \times |C_2|$, where $a_{ij} = 1$, if there exists an alternating path from τ_j to α_i and 0 if no such path exists. Using Lemma 22 each column of A contains exactly two non zero elements which are 1. The rows of the matrix A are denoted by R_{α_i} and the columns are denoted by C_{τ_i}.

Lemma 25. *The set $B = \{R_{\alpha_2}, \ldots, R_{\alpha_{n!}}\}$ is a basis for the row space of A over the field $\mathbb{Z}_2$.*

Proof. In each column exactly two entries are 1 and all other entries are 0 and thus column sum is zero (mod 2) and hence rank$(A) < n!$.

Assume that $\sum_{i=2}^{n!} a_i R_{\alpha_i} = 0$, $a_i \in \{0, 1\}$. For $1 \leqslant i \leqslant n! - 1$, let τ_i be the critical 2-cell which has alternating paths to α_i and α_{i+1}. The column C_{τ_i} has the i and $(i+1)^{th}$ entry equal to 1 and all other entries equal to zero. $\sum_{i=2}^{n!} a_i R_{\alpha_i} = 0$, implies $a_2 = a_2 + a_3 = a_3 + a_4 = \ldots = a_{(n-1)!} + a_{n!} = 0$. Hence $a_2 = a_3 = \ldots a_{n!} = 0$ and B is a basis for the row space of A. ☐

Let the Discrete Morse Complex corresponding to the acyclic matching μ on Δ be $\mathcal{M} = (\mathcal{M}_n, \partial_n)$, $n \geqslant 0$ where $\mathcal{M}_i$ denotes the free abelian groups over $\mathbb{Z}_2$ generated by the critical i-cells. The only non trivial groups are $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_{n-1}$. For any two critical cells τ and σ such that $\dim(\tau) = \dim(\sigma) + 1$, the incidence number $[\tau : \sigma]$ is either 0 or 1.

We have developed all the necessary tools to prove the main results.

Proof of Theorem 2. The graph $K_{n+1}^{K_n}$ folds to graph G, by Lemma 11 and therefore Hom $(K_2, K_{n+1}^{K_n}) \simeq$ Hom (K_2, G). Further since Hom $(K_2, G) \simeq \mathcal{N}(G)$ and $\mathcal{N}(G) \simeq \Delta$, from Proposition 20, it is sufficient to compute the homology groups of the Morse Complex $\mathcal{M}$.

For all $y \neq z \in [n]$, $\{\langle y\rangle, \langle z\rangle\} \in \mathcal{N}(G)$, thereby showing that $\{\langle y\rangle, \langle z\rangle\}$ is an edge in G. If $f \in V(G)$ such that $|\mathrm{Im}\, f| = n$, then $\{f, \langle x\rangle\}$ is an edge, where $x \notin \mathrm{Im}\, f$. Since $\{\langle x\rangle, \langle y\rangle\}$ is an edge for all $x, y \in [n+1]$, any two vertices of G are connected by an edge path and therefore $\mathcal{N}(G)$ is connected which implies $H_0(\mathcal{N}(G); \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Since $H_0(\mathcal{N}(G); \mathbb{Z}_2) \cong \mathbb{Z}_2 \cong (\mathcal{M}_0, \mathbb{Z}_2)$, $\mathrm{Ker}\, \partial_1 \cong \mathbb{Z}_2^{n!}$, where $\partial_1 : \mathcal{M}_1 \to \mathcal{M}_0$ is a boundary map.

Since $n \geqslant 4$, any critical 2-cell belongs to C_2. Further since any critical 2-cell is connected by alternating paths to exactly two 1-cells, from Lemma 25, the rank of the group homomorphism $\partial_2 : \mathbb{Z}_2^p \longrightarrow \mathbb{Z}_2^{n!}$ is $n! - 1$. Therefore $H_1(\mathcal{M}; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

If $n = 4$, then $\tau = \{\langle 2\rangle, \langle 3\rangle, \langle 4\rangle, \langle 5\rangle\}$ is the only critical 3-cell and $\mathcal{M}_3 \cong \mathbb{Z}_2$. Since each facet of τ belongs to S_1, there will be no alternating path from τ to any critical 2-cell which implies that the incidence number $[\tau : \alpha] = 0$, for any critical 2-cell α and $\partial_3 : \mathcal{M}_3 \to \mathcal{M}_2$ is the zero map. Rank$(\partial_2) = n! - 1$ and therefore $\mathrm{Ker}\, \partial_2 \cong \mathbb{Z}_2^{p-n!+1}$. Thus $H_2(\mathcal{M}, \mathbb{Z}_2) \cong \mathbb{Z}_2^{p-n!+1}$.

If $n > 4$, $\tau = \{\langle 2\rangle, \langle 3\rangle, \ldots, \langle n+1\rangle\}$ is the only $n-1$ critical cell. $\mathcal{M}_{n-2}$ and $\mathcal{M}_n$ are trivial groups and therefore $H_{n-1}(\mathcal{M}, \mathbb{Z}_2) \cong \mathbb{Z}_2$. □

Corollary 26.

$$H_k(\mathcal{N}(K_4^{K_3}); \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & \textit{if} \quad k = 0, 1 \\ \mathbb{Z}_2^{14}, & \textit{if} \quad k = 2 \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Since $n = 3$ in this case, $|C_1| = 6$, $|C_2| = 18$ and $C_3 = \{\{\langle 2\rangle, \langle 3\rangle, \langle 4\rangle\}\}$. There exist 19 critical 2-cells and therefore $\mathcal{M}_2 \cong \mathbb{Z}_2^{19}$. Since $\mathcal{N}(K_4^{K_3})$ is path connected, $H_0(\mathcal{N}(K_4^{K_3}); \mathbb{Z}_2) \cong \mathbb{Z}_2$.

Each facet of $\tau = \{\langle 2\rangle, \langle 3\rangle, \langle 4\rangle\}$ belongs to S_1 and therefore there exists no path from τ to any critical 1-cell α and therefore the incidence number $[\tau : \alpha] = 0$ for any critical 1-cell α. Hence $\partial_2(\tau) = 0$ i.e. $\tau \in$ Ker ∂_2. From Lemma 25, rank $(\partial_2) = 5$. Since $H_0(\mathcal{N}(K_4^{K_3}); \mathbb{Z}_2) \cong \mathbb{Z}_2 = \mathcal{M}_0$, $\partial_1 = 0$. Therefore $H_1(\mathcal{N}(K_4^{K_3}); \mathbb{Z}_2) \cong \mathbb{Z}_2$.

The rank of $\partial_2 = 5$ shows that Ker $\partial_2 \cong \mathbb{Z}_2^{14}$. Further, there is no critical cell of dimension greater than 2, $\mathcal{M}_i = 0$, for all $i > 2$. Hence, $H_2(\mathcal{N}(K_4^{K_3}); \mathbb{Z}_2) \cong \mathbb{Z}_2^{14}$ and $H_k(\mathcal{N}(K_4^{K_3}); \mathbb{Z}_2) = 0$, for all $k > 2$. □

We recall the following result to prove Theorem 3.

Proposition 27. *(Theorem 3A.3, [7])*
If C is a chain complex of free abelian groups, then there exist short exact sequences

$$0 \longrightarrow H_n(C;\mathbb{Z}) \otimes \mathbb{Z}_2 \longrightarrow H_n(C;\mathbb{Z}_2) \longrightarrow Tor(H_{n-1}(C;\mathbb{Z}),\mathbb{Z}_2) \longrightarrow 0$$

for all n and these sequences split.

Proof of Theorem 3. Since $\mathcal{N}(K_{n+1}^{K_n})$ is path connected, we only need to show that $\pi_1(\mathcal{N}(K_{n+1}^{K_n})) \neq 0$. If $n = 2$, then $\mathcal{N}(K_3^{K_2}) \simeq \text{Hom}\ (K_2 \times K_2, K_3) \simeq \text{Hom}\ (K_2 \sqcup K_2, K_3) \simeq \text{Hom}\ (K_2, K_3) \times \text{Hom}\ (K_2, K_3) \simeq S^1 \times S^1$. Hence $\pi_1(\mathcal{N}(K_3^{K_2}) \cong \mathbb{Z} \times \mathbb{Z}$.

Let $n \geqslant 3$. Since $H_1(\mathcal{N}(K_{n+1}^{K_n});\mathbb{Z})$ is the abelianization of $\pi_1(\mathcal{N}(K_{n+1}^{K_n}))$, it is enough to show that $H_1(\mathcal{N}(K_{n+1}^{K_n});\mathbb{Z}) \neq 0$. From Proposition 27, $H_1(\mathcal{N}(K_{n+1}^{K_n});\mathbb{Z}_2) \cong H_1(\mathcal{N}(K_{n+1}^{K_n});\mathbb{Z})\otimes\mathbb{Z}_2 \oplus \text{Tor}(H_0(\mathcal{N}(K_{n+1}^{K_n});\mathbb{Z}_2)$. Since $H_0(\mathcal{N}(K_{n+1}^{K_n});\mathbb{Z}) \cong \mathbb{Z}$, $\text{Tor}(H_0(\mathcal{N}(K_{n+1}^{K_n}));\mathbb{Z}_2) = 0$. So $H_1(\mathcal{N}\ (K_{n+1}^{K_n});\mathbb{Z}) = 0$, implies that $H_1(\mathcal{N}(K_{n+1}^{K_n});\mathbb{Z}_2) = 0$, which is a contradiction to Theorem 2 and Corollary 26. Therefore, $H_1(\mathcal{N}(K_{n+1}^{K_n});\mathbb{Z}) \neq 0$. ☐

The maximum degree d of the graph $K_2 \times K_n$ is $n-1$ and $\text{Hom}\ (K_2 \times K_n, K_{n+1}) \simeq \mathcal{N}(K_{n+1}^{K_n})$. Hence $\text{Hom}\ (K_2 \times K_n, K_{n+1})$ is exactly $(n+1-d-2)$-connected.

Proof of Corollary 4. Theorem 3 gives the result for the case $m = n+1$. If $m = n$, then for any $f \in V(K_n^{K_n})$ with $\text{Im}\, f = [n]$, $N(f) = \{f\}$. Since $n \geqslant 2$, $\mathcal{N}(K_n^{K_n})$ is disconnected.

If $m < n$, Lemma 11 shows that $K_m^{K_n}$ can be folded to the graph G, where $V(G) = \{\langle x\rangle\ |x \in [m]\}$. Then $N(\langle x\rangle) = \{\langle y\rangle \mid y \in [m] \setminus \{x\}\}$, for all $\langle x\rangle\ \in V(G)$ and therefore $\mathcal{N}(G)$ is homotopic to the simplicial boundary of $(m-1)$-simplex. Hence, $\mathcal{N}(K_m^{K_n}) \simeq \mathcal{N}(G) \simeq S^{m-2}$. Therefore, $\text{conn}(\text{Hom}(K_m^{K_n})) = m-3$. ☐

References

[1] E. Babson and D. N. Kozlov. Complexes of graph homomorphisms. *Israel J. Math.* 152:285–312, 2006.

[2] E. Babson and D. N. Kozlov. Proof of the Lovaśz conjecture. *Annals of Math.* 165:965–1007, 2007.

[3] Anders Björner. *Topological methods.* Handbook of Combinatorics. Elsevier, Amsterdam, 1,2:1819-1872, 1995.

[4] S. Čukić and D.N. Kozlov. Higher Connectivity of Graph Coloring Complexes. *Int. Math. Res. Not.* 25:1543–1562, 2005.

[5] Anton Dochtermann. Hom complexes and homotopy type in the category of graphs. *European Journal of Combinatorics.* 30:490-509, 2009.

[6] R. Forman. Morse Theory for Cell complexes. *Adv. Math.* 134(1):90–145, 1998.

[7] A. Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[8] P. Hell and J. Nešetřil. *Graphs and Homomorphisms.* Oxford Lecture Series in Mathematics and its Applications, no. 28. Oxford University Press, Oxford, 2004.

[9] C. Godsil and G. Royle. *Algebraic Graph Theory.* Graduate Texts in Mathematics, no. 207. Springer Verlag, New York, 2001.

[10] J. Jonsson. *Simplicial Complexes of graphs.* Lecture Notes in Mathematics, no. 1928. Springer Verlag, Berlin, 2008.

[11] D. N. Kozlov. Chromatic numbers, morphism complexes and Steifel Whitney Classes. *Geometric Combinatorics*, IAS Park City Math, Amer. Math. Soc, Ser 13:249–315, 2007.

[12] D. N. Kozlov. Cohomology of colorings of cycles. *American Journal of Mathematics.* 130:829–857, 2008.

[13] D. N. Kozlov. *Combinatorial Algebraic Topology.* Springer Verlag, Berlin, 2008.

[14] L. Lovász. Kneser's conjecture, chromatic number and homotopy. *J. Combinatorial Theory Ser. B*, 25:319–324, 1978.

[15] S. Maclane. *Categories for the Working Mathematician, second edition.* Graduate Texts in Mathematics, no. 5. Springer-Verlag, New York, 1998.