A family of symmetric graphs with complete quotients

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Abstract

A finite graph Γ is G-symmetric if it admits G as a group of automorphisms acting transitively on $V(\Gamma)$ and transitively on the set of ordered pairs of adjacent vertices of Γ. If $V(\Gamma)$ admits a nontrivial G-invariant partition $\mathcal{B}$ such that for blocks $B, C \in \mathcal{B}$ adjacent in the quotient graph $\Gamma_{\mathcal{B}}$ relative to $\mathcal{B}$, exactly one vertex of B has no neighbour in C, then we say that Γ is an almost multicover of $\Gamma_{\mathcal{B}}$. In this case there arises a natural incidence structure $\mathcal{D}(\Gamma, \mathcal{B})$ with point set $\mathcal{B}$. If in addition $\Gamma_{\mathcal{B}}$ is a complete graph, then $\mathcal{D}(\Gamma, \mathcal{B})$ is a $(G, 2)$-point-transitive and G-block-transitive 2-$(|\mathcal{B}|, m+1, \lambda)$ design for some $m \geqslant 1$, and moreover either $\lambda = 1$ or $\lambda = m+1$. In this paper we classify such graphs in the case when $\lambda = m+1$; this together with earlier classifications when $\lambda = 1$ gives a complete classification of almost multicovers of complete graphs.

Key words: Symmetric graph; arc-transitive graph; almost multicover

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1 Introduction

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite graph, and G a finite group acting on $V(\Gamma)$ as a group of automorphisms of Γ (that is, G preserves the adjacency and non-adjacency relations of Γ). If G is transitive on $V(\Gamma)$ and transitive on the set of arcs of Γ, then Γ is said to be *G-symmetric* or *G-arc-transitive*, where an *arc* is an ordered pair of adjacent vertices. Beginning with Tutte's seminal work [30], the study of symmetric graphs has long been one of the central topics in algebraic graph theory. See [24, 25] for two useful surveys in this area.

A G-symmetric graph Γ is called an *imprimitive G-symmetric graph* if $V(\Gamma)$ admits a nontrivial G-invariant partition $\mathcal{B}$, that is, $1 < |B| < |V(\Gamma)|$ and $B^g := \{\alpha^g : \alpha \in B\} \in \mathcal{B}$ for any $B \in \mathcal{B}$ and $g \in G$. In this case the *quotient graph* $\Gamma_{\mathcal{B}}$ of Γ relative to $\mathcal{B}$ is defined to be the graph with vertex set $\mathcal{B}$ in which $B, C \in \mathcal{B}$ are adjacent if and only if there exists an edge of Γ joining a vertex of B and a vertex of C. We assume without mentioning explicitly that $\Gamma_{\mathcal{B}}$ has at least one edge, so that each block of $\mathcal{B}$ is an independent set of Γ. Denote by $B(\alpha)$ the block of $\mathcal{B}$ containing α. Since $\mathcal{B}$ is G-invariant, $B(\alpha^g) = (B(\alpha))^g$ for any $\alpha \in V(\Gamma)$ and $g \in G$. For each $B \in \mathcal{B}$, define [14] $\mathcal{D}(B)$ to be the 1-design with point set B and blocks $\Gamma(C) \cap B$ (with possible repetitions) for all $C \in \Gamma_{\mathcal{B}}(B)$, where $\Gamma(C) := \cup_{\alpha \in C}\Gamma(\alpha)$ with $\Gamma(\alpha)$ the neighbourhood of α in Γ, and $\Gamma_{\mathcal{B}}(B)$ is the neighbourhood of B in $\Gamma_{\mathcal{B}}$. As in [14], for adjacent blocks B, C of $\mathcal{B}$, we use $\Gamma[B, C]$ to denote the induced bipartite subgraph of Γ with bipartition $\{\Gamma(C) \cap B, \Gamma(B) \cap C\}$. Since Γ is G-symmetric, up to isomorphism, $\mathcal{D}(B)$ and $\Gamma[B, C]$ are independent of the choice of $B \in \mathcal{B}$ and $C \in \Gamma_{\mathcal{B}}(B)$. Thus the block size $k := |\Gamma(C) \cap B|$ of $\mathcal{D}(B)$ and the number of times each block of $\mathcal{D}(B)$ is repeated are independent of the choice of B; denote this number by m and call it the *multiplicity* of $\mathcal{D}(B)$. We use $v := |B|$ to denote the block size of the partition $\mathcal{B}$.

Various possibilities for $\Gamma[B, C]$ can happen. In the "densest" case where $\Gamma[B, C] \cong K_{v,v}$ is a complete bipartite graph, Γ is uniquely determined by $\Gamma_{\mathcal{B}}$, namely, $\Gamma \cong \Gamma_{\mathcal{B}}[K_v]$ is the lexicographic product of $\Gamma_{\mathcal{B}}$ by the complete graph K_v. The "sparsest" case where $\Gamma[B, C] \cong K_2$ (that is, $k = 1$) can also happen; in this case Γ is called a *spread* of $\Gamma_{\mathcal{B}}$ in [16], where it was shown that spreads play a significant role in the study of edge-primitive graphs. See [14, Section 4], [32, Section 4] and [21, 31, 33] for discussions on spreads, and [13] for a recent classification of spreads of complete graphs. As the dual of spreads in some sense [21], the case when $v = k+1 \geqslant 3$ is also of considerable interest; in this case we call Γ an *almost multicover* of $\Gamma_{\mathcal{B}}$. This case was first studied in [21], where it was proved that G is transitive on the set of 2-arcs (that is, oriented paths of length 2) of $\Gamma_{\mathcal{B}}$ if and only if $\mathcal{D}(B)$ has no repeated blocks. It was proved in [31] that if in addition $\Gamma_{\mathcal{B}}$ is not a complete graph and $\Gamma[B, C]$ is a matching then $\Gamma_{\mathcal{B}}$ is a near polygonal graph. In the case when $\mathcal{D}(B)$ has no repeated blocks and $\Gamma_{\mathcal{B}}$ is a complete graph, all graphs Γ have been classified in [15, Theorem 1.1(b)(ii)(iii)(iv)] (and independently in [33, Theorem 3.19] by using a different approach).

In the case when Γ is an almost multicover of $\Gamma_{\mathcal{B}}$, a certain 1-design $\mathcal{D}(\Gamma, \mathcal{B})$ with point set $\mathcal{B}$ arises naturally (see Section 2.2), and on the other hand Γ can be reconstructed from this 1-design by using the flag graph construction introduced in [33] (see Theorem 2.2). If in addition $\Gamma_{\mathcal{B}}$ is a complete graph, then $\mathcal{D}(\Gamma, \mathcal{B})$ is a 2-$(mv+1, m+1, \lambda)$ design with $\lambda = 1$ or $m+1$ admitting G as a 2-point-transitive and block-transitive group of automorphisms (see Corollary 2.3). In the case when $\lambda = 1$, $\mathcal{D}(\Gamma, \mathcal{B})$ is a $(G, 2)$-point-transitive and G-block-transitive linear space, and the corresponding graphs Γ have been classified in [15, Theorem 1.1(b)(ii)(iii)(iv)] (see also [33, Theorem 3.19]), [17] and [6] together. These three papers deal with the cases when the linear space $\mathcal{D}(\Gamma, \mathcal{B})$ is trivial (that is, with block size two), nontrivial with G almost simple, and nontrivial with G affine, respectively. The purpose of the present paper is to classify almost multicovers of complete graphs in the case when $\lambda = m+1$ and thus complete the classification of all almost multicovers of complete graphs. The main result is as follows.

Theorem A. *Let Γ be a G-symmetric graph whose vertex set admits a nontrivial G-invariant partition $\mathcal{B}$ such that the quotient $\Gamma_{\mathcal{B}}$ is a complete graph and is almost multi-covered by Γ. In the case when $\mathcal{D}(\Gamma, \mathcal{B})$ is a 2-$(mv+1, m+1, m+1)$ design with $m > 1$, all graphs Γ are classified and will be described in Sections 3 and 4.*

A major tool for the proof of Theorem A is the flag graph construction introduced in [33]. By this construction, the problem of classifying the graphs in Theorem A is equivalent to the one of classifying all $(G, 2)$-point-transitive and G-block-transitive 2-$(mv+1, m+1, m+1)$ designs that admit a "feasible" G-orbit Ω on their sets of flags together with all self-paired G-orbitals on Ω "compatible" with Ω in some sense. (See Definition 2.1 for the definitions involved.) The next theorem gives the latter classification, which seems to be of interest for its own sake, from which Theorem A follows immediately.

Theorem B. *Let $\mathcal{D}$ be a $(G, 2)$-point-transitive and G-block-transitive* 2-$(|V|, m+1, m+1)$ *design with point set V, where $m > 1$ and $G \leqslant \mathrm{Sym}(V)$. Suppose that there exists a feasible G-orbit on the set of flags of $\mathcal{D}$. Then $(\mathcal{D}, G)$ is one of the following:*

(a) *$\mathcal{D}$ is a design with $|V| = q^2+1$ and $m = q = 2^{2e+1} > 2$ associated with the Suzuki group $\mathrm{Sz}(q)$, and G can be any subgroup of $\mathrm{Sym}(V)$ containing $\mathrm{Sz}(q)$ as a normal subgroup;*

(b) *$\mathcal{D}$ is a design with $|V| = q^3+1$ and $m = q^2$ associated with the Ree group $\mathrm{R}(q)$, $q = 3^{2e+1} \geqslant 3$, and G can be any subgroup of $\mathrm{Sym}(V)$ containing $\mathrm{R}(q)$ as a normal subgroup;*

(c) *$G \leqslant \mathrm{A\Gamma L}(1, p^d)$ with p prime and $d \geqslant 1$, and $(\mathcal{D}, G)$ is determined by an admissible quintuple (see Definition 4.4);*

(d) *$V = \mathbb{F}_p^2$, $G \leqslant \mathrm{AGL}(2, p)$, $p = 5, 7$ or 11, $G_{\mathbf{0}} \trianglerighteq \mathrm{SL}(2, 3)$ or $G_{\mathbf{0}} \trianglerighteq \mathrm{SL}(2, 5)$, where $G_{\mathbf{0}}$ is the stabiliser in G of the zero vector $\mathbf{0}$ of V, and each block of $\mathcal{D}$ is the union of at least two lines of the affine space $\mathrm{AG}(2, p)$;*

(e) *$V = \mathbb{F}_3^4$, $G \leqslant \mathrm{AGL}(4, 3)$, $G_{\mathbf{0}} \trianglerighteq E$, where E is an extraspecial group of order 32 with $G_{\mathbf{0}}/E \cong \mathrm{AGL}(1, 5)$, A_5 or S_5, and one of the blocks of $\mathcal{D}$ is the union of two 2-dimensional subspaces V_1 and V_2 such that $V_1 \oplus V_2 = V$.*

	G	$\mathcal{D}$	$\Gamma(\mathcal{D},\Omega,\Psi)$	Details
(a)	$\mathrm{soc}(G)=\mathrm{Sz}(q)$ $q=2^{2e+1}>2$	2-$(q^2+1,q+1,q+1)$	C, ord $=q(q^2+1)$ and val $=(q^2-q)i/\mathrm{gcd}(f,i)$	L3.7
(b)	$\mathrm{soc}(G)=\mathrm{R}(q)$ $q=3^{2e+1}\geqslant 3$	2-(q^3+1,q^2+1,q^2+1)	C, ord $=q(q^3+1)$ and val $=(q^3-q^2)i/\mathrm{gcd}(f,i)$	L3.12
(c)	$G\leqslant \mathrm{A\Gamma L}(1,q)$ $q=p^d$	2-$(q,\lvert L\rvert,\lvert L\rvert)$; $\mathcal{D}$ has a block $L=P\cup\{0\}$ with P a subgroup of $\mathbb{F}_q^{\times}$ and $\lvert\mathbb{F}_q^{\times}:P\rvert$ prime	C, ord $=q(q-1)/\lvert P\rvert$ and val $=q-\lvert L\rvert$ D, ord $=q(q-1)/\lvert P\rvert$ and val $=q-\lvert L\rvert$, $(q-1)/\lvert P\rvert$ components	L4.5 L4.7
(d)	$G\leqslant \mathrm{AGL}(2,p)$ $G_{\mathbf{0}}\trianglerighteq \mathrm{SL}(2,3)$ or $G_{\mathbf{0}}\trianglerighteq \mathrm{SL}(2,5)$ $p=5,7,11$ $V=\mathbb{F}_p^2$	2-$(p^2,m+1,m+1)$, $m=8$ when $p=5$; $m=12$ when $p=7$; $m=40$ or 20 when $p=11$	ord $=\frac{p^2(p^2-1)}{m}$ and val $=p^2-m-1$	Cases 1–3 in §4.9
(e)	$G\leqslant \mathrm{AGL}(4,3)$ $G_{\mathbf{0}}\trianglerighteq E$, $G_{\mathbf{0}}/E\cong$ $\mathrm{AGL}(1,5)$	2-$(81,17,17)$	ord $=405$ and val $=64$	Case 2 in §4.10
	$G\leqslant \mathrm{AGL}(4,3)$ $G_{\mathbf{0}}\trianglerighteq E$, $G_{\mathbf{0}}/E\cong$ A_5 or S_5	As above	ord $=405$, val $=64$ ord $=405$, val $=192$	Case 2 in §4.10

Table 1. Theorem B: Acronym: L = Lemma, C = Connected, D = Disconnected, ord = Order, val = Valency

Moreover, in each case the unique feasible G-orbit Ω on the flag set of $\mathcal{D}$ and all self-paired G-orbitals Ψ on Ω compatible with Ω are determined, the adjacency relations of the corresponding G-flag graphs $\Gamma(\mathcal{D},\Omega,\Psi)$ (see Definition 2.1) are given, and the connectedness of those G-flag graphs in (a), (b) and (c) is determined.

Information about $\mathcal{D}$, G and $\Gamma(\mathcal{D},\Omega,\Psi)$ in Theorem B is summarized in Table 1.

Several interesting families of graphs (that is, graphs in Theorem A up to isomorphism) arise from our classification. In particular, we obtain several infinite families of connected G-flag graphs (see Definition 2.1) with $\mathrm{soc}(G)=\mathrm{Sz}(q)$, $\mathrm{soc}(G)=\mathrm{R}(q)$, and G a certain 2-transitive subgroup of $\mathrm{A\Gamma L}(1,p^d)$, respectively. All these graphs as well as infinite families of disconnected graphs from (c) and the sporadic graphs from (d)-(e) in Theorem B will be given in the course of the proof of Theorem B; see Lemma 3.7, Lemma 3.12, Lemma 4.7, Cases 1-3 in Section 4.9 and Case 2 in Section 4.10, respectively.

Theorem A follows from Theorem B and Corollary 2.3. So we will prove Theorem B only. In Sections 2.1 and 2.2 we will set up notation and introduce the flag graph construction, respectively. Section 2.3 gives a few basic results on the flag graph construction that will be used later, and Section 2.4 outlines our method for the proof of Theorem B.

Since the group G in Theorem B is 2-transitive, it is almost simple or affine, and our proof in these two cases will be given in Sections 3 and 4 respectively, by using the classification of finite 2-transitive groups.

2 Preliminaries

2.1 Notation and definitions

The reader is referred to [10], [1] and [27] for notation and terminology on permutation groups, block designs and finite geometries, respectively. Unless stated otherwise, all designs in the paper are assumed to have no repeated blocks.

Let G be a group acting on a set Ω. That is, for any $\alpha \in \Omega$ and $g \in G$ there corresponds a point in Ω denoted by α^g, such that $\alpha^{1_G} = \alpha$ and $(\alpha^g)^h = \alpha^{gh}$ for any $\alpha \in \Omega$ and $g, h \in G$, where 1_G is the identity element of G. Let P_i be a point or subset of Ω for $i = 1, 2, \ldots, n$, where $n \geqslant 1$. Define $(P_1, P_2, \ldots, P_n)^g := (P_1^g, P_2^g, \ldots, P_n^g)$ for $g \in G$, where $P_i^g := \{\alpha^g : \alpha \in P_i\}$ if P_i is a subset of Ω. Let $P_i^G := \{P_i^g : g \in G\}$. In particular, α^G is the G-orbit on Ω containing α. Define $G_{P_1,P_2,\ldots,P_n} := \{g \in G : P_i^g = P_i,\ i = 1, \ldots, n\} \leqslant G$. In particular, if α is a point and P a subset of Ω, then G_α is the stabiliser of α in G, G_P is the setwise stabiliser of P in G, and $G_{\alpha,P}$ is the setwise stabiliser of P in G_α. The natural action of $\mathrm{Sym}(\Omega)$ on Ω is defined as $\alpha^g := g(\alpha)$ for $\alpha \in \Omega$ and $g \in \mathrm{Sym}(\Omega)$.

Let G and H be groups acting on Ω and Δ, respectively. These two actions are said to be *permutation isomorphic* if there exist a bijection $\rho : \Omega \to \Delta$ and an isomorphism $\eta : G \to H$ such that $\rho(\alpha^g) = (\rho(\alpha))^{\eta(g)}$ for $\alpha \in \Omega$ and $g \in G$. If in addition $G = H$ and η is the identity automorphism of G, then these two actions are said to be *permutation equivalent*. It is known that if $\varphi : G \to \mathrm{Sym}(\Omega)$ and $\psi : H \to \mathrm{Sym}(\Omega)$ are monomorphisms, then G and H are permutation isomorphic if and only if $\varphi(G)$ and $\psi(H)$ are conjugate in $\mathrm{Sym}(\Omega)$. Let Γ and Σ be G-symmetric graphs. If there exists a graph isomorphism $\rho : V(\Gamma) \to V(\Sigma)$ such that the actions of G on $V(\Gamma)$ and $V(\Sigma)$ are permutation equivalent with respect to ρ, then Γ and Σ are said to be *G-isomorphic* with respect to the *G-isomorphism* ρ, denoted by $\Gamma \cong_G \Sigma$.

2.2 Flag graphs

Let $\mathcal{D}$ be a 1-design with point set V. We identify each block L of $\mathcal{D}$ with the subset of V consisting of the points incident with L. Let Ω be a subset of (point-block) flags of $\mathcal{D}$, and let $\Psi \subseteq \Omega \times \Omega$. If Ψ is *self-paired*, that is, $((\sigma, L), (\tau, N)) \in \Psi$ implies $((\tau, N), (\sigma, L)) \in \Psi$, then we define [33] the *flag graph* of $\mathcal{D}$ with respect to (Ω, Ψ), denoted by $\Gamma(\mathcal{D}, \Omega, \Psi)$, to be the graph with vertex set Ω in which two "vertices" $(\sigma, L), (\tau, N) \in \Omega$ are adjacent if and only if $((\sigma, L), (\tau, N)) \in \Psi$. Given a point σ of $\mathcal{D}$, denote by $\Omega(\sigma)$ the set of flags of Ω with point entry σ. If Ω is a G-orbit on the flags of $\mathcal{D}$, for some group G of automorphisms of $\mathcal{D}$, then $\Omega(\sigma)$ is a G_σ-orbit on the flags of $\mathcal{D}$ with point entry σ. In this case $\Gamma(\mathcal{D}, \Omega, \Psi)$ is G-vertex-transitive and its vertex set Ω admits a natural G-invariant partition, namely,

$$\mathcal{B}(\Omega) := \{\Omega(\sigma) : \sigma \in V\}.$$

If in addition Ψ is a G-orbit on $\Omega \times \Omega$ (under the induced action), then $\Gamma(\mathcal{D}, \Omega, \Psi)$ is G-symmetric. Obviously, for a flag (σ, L) of $\mathcal{D}$, $G_{\sigma,L}$ is the stabiliser of (σ, L) in G.

Definition 2.1. ([33]) Let $\mathcal{D}$ be a 1-design that admits a point- and block-transitive group G of automorphisms. Let σ be a point of $\mathcal{D}$. A G-orbit Ω on the set of flags of $\mathcal{D}$ is said to be *feasible* if the following conditions are satisfied:

(a) $|\Omega(\sigma)| \geqslant 3$;

(b) $L \cap N = \{\sigma\}$, for distinct $(\sigma, L), (\sigma, N) \in \Omega(\sigma)$;

(c) $G_{\sigma,L}$ is transitive on $L \setminus \{\sigma\}$, for $(\sigma, L) \in \Omega$; and

(d) $G_{\sigma,\tau}$ is transitive on $\Omega(\sigma) \setminus \{(\sigma, L)\}$, for $(\sigma, L) \in \Omega$ and $\tau \in L \setminus \{\sigma\}$.

Denote

$$\begin{aligned} \mathrm{F}(\mathcal{D}, \Omega) := \{((\sigma, L), (\tau, N)) \in \Omega \times \Omega : \sigma \notin N, \tau \notin L, \\ \text{and } \sigma, \tau \in L' \cap N' \text{ for some } (\sigma, L'), (\tau, N') \in \Omega\}. \end{aligned} \tag{1}$$

If Ω is a feasible G-orbit on the set of flags of $\mathcal{D}$ and Ψ a self-paired G-orbit on $\mathrm{F}(\mathcal{D}, \Omega)$, then Ψ is said to be *compatible* with Ω and $\Gamma(\mathcal{D}, \Omega, \Psi)$ is called a *G-flag graph* of $\mathcal{D}$.

Since G is transitive on the points of $\mathcal{D}$, the validity of (a)-(d) above does not depend on the choice of σ. Note that $\mathrm{F}(\mathcal{D}, \Omega)$ is G-invariant, and is non-empty if $\mathcal{D}$ is $(G, 2)$-point-transitive.

Using the notation in Section 1, we will assume that $(\Gamma, G, \mathcal{B})$ is a triple such that Γ is an almost multicover of $\Gamma_{\mathcal{B}}$ with $v = k + 1 \geqslant 3$. Then, for each $\alpha \in V(\Gamma)$, $B(\alpha) \setminus \{\alpha\}$ appears m times as a block of $\mathcal{D}(B(\alpha))$, where m is the multiplicity of $\mathcal{D}(B(\alpha))$ as defined in Section 1. Set

$$\mathcal{B}(\alpha) := \{C \in \mathcal{B} : \Gamma(C) \cap B(\alpha) = B(\alpha) \setminus \{\alpha\}\}$$

so that $|\mathcal{B}(\alpha)| = m$. Define Γ' to be the graph with the same vertices as Γ in which α and β are adjacent if and only if $B(\alpha) \in \mathcal{B}(\beta)$ and $B(\beta) \in \mathcal{B}(\alpha)$. It was proved in [21, Proposition 3] that Γ' is a G-symmetric graph. One can check that for each $B \in \mathcal{B}$, $\mathbf{B}(B) := \{\mathcal{B}(\alpha) : \alpha \in B\}$ is a G_B-invariant partition of $\Gamma_{\mathcal{B}}(B)$, and hence G_B induces an action on $\mathbf{B}(B)$. Set

$$\mathcal{L}(\alpha) := \{B(\alpha)\} \cup \mathcal{B}(\alpha)$$

for each $\alpha \in V(\Gamma)$. Denote by $\mathbf{L}$ the set of all $\mathcal{L}(\alpha)$, $\alpha \in V(\Gamma)$, with repeated ones identified. Then the action of G on $\mathcal{B}$ induces a natural action on $\mathbf{L}$ defined by $(\mathcal{L}(\alpha))^g := \mathcal{L}(\alpha^g)$ for $\alpha \in V(\Gamma)$ and $g \in G$. The subset $\mathbf{L}(B) := \{\mathcal{L}(\alpha) : \alpha \in B\}$ of $\mathbf{L}$ is G_B-invariant under this action, and thus G_B induces an action on $\mathbf{L}(B)$. It can be verified that the action of G_B on B is permutation equivalent to the actions of G_B on $\mathbf{B}(B)$ and $\mathbf{L}(B)$ with respect to the bijections defined by $\alpha \mapsto \mathcal{B}(\alpha)$, $\alpha \mapsto \mathcal{L}(\alpha)$, $\alpha \in B$, respectively. Thus, $G_{B,\mathcal{B}(\alpha)} = G_{B,\mathcal{L}(\alpha)} = G_\alpha$, where $G_{B,\mathcal{B}(\alpha)}, G_{B,\mathcal{L}(\alpha)}$ are the setwise stabilisers of $\mathcal{B}(\alpha), \mathcal{L}(\alpha)$ in G_B, respectively. Define [33]

$$\mathcal{D}(\Gamma, \mathcal{B}) := (\mathcal{B}, \mathbf{L})$$

to be the incidence structure with point set $\mathcal{B}$ and block set $\mathbf{L}$ in which a "point" B is incident with a "block" $\mathcal{L}(\alpha)$ if and only if $B \in \mathcal{L}(\alpha)$. The flags of $\mathcal{D}(\Gamma, \mathcal{B})$ of the form $(B(\alpha), \mathcal{L}(\alpha))$ are pairwise distinct, and we define

$$\Omega(\Gamma, \mathcal{B}) := \{(B(\alpha), \mathcal{L}(\alpha)) : \alpha \in V(\Gamma)\}$$

to be the set of all such flags. Then by [33, Lemma 2.1(c), Lemma 2.2], $\Omega(\Gamma, \mathcal{B})$ is a feasible G-orbit on the set of flags of $\mathcal{D}(\Gamma, \mathcal{B})$.

The following is a slight extension of [33, Theorem 1.1], the only difference being the specification of the parameters of $\mathcal{D}$ that can be easily worked out by using [33, Lemma 2.1(d)] and a similar argument as in the proof of [32, Theorem 4.3].

Theorem 2.2. *Suppose that Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ such that $v = k+1 \geqslant 3$. Then $\Gamma \cong_G \Gamma(\mathcal{D}, \Omega, \Psi)$ for a certain G-point-transitive and G-block-transitive 1-design $\mathcal{D}$ with point set $\mathcal{B}$ and block size $m+1$, a certain feasible G-orbit Ω on the flags of $\mathcal{D}$, and a certain self-paired G-orbit Ψ on $\mathrm{F}(\mathcal{D}, \Omega)$, where m is the multiplicity of $\mathcal{D}(B)$. Moreover, $\mathcal{D}$ is either a 1-$(|\mathcal{B}|, m+1, v)$ design or a 1-$(|\mathcal{B}|, m+1, (m+1)v)$ design.*

Conversely, for any G-point-transitive and G-block-transitive 1-design $\mathcal{D}$ with block size $m+1$, any feasible G-orbit Ω on the flags of $\mathcal{D}$, and any self-paired G-orbit Ψ on $\mathrm{F}(\mathcal{D}, \Omega)$, the graph $\Gamma = \Gamma(\mathcal{D}, \Omega, \Psi)$, group G and partition $\mathcal{B} = \mathcal{B}(\Omega)$ satisfy all the conditions above. Moreover, the multiplicity of the 1-design $\mathcal{D}(B)$ (where $B \in \mathcal{B}$) is equal to m.

As noted in [33], in both parts of this theorem, G is faithful on the vertices of Γ if and only if it is faithful on the points of $\mathcal{D}$. In the first part of the theorem, we have $\mathcal{D} = \mathcal{D}(\Gamma, \mathcal{B})$, $\Omega = \Omega(\Gamma, \mathcal{B})$ and $\Psi = \{((B(\alpha), \mathcal{L}(\alpha)), (B(\beta), \mathcal{L}(\beta))) : (\alpha, \beta) \in \mathrm{Arc}(\Gamma)\}$, where $\mathrm{Arc}(\Gamma)$ is the set of arcs of Γ.

In the case when in addition $\Gamma_{\mathcal{B}}$ is a complete graph, we have $\Gamma_{\mathcal{B}} \cong K_{mv+1}$ as $\mathrm{val}(\Gamma_{\mathcal{B}}) = mv$ ([21, Theorem 5(a)]). Since $\Gamma_{\mathcal{B}}$ is G-symmetric, this occurs precisely when G is 2-transitive on $\mathcal{B}$. Hence in this case $\mathcal{D}(\Gamma, \mathcal{B})$ is a $(G, 2)$-point-transitive and G-block-transitive 2-$(mv+1, m+1, \lambda)$ design for some integer $\lambda \geqslant 1$. Conversely, if $\mathcal{D}$ is a $(G, 2)$-point-transitive and G-block-transitive 2-$(mv+1, m+1, \lambda)$ design, then for any G-flag graph $\Gamma = \Gamma(\mathcal{D}, \Omega, \Psi)$ of $\mathcal{D}$, we have $\Gamma_{\mathcal{B}(\Omega)} \cong K_{mv+1}$. Thus Theorem 2.2 has the following consequence, which is a slight extension of [33, Corollary 2.6].

Corollary 2.3. *Let $v \geqslant 3$ and $m \geqslant 1$ be integers, and let G be a group. Then the following statements are equivalent.*

(a) *Γ is a G-symmetric graph admitting a nontrivial G-invariant partition $\mathcal{B}$ of block size v such that $\mathcal{D}(B)$ has block size $v-1$ and $\Gamma_{\mathcal{B}} \cong K_{mv+1}$.*

(b) *$\Gamma \cong_G \Gamma(\mathcal{D}, \Omega, \Psi)$, for a $(G, 2)$-point-transitive and G-block-transitive 2-$(mv+1, m+1, \lambda)$ design $\mathcal{D}$, a feasible G-orbit Ω on the flags of $\mathcal{D}$, and a self-paired G-orbit Ψ on $\mathrm{F}(\mathcal{D}, \Omega)$.*

Moreover, either $\lambda = 1$ or $\lambda = m + 1$, and the set of points of $\mathcal{D}$ other than a fixed point σ admits a G_σ-invariant partition of block size m, namely, $\{L \setminus \{\sigma\} : (\sigma, L) \in \Omega\}$. In particular, $\mathcal{D}$ is not $(G, 3)$-point-transitive when $m \geqslant 2$.

As in Theorem 2.2, the integer m above is equal to the multiplicity of $\mathcal{D}(B)$, and G is faithful on $V(\Gamma)$ if and only if it is faithful on the points of $\mathcal{D}$. The statements in the last paragraph of Corollary 2.3 follow from Theorem 2.2 and basic relations [1, 2.10, Chapter I] among parameters of a 2-design (and also from [32, Corollary 4.4] since $(\Gamma', G, \mathcal{D})$ satisfies all conditions of [32, Corollary 4.4]). As mentioned earlier, the G-symmetric graphs Γ in Corollary 2.3 have been classified when $\lambda = 1$.

In the rest of this paper, we will classify all graphs in part (a) of Corollary 2.3 by classifying all $\Gamma(\mathcal{D}, \Omega, \Psi)$ with $\lambda = m + 1 > 2$ in part (b), thus proving Theorems A and B.

2.3 Orbits and feasible orbits on the set of flags

In this section we assume that $\mathcal{D}$ is a $(G, 2)$-point-transitive and G-block-transitive 2-$(|V|, m + 1, \lambda)$ design with point set V.

Let $\sigma, \tau \in V$ be distinct points. Denote by $L_1, \ldots, L_\lambda$ the λ blocks of $\mathcal{D}$ containing σ and τ. Since $\sigma, \tau \in L_i$ for each i, any G-orbit on the flag set of $\mathcal{D}$ satisfying (b) in Definition 2.1 contains at most one flag (σ, L_i) for some $i = 1, 2, \ldots, \lambda$. Denote

$$\Omega_i := (\sigma, L_i)^G, \quad i = 1, 2, \ldots, \lambda.$$

Proposition 2.4. *$\Omega_1, \ldots, \Omega_\lambda$ are all possible G-orbits on the flag set of $\mathcal{D}$ (possibly with $\Omega_i = \Omega_j$ for distinct i and j).*

Proof. In fact, let (ξ, N) be any flag of $\mathcal{D}$ and $\eta \in N \setminus \{\xi\}$. Since G is 2-transitive on V, there exists $g \in G$ such that $(\xi, \eta)^g = (\sigma, \tau)$. Since $(\xi, N)^g = (\sigma, N^g)$ and $\sigma, \tau = \eta^g \in N^g$, we have $N^g = L_i$ for some i and hence $(\xi, N)^G = (\sigma, L_i)^G$. □

Proposition 2.5. *If G_L is transitive on L for some block L of $\mathcal{D}$, then G is transitive on the flag set of $\mathcal{D}$ (that is, $\Omega_1 = \cdots = \Omega_\lambda$ is the flag set of $\mathcal{D}$). If in addition the flag set of $\mathcal{D}$ satisfies (b) in Definition 2.1, then $\lambda = 1$.*

Proof. Suppose that G_L is transitive on L for some block L of $\mathcal{D}$. Let N be any block of $\mathcal{D}$. Then G_N is transitive on N and there exists $g \in G$ such that $(\sigma^g, N) = (\sigma, L_1)^g \in \Omega_1$ by the G-block-transitivity of $\mathcal{D}$. Hence $(\eta, N) \in \Omega_1$ for any $\eta \in N$, which implies that G is transitive on the set of flags of $\mathcal{D}$. Consequently, if in addition the flag set Ω_1 of $\mathcal{D}$ satisfies (b) in Definition 2.1, then we must have $\lambda = 1$. □

Proposition 2.6. *If there exists a G-orbit $\Omega = (\xi, L)^G$ on the flag set of $\mathcal{D}$ satisfying (b) and (c) in Definition 2.1 and G_L is not transitive on L, then $\lambda = m + 1$.*

Proof. By (b) in Definition 2.1 we have $|V| = mv+1$ for some integer v. Let η be a fixed point of V. For each $\pi \in V \setminus \{\eta\}$, by (b) in Definition 2.1 there is only one flag in $\Omega(\pi)$ whose block entry contains η.

On the other hand, if there are two distinct flags (τ_1, M), (τ_2, M) in Ω for some $M \in L^G$, then there is some $g \in G$ such that $(\tau_1, M) = (\tau_2, M)^g$. Thus $g \in G_M$ and $\tau_1 = \tau_2^g$. Since Ω satisfies (c) in Definition 2.1, G_M is transitive on M, which contradicts our assumption. Hence the block entries of the flags in $\Omega(\eta)$ and the block entries containing η of the flags in $\Omega(\pi)$ with $\pi \in V \setminus \{\eta\}$ are pairwise distinct, and there are $|\Omega(\eta)|+(|V|-1) = v+mv = (m+1)v$ blocks of $\mathcal{D}$ containing η. By the relations between parameters of the 2-design $\mathcal{D}$, we get $\lambda = m+1$. □

Proposition 2.7. *If $m > 1$, then there is at most one G-orbit on the flag set of $\mathcal{D}$ that satisfies* (*b*) *and* (*c*) *in Definition 2.1.*

Proof. Suppose $\Omega_i \neq \Omega_j$ and each of them satisfies (b) and (c) in Definition 2.1. Since $\mathcal{D}$ is G-block-transitive, there exists a point ξ of $\mathcal{D}$ such that $(\xi, L_j) \in \Omega_i$. The assumption $\Omega_i \neq \Omega_j$ implies $\sigma \neq \xi$, and by (c) in Definition 2.1 we obtain $G_{L_j} = G_{\xi,L_j} \leqslant G_\xi$ (for otherwise G_{L_j} is transitive on L_j and thus $\Omega_i = \Omega_j$ by Proposition 2.5). Since $\xi \in L_j \setminus \{\sigma\}$, $G_{\sigma,L_j} \leqslant G_{L_j} \leqslant G_\xi$ and $|L_j| = m+1 \geqslant 3$, G_{σ,L_j} cannot be transitive on $L_j \setminus \{\sigma\}$, which contradicts the assumption that Ω_j satisfies (c) in Definition 2.1. □

The results above imply the following:

Lemma 2.8. *Let $\mathcal{D}$ be a $(G,2)$-point-transitive and G-block-transitive* 2-$(|V|, m+1, \lambda)$ *design with point set V and $m > 1$. Then there is at most one feasible G-orbit on the flag set of $\mathcal{D}$. Moreover, if such an orbit exists, say, $\Omega = (\xi, L)^G$, then either (a) G_L is transitive on L (or equivalently $G_L \nleqslant G_\xi$), $\lambda = 1$, and Ω is the set of all flags of $\mathcal{D}$; or (b) G_L is not transitive on L (or equivalently $G_L \leqslant G_\xi$) and $\lambda = m+1$.*

The following result enables us to check whether a G-orbit on the flag set of $\mathcal{D}$ is feasible in another way.

Lemma 2.9. *Suppose that $\mathcal{D}$ is a $(G,2)$-point-transitive and G-block-transitive* 2-$(|V|, m+1, \lambda)$ *design with point set V and $m > 1$. Let $\Omega = (\sigma, L)^G$ be a G-orbit on the flag set of $\mathcal{D}$. Then Ω is feasible if and only if the following hold:*

(a) $|\Omega(\sigma)| \geqslant 3$*;*

(b*) $L \setminus \{\sigma\}$ *is an imprimitive block for the action of G_σ on $V \setminus \{\sigma\}$; and*

(d*) $G_{\sigma,L}$ *is transitive on $V \setminus L$.*

Proof. Since G is 2-transitive on V, G_σ is transitive on $V \setminus \{\sigma\}$. Suppose Ω satisfies (b) in Definition 2.1. If $(L \setminus \{\sigma\})^g \cap (L \setminus \{\sigma\}) \neq \emptyset$ for some $g \in G_\sigma$, then $(L^g \cap L) \setminus \{\sigma\} \neq \emptyset$ and hence $L^g = L$ by (b). Therefore, (b) in Definition 2.1 implies (b*). The converse can be easily seen, and so (b) in Definition 2.1 is equivalent to (b*). We can see that (b*) implies (c) in Definition 2.1 as $G_{\sigma,L} = (G_\sigma)_{L\setminus\{\sigma\}}$.

Now suppose that Ω satisfies (a) and (b) in Definition 2.1 so that it also satisfies (b*) (we have $|V| = mv + 1$ for some integer v). We aim to prove that (d) in Definition 2.1 is equivalent to (d*). Define $\mathcal{P} := \{N\setminus\{\sigma\} : (\sigma, N) \in \Omega\} = \{L^g\setminus\{\sigma\} : g \in G_\sigma\}$ and $P := L\setminus\{\sigma\}$ so that $G_{\sigma,L} = G_{\sigma,P}$. By (b*), $G_{\sigma,\eta} \leqslant G_{\sigma,L}$ for $\eta \in P$, $|G_{\sigma,P}| = |P||G_{\sigma,\eta}| = m|G_{\sigma,\eta}|$ and $|L^{G_\sigma}| = |\mathcal{P}| = v$. We then have: (d) in Definition 2.1 holds $\Leftrightarrow$ $G_{\sigma,\eta}$ is transitive on $\mathcal{P}\setminus\{P\}$ $\Leftrightarrow$ for any $Q \in \mathcal{P} \setminus \{P\}$ (so $\eta \notin Q$), $v - 1 = |Q^{G_{\sigma,\eta}}| = |G_{\sigma,\eta}|/|G_{\sigma,\eta,Q}| = |G_{\sigma,P}|/(m|G_{\sigma,Q,\eta}|)$ $\Leftrightarrow$ $|G_{\sigma,P}| = m(v-1)|G_{\sigma,Q,\eta}| = m(v-1)|G_{\sigma,Q}|/|\eta^{G_{\sigma,Q}}| = m(v-1)|G_{\sigma,P}|/|\eta^{G_{\sigma,Q}}|$ (as the transitivity of G_σ on $\mathcal{P}$ implies $|G_{\sigma,P}| = |G_{\sigma,Q}|$) $\Leftrightarrow$ $|\eta^{G_{\sigma,Q}}| = m(v-1) = |(V \setminus \{\sigma\}) \setminus Q|$ $\Leftrightarrow$ $G_{\sigma,Q}$ is transitive on $V \setminus (\{\sigma\} \cup Q)$ $\Leftrightarrow$ $G_{\sigma,L}$ is transitive on $V \setminus L$ (as G_σ is transitive on $\mathcal{P}$) $\Leftrightarrow$ (d*) holds. □

Lemma 2.10. *Suppose that $\mathcal{D}$ is a $(G, 2)$-point-transitive and G-block-transitive 2-$(|V|, m + 1, \lambda)$ design with point set V and $m > 1$ such that there is a feasible G-orbit $\Omega = (\sigma, L)^G$ on the flags of $\mathcal{D}$. Let $P := L \setminus \{\sigma\}$. Then the following hold:*

(a) *for any subgroup H of G_σ transitive on $V \setminus \{\sigma\}$, P is an imprimitive block of H on $V \setminus \{\sigma\}$ and P is the union of some H_η-orbits (including the H_η-orbit $\{\eta\}$ of length 1), where $\eta \in P$;*

(b) *G_σ is 2-transitive on $\mathcal{P} := \{N \setminus \{\sigma\} : (\sigma, N) \in \Omega\}$ and $G_{\sigma,L} = G_{\sigma,P}$ is a maximal subgroup of G_σ; moreover, $v := |G_\sigma : G_{\sigma,L}| = |\mathcal{P}|$, $v - 1$ divides $|G_\sigma|/(|V| - 1)$, and $G_{\sigma,L}$ is self-normalizing in G_σ.*

Proof. (a) The first statement follows from Lemma 2.9 (b*) and the assumption that $H \leqslant G_\sigma$, and the second statement follows from the first one and the fact that H_η stabilises P as $\eta \in P$.

(b) Since G_σ is transitive on $\mathcal{P}$ and $G_{\sigma,L}$ ($\geqslant G_{\sigma,\eta}$ for $\eta \in P$) is transitive on $\mathcal{P} \setminus \{P\}$, G_σ acts 2-transitively on $\mathcal{P}$. In addition, since $G_{\sigma,L}$ contains the kernel K of the action of G_σ on $\mathcal{P}$, the point stabiliser $G_{\sigma,L}/K$ is maximal in the primitive permutation group G_σ/K on $\mathcal{P}$, and thus $G_{\sigma,L}$ is maximal in G_σ. If $G_{\sigma,L}$ is not self-normalizing in G_σ, then $G_{\sigma,L}$ is a normal subgroup of G_σ, which implies $G_{\sigma,L} \leqslant K$ and so $G_{\sigma,L}$ is not transitive on $\mathcal{P} \setminus \{P\}$ as $|\mathcal{P} \setminus \{P\}| \geqslant 2$, a contradiction. Hence $G_{\sigma,P}$ is self-normalizing in G_σ and $v = |\{(G_{\sigma,P})^g : g \in G_\sigma\}| = |\{G_{\sigma,Q} : Q \in \mathcal{P}\}|$. Let $Q \in \mathcal{P} \setminus \{P\}$. By Lemma 2.9 (d*), $G_{\sigma,Q} \neq G_{\sigma,P}$ and thus $v = |\mathcal{P}|$. Since $G_{\sigma,\eta}$ is transitive on $\mathcal{P} \setminus \{P\}$, where $\eta \in P$, $v - 1 = |\mathcal{P} \setminus \{P\}|$ is a divisor of $|G_{\sigma,\eta}| = |G_\sigma|/(|V| - 1)$. □

2.4 Overview of the proof of Theorem B

We will use the set-up below in the next two sections. Without loss of generality we may assume that the group G in Theorem B is faithful on V. Thus in the rest of this paper we assume that $G \leqslant \mathrm{Sym}(V)$ is 2-transitive on V with degree $u := |V|$. Then the socle of G, $\mathrm{soc}(G)$, is either a nonabelian simple group (almost simple case) or an abelian group (affine case). We will deal with these two cases in Sections 3 and 4, respectively.

Let σ be a point in V. Using Lemma 2.10, we will search for an imprimitive block of G_σ on $V \setminus \{\sigma\}$ by using the following approaches.

(i) Suppose H is a subgroup of G_σ that is transitive on $V \setminus \{\sigma\}$. For each imprimitive block P of H on $V \setminus \{\sigma\}$ satisfying $(|V|-1)/|P| \geqslant 3$ and $|P| \geqslant 2$, we need to check that P is also an imprimitive block of G_σ on $V \setminus \{\sigma\}$. By Lemma 2.10(a), P is the union of some H_τ-orbits on $V \setminus \{\sigma\}$, where $\tau \in P$.

(ii) Suppose H is a subgroup of G_σ. If there is a point $\tau \in V \setminus \{\sigma\}$ such that $H_\tau = G_{\sigma,\tau}$, then $P := \tau^H$ is an imprimitive block of G_σ on $V \setminus \{\sigma\}$ by [10, Theorem 1.5A].

For each imprimitive block P of G_σ on $V \setminus \{\sigma\}$ from (i) or (ii), define

$$\mathcal{D} := (V, L^G), \text{ where } L := P \cup \{\sigma\},$$

to be the incidence structure with point set V and block set L^G. Then σ $(\in V)$ and N $(\in L^G)$ are incident if and only if $\sigma \in N$. By [1, Proposition III.4.6], $\mathcal{D}$ is a 2-$(|V|, |L|, \lambda)$ design admitting G as an automorphism group. By Proposition 2.7, the only possible feasible G-orbit on the flag set of $\mathcal{D}$ is $\Omega := (\sigma, L)^G$. We will test whether Ω is feasible with the help of Lemma 2.9. If Ω is indeed feasible, then we will move on to determine all self-paired G-orbits on $\mathrm{F}(\mathcal{D}, \Omega)$ (see (1)). Suppose Ψ is a self-paired G-orbit on $\mathrm{F}(\mathcal{D}, \Omega)$. Then by the definition of $\Gamma(\mathcal{D}, \Omega, \Psi)$, for each $\eta \in V \setminus L$, (σ, L) has a neighbour in $\Omega(\eta)$, and (σ, L) has no neighbour in $\Omega(\xi)$ when $\xi \in L$. Hence the valency of $\Gamma(\mathcal{D}, \Omega, \Psi)$ is $(|V| - |L|)n$, where n is the valency of $\Gamma[\Omega(\delta), \Omega(\pi)]$ for distinct $\delta, \pi \in V$.

In order to obtain the connectedness of $\Gamma(\mathcal{D}, \Omega, \Psi)$, we need the following construction. Given a group G, a subgroup T of G, and an element $g \in G$ with $g \notin N_G(T)$ and $g^2 \in T \cap T^g$, define the coset graph $\mathrm{Cos}(G, T, TgT)$ to be the graph with vertex set $[G : T] := \{Tx : x \in G\}$ and edge set $\{\{Tx, Ty\} : xy^{-1} \in TgT\}$. It is well known (see e.g. [24]) that $\mathrm{Cos}(G, T, TgT)$ is a G-symmetric graph with G acting on $[G : T]$ by right multiplication, and $\mathrm{Cos}(G, T, TgT)$ is connected if and only if $\langle T, g\rangle = G$. Conversely, any G-symmetric graph Γ is G-isomorphic to $\mathrm{Cos}(G, T, TgT)$ (see e.g. [24]), where g is an element of G interchanging two adjacent vertices α and β of Γ and $T := G_\alpha$, and the required G-isomorphism is given by $V(\Gamma) \to [G : T], \gamma \mapsto Tx$, with $x \in G$ satisfying $\alpha^x = \gamma$. Based on this one can prove the following result.

Lemma 2.11. *Let $((\sigma, L), (\tau, N)) \in \Psi$ and $T := G_{\sigma,L}$. Let $g \in G$ interchange (σ, L) and (τ, N), and set $H := \langle T, g\rangle$. Then $\rho : \Omega \to [G : T], \gamma \mapsto Tx$, with $x \in G$ satisfying $(\sigma, L)^x = \gamma$, defines a G-isomorphism from $\Gamma(\mathcal{D}, \Omega, \Psi)$ to $\mathrm{Cos}(G, T, TgT)$, under which the preimage of the subgraph $\mathrm{Cos}(H, T, TgT)$ of $\mathrm{Cos}(G, T, TgT)$ is the connected component of $\Gamma(\mathcal{D}, \Omega, \Psi)$ containing the vertex (σ, L).*

By Lemma 2.8, the parameter λ of $\mathcal{D}$ is equal to 1 or $|P| + 1$. We will repeatedly use the following result to exclude those $\mathcal{D}$ with $\lambda = 1$.

Lemma 2.12. *([23, Theorem B]) Let G be a 2-transitive permutation group on a finite set V. Suppose that, for $\sigma \in V$, G_σ has a system $\Sigma := \{P_1, \dots, P_v\}$ of blocks of imprimitivity in $V \setminus \{\sigma\}$, where $|\Sigma| = v > 1$ and $|P_i| = m > 1$. If $m < v$ and for $\tau \in P_1$, $G_{\sigma,\tau}$ is transitive on $\Sigma \setminus \{P_1\}$, then G is a group of automorphisms of a 2-design with $\lambda = 1$, the blocks of which are the images under G of the set $P_1 \cup \{\sigma\}$.*

3 Almost simple case

In this section we deal with the case when $G \leqslant \mathrm{Sym}(V)$ is 2-transitive on V of degree $u := |V|$ with $\mathrm{soc}(G)$ a nonabelian simple group. Then $\mathrm{soc}(G)$ and u are as follows ([19], [5, p.196], [4]):

(i) $\mathrm{soc}(G) = A_u$, $u \geqslant 5$;

(ii) $\mathrm{soc}(G) = \mathrm{PSL}(d,q)$, $d \geqslant 2$, q is a prime power and $u = (q^d - 1)/(q-1)$, where $(d,q) \neq (2,2),(2,3)$;

(iii) $\mathrm{soc}(G) = \mathrm{PSU}(3,q)$, $q \geqslant 3$ is a prime power and $u = q^3 + 1$;

(iv) $\mathrm{soc}(G) = \mathrm{Sz}(q)$, $q = 2^{2e+1} > 2$ and $u = q^2 + 1$;

(v) $\mathrm{soc}(G) = \mathrm{R}(q)'$, $q = 3^{2e+1}$ and $u = q^3 + 1$;

(vi) $G = \mathrm{Sp}_{2d}(2)$, $d \geqslant 3$ and $u = 2^{2d-1} \pm 2^{d-1}$;

(vii) $G = \mathrm{PSL}(2,11)$, $u = 11$;

(viii) $\mathrm{soc}(G) = M_u$, $u = 11, 12, 22, 23, 24$;

(ix) $G = M_{11}$, $u = 12$;

(x) $G = A_7$, $u = 15$;

(xi) $G = \mathrm{HS}$, $u = 176$;

(xii) $G = \mathrm{Co}_3$, $u = 276$.

We will show that, in all cases above except (iv) and (v), there is no 2-design as in Lemma 2.10 admitting G as a group of automorphisms, or there is such a 2-$(u, m+1, \lambda)$ design but its parameter λ is equal to 1.

In fact, in cases (i), (viii) and (ix), $\mathrm{soc}(G)$ is 3-transitive and so a 2-design as in Lemma 2.10 does not exist. In case (x), $G_{\sigma,\tau}$ has orbit-lengths 1 and 12 on $V \setminus \{\sigma,\tau\}$ ([19]). If there exists a 2-$(15, m+1, \lambda)$ design as in Lemma 2.10, then $\lambda = 1$ by Lemma 2.12. In case (vii), $G_{\sigma,\tau}$ has orbit-lengths 3 and 6 on $V \setminus \{\sigma,\tau\}$ ([19]), and hence there is no 2-design as in Lemma 2.10. In case (xi), $G_{\sigma,\tau}$ has orbit-lengths 12, 72 and 90 on $V \setminus \{\sigma,\tau\}$ by [19], and similarly in case (xii), $G_{\sigma,\tau}$ has orbit-lengths 112 and 162 on $V \setminus \{\sigma,\tau\}$. Thus there is no 2-design as in Lemma 2.10 in these two cases.

In case (ii), if $d = 2$ and $q \geqslant 5$, then all $G_{\sigma,\tau}$-orbits on $V \setminus \{\sigma,\tau\}$ have lengths at least $(q-1)/2$, and so a 2-design as in Lemma 2.10 does not exist. If $d \geqslant 3$, then $G_{\sigma,\tau}$ has orbit-lengths $q-1$ and $u-(q+1)$ on $V \setminus \{\sigma,\tau\}$, and so by Lemma 2.12 any 2-$(u, m+1, \lambda)$ design as in Lemma 2.10 must have parameter $\lambda = 1$.

In case (vi), G_σ acts on $V \setminus \{\sigma\}$ as $O^{\pm}(2d,2)$ does on its singular vectors ([19]), and $G_{\sigma,\tau}$ has orbit-lengths $2(2^{d-1} \mp 1)(2^{d-2} \pm 1)$ and 2^{2d-2} on $V \setminus \{\sigma,\tau\}$. Since the length of an orbit of $G_{\sigma,\tau}$ on $V \setminus \{\sigma,\tau\}$ plus 1 cannot divide $u-1$, a 2-design as in Lemma 2.10 does not exist.

3.1 $\mathrm{soc}(G) = \mathrm{PSU}(3,q)$, $u = q^3+1$, $q \geqslant 3$ a prime power

We prove that a 2-$(u, m+1, \lambda)$ design as in Lemma 2.10 with $\lambda > 1$ does not exist in this case. We need the following lemma whose proof is straightforward and hence omitted.

Lemma 3.1. *Suppose that $q \geqslant 3$ is a prime power with $3 \mid (q+1)$ and ℓ a nonnegative integer.*

(a) *If $(\ell(q^2-1)/3+q) \mid q^3$, then $\ell = 0$ or $3q$;*

(b) *if $(\ell(q^2-1)/3+1) \mid q^3$, then $\ell = 0$ or 3.*

We take the advantage of the following permutation representation of $\mathrm{PSU}(3,q)$ (see [10, pp.248–249]). Denote by W the 3-dimensional vector space over $\mathbb{F}_{q^2}$. The mapping $f : \xi \mapsto \xi^q$ is an automorphism of $\mathbb{F}_{q^2}$ and $f^2 = 1$. Let $w = (\xi_1, \xi_2, \xi_3)$ and $z = (\eta_1, \eta_2, \eta_3)$ be arbitrary vectors in W. Using $\xi \mapsto \overline{\xi} = \xi^q$ to denote the automorphism of $\mathbb{F}_{q^2}$ of order 2, we define a hermitian form $\varphi : W \times W \to \mathbb{F}_{q^2}$, $\varphi(w,z) = \xi_1\overline{\eta_3} + \xi_2\overline{\eta_2} + \xi_3\overline{\eta_1}$. It is straightforward to calculate that for this hermitian form the set of 1-dimensional isotropic subspaces is

$$V = \{\langle(1,0,0)\rangle\} \cup \{\langle(\alpha,\beta,1)\rangle : \alpha + \overline{\alpha} + \beta\overline{\beta} = 0, \alpha, \beta \in \mathbb{F}_{q^2}\}.$$

(A vector $w \in W$ is called isotropic if $\varphi(w,w) = 0$.) Thus $|V| = q^3 + 1$.

Let

$$t_{\alpha,\beta} := \begin{bmatrix} 1 & -\overline{\beta} & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad h_{\gamma,\delta} := \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \overline{\gamma}^{-1} \end{bmatrix}.$$

If $\alpha, \beta, \gamma, \delta \in \mathbb{F}_{q^2}$ satisfy $\delta\overline{\delta} = 1$, $\gamma \neq 0$ and $\alpha + \overline{\alpha} + \beta\overline{\beta} = 0$, then they define elements of $\mathrm{PGU}(3,q)$, to which we give the same names. There are q^3 matrices of type $t_{\alpha,\beta}$ and $(q^2-1)(q+1)$ of type $h_{\gamma,\delta}$. Let $e_1 = (1,0,0)$ and $e_3 = (0,0,1)$. Then the stabiliser $\mathrm{PGU}(3,q)_{\langle e_1\rangle}$ of the subspace spanned by e_1 consists of the elements of the form $x = h_{\gamma,\delta}t_{\alpha,\beta}$ (where $\delta\overline{\delta} = 1$, $\gamma \neq 0$, $\alpha + \overline{\alpha} + \beta\overline{\beta} = 0$). The stabiliser in $\mathrm{GU}(3,q)$ of two points $\langle e_1\rangle$ and $\langle e_3\rangle$ is $\mathrm{GU}(3,q)_{\langle e_1\rangle,\langle e_3\rangle} = \{h_{\gamma,\delta} : \delta\overline{\delta} = 1, \gamma \neq 0\}$. Obviously, $t_{\alpha,\beta} \in \mathrm{SU}(3,q)$, and $h_{\gamma,\delta} \in \mathrm{SU}(3,q)$ if and only if $\delta = \gamma^{q-1}$. Moreover, $h_{\gamma,\delta} \in \mathrm{SU}(3,q)$ is a scalar matrix if and only if $\gamma^{q-2} = 1$.

In the rest of this section we set $J := \mathrm{PSU}(3,q)$ and $Z := V \setminus \{\langle e_1\rangle\}$.

Lemma 3.2. *Let $\langle(\eta_1,\eta_2,1)\rangle \in V \setminus \{\langle e_1\rangle, \langle e_3\rangle\}$. Denote by Q the $J_{\langle e_1\rangle,\langle e_3\rangle}$-orbit containing $\langle(\eta_1,\eta_2,1)\rangle$. If $\eta_2 = 0$, then $|Q| = q-1$. If $\eta_2 \neq 0$, then*

$$|Q| = |J_{\langle e_1\rangle,\langle e_3\rangle}| = \begin{cases} q^2-1, & \text{if } 3 \nmid (q+1), \\ (q^2-1)/3, & \text{if } 3 \mid (q+1). \end{cases} \tag{2}$$

Proof. The action of $J_{\langle e_1\rangle}$ on Z can be represented as follows:

$$\langle(\xi_1,\xi_2,1)\rangle^{t_{\alpha,\beta}}=\langle(\xi_1+\alpha-\overline{\beta}\xi_2,\xi_2+\beta,1)\rangle,\ \ \langle(\xi_1,\xi_2,1)\rangle^{h_{\gamma,\delta}}=\langle(\gamma\overline{\gamma}\xi_1,\delta\overline{\gamma}\xi_2,1)\rangle.$$

Since $\delta\overline{\delta}=1$, $\gamma\neq 0$ and $\alpha+\overline{\alpha}+\beta\overline{\beta}=0$, setting $a=(\gamma/\delta)^q$ and $g_a:=h_{\gamma,\delta}$, we can write

$$\langle(\xi_1,\xi_2,1)\rangle^{g_a}=\langle(a\overline{a}\xi_1,a\xi_2,1)\rangle.$$

Hence $J_{\langle e_1\rangle,\langle e_3\rangle}=\langle g_a\mid a=r^{2q-1},r\in\mathbb{F}_{q^2}^{\times}\rangle$ (since $h_{\gamma,\delta}\in\mathrm{SU}(3,q)$ if and only if $\delta=\gamma^{q-1}$, we have $a=\gamma^{2q-1}$), and

$$\langle(a\overline{a}\eta_1,a\eta_2,1)\rangle=\langle(b\overline{b}\eta_1,b\eta_2,1)\rangle\Leftrightarrow\begin{cases}a=b, & \text{if } \ \eta_2\neq 0,\\ a^{q+1}=b^{q+1}, & \text{if } \ \eta_2=0.\end{cases}$$

Moreover, $|\{(\alpha,0,1):(\alpha,0,1)\in V\}|=q$ and each orbit of $J_{\langle e_1\rangle,\langle e_3\rangle}$ on $V\setminus\{\langle e_1\rangle,\langle e_3\rangle\}$ has length $q-1$ or at least $(q^2-1)/3$ ([19, p.69]). Therefore, if $\eta_2=0$, then $|Q|=q-1$; if $\eta_2\neq 0$, then $|Q|=|J_{\langle e_1\rangle,\langle e_3\rangle}|$. Since $\gcd(2q-1,q^2-1)=\gcd(q+1,3)$, in the latter case we obtain (2). □

Now suppose P is an imprimitive block of $J_{\langle e_1\rangle}$ on Z containing $\langle e_3\rangle$ with $|P|>1$ and $|Z|/|P|\geqslant 3$. We know that $P\setminus\{\langle e_3\rangle\}$ is the union of some $J_{\langle e_1\rangle,\langle e_3\rangle}$-orbits on $Z\setminus\{\langle e_3\rangle\}$. By Lemma 3.1, we have $|P|=q$ or $|P|=q^2$. By Lemma 2.12, we may assume $|P|=q^2$ in the following.

Denote the q solutions in $\mathbb{F}_{q^2}$ of the equation $x+\overline{x}=0$ by $\varepsilon_0=0$, ε_1, …, ε_{q-1}. We know that $\langle(\varepsilon_1,0,1)\rangle$, …, $\langle(\varepsilon_{q-1},0,1)\rangle$ form a $J_{\langle e_1\rangle,\langle e_3\rangle}$-orbit on $Z\setminus\{\langle e_3\rangle\}$. By Lemma 3.1, $\langle(\varepsilon_i,0,1)\rangle$ is not contained in P for $i>0$.

Now $\Sigma:=\{P^g:g\in J_{\langle e_1\rangle}\}$ is a system of blocks of $J_{\langle e_1\rangle}$ on Z with $|\Sigma|=q$, and $T:=\langle t_{\alpha,\beta}\mid\alpha+\overline{\alpha}+\beta\overline{\beta}=0\rangle$ is transitive on Σ. Actually T is a normal subgroup of $J_{\langle e_1\rangle}$ acting regularly on Z (see [10, p.249]). Hence the stabiliser of P in T has order q^2, that is, $|T_P|=q^2$. Let $t_{\alpha_1,\beta}$, $t_{\alpha_2,\beta}\in T_P$. Then $\langle(0,0,1)\rangle^{t_{\alpha_1,\beta}t^{-1}_{\alpha_2,\beta}}=\langle(\alpha_1,\beta,1)\rangle^{t_{-\alpha_2-\beta\overline{\beta},-\beta}}=\langle(\alpha_1-\alpha_2,0,1)\rangle\in P$. Since $\langle(\varepsilon_i,0,1)\rangle$ is not contained in P for $i>0$, we have $\alpha_1=\alpha_2$ and $t_{\alpha_1,\beta}=t_{\alpha_2,\beta}$. Therefore,

$$\{\beta:\langle(\alpha,\beta,1)\rangle\in P\}=\{\beta:t_{\alpha,\beta}\in T_P\}=\mathbb{F}_{q^2}. \tag{3}$$

For any $\langle(\eta_1,\eta_2,1)\rangle$, $\langle(\xi_1,\xi_2,1)\rangle\in P$, η_2, $\xi_2\neq 0$, since by our assumption P is an imprimitive block of $J_{\langle e_1\rangle}$ on Z, both t_{η_1,η_2} and t_{ξ_1,ξ_2} fix P setwise. Thus

$$\langle(0,0,1)\rangle^{t_{\eta_1,\eta_2}t_{\xi_1,\xi_2}}=\langle(\eta_1,\eta_2,1)\rangle^{t_{\xi_1,\xi_2}}=\langle(\eta_1+\xi_1-\overline{\xi_2}\eta_2,\eta_2+\xi_2,1)\rangle\in P,$$

$$\langle(0,0,1)\rangle^{t_{\xi_1,\xi_2}t_{\eta_1,\eta_2}}=\langle(\xi_1,\xi_2,1)\rangle^{t_{\eta_1,\eta_2}}=\langle(\xi_1+\eta_1-\overline{\eta_2}\xi_2,\xi_2+\eta_2,1)\rangle\in P.$$

Hence by (3) we have $\eta_1+\xi_1-\overline{\xi_2}\eta_2=\xi_1+\eta_1-\overline{\eta_2}\xi_2$, that is, $(\xi_2/\eta_2)^{q-1}=1$, which implies $(\xi_2/\eta_2)\in F_0:=\mathrm{Fix}_f(\mathbb{F}_{q^2})$. Fix $\eta_2=1$. Then $\xi_2\in F_0$ and thus $\mathbb{F}_{q^2}\subseteq F_0$, a contradiction. Hence there is no 2-$(u,m+1,\lambda)$ design as in Lemma 2.10 with $\lambda>1$ admitting G as a group of automorphisms with $\mathrm{soc}(G)=\mathrm{PSU}(3,q)$.

3.2 $\mathrm{soc}(G) = \mathrm{Sz}(q)$, $q = 2^{2e+1} > 2$ and $u = q^2 + 1$

We need the following two lemmas that can be easily proved.

Lemma 3.3. *Suppose that ℓ and n are positive integers, and $q > 1$ is a power of prime. If $(\ell(q-1)+1) \mid q^n$, then $\ell = (q^i - 1)/(q-1)$ for some $i = 1, 2, \ldots, n$.*

Lemma 3.4. *Let $\mathbb{F}$ be a field with characteristic $p > 0$ and let $\kappa \in \mathbb{F}$. If $\kappa^{p^a} = \kappa = \kappa^{p^b}$ for some positive integers a and b, then $\kappa^{p^{\gcd(a,b)}} = \kappa$.*

We use the permutation representation of $\mathrm{Sz}(q)$ in [10, p.250]. The mapping $\sigma : \xi \mapsto \xi^{2^{e+1}}$ is an automorphism of $\mathbb{F}_q$ and σ^2 is the Frobenius automorphism $\xi \mapsto \xi^2$. Define

$$V := \{(\eta_1, \eta_2, \eta_3) \in \mathbb{F}_q^3 : \eta_3 = \eta_1\eta_2 + \eta_1^{\sigma+2} + \eta_2^{\sigma}\} \cup \{\infty\}. \qquad (4)$$

Thus $|V| = q^2 + 1$. For $\alpha, \beta, \kappa \in \mathbb{F}_q$ with $\kappa \neq 0$, define the following permutations of V fixing ∞:

$$t_{\alpha,\beta} : (\eta_1, \eta_2, \eta_3) \mapsto (\eta_1 + \alpha, \eta_2 + \beta + \alpha^{\sigma}\eta_1, \mu),$$
$$n_{\kappa} : (\eta_1, \eta_2, \eta_3) \mapsto (\kappa\eta_1, \kappa^{\sigma+1}\eta_2, \kappa^{\sigma+2}\eta_3),$$

where $\mu = \eta_3 + \alpha\beta + \alpha^{\sigma+2} + \beta^{\sigma} + \alpha\eta_2 + \alpha^{\sigma+1}\eta_1 + \beta\eta_1$. Define the involution w fixing V by

$$w : (\eta_1, \eta_2, \eta_3) \leftrightarrow \left(\frac{\eta_2}{\eta_3}, \frac{\eta_1}{\eta_3}, \frac{1}{\eta_3}\right) \text{ for } \eta_3 \neq 0, \ \ \infty \leftrightarrow \mathbf{0} := (0, 0, 0).$$

The Suzuki group $\mathrm{Sz}(q)$ is the group generated by w and all $t_{\alpha,\beta}$ and n_κ. The stabiliser of ∞ is $\mathrm{Sz}(q)_\infty = \langle t_{\alpha,\beta}, n_\kappa \mid \alpha, \beta, \kappa \in \mathbb{F}_q, \kappa \neq 0\rangle$. The stabiliser of ∞ and $\mathbf{0}$ is the cyclic group $\langle n_\kappa \mid \kappa \in \mathbb{F}_q, \kappa \neq 0\rangle$.

Lemma 3.5. *Each orbit of $\mathrm{Sz}(q)_{\infty,\mathbf{0}}$ on $V \setminus \{\infty, \mathbf{0}\}$ has length $q - 1$.*

Proof. Since $\gcd(2^{e+1}+1, 2^{2e+1}-1) = 1$ and $\mathbb{F}_q^\times$ is a cyclic group of order $q-1 = 2^{2e+1}-1$, the mapping $\mathbb{F}_q^\times \to \mathbb{F}_q^\times$, $z \mapsto z^{2^{e+1}+1} = z^{\sigma+1}$ is a group automorphism. Thus, if $\eta_1 \neq 0$ or $\eta_2 \neq 0$, then $(a\eta_1, a^{\sigma+1}\eta_2, a^{\sigma+2}\eta_3) = (b\eta_1, b^{\sigma+1}\eta_2, b^{\sigma+2}\eta_3) \Leftrightarrow a = b$. Therefore each orbit of $\mathrm{Sz}(q)_{\infty,\mathbf{0}}$ on $V \setminus \{\infty, \mathbf{0}\}$ has length $q - 1$. □

Lemma 3.6. *Suppose that P is an imprimitive block of $\mathrm{Sz}(q)_\infty$ on $V \setminus \{\infty\}$ containing $\mathbf{0}$, and $1 < |P| < q^2$. Then $P = \{(0, \eta, \eta^\sigma) \in V : \eta \in \mathbb{F}_q\}$ and $\mathrm{Sz}(q)_{\infty,P} = \langle t_{0,\xi}, n_\kappa \mid \kappa \in \mathbb{F}_q^\times, \xi \in \mathbb{F}_q\rangle$.*

Proof. By Lemma 3.3 we can assume that $P \setminus \{\mathbf{0}\}$ is a $\mathrm{Sz}(q)_{\infty,\mathbf{0}}$-orbit on $V \setminus \{\infty, \mathbf{0}\}$. The elements of P have the form $(\kappa\eta_1, \kappa^{\sigma+1}\eta_2, \kappa^{\sigma+2}\eta_3)$, where $\kappa \in \mathbb{F}_q$ and (η_1, η_2, η_3) is a fixed point in P. Suppose $P^{t_{\alpha,\beta}} \cap P \neq \emptyset$ for some $\alpha, \beta \in \mathbb{F}_q$, that is,

$$\left(\kappa_1\eta_1, \kappa_1^{\sigma+1}\eta_2, \kappa_1^{\sigma+2}\eta_3\right)^{t_{\alpha,\beta}} = \left(\kappa_0\eta_1, \kappa_0^{\sigma+1}\eta_2, \kappa_0^{\sigma+2}\eta_3\right) \text{ for some } \kappa_0, \kappa_1 \in \mathbb{F}_q. \qquad (5)$$

Then we have the following equations (since the third coordinate of each element in V is determined by the first two, we can just consider the equations given by the first two coordinates):

$$\alpha=(\kappa_0+\kappa_1)\eta_1,\ \ \beta=(\kappa_1^{\sigma+1}+\kappa_0^{\sigma+1})\eta_2+(\kappa_0^\sigma+\kappa_1^\sigma)\kappa_1\eta_1^{\sigma+1}. \tag{6}$$

Hence, if α,β are as in (6) with respect to η_1 and η_2, then (5) holds. Since by our assumption P is an imprimitive block of $\mathrm{Sz}(q)_\infty$ on $V\setminus\{\infty\}$, we need to verify that $P^{t_{\alpha,\beta}}=P$, that is, for any $\ell\in\mathbb{F}_q$ there exists $\ell_0\in\mathbb{F}_q$ such that $(\ell\eta_1,\ell^{\sigma+1}\eta_2,\ell^{\sigma+2}\eta_3)^{t_{\alpha,\beta}}=(\ell_0\eta_1,\ell_0^{\sigma+1}\eta_2,\ell_0^{\sigma+2}\eta_3)$. This is to say that, for any $\ell\in\mathbb{F}_q$, the equation system

$$(\ell+x)\eta_1=\alpha,\ \ (\ell^{\sigma+1}+x^{\sigma+1})\eta_2+(\ell^\sigma+x^\sigma)\ell\eta_1^{\sigma+1}=\beta \tag{7}$$

has a solution $x\in\mathbb{F}_q$. We claim that this happens only when $\eta_1=0$. In fact, if $P^{t_{\xi,\theta}}\cap P=\emptyset$ for any $t_{\xi,\theta}\neq\mathrm{id}$, then different $t_{\xi,\theta}$ must map P to different elements in $P^{\mathrm{Sz}(q)_\infty}$, and thus $q^2=|\langle t_{\xi,\theta}\mid \xi,\theta\in\mathbb{F}_q\rangle|\leqslant|P^{\mathrm{Sz}(q)_\infty}|=q$, a contradiction. Hence we can assume that at most one of α,β is 0 in (5). If $\eta_1\neq 0$, then $x=\alpha/\eta_1-\ell$. The second equation of (7) becomes $\frac{\alpha\eta_2}{\eta_1}\ell^\sigma+\left(\frac{\alpha^\sigma\eta_2}{\eta_1^\sigma}+\alpha^\sigma\eta_1\right)\ell+\frac{\alpha^{\sigma+1}\eta_2}{\eta_1^{\sigma+1}}-\beta=0$, and it holds for every $\ell\in\mathbb{F}_q$. From the knowledge of polynomials over fields we have $\alpha\eta_2/\eta_1=0$, $\alpha^\sigma\eta_2/\eta_1^\sigma+\alpha^\sigma\eta_1=0$ since $q>2$. If $\alpha=0$, then from (6) we have $\beta=0$, which contradicts our assumption. Thus $\alpha\neq 0,\eta_2=0$, $\alpha^\sigma\eta_1=0$, the latter being a contradiction. Therefore, $\eta_1=0$.

By Lemma 3.5, $P=\{\mathbf{0}\}\cup(0,1,1)^{\mathrm{Sz}(q)_{\infty,\mathbf{0}}}=\{(0,\eta,\eta^\sigma)\in V:\eta\in\mathbb{F}_q\}$. This P is indeed an imprimitive block of $\mathrm{Sz}(q)_\infty$ on $V\setminus\{\infty\}$, and $\mathrm{Sz}(q)_{\infty,P}=\langle t_{0,\xi},n_\kappa\mid\kappa\in\mathbb{F}_q^\times,\xi\in\mathbb{F}_q\rangle$. $\square$

Let G be a subgroup of $\mathrm{Sym}(V)$ containing $\mathrm{Sz}(q)$ as a normal subgroup. Since $\mathrm{Sz}(q)$ has index $2e+1$ in its normalizer Q in $\mathrm{Sym}(V)$ (see [5, Table 7.4]), $Q/\mathrm{Sz}(q)$ is a cyclic group of order $2e+1$ and $G=\langle\mathrm{Sz}(q),\zeta\rangle$, where ζ is an automorphism of $\mathbb{F}_q$ inducing a permutation of V with ζ fixing ∞ and acting on elements of $V\setminus\{\infty\}$ componentwise. Hence the group G has b possibilities, where b is the number of divisors of $2e+1$.

Lemma 3.7. *Let $\mathcal{D}:=(V,L^{\mathrm{Sz}(q)})$ and $\Omega:=(\infty,L)^{\mathrm{Sz}(q)}$, where V is as in (4) and $L:=P\cup\{\infty\}$ with $P=\{(0,\eta,\eta^\sigma)\in V:\eta\in\mathbb{F}_q\}$. Let G be a subgroup of $\mathrm{Sym}(V)$ containing $\mathrm{Sz}(q)$ as a normal subgroup with $|G/\mathrm{Sz}(q)|=(2e+1)/f$ for some integer f. Then the following hold:*

- (a) *$\mathcal{D}$ is a 2-$(q^2+1,q+1,q+1)$ design admitting G as a 2-point-transitive and block-transitive group of automorphisms, and Ω is a feasible G-orbit on the set of flags of $\mathcal{D}$;*
- (b) *any G-orbit $\Psi=((\infty,M),(\mathbf{0},N))^G$ on $\mathrm{F}(\mathcal{D},\Omega)$ is self-paired, and the corresponding G-flag graph $\Gamma(\mathcal{D},\Omega,\Psi)$ is connected with order $|\Omega|=q(q^2+1)$; moreover, by (d) in Definition 2.1 we may assume $M=L^{t_{1,0}}=\{(1,\eta,\eta+1+\eta^\sigma)\in V:\eta\in\mathbb{F}_q\}\cup\{\infty\}$ and $N=M^{n_{\kappa_0}w}$ for some $\kappa_0\in\mathbb{F}_q^\times$; the valency of $\Gamma(\mathcal{D},\Omega,\Psi)$ is equal to $(q^2-q)i/\gcd(f,i)$, where i is the smallest positive integer satisfying $\kappa_0^{2^i}=\kappa_0$.*

Proof. From the discussion above we see that $\mathcal{D}$ is a 2-$(q^2+1, q+1, \lambda)$ design admitting $\mathrm{Sz}(q)$ as a 2-point-transitive and block-transitive group of automorphisms. Let $\tau \in V \setminus L$. Then $|\tau^{\mathrm{Sz}(q)_{\infty,L}}| = |\mathrm{Sz}(q)_{\infty,L}|/|\mathrm{Sz}(q)_{\infty,L,\tau}| = |\mathrm{Sz}(q)_{\infty,L}| = q(q-1)$. Hence $\mathrm{Sz}(q)_{\infty,L}$ is transitive on $V \setminus L$, and by Lemma 2.9, Ω is a feasible $\mathrm{Sz}(q)$-orbit on the flag set of $\mathcal{D}$. Since w does not stabilise L, $\lambda \neq 1$ and thus $\lambda = q+1$.

Let $G = \langle \mathrm{Sz}(q), \zeta \rangle$, where $\zeta : \mathbb{F}_q \to \mathbb{F}_q$, $\xi \mapsto \xi^{2^f}$. One can verify that $G_\infty = \langle \mathrm{Sz}(q)_\infty, \zeta \rangle$ and P is an imprimitive block of G_∞ on $V \setminus \{\infty\}$. Moreover, $(\infty, L)^G = (\infty, L)^{\mathrm{Sz}(q)}$ and $L^G = L^{\mathrm{Sz}(q)}$. By Lemma 2.9, Ω is a feasible G-orbit on the flag set of $\mathcal{D}$. Since $N^{n_{\kappa_0} w} = M^{n_{\kappa_0} w n_{\kappa_0} w} = M^{n_{\kappa_0} n_{\kappa_0}^{-1}} = M$, $n_{\kappa_0} w$ interchanges (∞, M) and $(\mathbf{0}, N)$. Therefore, Ψ is self-paired and so produces the G-flag graph $\Gamma(\mathcal{D}, \Omega, \Psi)$.

Set $L_\kappa := P_\kappa \cup \{\infty\}$ for each $\kappa \in \mathbb{F}_q$, where $P_\kappa = \{(\kappa, \eta, \kappa\eta + \kappa^{\sigma+2} + \eta^\sigma) \in V : \eta \in \mathbb{F}_q\}$. Consider the set $(\mathbf{0}, N)^{G_{\infty,M,\mathbf{0}}}$ of neighbours of (∞, M) in $\Gamma(\mathcal{D}, \Omega, \Psi)$ contained in $\Omega(\mathbf{0})$. Since $\zeta w = w\zeta$ and $G_{\infty,M,\mathbf{0}} = \langle n_\kappa, \zeta \mid \kappa \in \mathbb{F}_q^\times \rangle_M = \langle \zeta \rangle$, we have $N^{G_{\infty,M,\mathbf{0}}} = M^{n_{\kappa_0} w \langle\zeta\rangle} = M^{n_{\kappa_0} \langle\zeta\rangle w}$ and $M^{n_{\kappa_0}\varphi} = M^{\varphi\varphi^{-1} n_{\kappa_0} \varphi} = M^{n_{\kappa_0^\varphi}} = L_{\kappa_0^\varphi}$ for any $\varphi \in \langle \zeta \rangle$. It follows that $N^{G_{\infty,M,\mathbf{0}}} = (L_{\kappa_0^{\langle\zeta\rangle}})^w$, and in particular $|(\mathbf{0}, N)^{G_{\infty,M,\mathbf{0}}}| = |\kappa_0^{\langle\zeta\rangle}|$.

By Lemma 3.4 we have $|\kappa_0^{\langle\zeta\rangle}| = \mathrm{lcm}(f,i)/f = i/\gcd(f,i)$. Therefore, (∞, M) is adjacent to $i/\gcd(f,i)$ vertices in $\Omega(\mathbf{0})$, namely, $(\mathbf{0}, (L_{\kappa_0^{\zeta^\ell}})^w)$, $\ell = 1, 2, \ldots, i/\gcd(f,i)$. By the discussion in Section 2.4, $\Gamma(\mathcal{D}, \Omega, \Psi)$ has valency $(q^2-q)i/\gcd(f,i)$.

Denote $H := \langle t_{0,\xi}, n_{\kappa_0} w : \xi \in \mathbb{F}_q \rangle$. For any $(\eta_1, \eta_2, \eta_3) \in V \setminus \{\infty, \mathbf{0}\}$, if $\eta_1 = 0$ then $\mathbf{0}^{t_{0,\eta_2}} = (\eta_1, \eta_2, \eta_3)$, and if $\eta_1 \neq 0$ then $\mathbf{0}^{t_{0,\theta} n_{\kappa_0} w t_{0,\eta_2}} = (\eta_1, \eta_2, \eta_3)$, where $\theta/\theta^\sigma = \eta_1 \kappa_0$. Hence H is transitive on V, and thus $(q^2+1)q$ divides $|H|$. So $|H|$ does not divide $q^2(q-1)$, $2(q-1)$, $4(q+\sqrt{2q}+1)$ or $4(q-\sqrt{2q}+1)$. Thus, by [26, p.137, Theorem 9], $|H| = (s^2+1)s^2(s-1)$, where $s^j = q$ for some positive integer j. It follows that $j = 1$, $|H| = (q^2+1)q^2(q-1)$, and thus $\mathrm{Sz}(q) = H$. Therefore, $\mathrm{Sz}(q) = \langle \mathrm{Sz}(q)_{\infty,M}, n_{\kappa_0} w \rangle$ as $\langle t_{0,\xi} : \xi \in \mathbb{F}_q \rangle \leqslant \mathrm{Sz}(q)_{\infty,M}$, and so $\Gamma(\mathcal{D}, \Omega, \Psi)$ is connected by Lemma 2.11. □

Example 3.8. *Suppose that* $G = \langle \mathrm{Sz}(8), \zeta \rangle$*, where* $\zeta : \mathbb{F}_8 \to \mathbb{F}_8$*,* $\xi \mapsto \xi^2$ *is the Frobenius map. Let* $\Psi := ((\infty, M), (\mathbf{0}, N))^G$*, where* $M = L_1 = \{(1, \eta, 1+\eta+\eta^4) \in V : \eta \in \mathbb{F}_8\} \cup \{\infty\}$*,* $N = M^{n_{\kappa_0} w}$*, and* κ_0 *is a generator of* $\mathbb{F}_8^\times$*. Then the edges of the* G*-flag graph* $\Gamma(\mathcal{D}, \Omega, \Psi)$ *between* $\Omega(\infty)$ *and* $\Omega(\mathbf{0})$ *are as shown in Figure 1.*

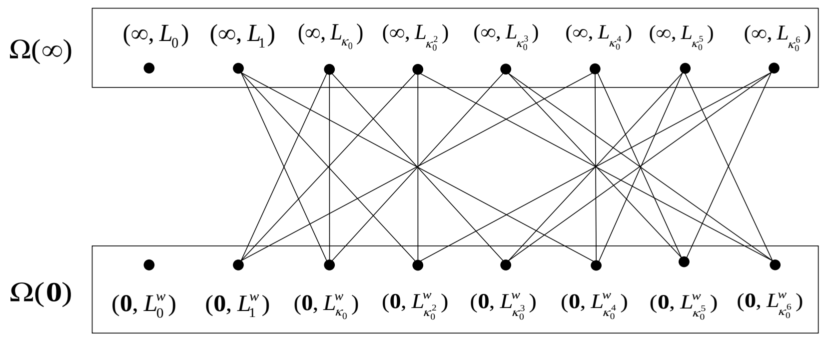

Figure 1

3.3 $\mathrm{soc}(G) = \mathrm{R}(q)$, $q = 3^{2e+1} > 3$, $u = q^3 + 1$; or $G = \mathrm{R}(3)$, $\mathrm{R}(3)' \cong \mathrm{PSL}(2,8)$, $u = 28$

We will use the following lemma that can be easily proved.

Lemma 3.9. *Suppose that $\ell \geqslant 0$ is an integer, n a positive integer, and q an odd power of 3. Then $(\ell(q-1) + (q-1)/2 + 1) \nmid q^n$.*

We use the permutation representation of $\mathrm{R}(q)$ in [10, p.251]. The mapping $\sigma : \xi \mapsto \xi^{3^{e+1}}$ is an automorphism of $\mathbb{F}_q$ and σ^2 is the Frobenius automorphism $\xi \mapsto \xi^3$. The set V of points on which $\mathrm{R}(q)$ acts consists of ∞ and the set of sixtuples $(\eta_1, \eta_2, \eta_3, \lambda_1, \lambda_2, \lambda_3)$ with $\eta_1, \eta_2, \eta_3 \in \mathbb{F}_q$ and

$$\begin{cases} \lambda_1 = \eta_1^2\eta_2 - \eta_1\eta_3 + \eta_2^\sigma - \eta_1^{\sigma+3}, \\ \lambda_2 = \eta_1^\sigma\eta_2^\sigma - \eta_3^\sigma + \eta_1\eta_2^2 + \eta_2\eta_3 - \eta_1^{2\sigma+3}, \\ \lambda_3 = \eta_1\eta_3^\sigma - \eta_1^{\sigma+1}\eta_2^\sigma + \eta_1^{\sigma+3}\eta_2 + \eta_1^2\eta_2^2 - \eta_2^{\sigma+1} - \eta_3^2 + \eta_1^{2\sigma+4}. \end{cases} \tag{8}$$

Thus $|V| = q^3 + 1$. For $\alpha, \beta, \gamma, \kappa \in \mathbb{F}_q$ with $\kappa \neq 0$, define the following permutations of V fixing ∞:

$$\begin{aligned} t_{\alpha,\beta,\gamma} &: (\eta_1, \eta_2, \eta_3, \lambda_1, \lambda_2, \lambda_3) \mapsto \\ &\quad (\eta_1 + \alpha, \eta_2 + \beta + \alpha^\sigma\eta_1, \eta_3 + \gamma - \alpha\eta_2 + \beta\eta_1 - \alpha^{\sigma+1}\eta_1, \mu_1, \mu_2, \mu_3), \\ n_\kappa &: (\eta_1, \eta_2, \eta_3, \lambda_1, \lambda_2, \lambda_3) \mapsto (\kappa\eta_1, \kappa^{\sigma+1}\eta_2, \kappa^{\sigma+2}\eta_3, \kappa^{\sigma+3}\lambda_1, \kappa^{2\sigma+3}\lambda_2, \kappa^{2\sigma+4}\lambda_3), \end{aligned}$$

where μ_1, μ_2 and μ_3 can be calculated from the formulas in (8). Define the involution w fixing V by

$$w : (\eta_1, \eta_2, \eta_3, \lambda_1, \lambda_2, \lambda_3) \leftrightarrow \left(\frac{\lambda_2}{\lambda_3}, \frac{\lambda_1}{\lambda_3}, \frac{\eta_3}{\lambda_3}, \frac{\eta_2}{\lambda_3}, \frac{\eta_1}{\lambda_3}, \frac{1}{\lambda_3}\right) \text{ for } \lambda_3 \neq 0,$$
$$\infty \leftrightarrow \mathbf{0} := (0,0,0,0,0,0).$$

(We correct the action of w on V in [10, p.251] according to [11].) The Ree group $\mathrm{R}(q)$ is the group generated by w and all $t_{\alpha,\beta,\gamma}$ and n_κ. We have $\mathrm{R}(q)_\infty = \langle t_{\alpha,\beta,\gamma}, n_\kappa \mid \alpha, \beta, \gamma, \kappa \in \mathbb{F}_q, \kappa \neq 0\rangle$ and $\mathrm{R}(q)_{\infty,\mathbf{0}}$ is the cyclic group $\langle n_\kappa \mid \kappa \in \mathbb{F}_q, \kappa \neq 0\rangle$. Since the first three coordinates in each element of V determine the last three, in the following we simply present an element of V in the form $(\eta_1, \eta_2, \eta_3, \ldots)$.

Lemma 3.10. *Let $(\eta_1, \eta_2, \eta_3, \ldots) \in V \setminus \{\infty, \mathbf{0}\}$. Then*

$$|(\eta_1, \eta_2, \eta_3, \ldots)^{\mathrm{R}(q)_{\infty,\mathbf{0}}}| = \begin{cases} q-1, & \text{if } \eta_1 \neq 0 \text{ or } \eta_3 \neq 0, \\ (q-1)/2, & \text{if } \eta_1 = \eta_3 = 0. \end{cases}$$

Proof. Since $\mathrm{id} : \mathbb{F}_q^\times \to \mathbb{F}_q^\times$, $\xi \mapsto \xi$ and $\varphi : \mathbb{F}_q^\times \to \mathbb{F}_q^\times$, $\xi \mapsto \xi^{\sigma+2}$ are both group automorphisms, if $\eta_1 \neq 0$ or $\eta_3 \neq 0$, then $(a\eta_1, a^{\sigma+1}\eta_2, a^{\sigma+2}\eta_3, \ldots) = (b\eta_1, b^{\sigma+1}\eta_2, b^{\sigma+2}\eta_3, \ldots) \Leftrightarrow a = b$.

Let δ be a generator of the cyclic group $\mathbb{F}_q^\times$. Since $\delta^{\sigma+1} = \delta^{3^{e+1}+1}$ and $\gcd(3^{e+1}+1, q-1) = 2$, we have $|\delta^{\sigma+1}| = (q-1)/2$, and thus

$$L_1 := (0, 1, 0, \dots)^{\mathrm{R}(q)_{\infty,\mathbf{0}}} \text{ and } L_2 := (0, \delta, 0, \dots)^{\mathrm{R}(q)_{\infty,\mathbf{0}}} \tag{9}$$

are two orbits of length $(q-1)/2$ of $\mathrm{R}(q)_{\infty,\mathbf{0}}$ on $V \setminus \{\infty, \mathbf{0}\}$. □

By the result above we know that $\mathrm{R}(q)_{\infty,\mathbf{0}}$ has two orbits of length $(q-1)/2$ and $q(q+1)$ orbits of length $q-1$ on $V \setminus \{\infty, \mathbf{0}\}$.

Let G be a subgroup of $\mathrm{Sym}(V)$ containing $\mathrm{R}(q)$ as a normal subgroup. Since $\mathrm{R}(q)$ has index $2e+1$ in its normalizer Q in $\mathrm{Sym}(V)$ ([5, Table 7.4]), $Q/\mathrm{R}(q)$ is a cyclic group of order $2e+1$ and $G = \langle \mathrm{R}(q), \zeta \rangle$, where ζ is an automorphism of $\mathbb{F}_q$ inducing a permutation of V with ζ fixing ∞ and acting on elements of $V \setminus \{\infty\}$ componentwise.

Lemma 3.11. *Let $G = \langle \mathrm{R}(q), \zeta \rangle$ be a subgroup of $\mathrm{Sym}(V)$ containing $\mathrm{R}(q)$ as a normal subgroup, where ζ is an automorphism of $\mathbb{F}_q$. Suppose that P is an imprimitive block of G_∞ on $V \setminus \{\infty\}$ containing $\mathbf{0}$ with $1 < |P| < q^3$. Then $|P| = q$ or $|P| = q^2$. Moreover, if $|P| = q^2$ and $G_{\infty,\mathbf{0}}$ is transitive on $P^{G_\infty} \setminus \{P\}$, then*

$$P = \{(0, \eta_2, \eta_3, \dots) : \eta_2, \eta_3 \in \mathbb{F}_q\}, \tag{10}$$

$$G_{\infty,P} = \langle t_{0,\beta,\gamma}, n_\kappa, \zeta \mid \beta, \gamma \in \mathbb{F}_q, \kappa \in \mathbb{F}_q^\times \rangle. \tag{11}$$

Proof. By Lemma 2.10 (a), $P \setminus \{\mathbf{0}\}$ is the union of some $\mathrm{R}(q)_{\infty,\mathbf{0}}$-orbits on $V \setminus \{\infty, \mathbf{0}\}$. By Lemmas 3.3 and 3.9, we have $|P| = q$ or $|P| = q^2$.

Suppose $|P| = q^2$ and $G_{\infty,\mathbf{0}}$ is transitive on $P^{G_\infty} \setminus \{P\}$. Let L_1 and L_2 be as in the proof of Lemma 3.10. Then by Lemma 3.9 either $L_1 \cup L_2 \subseteq P$ or $(L_1 \cup L_2) \cap P = \emptyset$. Since $G_{\infty,\mathbf{0}} = \langle n_\kappa, \zeta \mid \kappa \in \mathbb{F}_q^\times \rangle$, L_1 and L_2 are $G_{\infty,\mathbf{0}}$-orbits on $V \setminus \{\infty, \mathbf{0}\}$. If $(L_1 \cup L_2) \cap P = \emptyset$, then $G_{\infty,\mathbf{0}}$ has an orbit of length at most $(q-1)/2$ on $P^{G_\infty} \setminus \{P\}$ and $G_{\infty,\mathbf{0}}$ is not transitive on $P^{G_\infty} \setminus \{P\}$. Hence $L_1 \cup L_2 \subseteq P$ and thus $\{(0, \eta, 0, \dots) : \eta \in \mathbb{F}_q\} \subseteq P$. Since $(0, \eta, 0, \dots)^{t_{\alpha,\beta,\gamma}} = (\alpha, \eta+\beta, \gamma - \alpha\eta, \dots)$, we have $\langle t_{0,\beta,0} \mid \beta \in \mathbb{F}_q \rangle \leqslant G_{\infty,P}$ and $H := \langle t_{0,\beta,0}, n_\kappa \mid \beta \in \mathbb{F}_q, \kappa \in \mathbb{F}_q^\times \rangle \leqslant G_{\infty,P}$.

If P has a point $(\eta_1, \eta_2, \eta_3, \dots)$ with $\eta_1 \neq 0$, then by the action of H, we can assume that $\rho := (1, 0, \varepsilon_0, \dots) \in P$ for some $\varepsilon_0 \in \mathbb{F}_q$. Since $|\rho^H| = |H|/|H_\rho| = |H| = q(q-1)$ and $\rho^H \cap \{(0, \eta, 0, \dots) : \eta \in \mathbb{F}_q\} = \emptyset$, we have

$$P = \rho^H \cup \{(0, \eta, 0, \dots) : \eta \in \mathbb{F}_q\} = \{(\eta_1, \eta_2, \eta_3, \dots) \in V : \eta_3 = \eta_1^{\sigma+2}\varepsilon_0 + \eta_1\eta_2\}. \tag{12}$$

However, this P is not an imprimitive block of $\mathrm{R}(q)_\infty$ on $V \setminus \{\infty\}$. In fact, if $(0, 1, 0, \dots)^{t_{a,b,c}} = (1, 0, \varepsilon_0, \dots)$, then $a = 1, b = -1, c = 1 + \varepsilon_0$. On the other hand, $(0, -1, 0, \dots) \in P$, and $(0, -1, 0, \dots)^{t_{1,-1,1+\varepsilon_0}} = (1, -2, 2+\varepsilon_0, \dots) = (1, 1, 2+\varepsilon_0, \dots)$. We can check that the first three coordinates of $(1, 1, 2+\varepsilon_0, \dots)$ do not satisfy the equation (see (12)) for the elements of P. Hence $(0, -1, 0, \dots)^{t_{1,-1,1+\varepsilon_0}} \notin P$, and P given in (12) is not an imprimitive block of $\mathrm{R}(q)_\infty$ on $V \setminus \{\infty\}$. Therefore, every element in P must have 0 as the first coordinate. It follows that P is as given in (10). It is straightforward to check that P is indeed an imprimitive block of $G_\infty = \langle \mathrm{R}(q)_\infty, \zeta \rangle$ on $V \setminus \{\infty\}$ and $G_{\infty,P}$ is as shown in (11). □

We will ignore the case $|P| = q$ in Lemma 3.11, since in this case the design in Lemma 2.10 (if it exists) is a linear space by Lemma 2.12.

Lemma 3.12. *Let $\mathcal{D} := (V, L^{\mathrm{R}(q)})$ and $\Omega := (\infty, L)^{\mathrm{R}(q)}$, where $L := P \cup \{\infty\}$ with P as defined in (10). Let G be a subgroup of $\mathrm{Sym}(V)$ containing $\mathrm{R}(q)$ as a normal subgroup such that $|G/\mathrm{R}(q)| = (2e+1)/f$ for some integer f. Then the following hold:*

(a) *$\mathcal{D}$ is a 2-(q^3+1, q^2+1, q^2+1) design admitting G as a 2-point-transitive and block-transitive group of automorphisms, and Ω is a feasible G-orbit on the set of flags of $\mathcal{D}$;*

(b) *any G-orbit $\Psi = ((\infty, M), (\mathbf{0}, N))^G$ on $\mathrm{F}(\mathcal{D}, \Omega)$ is self-paired, and the G-flag graph $\Gamma(\mathcal{D}, \Omega, \Psi)$ is connected with order $|\Omega| = q(q^3+1)$; moreover, by (d) in Definition 2.1, we may assume $M = L^{t_{1,0,0}} = \{(1, \eta_2, \eta_3, \ldots) \in V : \eta_2, \eta_3 \in \mathbb{F}_q\} \cup \{\infty\}$ and $N = M^{n_{\kappa_0} w}$ for some $\kappa_0 \in \mathbb{F}_q^{\times}$; the valency of $\Gamma(\mathcal{D}, \Omega, \Psi)$ is equal to $(q^3 - q^2)i/\gcd(f, i)$, where i is the smallest positive integer satisfying $\kappa_0^{3^i} = \kappa_0$.*

Proof. Using the notation above, we have $G = \langle \mathrm{R}(q), \zeta \rangle$, where $\zeta : \mathbb{F}_q \to \mathbb{F}_q$, $\xi \mapsto \xi^{2f}$. Then $(\infty, L)^G = (\infty, L)^{\mathrm{R}(q)}$, $L^G = L^{\mathrm{R}(q)}$, and $\mathcal{D}$ is a 2-(q^3+1, q^2+1, λ) design admitting G as a 2-point-transitive and block-transitive group of automorphisms. Let $\theta := (1, 0, 0, \ldots) \in V \setminus L$. Since $|\theta^{\mathrm{R}(q)_{\infty,L}}| = |\mathrm{R}(q)_{\infty,L}|/|\mathrm{R}(q)_{\infty,L,\theta}| = |\mathrm{R}(q)_{\infty,L}| = q^2(q-1)$, $\mathrm{R}(q)_{\infty,L}$ and $G_{\infty,L}$ are transitive on $V \setminus L$ and by Lemma 2.9, Ω is a feasible G-orbit on the flag set of $\mathcal{D}$. Since w does not stabilise L, $\lambda \neq 1$ and thus $\lambda = q^2+1$.

Since $N^{n_{\kappa_0} w} = M^{n_{\kappa_0} w n_{\kappa_0} w} = M^{n_{\kappa_0} n_{\kappa_0}^{-1}} = M$, $n_{\kappa_0} w$ interchanges (∞, M) and $(\mathbf{0}, N)$. Therefore, Ψ is self-paired and so produces the G-flag graph $\Gamma(\mathcal{D}, \Omega, \Psi)$.

Set $L_\kappa := P_\kappa \cup \{\infty\}$ for each $\kappa \in \mathbb{F}_q$, where $P_\kappa = \{(\kappa, \eta_2, \eta_3, \ldots) \in V : \eta_2, \eta_3 \in \mathbb{F}_q\}$. Note that $(\mathbf{0}, N)^{G_{\infty,M,\mathbf{0}}}$ is the set of neighbours of (∞, M) in $\Gamma(\mathcal{D}, \Omega, \Psi)$ contained in $\Omega(\mathbf{0})$. Since $\zeta w = w\zeta$ and $G_{\infty,M,\mathbf{0}} = \langle n_\kappa, \zeta \mid \kappa \in \mathbb{F}_q^{\times} \rangle_M = \langle \zeta \rangle$, we have $N^{G_{\infty,M,\mathbf{0}}} = M^{n_{\kappa_0} w \langle \zeta \rangle} = M^{n_{\kappa_0} \langle \zeta \rangle w}$ and $M^{n_{\kappa_0} \varphi} = M^{\varphi \varphi^{-1} n_{\kappa_0} \varphi} = M^{n_{\kappa_0^{\varphi}}} = L_{\kappa_0^{\varphi}}$ for any $\varphi \in \langle \zeta \rangle$. It follows that $N^{G_{\infty,M,\mathbf{0}}} = (L_{\kappa_0^{\langle \zeta \rangle}})^w$, and in particular $|(\mathbf{0}, N)^{G_{\infty,M,\mathbf{0}}}| = |\kappa_0^{\langle \zeta \rangle}|$.

By Lemma 3.4 we have $|\kappa_0^{\langle \zeta \rangle}| = \mathrm{lcm}(f, i)/f = i/\gcd(f, i)$. Therefore, (∞, M) is adjacent to $i/\gcd(f, i)$ vertices in $\Omega(\mathbf{0})$, namely, $(\mathbf{0}, (L_{\kappa_0^{\zeta^\ell}})^w)$, $\ell = 1, 2, \ldots, i/\gcd(f, i)$. By the discussion in Section 2.4, $\Gamma(\mathcal{D}, \Omega, \Psi)$ has valency $(q^3 - q^2)i/\gcd(f, i)$.

Recall the following known result (see [20, p.60, Theorem C] or [12, p.3758, Lemma 2.2]): For any subgroup H of $\mathrm{R}(q)$, either $|H| = (s^3+1)s^3(s-1)$, where $s^j = q$ for some positive integer j, or $|H|$ divides $q^3(q-1)$, $12(q+1)$, $q^3 - q$, $6(q + \sqrt{3q} + 1)$, $6(q - \sqrt{3q} + 1)$, 504 or 168. By Lemma 2.11, in order to prove $\Gamma(\mathcal{D}, \Omega, \Psi)$ is connected, it suffices to prove $\mathrm{R}(q) = H := \langle t_{0,\xi,\eta}, n_{\kappa_0} w : \xi, \eta \in \mathbb{F}_q \rangle$ as $\langle t_{0,\xi,\eta} : \xi, \eta \in \mathbb{F}_q \rangle \leqslant \mathrm{R}(q)_{\infty,M}$. For any $(\eta_1, \eta_2, \eta_3, \ldots) \in V \setminus \{\infty, \mathbf{0}\}$, if $\eta_1 = 0$ then $\mathbf{0}^{t_{0,\eta_2,\eta_3}} = (\eta_1, \eta_2, \eta_3, \ldots)$, and if $\eta_1 \neq 0$, then similar to the proof of Lemma 3.7, there exist some $\delta, \xi, \eta \in \mathbb{F}_q$ such that $\mathbf{0}^{t_{0,0,\delta} n_{\kappa_0} w t_{0,\xi,\eta}} = (\eta_1, \eta_2, \eta_3, \ldots)$. Hence H is transitive on V, and thus $|H| = |V||H_\infty|$ is divisible by $(q^3+1)q^2$. When $q \geqslant 27$, we have $|H| = (s^3+1)s^3(s-1)$, where $s^j = q$ for some odd positive integer j. It follows that $j = 1$ and $H = \mathrm{R}(q)$. When $q = 3$, we use the permutation representation of $\mathrm{R}(3)$ as a primitive group of degree 28 in the database of primitive groups in Magma [3]. Now $\mathrm{R}(3)$ acts on $\Delta := \{1, 2, \ldots, 28\}$, and the two actions of $\mathrm{R}(3)$ on V and Δ are permutation isomorphic. Let Q be the normal subgroup of $\mathrm{R}(3)_1$ (the stabiliser of $1 \in \Delta$ in R(3)) which is regular on $\Delta \setminus \{1\}$. Q has two subgroups of order 9 which are normal in $\mathrm{R}(3)_1$. One of them, say X, is elementary abelian, while the other is cyclic. So H is (permutation) isomorphic to $\widetilde{H} := \langle X, \tau \rangle$ for some involution $\tau \in \mathrm{R}(3)$ as $n_{\kappa_0} w$ is an involution. Computation in MAGMA shows that $|\widetilde{H}| = 18$ or 1512 for any involution τ in R(3). Since $|H| \geqslant 28 \cdot 9$, it follows that $H = \mathrm{R}(3)$. □

4 Affine case

In this section we deal with the case where G is a finite 2-transitive group with an abelian socle acting on a point set V, which we always assume to be some vector space over a finite field. Let $u := |V| = p^d$ be the degree of G, where p is a prime and $d \geqslant 1$. Then u and the stabiliser $G_{\mathbf{0}}$ in G of the zero vector $\mathbf{0}$ are as follows ([19], [5, p.194], [4], [18, p.386]):

(i) $G_{\mathbf{0}} \leqslant \Gamma\mathrm{L}(1, q)$, $q = p^d$;

(ii) $G_{\mathbf{0}} \trianglerighteq \mathrm{SL}(n, q)$, $n \geqslant 2$, $q^n = p^d$;

(iii) $G_{\mathbf{0}} \trianglerighteq \mathrm{Sp}(n, q)$, $n \geqslant 4$, n is even, $q^n = p^d$;

(iv) $G_{\mathbf{0}} \trianglerighteq G_2(q)$, $q^6 = p^d$, $q > 2$, q is even;

(v) $G_{\mathbf{0}} = G_2(2)' \cong \mathrm{PSU}(3, 3)$, $u = 2^6$;

(vi) $G_{\mathbf{0}} \cong A_6$ or A_7, $u = 2^4$;

(vii) $G_{\mathbf{0}} \cong \mathrm{SL}(2, 13)$, $u = 3^6$;

(viii) $G_{\mathbf{0}} \trianglerighteq \mathrm{SL}(2, 5)$ or $G_{\mathbf{0}} \trianglerighteq \mathrm{SL}(2, 3)$, $d = 2$, $p = 5, 7, 11, 19, 23, 29$ or 59;

(ix) $d = 4$, $p = 3$, $G_{\mathbf{0}} \trianglerighteq \mathrm{SL}(2, 5)$ or $G_{\mathbf{0}} \trianglerighteq E$, where E is an extraspecial group of order 32.

4.1 $G_{\mathbf{0}} \leqslant \Gamma\mathrm{L}(1, q)$, $q = p^d$

Now G acts on $V = \mathbb{F}_q$, and a typical element in G is of the form

$$\tau(a, b, \varphi) : \mathbb{F}_q \to \mathbb{F}_q, z \mapsto a z^{\varphi} + b,$$

where $a \in \mathbb{F}_q^{\times}$, $b \in \mathbb{F}_q$ and $\varphi \in \mathrm{Aut}(\mathbb{F}_q) = \langle \zeta \rangle$. Here $\zeta : \mathbb{F}_q \to \mathbb{F}_q$, $z \mapsto z^p$ is the Frobenius map. For convenience, we also use $t(a, j)$ to denote $\tau(a, 0, \zeta^j)$, where j is an integer. For $\delta = \zeta^n$ and an integer $i \geqslant 0$, where $n = \min\{n_1 > 0 : \delta = \zeta^{n_1}\}$, we use $[\delta, i]$ to denote $(p^{ni} - 1)/(p^n - 1)$, and $\delta - 1$ to denote $p^n - 1$. Thus, for $i > 0$ and $c \in \mathbb{F}_q^{\times}$, $c^{[\delta,i]}$ is the product of $c^{\delta^{i-1}}$, $c^{\delta^{i-2}}$, …, c^{δ}, c in $\mathbb{F}_q^{\times}$.

Lemma 4.1. *Suppose that H is a subgroup of $\mathbb{F}_q^{\times}$, $b \in \mathbb{F}_q^{\times} \setminus H$, and δ is a field automorphism of $\mathbb{F}_q$. In the sequence: H, $Hb^{[\delta,1]}$, $Hb^{[\delta,2]}$, …, $Hb^{[\delta,n]}$, …, if j is the smallest positive integer such that $Hb^{[\delta,j]}$ equals some previous term, then $Hb^{[\delta,j]} = H$.*

Proof. If $Hb^{[\delta,j]} \neq H$, then $Hb^{[\delta,j]} = Hb^{[\delta,i]}$ for some i with $1 \leqslant i < j$. Thus $H = H(b^{[\delta,j-i]})^{\delta^i} = (Hb^{[\delta,j-i]})^{\delta^i}$ and $H = H^{\delta^{-i}} = Hb^{[\delta,j-i]}$, contradicting the definition of j. $\square$

Lemma 4.2. *Suppose that H is a subgroup of $\mathbb{F}_q^\times$ and $x \in \mathbb{F}_q^\times$. Then $x \in H$ if and only if $x^{|H|} = 1$.*

Proof. This follows from the fact that the polynomial $\alpha^{|H|} - 1$ with indeterminate α has at most (actually exactly) $|H|$ solutions in $\mathbb{F}_q^\times$. $\square$

Lemma 4.3. *Let $G \leqslant \mathrm{A\Gamma L}(1,q)$ act 2-transitively on $\mathbb{F}_q$, where $q = p^d$ and p is a prime. Suppose that P is an imprimitive block of G_0 on $\mathbb{F}_q^\times$ containing 1 such that $(q-1)/|P| \geqslant 3$ and $G_{0,1}$ is transitive on $P^{G_0} \setminus \{P\}$. Then P is a subgroup of $\mathbb{F}_q^\times$ and $|\mathbb{F}_q^\times/P| = (q-1)/|P|$ is a prime.*

Proof. Set $Y := \{\ell > 0 : t(a,\ell) \in G_0 \text{ for some } a \in \mathbb{F}_q^\times\}$. Let s be the smallest integer in Y and $\varphi := \zeta^s$. For $t(a_i,\ell_i) \in G_0$, $i = 1,2$, we have $t(a_1,\ell_1)t(a_2,\ell_2) = t(a_2a_1^{\zeta^{\ell_2}}, \ell_1+\ell_2) \in G_0$ and $t(a_i,\ell_i)^{-1} = t((1/a_i)^{\zeta^{-\ell_i}}, -\ell_i) \in G_0$. Hence $s \mid d$ and $Y = \{js : j = 1,2,\ldots\}$. If $s = d$, then $G_0 \leqslant \mathrm{GL}(1,q)$, $G_{0,1} = \{1\}$ and $G_{0,1}$ would not be transitive on $P^{G_0} \setminus \{P\}$ as $|P^{G_0} \setminus \{P\}| = (q-1)/|P| - 1 \geqslant 2$. Thus s is a proper divisor of d. For each integer i, set

$$A_i := \{t(a,si) : t(a,si) \in G_0\}, \text{ and } H_i := \{a : t(a,si) \in G_0\}.$$

Let $H := H_0$. Then A_0 is a normal cyclic subgroup of G_0, H is a cyclic subgroup of $\mathbb{F}_q^\times$, and $A_i = A_j$ if and only if $d \mid (i-j)s$. Let $t(b,s)$ be an arbitrary element of A_1. Since $A_i t(b,s)^j \subseteq A_{i+j}$ for any two integers i and j, $|A_i|$ is a constant and thus $A_i t(b,s)^j = A_{i+j}$. Hence, for $i = 1,2,\ldots,d/s-1$,

$$A_i = A_0 t(b,s)^i, \; H_i = Hb^{[\varphi,i]},$$

and $A_{d/s} = A_0 t(b,s)^{d/s} = A_0$ and $H_{d/s} = Hb^{[\varphi,d/s]} = H$. Since G_0 is the (disjoint) union $G_0 = A_0 \cup A_1 \cup \cdots \cup A_{d/s-1}$ and G_0 is transitive on $\mathbb{F}_q^\times$, we have $\mathbb{F}_q^\times = H \cup H_1 \cup H_2 \cup \cdots \cup H_{d/s-1}$.

If $b \in H$, then $H = \mathbb{F}_q^\times$, which means $\mathrm{GL}(1,q) \leqslant G_0$. Hence, for any $a \in P$, since $t(a,0) \in G_0$ and $1^{t(a,0)} = a \in P$, we have $Pa = P^{t(a,0)} = P$. Therefore P is closed under multiplication and thus P is a subgroup of $\mathbb{F}_q^\times$. In the rest of the proof we assume $b \notin H$.

Let $r := \min\{n > 0 : t(1,ns) \in G_{0,1}\}$. Then $r \leqslant d/s$, $G_{0,1} = \langle t(1,rs) \rangle$ and $|G_{0,1}| = d/(rs)$. Let $b \in H_1$. Since $1 \in H_r = Hb^{[\varphi,r]}$, we have $Hb^{[\varphi,r]} = H$. In the case when $r > 1$, if $H_j = H$ for some positive integer $j < r$, then $t(1,js) \in A_j \subseteq G_0$, which contradicts the definition of r. Hence by Lemma 4.1, in the sequence: H, $Hb^{[\varphi,1]}$, $Hb^{[\varphi,2]}$, …, $Hb^{[\varphi,r-1]}$, $Hb^{[\varphi,r]}$, …, the first r terms are pairwise distinct, and the subsequent terms repeat the previous ones. Since G_0 is transitive on $\mathbb{F}_q^\times$, we have

$$\mathbb{F}_q^\times = H \cup Hb^{[\varphi,1]} \cup \cdots \cup Hb^{[\varphi,r-1]}, \quad |\mathbb{F}_q^\times : H| = r, \quad r \mid [\varphi,r]. \tag{13}$$

Now $G_{0,1} \leqslant G_{0,P} \leqslant G_0 \leqslant \Gamma\mathrm{L}(1,q)$. If $G_{0,P} \leqslant \mathrm{GL}(1,q)$, then $G_{0,1} = \{1\}$ and is not transitive on $P^{G_0} \setminus \{P\}$. Therefore $G_{0,P} \nleqslant \mathrm{GL}(1,q)$. Set $e := \min\{j > 0 : t(c,js) \in G_{0,P} \text{ for some } c \in \mathbb{F}_q^\times\}$, and $\psi := \varphi^e = \zeta^{se}$. Then $G_{0,P} \subseteq \cup_{i \geqslant 0} A_{ie}$. For each integer i, set

$$C_i := \{t(a,ies) : t(a,ies) \in G_{0,P}\}, \text{ and } K_i := \{a : t(a,ies) \in G_{0,P}\}.$$

Then $K := K_0 \leqslant H$. Let $t(w, es)$ be an element of $G_{0,P}$. For $j = 1, 2, \ldots, r/e - 1$, we have

$$A_{je} = A_0 t(w, es)^j, \ H_{je} = Hw^{[\psi,j]}, \ C_j = C_0 t(w, es)^j, \text{ and } K_j = Kw^{[\psi,j]}. \tag{14}$$

Let i_0 be the smallest positive integer such that $Kw^{[\psi,i_0]} = K$. Then $t(1, i_0 es) \in G_{0,1}$. Since $G_{0,1} \leqslant G_{0,P}$, by the definition of r we have $r = ei_0$. By Lemma 4.1, in the sequence: K, $Kw^{[\psi,1]}$, $Kw^{[\psi,2]}$, …, $Kw^{[\psi,r/e-1]}$, $Kw^{[\psi,r/e]}$, …, the first r/e terms must be pairwise distinct, and the subsequent terms repeat the previous ones. Since $G_{0,P}$ is transitive on P, we have

$$P = K \cup Kw^{[\psi,1]} \cup Kw^{[\psi,2]} \cup \cdots \cup Kw^{[\psi,r/e-1]}, \text{ and } Kw^{[\psi,r/e]} = K. \tag{15}$$

Suppose that $e > 1$. Let $t(b, s) \in A_1$. Since $P \subseteq H \cup H_e \cup H_{2e} \cup \cdots \cup H_{r-e}$, we have $P^{t(b,s)} \subseteq H_1 \cup H_{e+1} \cup \cdots \cup H_{r-e+1}$ and thus $P^{t(b,s)} \in P^{G_0} \setminus \{P\}$. Since $A_j t(1, rs) = A_{j+r}$ and $H_j^{t(1,rs)} = H_{j+r} = H_j$ ($j = 1, 2, \ldots$), $t(1, rs)$ stabilises each term in the sequence: H, Hb, $Hb^{[\varphi,2]}$, $Hb^{[\varphi,3]}$, ….

If $K = H$, then by (14) and (15) we have $e = (q-1)/|P| \geqslant 3$ and $P^{t(b,s)} = H_1 \cup H_{e+1} \cup \cdots \cup H_{r-e+1}$. Hence $t(1, rs)$ stabilises $P^{t(b,s)}$ and $G_{0,1} = \langle t(1, rs) \rangle$ is not transitive on $P^{G_0} \setminus \{P\}$, a contradiction.

If $K \neq H$, then take $a \in H \setminus K$ and $t(a, 0) \in G_0$. We have $Pa = P^{t(a,0)} \in P^{G_0} \setminus \{P\}$ and $Pa \subseteq H \cup H_e \cup H_{2e} \cup \cdots \cup H_{r-e}$. Hence Pa can not be mapped to $P^{t(b,s)}$ by elements of $G_{0,1}$, a contradiction.

Therefore, $e = 1$, $\psi = \varphi$, and $|H/K| = (q-1)/|P| \geqslant 3$. Moreover, set $\pi := (q-1)/|P|$ and let $\{h_1 = 1, h_2, \ldots, h_\pi\}$ be a transversal of K in H. Then $P^{G_0} \setminus \{P\} = \{Ph_2, Ph_3, \ldots, Ph_\pi\}$, and thus $G_{0,1}$ is transitive on $P^{G_0} \setminus \{P\}$ if and only if the induced action of $G_{0,1}$ on the quotient group H/K is transitive on the set of non-identity elements of H/K. $G_{0,1}$ induces an automorphism group $\widehat{G}_{0,1} := \{\widehat{\tau}(1, 0, \delta) : \tau(1, 0, \delta) \in G_{0,1}\}$ on H/K, where $\widehat{\tau}(1, 0, \delta) : H/K \to H/K$, $Kb \mapsto Kb^\delta$. If $\widehat{\tau}(1, 0, \delta) = \mathrm{id}_{H/K}$, that is, $Kb^\delta = Kb$ for any $b \in H$, then $b^{\delta-1} \in K$ for any $b \in H$. By Lemma 4.2, this is equivalent to saying that $b^{(\delta-1)|K|} = 1$ for any $b \in H$. In particular, for a generator y of H, $y^{(\delta-1)|K|} = 1$. Hence $|H|$ divides $(\delta - 1)|K|$, or equivalently $\pi \mid (\delta - 1)$, and

$$\widehat{\tau}(1, 0, \delta) = \mathrm{id}_{H/K} \Leftrightarrow \pi \mid (\delta - 1). \tag{16}$$

Since the automorphism group $\widehat{G}_{0,1}$ is transitive on the set of non-identity elements of H/K, H/K must be elementary abelian (see [28, Theorem 11.1]). In addition, since H/K is cyclic, $\pi = |H/K|$ has to be a prime.

Now $P = K \cup Kw \cup Kw^{[\varphi,2]} \cup \cdots \cup Kw^{[\varphi,r-1]}$ and $|P| = |K|r$. Let $w = \rho^j$, where ρ is a generator of $\mathbb{F}_q^\times$ and $j \geqslant 1$.

If $w^{|K|r} \neq 1$, then $|\rho^{|K|r}| = |H|r/(|K|r) = \pi$ is a prime, and $|w^{|K|r}| = |(\rho^{|K|r})^j| = \pi/\gcd(j, \pi) = \pi$. Since $Kw^{[\varphi,r]} = K$ by (15), we have $w^{[\varphi,r]|K|} = 1$. Also, $r \mid [\varphi, r]$ by (13), and thus $1 = (w^{|K|r})^{[\varphi,r]/r}$. Hence $\pi = |w^{|K|r}|$ is a divisor of $[\varphi, r]/r$, and $\pi \mid (\varphi^r - 1)$. By (16) we have $\widehat{\tau}(1, 0, \varphi^r) = \mathrm{id}_{H/K}$, and $\widehat{G}_{0,1} = \{1\}$ as $G_{0,1} = \langle \tau(1, 0, \varphi^r) \rangle$. Thus $G_{0,1}$ is not transitive on $P^{G_0} \setminus \{P\}$.

Therefore $w^{|K|r} = 1$, $w^r \in K$, which means $(Kw)^r = K$. Thus $P/K = \langle Kw \rangle$ is a subgroup of order r of the quotient group $\mathbb{F}_q^\times/K$, and P is a subgroup of $\mathbb{F}_q^\times$. This completes the proof of Lemma 4.3. $\square$

The following notion will be used in our construction of all G-flag graphs (see Lemmas 4.5 and 4.7).

Definition 4.4. A quintuple of positive integers (p, d, π, r, s) is called *admissible* if the following conditions are satisfied:

(a) p is a prime, d is a positive integer, and π is an odd prime;

(b) $p \pmod{\pi}$ is a generator of the multiplication group $\mathbb{F}_\pi^\times$;

(c) $\gcd(rs, \pi - 1) = 1$ and $rs(\pi - 1) \mid d$; and

(d) $r = 1$ or $r \nmid (p^{si} - 1)/(p^s - 1)$ for $i = 1, 2, \ldots, r - 1$, and $r \mid (p^{sr} - 1)/(p^s - 1)$.

With the help of Dirichlet's theorem about primes in an arithmetic progression, it can be proved that there are infinitely many admissible quintuples (p, d, π, r, s) with $r > 1$.

Lemma 4.5. *Let $q = p^d$ with p a prime and $d \geqslant 1$. Then there exist a group $G \leqslant \mathrm{A\Gamma L}(1, q)$ and a subset P of $\mathbb{F}_q^\times$ containing 1 such that*

(a) *G is 2-transitive on $\mathbb{F}_q$,*

(b) *P is an imprimitive block of G_0 on $\mathbb{F}_q^\times$ and $(q-1)/|P| \geqslant 3$, and*

(c) *$G_{0,1}$ is transitive on $P^{G_0} \setminus \{P\}$*

if and only if $(p, d, (q-1)/|P|, r, s)$ is an admissible quintuple for some positive integers r and s.

Proof. Let G and P satisfy (a)-(c). Then by Lemma 4.3 $P \leqslant \mathbb{F}_q^\times$ and $\pi := |\mathbb{F}_q^\times/P|$ is an odd prime. P^{G_0} is the set of right cosets of P in $\mathbb{F}_q^\times$. Let s, r and φ be defined as in the proof of Lemma 4.3, and let $x = Ph$ and $h \in \mathbb{F}_q^\times \setminus P$. Then $G_{0,1} = \langle \tau(1, 0, \theta) \rangle$ $(\theta = \zeta^{sr})$ is transitive on $P^{G_0} \setminus \{P\}$ if and only if in the sequence: x, x^θ, x^{θ^2}, $\ldots$, x^{θ^i}, $x^{\theta^{i+1}}$, $\ldots$, the first $\pi - 1$ terms are pairwise distinct (that is, they are in the same cycle of the permutation induced by $\tau(1, 0, \theta)$ on $\mathbb{F}_q^\times/P$). By a similar analysis as in the proof of Lemma 4.3 leading to (16), we have $x^{\theta^i} = x$ if and only if $\pi \mid (\theta^i - 1)$. Hence the following statements are equivalent:

(T_1) $G_{0,1}$ is transitive on $P^{G_0} \setminus \{P\}$;

(T_2) $x^{\theta^i} \neq x$, $i = 1, 2, \ldots, \pi - 2$ and $x^{\theta^{\pi-1}} = x$;

(T_3) $\pi \nmid (p^{sri} - 1)$, $i = 1, 2, \ldots, \pi - 2$ and $\pi \mid (p^{sr(\pi-1)} - 1)$;

(T_4) $\gcd(sr, \pi - 1) = 1$, and $p \pmod{\pi}$ is a generator of $\mathbb{F}_\pi^\times$.

Thus $(\pi-1)\mid d$ and $rs(\pi-1)\mid d$ by (T_4). By the proof of Lemma 4.3, we know G_0 is generated by $\{t(a,0): a\in H\}$ and $t(b,s)$, where H is the subgroup of $\mathbb{F}_q^{\times}$ of index r and b is some element of $\mathbb{F}_q^{\times}$, and (13) holds.

(i) If $r=1$, then $H=\mathbb{F}_q^{\times}$ and G_0 is the group generated by $\mathrm{GL}(1,q)$ and $\tau(1,0,\varphi)$.

(ii) If $r>1$, then by (13) and Lemma 4.1, we have $Hb\neq H$, $Hb^{[\varphi,2]}\neq H$, ..., $Hb^{[\varphi,r-1]}\neq H$. This is equivalent to saying that $|H|=(q-1)/r$ and $b^{|H|}\neq 1$, $b^{[\varphi,2]|H|}\neq 1$, ..., $b^{[\varphi,r-1]|H|}\neq 1$ by Lemma 4.2. Denote the set of solutions in $\mathbb{F}_q^{\times}$ of each of the equations

$$\alpha^{|H|}=1, \alpha^{[\varphi,2]|H|}=1, \ldots, \alpha^{[\varphi,r-1]|H|}=1$$

by E_1, E_2, ..., E_{r-1}, respectively. Then E_i $(i=1,2,\ldots,r-1)$ is a cyclic subgroup of $\mathbb{F}_q^{\times}$ with $|E_i|=\gcd(p^d-1,[\varphi,i]|H|)=|H|\cdot\gcd(r,[\varphi,i])$, and E_i/H is a subgroup of $\mathbb{F}_q^{\times}/H$ of order $\gcd(r,[\varphi,i])$. Hence the existence of b satisfying (13) implies $\cup_{i=1}^{r-1}E_i\neq\mathbb{F}_q^{\times}$, and so $r\nmid(p^{si}-1)/(p^s-1)$, $i=1,2,\ldots,r-1$, and $r\mid(p^{sr}-1)/(p^s-1)$. Thus (p,d,π,r,s) is an admissible quintuple.

Conversely, suppose that (p,d,π,r,s) is an admissible quintuple. Let P be the subgroup of $\mathbb{F}_q^{\times}$ of index π and let $\varphi:=\zeta^s$. If $r=1$, then choose G to be the group generated by $\mathrm{GL}(1,q)$ and $\tau(1,0,\varphi)$. If $r>1$, then choose G to be the group generated by $\{t(a,0): a\in H\}$ and $t(b,s)$, where H is the subgroup of $\mathbb{F}_q^{\times}$ of index r and b is a generator of $\mathbb{F}_q^{\times}$. Then (13) together with (T_1)-(T_4) above implies that G and P satisfy (a)-(c). □

Remark 4.6. For an admissible quintuple (p,d,π,r,s), there are $\phi(r)$ different subgroups G of $\mathrm{A\Gamma L}(1,q)$ such that $s=\min\{\ell>0: t(a,\ell)\in G_0$ for some $a\in\mathbb{F}_q^{\times}\}$ and $r=\min\{n>0: t(1,ns)\in G_{0,1}\}$, where $q:=p^d$ and $\phi(r):=|\{\ell>0:\ell\leqslant r,\ \gcd(\ell,r)=1\}|$. In fact, if $r=1$, then G_0 is the group generated by $\mathrm{GL}(1,q)$ and $\tau(1,0,\zeta^s)$. Assume $r>1$. Let $\varphi:=\zeta^s$, and let H and E_i $(1\leqslant i\leqslant r-1)$ be as in the proof of Lemma 4.5. Then $\{[\varphi,1],\ldots,[\varphi,r]\}$ is a complete residue system modulo r by (13). It follows that $\cup_{i=1}^{r-1}(E_i/H)$ is the set of all non-generators of $\mathbb{F}_q^{\times}/H$. Let ξ be a fixed generator of $\mathbb{F}_q^{\times}$. Then $\mathbb{F}_q^{\times}\setminus\cup_{i=1}^{r-1}E_i=\cup_{i=1}^{\phi(r)}H\xi^{\ell_i}$, where $\{\ell_1=1,\ell_2,\ldots,\ell_{\phi(r)}\}$ is a reduced residue system modulo r, and hence G_0 is the group generated by $\{t(a,0): a\in H\}$ and $t(\xi^{\ell_i},s)$ for some $i\in\{1,2,\ldots,\phi(r)\}$.

Lemma 4.7. *Assume that G and P satisfy (a)-(c) in Lemma 4.5 with $|P|>1$. Let H,K,s,r be defined as in the proof of Lemma 4.3 and $\pi:=(q-1)/|P|$. Set $\mathcal{D}:=(\mathbb{F}_q,L^G)$ and $\Omega:=(0,L)^G$, where $L:=P\cup\{0\}$. Then $\mathcal{D}$ is a 2-$(q,|P|+1,\lambda)$ design.*

(a) *If $G\neq\mathrm{A\Gamma L}(1,16)$ or $|P|\neq 3$, then $\mathcal{D}$ is a 2-$(q,|P|+1,|P|+1)$ design admitting G as an automorphism group, Ω is a feasible orbit of G on the flag set of $\mathcal{D}$, and there are exactly two distinct self-paired G-orbits on $\mathrm{F}(\mathcal{D},\Omega)$.*

(b) *Assume $\lambda>1$. Denote the two distinct self-paired G-orbits on $\mathrm{F}(\mathcal{D},\Omega)$ by Ψ_1 and Ψ_2, and denote $\Gamma_i=\Gamma(\mathcal{D},\Omega,\Psi_i)$ for $i=1,2$. Then $\Gamma_i[\Omega(0),\Omega(1)]\cong(\pi-1)\cdot K_2$,*

$i = 1, 2$. *Moreover,* Γ_1 *has* π *connected components each with order* $|\Omega|/\pi = q$ *and valency* $(\pi-1)(q-1)/\pi$*, and* Γ_2 *is connected with order* $|\Omega| = \pi q$ *and valency* $(\pi-1)(q-1)/\pi$.

Proof. (a) By Lemma 4.3 P is a nontrivial subgroup of $\mathbb{F}_q^\times$. If $\lambda = 1$, then L is a subfield of $\mathbb{F}_q$ by [19, Section 4]. Conversely, if L is a subfield of $\mathbb{F}_q$, then each element in G interchanging 0 and 1 must stabilise L, and thus $\lambda = 1$. Moreover, let $|L| = p^t$. Then $(p^d-1)/(p^t-1) - 1 = |P^{G_0} \setminus \{P\}| \leqslant |G_{0,1}| \leqslant d$ as $G_{0,1}$ is transitive on $P^{G_0} \setminus \{P\}$. Since $|P| > 1$, this can happen only when $(p, d, t) = (2, 4, 2)$, or equivalently $(p, d, |P|) = (2, 4, 3)$. Therefore $\lambda = 1$ implies $G = \mathrm{A\Gamma L}(1, 16)$ (by Remark 4.6) and $|P| = 3$.

Let $P^{G_0} = \{P_1 = P, P_2, \ldots, P_\pi\}$ and $L_i := P_i \cup \{0\}$, $i = 1, 2, \ldots, \pi$. Since $\gcd(r, \pi-1) = 1$, r is odd and $|H| = (q-1)/r$ is even when $p > 2$. Thus $-1 \in H$ and $\gamma := \tau(-1, 0, \mathrm{id}) \in G_0$. Similarly, we have $-1 \in P$ since $|P| = (q-1)/\pi$ is even when $p > 2$.

Let $\Psi = ((0, M), (1, N))^G$ be a G-orbit on $\mathrm{F}(\mathcal{D}, \Omega)$, where $M = L_2 = Px \cup \{0\}$ and $N = L_j + 1$, for some $x \in \mathbb{F}_q^\times \setminus P$ and $j \geqslant 2$. Then Ψ is self-paired if and only if there is some $g \in G$ interchanging $(0, M)$ and $(1, N)$. Hence $g = h\widetilde{1}$, where $\widetilde{1}$ is the translation induced by 1, that is, $\widetilde{1} : \mathbb{F}_q \to \mathbb{F}_q$, $z \mapsto z + 1$, and $h \in G_0$ is such that $1^h = -1$ and h interchanges P_2 and P_j. Thus $h \in \gamma G_{0,1} = G_{0,1}\gamma$ and the action of h on $P^{G_0} \setminus \{P\}$ has a cycle $(P_2\ P_j)$, possibly with $P_2 = P_j$. Since γ stabilises each element in $P^{G_0} \setminus \{P\}$, we just need $h\gamma$ $(\in G_{0,1})$ to have a cycle $(P_2\ P_j)$ on $P^{G_0} \setminus \{P\}$. Since $G_{0,1} = \langle \tau(1, 0, \theta) \rangle$ $(\theta = \zeta^{sr})$ induces a regular permutation group on $P^{G_0} \setminus \{P\}$, $\tau(1, 0, \theta)^{\frac{\pi-1}{2}}$ induces the unique permutation on $P^{G_0} \setminus \{P\}$ which has a 2-cycle, and its cycle decomposition on $P^{G_0} \setminus \{P\}$ is $(P_2\ P_2^\varepsilon) \cdots$, where $\varepsilon := \theta^{\frac{\pi-1}{2}}$. Thus Ψ is self-paired if and only if $P_j = P_2$ or P_2^ε.

(b) First assume $P_j = P_2$, and let $\Psi_1 := ((0, L_2), (1, L_2 + 1))^G$. One can verify that the set $(1, N)^{G_{0,1,Px}}$ of vertices in $\Omega(1)$ adjacent to $(0, M)$ in Γ_1 is $\{(1, N)\}$, and the set of vertices in $\Omega(1)$ adjacent to $(0, L_i)$ is $\{(1, L_i + 1)\}$, $i = 2, 3, \ldots, \pi$, which implies $\Gamma_1[\Omega(0), \Omega(1)] \cong (\pi - 1) \cdot K_2$.

Set $J := \langle G_{0,Px}, \kappa \rangle$, where $\kappa := \tau(-1, 1, \mathrm{id})$ interchanges $(0, M)$ and $(1, N)$. If $(Px)^\kappa = Px$, then $(1, \widetilde{N}) = (0, M)^\kappa \in \Omega(1)$, where $\widetilde{N} = Px \cup \{1\}$. Suppose $(1, \widetilde{L})$ is the flag in $\Omega(1)$ such that $0 \in \widetilde{L}$, and let $\widetilde{P} := \widetilde{L} \setminus \{1\}$. Then $\widetilde{P}^{G_1} \setminus \{\widetilde{P}\} = (Px)^{G_{1,0}} = P^{G_0} \setminus \{P\}$ as Ω is feasible. It follows that $\widetilde{L} = L_1$ and G_{L_1} is transitive on L_1, which is a contradiction by Lemma 2.8. Therefore κ does not stabilise Px, and J is transitive on $\mathbb{F}_q$ as $G_{0,Px}$ is transitive on $\mathbb{F}_q^\times \setminus Px$ by Lemma 2.9. Since $\tau(-1, c, \mathrm{id})\tau(a, 0, \delta) = \tau(a, 0, \delta)\tau(-1, ac^\delta, \mathrm{id})$ for $c \in \mathbb{F}_q$ and $\tau(a, 0, \delta) \in G_{0,Px}$, one can see that $J_0 = G_{0,Px}$. By Lemmas 2.11 and 2.10, the number of connected components of Γ_1 is equal to $|G : J| = |G_0 : J_0| = \pi$.

Next assume $P_j = P_2^\varepsilon$, and let $\Psi_2 := ((0, L_2), (1, L_2^\varepsilon + 1))^G$. One can verify that $(1, N)^{G_{0,1,Px}} = \{(1, N)\}$, and the set of vertices in $\Omega(1)$ adjacent to $(0, L_i)$ is $\{(1, L_i^\varepsilon + 1)\}$, $i = 2, 3, \ldots, \pi$, which implies $\Gamma_2[\Omega(0), \Omega(1)] \cong (\pi - 1) \cdot K_2$.

Set $\widetilde{J} := \langle G_{0,Px}, \eta \rangle$, where $\eta := \tau(-1, 1, \varepsilon)$ interchanges $(0, M)$ and $(1, N)$. Similar to J, $\widetilde{J}$ is transitive on $\mathbb{F}_q$. If $a \in \mathbb{F}_q^\times \setminus Px$, then by the transitivity of $G_{0,Px}$ on $\mathbb{F}_q^\times \setminus Px$, there is some $\tau(a, 0, \delta) \in G_{0,Px}$, and thus $\tau(a, 0, \delta)^{-1}\eta^{-1}\tau(a, 0, \delta)\eta = \tau(a^{\varepsilon-1}, -a^\varepsilon + 1, \mathrm{id}) \in \widetilde{J}$. In particular, we have $\tau(a^{\varepsilon-1}, -a^\varepsilon + 1, \mathrm{id}) = \tau(1, -a + 1, \mathrm{id}) \in \widetilde{J}$ for any $a \in \mathbb{F}_\varepsilon^\times \setminus Px$, where $\mathbb{F}_\varepsilon$ is the subfield of $\mathbb{F}_q$ such that $|\mathbb{F}_\varepsilon| = p^{sr(\pi-1)/2}$.

Case 1: $p > 2$. Since $|P^{G_0} \setminus \{P\}| \geqslant 2$, we can choose $Px \in P^{G_0} \setminus \{P\}$ such that $2 \notin Px$. Then $\tau(2^{\varepsilon-1}, -2^{\varepsilon} + 1, \mathrm{id}) = \tau(1, -1, \mathrm{id}) \in \widetilde{J}$. It follows that $\tau(1, 0, \varepsilon) \in \widetilde{J}_0 \setminus G_{0,Px}$. By Lemma 2.10, $G_{0,Px}$ is maximal in G_0 and hence $\widetilde{J}_0 = G_0$. Therefore $\widetilde{J} = G$ and Γ_2 is connected.

Case 2: $p = 2$. First assume $\varepsilon - 1 \nmid \frac{q-1}{\pi}$. Then $sr(\pi - 1)/2 > 1$ as $\varepsilon = \zeta^{sr(\pi-1)/2}$. Since $\tau(a^{\varepsilon-1}, -a^{\varepsilon} + 1, \mathrm{id}) = \tau(1, -a + 1, \mathrm{id}) \in \widetilde{J}$ for any $a \in \mathbb{F}_{\varepsilon}^{\times} \setminus Px$ and $|\mathbb{F}_{\varepsilon}^{\times} \cap Py| = (\varepsilon - 1)/\pi$ for any $y \in \mathbb{F}_q^{\times}$, we have $|\widetilde{T}| \geqslant (\varepsilon-1)(\pi-1)/\pi$, where $\widetilde{T} := \langle \tau(1, -a+1, \mathrm{id}) \mid a \in \mathbb{F}_{\varepsilon}^{\times} \setminus Px \rangle$. One can see that $|\widetilde{T}|$ is a divisor of $|\mathbb{F}_{\varepsilon}| = 2^{sr(\pi-1)/2}$. If $|\widetilde{T}| \neq |\mathbb{F}_{\varepsilon}|$, then $2 \leqslant |\mathbb{F}_{\varepsilon}|/|\widetilde{T}| \leqslant |\mathbb{F}_{\varepsilon}|/(|\mathbb{F}_{\varepsilon}^{\times}|(\pi - 1)/\pi)$, or equivalently $1/2 \geqslant (\pi - 1)|\mathbb{F}_{\varepsilon}^{\times}|/(\pi|\mathbb{F}_{\varepsilon}|)$. This happens only when $\pi = 3$ and $sr = 2$, which is impossible as $\gcd(sr, \pi - 1) = 1$ by (T_4) in the proof of Lemma 4.5. Therefore, $|\widetilde{T}| = |\mathbb{F}_{\varepsilon}|$ and $\tau(1, 1, \mathrm{id}) \in \widetilde{T} \leqslant \widetilde{J}$, which implies $\tau(1, 0, \varepsilon) \in \widetilde{J}_0 \setminus G_{0,Px}$ and $\widetilde{J}_0 = G_0$ by the maximality of $G_{0,Px}$ in G_0. Hence $\widetilde{J} = G$ and Γ_2 is connected.

Next assume $\varepsilon - 1 \mid \frac{q-1}{\pi} = |P|$. Then $\mathbb{F}_{\varepsilon}^{\times} \leqslant P$. If $sr(\pi - 1)/2 > 1$, then there are $a, b \in \mathbb{F}_{\varepsilon}^{\times}$ such that $a + b = 1$, and thus $\tau(1, 1, \mathrm{id}) = \tau(1, a + 1, \mathrm{id})\tau(1, b + 1, \mathrm{id}) \in \widetilde{J}$. Therefore, similar to the above discussion we have $\widetilde{J} = G$ and Γ_2 is connected. If $sr(\pi - 1)/2 = 1$, then $s = r = 1$, $\pi = 3$ and $\varepsilon = \zeta$. It follows that $G = \mathrm{A\Gamma L}(1, 2^d)$ (by Remark 4.6) with d even. Now $\tau(a^{\varepsilon-1}, -a^{\varepsilon}+1, \mathrm{id}) = \tau(a, a^2+1, \mathrm{id})$ for $a \in \mathbb{F}_q^{\times} \setminus Px$. One can verify that $G_{0,Px}$ normalizes $\widehat{T} := \{\tau(a, b, \mathrm{id}) : \tau(a, b, \mathrm{id}) \in \widetilde{J}\} \leqslant \widetilde{J}$, $G_{0,Px} \cap \widehat{T} = \{\tau(a, 0, \mathrm{id}) : a \in P\}$, and η normalizes $\widehat{T}G_{0,Px}$. Moreover, since $\eta^2 = \tau(1, 0, \varepsilon^2) \in G_{0,Px}$, $\langle \eta \rangle \cap \widehat{T}G_{0,Px}$ is of index f in $\langle \eta \rangle$, where $f = 1$ or 2. Hence $|\widetilde{J}| = |(\widehat{T}G_{0,Px})\langle \eta \rangle| = |\widehat{T}G_{0,Px}|f = |\widehat{T}||G_{0,Px}|f/|P|$. We can see that $|\widehat{T}| = (q - 1)n$, where n is the order of the group $\{\tau(1, c, \mathrm{id}) : \tau(1, c, \mathrm{id}) \in \widetilde{J}\}$. Hence $n \mid q = 2^d$, and $|G : \widetilde{J}| = 2^d|G_0|/(\pi n f|G_{0,Px}|) = 2^d/(nf)$. Since $G_{0,Px} \leqslant \widetilde{J}_0$ and $G_{0,Px}$ is maximal in G_0 by Lemma 2.10, $|G : \widetilde{J}|$ is equal to 1 or π. Therefore $|G : \widetilde{J}| = 1$, and Γ_2 is connected. □

4.2 $G_0 \trianglerighteq \mathrm{Sp}(n, q)$, $n \geqslant 4$ even, $u = q^n = p^d$

We denote the underlying symplectic space by (V, φ), where $V = \mathbb{F}_q^n$ and φ is a symplectic form. Set $H := \mathrm{Sp}(n, q) \trianglelefteq G_{\mathbf{0}}$. Suppose that P is an imprimitive block of $G_{\mathbf{0}}$ on $V \setminus \{\mathbf{0}\}$ and let $\mathbf{x} \in P$. Define $C_i := \{\mathbf{z} \in V \setminus \langle \mathbf{x} \rangle : \varphi(\mathbf{z}, \mathbf{x}) = i\}$, $i \in \mathbb{F}_q$. By Witt's Lemma, each C_i is an orbit of $H_{\mathbf{x}}$ on $V \setminus \langle \mathbf{x} \rangle$. Moreover, $|C_i| = q^{n-1}$ for $i \in \mathbb{F}_q^{\times}$ and $|C_0| = q^{n-1} - q$.

First assume that $C_0 \nsubseteq P$. Suppose that P includes j orbits of $H_{\mathbf{x}}$ of length q^{n-1} ($0 \leqslant j < q$) and P contains ℓ elements in $\langle \mathbf{x} \rangle$ ($1 \leqslant \ell < q$). Then $|P| = jq^{n-1} + \ell$ and $jq^{n-1} + \ell = \gcd(jq^{n-1} + \ell, q^n - 1) = \gcd(q^n - 1, \ell q + j) \leqslant \ell q + j$. This implies $j = 0$ and $P \subseteq \langle \mathbf{x} \rangle$ as $n \geqslant 4$. If there is a feasible G-orbit on the flag set of the 2-$(u, |P| + 1, \lambda)$ design $\mathcal{D} := (V, L^G)$, where $L := P \cup \{\mathbf{0}\}$, then by Lemma 2.12, we have $\lambda = 1$.

Next assume that $C_0 \subseteq P$. Suppose that P includes $j - 1$ orbits of $H_{\mathbf{x}}$ of length q^{n-1} ($1 \leqslant j < q+1$) and P contains ℓ elements in $\langle \mathbf{x} \rangle$ ($1 \leqslant \ell < q$). Then $|P| = jq^{n-1}+\ell-q$ and $jq^{n-1} + \ell - q = \gcd(jq^{n-1} + \ell - q, q^n - 1) = \gcd(q^n - 1, q^2 - \ell q - j)$. If $q^2 - \ell q - j \neq 0$, then $jq^{n-1} + \ell - q \leqslant q^2 - \ell q - j$, which is impossible as $n \geqslant 4$. If $q^2 - \ell q - j = 0$, then $j = q$, $\ell = q - 1$, and thus $P = V \setminus \{\mathbf{0}\}$, violating the condition $(u - 1)/|P| \geqslant 3$.

Therefore, there is no 2-$(u, m+1, \lambda)$ design as in Lemma 2.10 with $\lambda > 1$ admitting G as a group of automorphisms.

4.3 $\mathrm{SL}(2,q) = \mathrm{Sp}(2,q) \trianglelefteq G_{\mathbf{0}}$, $u = q^2 = p^d$

Denote the underlying symplectic space by (V, φ), where $V = \mathbb{F}_q^2$ and φ is a symplectic form. Let $H := \mathrm{Sp}(2,q) = \mathrm{SL}(2,q) \trianglelefteq G_{\mathbf{0}}$. Suppose that P is an imprimitive block of $G_{\mathbf{0}}$ on $V \setminus \{\mathbf{0}\}$ and $\mathbf{x} \in P$. Define C_i for $i \in \mathbb{F}_q$ as in Section 4.2. Then $C_0 = \emptyset$ and $C_i = \langle \mathbf{x} \rangle + \mathbf{z}_i$ for each $i \in \mathbb{F}_q^\times$, where $\mathbf{z}_i \in C_i$. By Witt's Lemma, each C_i is an orbit of $H_{\mathbf{x}}$ on $V \setminus \langle \mathbf{x} \rangle$. Denote all 1-subspaces of V by $U = \langle \mathbf{x} \rangle$, $U_1, \ldots, U_q$.

If $P \subseteq \langle \mathbf{x} \rangle$ and there is a feasible G-orbit on the flag set of the 2-$(u, |P|+1, \lambda)$ design $\mathcal{D} := (V, L^G)$, where $L := P \cup \{\mathbf{0}\}$, then $\lambda = 1$ by Lemma 2.12. So we assume that $P \nsubseteq \langle \mathbf{x} \rangle$ and $P = (U + \mathbf{z}_{t_1}) \cup \cdots \cup (U + \mathbf{z}_{t_j}) \cup E$ $(1 \leqslant j < q)$, where E is a subset of $\langle \mathbf{x} \rangle$ of size ℓ $(1 \leqslant \ell < q)$, $t_1, \ldots, t_j$ are pairwise distinct elements of $\mathbb{F}_q^\times$, and $\mathbf{z}_{t_n} \in C_{t_n}$, $n = 1, 2, \ldots, j$.

Since H is transitive on the set of 1-subspaces of V, there is some $\gamma \in H$ such that $U^\gamma = U_1$. Hence $P^\gamma = (U_1 + \mathbf{z}_{t_1}^\gamma) \cup \cdots \cup (U_1 + \mathbf{z}_{t_j}^\gamma) \cup E^\gamma$. Since U and U_1 are not parallel, $P^\gamma \cap P \neq \emptyset$ and thus $P = P^\gamma \supseteq U_1 + \mathbf{z}_{t_1}^\gamma$. Since $|(U_1 + \mathbf{z}_{t_1}^\gamma) \cap (U + \mathbf{z}_{t_n})| = 1$, $n = 1, 2, \ldots, j$, and $|(U_1 + \mathbf{z}_{t_1}^\gamma) \cap U| = 1$, we have $j + 1 \geqslant |U_1 + \mathbf{z}_{t_1}^\gamma| = q$ and thus $j = q - 1$. Now $|P| = q^2 - q + \ell$ is a divisor of $q^2 - 1$, that is, $q^2 - q + \ell = \gcd(q^2 - q + \ell, q^2 - 1) = \gcd(q^2 - 1, q - \ell - 1)$. Thus $\ell = q - 1$ and $P = V \setminus \{\mathbf{0}\}$, violating the condition $(u-1)/|P| \geqslant 3$. Hence there is no 2-$(u, m+1, \lambda)$ design as in Lemma 2.10 with $\lambda > 1$.

4.4 $G_{\mathbf{0}} \trianglerighteq \mathrm{SL}(n,q)$, $n \geqslant 3$, $u = q^n = p^d$

Suppose that P is an imprimitive block of $G_{\mathbf{0}}$ on $V \setminus \{\mathbf{0}\}$ and $\mathbf{x} \in P$, where $V = \mathbb{F}_q^n$. Since $V \setminus \langle \mathbf{x} \rangle$ is a $G_{\mathbf{0},\mathbf{x}}$-orbit of length $q^n - q$, if P does not include this orbit, then $P \subseteq \langle \mathbf{x} \rangle$; if in addition there is a feasible G-orbit on the flag set of the 2-$(u, |P|+1, \lambda)$ design $\mathcal{D} := (V, L^G)$, where $L := P \cup \{\mathbf{0}\}$, then $\lambda = 1$ by Lemma 2.12. If P contains $V \setminus \langle \mathbf{x} \rangle$, then since $|P|$ is a divisor of $|V \setminus \{\mathbf{0}\}| = q^n - 1$, we have $P = V \setminus \{\mathbf{0}\}$, violating the condition $(u-1)/|P| \geqslant 3$. Therefore, there is no 2-$(u, m+1, \lambda)$ design as in Lemma 2.10 with $\lambda > 1$.

4.5 $G_{\mathbf{0}} \trianglerighteq G_2(q)$, $u = q^6 = p^d$, $q > 2$ even

Suppose that P is an imprimitive block of $G_{\mathbf{0}}$ on $V \setminus \{\mathbf{0}\}$ and $\mathbf{a} \in P$, where $V = \mathbb{F}_q^6$. Then P is also an imprimitive block of $G_2(q)$ on $V \setminus \{\mathbf{0}\}$ and P is the union of some orbits of $G_2(q)_{\mathbf{a}}$ on $V \setminus \{\mathbf{0}\}$. We will determine all possible lengths of the $G_2(q)_{\mathbf{a}}$-orbits on $V \setminus \{\mathbf{0}\}$, with the help of the knowledge about $G_2(q)$ from [29, Section 4.3.4].

Now take a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_8\}$ of the octonion algebra $\mathbb{O}$ over $\mathbb{F}_q$ with the multiplication given by Table 2, or equivalently by Table 3, where $\mathbf{e} := \mathbf{x}_4 + \mathbf{x}_5$ is the identity element of $\mathbb{O}$ (since the characteristic is 2, we omit the signs).

	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_8$
$\mathbf{x}_1$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$
$\mathbf{x}_2$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{x}_5$	$\mathbf{x}_6$
$\mathbf{x}_3$	$\mathbf{0}$	$\mathbf{x}_1$	$\mathbf{0}$	$\mathbf{x}_3$	$\mathbf{0}$	$\mathbf{x}_5$	$\mathbf{0}$	$\mathbf{x}_7$
$\mathbf{x}_4$	$\mathbf{x}_1$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{x}_4$	$\mathbf{0}$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{0}$
$\mathbf{x}_5$	$\mathbf{0}$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{0}$	$\mathbf{x}_5$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{x}_8$
$\mathbf{x}_6$	$\mathbf{x}_2$	$\mathbf{0}$	$\mathbf{x}_4$	$\mathbf{0}$	$\mathbf{x}_6$	$\mathbf{0}$	$\mathbf{x}_8$	$\mathbf{0}$
$\mathbf{x}_7$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{x}_7$	$\mathbf{x}_8$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{x}_8$	$\mathbf{x}_5$	$\mathbf{x}_6$	$\mathbf{x}_7$	$\mathbf{x}_8$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

Table 2. Multiplication table of $\mathbb{O}$

	e	x_1	x_8	x_2	x_7	x_3	x_6	x_4
e	e	x_1	x_8	x_2	x_7	x_3	x_6	x_4
x_1	x_1	0	x_4	0	x_3	0	x_2	0
x_8	x_8	$\mathrm{e}+\mathrm{x}_4$	0	x_6	0	x_7	0	x_8
x_2	x_2	0	x_6	0	$\mathrm{e}+\mathrm{x}_4$	x_1	0	x_2
x_7	x_7	x_3	0	x_4	0	0	x_8	0
x_3	x_3	0	x_7	x_1	0	0	$\mathrm{e}+\mathrm{x}_4$	x_3
x_6	x_6	x_2	0	0	x_8	x_4	0	0
x_4	x_4	x_1	0	0	x_7	0	x_6	x_4

Table 3. Multiplication table of $\mathbb{O}$

There is a quadratic form N and an associated bilinear form f satisfying

$$N(\mathbf{x}_i) = 0 \text{ and } f(\mathbf{x}_i, \mathbf{x}_j) = \begin{cases} 0, & i+j \neq 9, \\ 1, & i+j = 9, \end{cases} \quad i, j = 1, 2, \ldots, 8.$$

$G_2(q)$ is the automorphism group of this octonion algebra, and since it preserves the multiplication table, a straightforward computation shows that $G_2(q)$ preserves N and f. Moreover, $G_2(q)$ induces a faithful action on $\mathbf{e}^\perp/\langle \mathbf{e} \rangle$, where $\mathbf{e}^\perp = \langle \mathbf{x}_1, \mathbf{x}_8, \mathbf{x}_2, \mathbf{x}_7, \mathbf{x}_3, \mathbf{x}_6, \mathbf{e} \rangle$. Hence $G_2(q)$ can be embedded into $\mathrm{Sp}(6, q)$.

Let $\langle \mathbf{x} \rangle$ denote the subspace of $\mathbb{O}$ spanned by $\mathbf{x}$, and let $\langle \overline{\mathbf{x}} \rangle$ denote the subspace of $\mathbf{e}^\perp/\langle \mathbf{e} \rangle$ spanned by $\overline{\mathbf{x}}$, where $\overline{\mathbf{x}} = \mathbf{x} + \langle \mathbf{e} \rangle$. The actions of $G_2(q)$ on $\mathbf{e}^\perp/\langle \mathbf{e} \rangle$ and V are permutation isomorphic.

We know that $G_2(q)_{\langle \overline{\mathbf{x}}_1 \rangle}$ has four orbits on the set of 1-subspaces of $\mathbf{e}^\perp/\langle \mathbf{e} \rangle$ ([7, Lemma 3.1], [19, p.72]), which are represented by $\langle \overline{\mathbf{x}}_1 \rangle$, $\langle \overline{\mathbf{x}}_8 \rangle$, $\langle \overline{\mathbf{x}}_2 \rangle$ and $\langle \overline{\mathbf{x}}_7 \rangle$ and have length 1, q^5, $q(q+1)$ and $q^3(q+1)$, respectively.

Actually, $\overline{\mathbf{x}}_8$ is not perpendicular to $\overline{\mathbf{x}}_1$, while $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_7$ are perpendicular to $\overline{\mathbf{x}}_1$. Hence the orbit of $\langle \overline{\mathbf{x}}_8 \rangle$ is different from the orbit of $\langle \overline{\mathbf{x}}_2 \rangle$ and the orbit of $\langle \overline{\mathbf{x}}_7 \rangle$ under $G_2(q)_{\langle \overline{\mathbf{x}}_1 \rangle}$. On the other hand, if there exists some $\varphi \in G_2(q)_{\langle \overline{\mathbf{x}}_1 \rangle}$ such that $\varphi(\langle \overline{\mathbf{x}}_2 \rangle) = \langle \overline{\mathbf{x}}_7 \rangle$, then $\varphi(\mathbf{x}_1) = a\mathbf{x}_1 + \ell \mathbf{e}$ and $\varphi(\mathbf{x}_2) = b\mathbf{x}_7 + s\mathbf{e}$, $a, b \neq 0$, and hence $\mathbf{0} = \varphi(\mathbf{x}_1)\varphi(\mathbf{x}_2) = (a\mathbf{x}_1 + \ell\mathbf{e})(b\mathbf{x}_7 + s\mathbf{e}) = ab\mathbf{x}_3 + \ell b\mathbf{x}_7 + as\mathbf{x}_1 + \ell s\mathbf{e}$, which is a contradiction as $\mathbf{x}_1$, $\mathbf{x}_3$, $\mathbf{x}_7$ and $\mathbf{e}$ are linearly independent.

Lemma 4.8. *Let $\mathbf{a} \in V \setminus \{\mathbf{0}\}$. Then $G_2(q)_{\mathbf{a}}$ has $q-1$ orbits of length 1, $q-1$ orbits of length q^5, one orbit of length $q(q^2-1)$ and one orbit of length $q^3(q^2-1)$ on $V \setminus \{\mathbf{0}\}$.*

Proof. Denote the $G_2(q)_{\overline{\mathbf{x}}_1}$-orbits containing $\overline{\mathbf{x}}_8$, $\overline{\mathbf{x}}_2$ and $\overline{\mathbf{x}}_7$ by Θ_8, Θ_2 and Θ_7, respectively. Since the actions of $G_2(q)$ on $\mathbf{e}^\perp/\langle \mathbf{e} \rangle$ and V are permutation isomorphic, it suffices to prove that $|\Theta_8| = q^5$, $|\Theta_2| = q(q^2-1)$ and $|\Theta_7| = q^3(q^2-1)$.

To prove $|\Theta_8| = q^5$, we first show that $\Theta_8 \cap \langle \overline{\mathbf{w}} \rangle \neq \emptyset$ for each $\langle \overline{\mathbf{w}} \rangle$ in the $G_2(q)_{\langle \overline{\mathbf{x}}_1 \rangle}$-orbit containing $\langle \overline{\mathbf{x}}_8 \rangle$. In fact, let $\varphi \in G_2(q)_{\langle \overline{\mathbf{x}}_1 \rangle}$, $\varphi(\overline{\mathbf{x}}_1) = a\overline{\mathbf{x}}_1$ for some $a \neq 0$ and $\varphi(\overline{\mathbf{x}}_8) = \overline{\mathbf{z}} \in \langle \overline{\mathbf{w}} \rangle$. Define a linear transformation ψ stabilising $\mathbf{e}$ and $\overline{\mathbf{x}}_1$ as follows:

$$\psi(\mathbf{x}_1, \mathbf{x}_8, \mathbf{x}_2, \mathbf{x}_7, \mathbf{x}_3, \mathbf{x}_6, \mathbf{x}_4) := \left(\frac{1}{a}\varphi(\mathbf{x}_1), a\varphi(\mathbf{x}_8), \varphi(\mathbf{x}_2), \varphi(\mathbf{x}_7), \frac{1}{a}\varphi(\mathbf{x}_3), a\varphi(\mathbf{x}_6), \varphi(\mathbf{x}_4) \right).$$

Then ψ preserves Table 3 and hence $\psi \in G_2(q)_{\overline{\mathbf{x}}_1}$. Now $\psi(\overline{\mathbf{x}}_8) = a\overline{\mathbf{z}} \in \Theta_8 \cap \langle \overline{\mathbf{w}} \rangle$.

On the other hand, if there are distinct $s, t \in \mathbb{F}_q^{\times}$ such that $\psi_1(\overline{\mathbf{x}}_8) = s\overline{\mathbf{w}}$ and $\psi_2(\overline{\mathbf{x}}_8) = t\overline{\mathbf{w}}$, where ψ_1, $\psi_2 \in G_2(q)_{\overline{\mathbf{x}}_1}$, then $sf(\overline{\mathbf{w}}, \overline{\mathbf{x}}_1) = f(\overline{\mathbf{x}}_8, \overline{\mathbf{x}}_1) = tf(\overline{\mathbf{w}}, \overline{\mathbf{x}}_1)$ and hence $s = t$ as $f(\overline{\mathbf{w}}, \overline{\mathbf{x}}_1) \neq 0$, a contradiction. Therefore, $|\Theta_8 \cap \langle \overline{\mathbf{w}} \rangle| = 1$ and thus $|\Theta_8| = q^5$. Similarly, for each $c \in \mathbb{F}_q^{\times}$, the length of the $G_2(q)_{\overline{\mathbf{x}}_1}$-orbit containing $c\overline{\mathbf{x}}_8$ is q^5.

To prove $|\Theta_2| = q(q^2 - 1)$, let $\langle \overline{\mathbf{y}} \rangle$ be the image of $\langle \overline{\mathbf{x}}_2 \rangle$ under some $\eta \in G_2(q)_{\langle \overline{\mathbf{x}}_1 \rangle}$ with $\eta(\overline{\mathbf{x}}_1) = b\overline{\mathbf{x}}_1$ $(b \neq 0)$ and $\eta(\overline{\mathbf{x}}_2) = \overline{\mathbf{y}}$. Then for each $c \in \mathbb{F}_q^{\times}$, there exists $\zeta_c \in G_2(q)_{\overline{\mathbf{x}}_1}$ stabilising $\mathbf{e}$ such that $\zeta_c(\overline{\mathbf{x}}_2) = c\overline{\mathbf{y}}$, say, ζ_c defined by

$$\zeta_c(\mathbf{x}_1, \mathbf{x}_8, \mathbf{x}_2, \mathbf{x}_7, \mathbf{x}_3, \mathbf{x}_6, \mathbf{x}_4) := \left(\frac{1}{b}\eta(\mathbf{x}_1), b\eta(\mathbf{x}_8), c\eta(\mathbf{x}_2), \frac{1}{c}\eta(\mathbf{x}_7), \frac{1}{bc}\eta(\mathbf{x}_3), bc\eta(\mathbf{x}_6), \eta(\mathbf{x}_4)\right).$$

Then ζ_c preserves Table 3 and hence $\zeta_c \in G_2(q)_{\overline{\mathbf{x}}_1}$. Thus $|\Theta_2| = q(q+1)(q-1) = q(q^2-1)$. Similarly, one can prove $|\Theta_7| = q^3(q^2 - 1)$. □

Since P is the union of some $G_2(q)_{\mathbf{a}}$-orbits on $V \setminus \{\mathbf{0}\}$, we have four possibilities to consider. First, if P includes neither the orbit of length $q(q^2 - 1)$ nor the orbit of length $q^3(q^2 - 1)$, then similar to the case $C_0 \nsubseteq P$ in Section 4.2, we have $P \subseteq \langle \mathbf{a} \rangle$, and moreover if there is a feasible G-orbit on the flag set of the 2-$(u, |P| + 1, \lambda)$ design $\mathcal{D} := (V, L^G)$, where $L := P \cup \{\mathbf{0}\}$, then $\lambda = 1$ by Lemma 2.12.

Next, if P includes the orbit of length $q(q^2 - 1)$ and the orbit of length $q^3(q^2 - 1)$, then similar to case $C_0 \subseteq P$ in Section 4.2, we have $P = V \setminus \{\mathbf{0}\}$, violating the condition $(u - 1)/|P| \geqslant 3$.

Next assume that P includes the orbit of length $q(q^2 - 1)$, i orbits of length q^5 $(0 \leqslant i < q)$ and ℓ orbits of length 1 $(1 \leqslant \ell < q)$, and P does not include the orbit of length $q^3(q^2 - 1)$. Then $|P| = iq^5 + q^3 - q + \ell$ and $iq^5 + q^3 - q + \ell = \gcd(|P|, q^6 - 1) = \gcd(\ell q^5 + iq^4 + q^2 - 1, q^4 - \ell q^3 - iq^2 - 1)$. Since $0 < q^2 - 1 \leqslant q^4 - \ell q^3 - iq^2 - 1 \leqslant q^4 - q^3 - 1$, we have $iq^5 + q^3 - q + \ell \leqslant q^4 - \ell q^3 - iq^2 - 1 \leqslant q^4 - q^3 - 1$, which implies $i = 0$. Thus $|P| = q^3 - q + \ell$ and $q^3 - q + \ell = \gcd(\ell q^5 + q^2 - 1, q^4 - \ell q^3 - 1) = \gcd(q^4 - \ell q^3 - 1, \ell q^3 - q^2 + \ell q + 1) = \gcd(q^3 - q + \ell, q^2 - 2\ell q + (\ell^2 - 1))$. Since $0 \leqslant q^2 - 2\ell q + (\ell^2 - 1) = (\ell - q)^2 - 1 \leqslant q^2 - 2q$, if $q^2 - 2\ell q + (\ell^2 - 1) \neq 0$, then $q^3 - q + \ell \leqslant q^2 - 2\ell q + (\ell^2 - 1) \leqslant q^2 - 2q$, which is impossible. Hence $q^2 - 2\ell q + (\ell^2 - 1) = 0$, $\ell = q - 1$ and $|P| = q^3 - 1$. Now $v = (q^6 - 1)/(q^3 - 1) = q^3 + 1 > |P|$, and thus if a feasible G-orbit on the flag set of the 2-$(u, |P| + 1, \lambda)$ design $\mathcal{D} := (V, L^G)$ exists, where $L := P \cup \{\mathbf{0}\}$, then $\lambda = 1$ by Lemma 2.12.

Finally, assume that P includes the orbit of length $q^3(q^2 - 1)$, $i - 1$ orbits of length q^5 $(1 \leqslant i < q + 1)$ and ℓ orbits of length 1 $(1 \leqslant \ell < q)$, and P does not include the orbit of length $q(q^2 - 1)$. Then $|P| = iq^5 - q^3 + \ell$ and $iq^5 - q^3 + \ell = \gcd(iq^5 - q^3 + \ell, q^6 - 1) = \gcd(q^6 - 1, \ell q^3 + iq^2 - 1)$. Since $0 < \ell q^3 + iq^2 - 1$, we have $iq^5 - q^3 + \ell \leqslant \ell q^3 + iq^2 - 1 \leqslant q^4 - 1$, which is impossible.

In summary, we have proved that there is no 2-$(q^6, m + 1, \lambda)$ design as in Lemma 2.10 with $\lambda > 1$ admitting G as a group of automorphisms.

4.6 $G_0 \cong \mathrm{SL}(2, 13)$, $u = 3^6$

Suppose that $G_{\mathbf{0}}$ has an imprimitive block P on $V \setminus \{\mathbf{0}\}$, where $V = \mathbb{F}_3^6$, and there is a feasible G-orbit Ω on the flag set of the 2-design $\mathcal{D} := (V, L^G)$, where $L := P \cup \{\mathbf{0}\}$. Then $H := G_{\mathbf{0},P}$ is maximal in $G_{\mathbf{0}}$ by Lemma 2.10(b), and $v := |G_{\mathbf{0}} : H|$ equals the size of $P^{G_{\mathbf{0}}}$. If the center Z of $G_{\mathbf{0}}$ is not contained in H, then $G_{\mathbf{0}} = ZH$ and $G_{\mathbf{0}} = G_{\mathbf{0}}' = (ZH)' = H' \leqslant H$, a contradiction. Thus $Z \leqslant H$ and H/Z is maximal in $G_{\mathbf{0}}/Z \cong \mathrm{PSL}(2, 13)$. By [8, p.8], each maximal subgroup of $\mathrm{PSL}(2, 13)$ is of index 14, 78 or 91 in $\mathrm{PSL}(2, 13)$.

Since $v := |G_{\mathbf{0}} : H| = |(G_{\mathbf{0}}/Z) : (H/Z)|$ is a divisor of $u - 1 = 728 = 8 \cdot 91$, we have $v = 14$ or 91. But by Lemma 2.10(b), $v - 1$ is a divisor of $|G_{\mathbf{0}}|/(u - 1) = 3$, which is a contradiction. Hence in this case there is no 2-design as in Lemma 2.10.

4.7 $G_{\mathbf{0}} = G_2(2)' \cong \mathrm{PSU}(3, 3)$, $u = 2^6$

Suppose that $G_{\mathbf{0}}$ has an imprimitive block P on $V \setminus \{\mathbf{0}\}$, where $V = \mathbb{F}_2^6$, and Ω is a feasible G-orbit on the flag set of $\mathcal{D} := (V, L^G)$, where $L := P \cup \{\mathbf{0}\}$. Let $H := G_{\mathbf{0},P}$ and $v := |G_{\mathbf{0}} : H|$. By [8, p.14], each maximal subgroup of $\mathrm{PSU}(3, 3)$ is of index 28, 36 or 63 in $\mathrm{PSU}(3, 3)$. Since v is a divisor of $u - 1 = 63$, we have $v = |G_{\mathbf{0}} : H| = 63$ and $|P| = (u - 1)/v = 1$. Hence there is no 2-design as in Lemma 2.10 in this case.

4.8 $G_{\mathbf{0}} \cong A_6$ or A_7, $u = 2^4$

Suppose that $G_{\mathbf{0}}$ has an imprimitive block P on $V \setminus \{\mathbf{0}\}$, where $V = \mathbb{F}_2^4$, and Ω is a feasible G-orbit on the flag set of $\mathcal{D} := (V, L^G)$, where $L := P \cup \{\mathbf{0}\}$. Let $H := G_{\mathbf{0},P}$ and $v := |G_{\mathbf{0}} : H|$. When $G_{\mathbf{0}} \cong A_6$, by [8, p.4] each maximal subgroup of A_6 is of index 6, 10 or 15 in A_6. By Lemma 2.10(b), $v - 1$ divides $|G_{\mathbf{0}}|/(u - 1) = 24$, which is a contradiction. When $G_{\mathbf{0}} \cong A_7$, by [8, p.10] each maximal subgroup of A_7 is of index 7, 15, 21 or 35 in A_7. Since v is a divisor of $u - 1 = 15$, we have $v = 15$ and $|P| = (u - 1)/v = 1$. Hence in this case there is no 2-design as in Lemma 2.10.

4.9 $d = 2$, $p = 5, 7, 11, 19, 23, 29$ or 59, and $G_{\mathbf{0}} \trianglerighteq \mathrm{SL}(2, 5)$ or $G_{\mathbf{0}} \trianglerighteq \mathrm{SL}(2, 3)$

In this case $G_{\mathbf{0}}$ has a normal subgroup $J = \langle \gamma \rangle$ of order 2 which is the center of the normal subgroup isomorphic to $\mathrm{SL}(2, 5)$ or $\mathrm{SL}(2, 3)$. Thus γ is central in $G_{\mathbf{0}}$. Let $\mathcal{L}_\gamma(V)$ denote the set of vectors in $V = \mathbb{F}_p^2$ fixed by γ. Then $\mathcal{L}_\gamma(V)$ is a subspace of V and is $G_{\mathbf{0}}$-invariant. Since $G_{\mathbf{0}}$ acts irreducibly on V and $\gamma \neq \mathrm{id}_V$, we have $\mathcal{L}_\gamma(V) = \{\mathbf{0}\}$ and thus $\gamma - \mathrm{id}_V$ is nonsingular. Moreover, since $(\gamma - \mathrm{id}_V)(\gamma + \mathrm{id}_V) = \gamma^2 - \mathrm{id}_V$ is the zero map, we have $\gamma = -\mathrm{id}_V$. Hence $G_{\mathbf{0}}$ contains $-\mathrm{id}_V$. Set $\mathbf{e}_1 := (1, 0)$ and $\mathbf{e}_2 := (0, 1)$.

Lemma 4.9. *Let P be an imprimitive block of $G_{\mathbf{0}}$ on $V \setminus \{\mathbf{0}\}$ such that $|P| \geqslant 2$ and $v := (p^2 - 1)/|P| \geqslant 3$. Suppose that $G_{\mathbf{0},\mathbf{y}}$ is transitive on $P^{G_{\mathbf{0}}} \setminus \{P\}$ for some $\mathbf{y} \in P$, and the 2-$(p^2, |L|, \lambda)$ design $\mathcal{D} := (V, L^G)$ has $\lambda > 1$, where $L := P \cup \{\mathbf{0}\}$. Then $v \mid (p + 1)$, $(v - 1) \mid (p - 1)$, and $G_{\mathbf{0},\mathbf{x}}$ is a nontrivial cyclic group with order dividing $p - 1$ for any $\mathbf{x} \in V \setminus \{\mathbf{0}\}$.*

Proof. Let $\mathbf{z} \in P$. If $a\mathbf{z} \notin P$ for some $a \in \mathbb{F}_p^\times$, then $a\mathbf{z} \in R$ for some $R \in P^{G_{\mathbf{0}}} \setminus \{P\}$ and $G_{\mathbf{0},\mathbf{z}}$ ($\leqslant \mathrm{GL}(2, p)$) stabilises R, which is a contradiction. Hence $\mathbf{z} \in P$ implies $\langle \mathbf{z} \rangle \setminus \{\mathbf{0}\} \subseteq P$, and thus $p - 1$ divides $|P|$ and $v \mid (p + 1)$.

Next we prove that $G_{\mathbf{0},\mathbf{x}}$ is cyclic and $|G_{\mathbf{0},\mathbf{x}}|$ divides $p-1$ for any $\mathbf{x} \in V \setminus \{\mathbf{0}\}$ ($G_{\mathbf{0},\mathbf{x}}$ is nontrivial as $|G_{\mathbf{0},\mathbf{x}}| = |G_{\mathbf{0},\mathbf{y}}| \geqslant |P^{G_\mathbf{0}} \setminus \{P\}| = v-1 > 1$). Since $G_\mathbf{0}$ is transitive on $V \setminus \{\mathbf{0}\}$, we may assume $\mathbf{x} = \mathbf{e}_1$.

For any $\varphi, \psi \in G_{\mathbf{0},\mathbf{e}_1}$ such that $\mathbf{e}_2^\varphi = (s,t)$ and $\mathbf{e}_2^\psi = (\ell, n)$, we have $(a,b)^\varphi = (a+bs, bt)$ and $\langle (a,1)\rangle^\varphi = \langle ((a+s)/t, 1)\rangle$. Moreover, $\mathbf{e}_2^{\varphi^{-1}} = (-s/t, 1/t)$, $\mathbf{e}_2^{\varphi\psi} = (s+t\ell, tn)$ and $\mathbf{e}_2^{\varphi^{-1}\psi} = ((\ell-s)/t, n/t)$. Hence $S := \{t \in \mathbb{F}_p^\times : (s,t) = \mathbf{e}_2^\varphi$ for some $\varphi \in G_{\mathbf{0},\mathbf{e}_1}\}$ is a subgroup of $\mathbb{F}_p^\times$ and $S = \langle c\rangle$ for some $c \in \mathbb{F}_p^\times$. Let $\varphi_c \in G_{\mathbf{0},\mathbf{e}_1}$ with $\mathbf{e}_2^{\varphi_c} = (s,c)$.

Suppose that $G_{\mathbf{0},\mathbf{e}_1} \neq \langle \varphi_c\rangle$. Then there exists $\theta \in G_{\mathbf{0},\mathbf{e}_1}$ such that $\mathbf{e}_2^\theta = (h,1)$ for some $h \in \mathbb{F}_p^\times$. If $|P| \leqslant p-1$, then $P \subseteq \langle \mathbf{y}\rangle$, where $\mathbf{y} \in P$, and $\lambda = 1$ by Lemma 2.12. Therefore $|P| > p-1$. Let $Q \in P^{G_\mathbf{0}}$ and $\mathbf{e}_1 \in Q$. Then $(a,1) \in Q$ for some $a \in \mathbb{F}_p$. Since $G_{\mathbf{0},\mathbf{e}_1}$ stabilises Q and $(a,1)^{\theta^j} = (a+jh, 1)$, $j = 1, 2, \ldots$, we have $\langle (b,1)\rangle \setminus \{\mathbf{0}\} \subseteq Q$ for any $b \in \mathbb{F}_p$, and thus $Q = V \setminus \{\mathbf{0}\}$, which contradicts our assumption that $(p^2-1)/|Q| \geqslant 3$. Therefore, $G_{\mathbf{0},\mathbf{e}_1} = \langle \varphi_c\rangle$ and $|G_{\mathbf{0},\mathbf{e}_1}| = |c|$ divides $p-1$ ($c \neq 1$, for otherwise $\varphi_c = \mathrm{id}_V$ and $G_{\mathbf{0},\mathbf{e}_1}$ is trivial, a contradiction). Since $G_{\mathbf{0},\mathbf{y}}$ is transitive on $P^{G_\mathbf{0}} \setminus \{P\}$, $v-1$ divides $|G_{\mathbf{0},\mathbf{y}}|$ and thus $(v-1) \mid (p-1)$. □

Next we search for all 2-$(p^2, m+1, \lambda)$ designs each with $\lambda > 1$ and with a feasible G-orbit on the set of flags, with the assistance of Magma [3]. Set $V^\sharp := V \setminus \{\mathbf{0}\}$. Denote the group consisting of all translations of V by T. Since G is 2-transitive on V, we have $G = TG_\mathbf{0}$ with $G_\mathbf{0}$ transitive on $V^\sharp$. We call a subgroup K of $\mathrm{GL}(2,p)$ *almost satisfactory* if K is transitive but not regular on $V^\sharp$, K contains a normal subgroup isomorphic to $\mathrm{SL}(2,5)$ or $\mathrm{SL}(2,3)$ and $K_\mathbf{x}$ is cyclic for some $\mathbf{x} \in V^\sharp$. In each case below, we will compute the conjugacy classes of subgroups by using Magma , choose one representative K from each of them that is almost satisfactory (or show that none exists), consider subgroups H of K of index v with $v \mid (p+1)$ and $(v-1) \mid (p-1)$, and then construct the corresponding 2-designs and flag graphs (or show that none exists) with the help of Lemma 2.10(b). (Note that, for conjugate K_1, K_2, say, $K_2 = \varphi^{-1}K_1\varphi$ for some $\varphi \in \mathrm{GL}(2,p)$, we have $\varphi^{-1}(TK_1)\varphi = TK_2$ and so TK_1 and TK_2 are permutation isomorphic on V.) Denote

$$G := TK \leqslant \mathrm{AGL}(2,p).$$

Then G is 2-transitive on V and $G_\mathbf{0} = K$.

Case 1: $p = 5$. There are three conjugacy classes of subgroups of $\mathrm{GL}(2,5)$, denoted by $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, such that every $K \in \mathcal{C}_i$ ($1 \leqslant i \leqslant 3$) is almost satisfactory.

When $i = 1$, we have $|K| = 48$ and $|G_{\mathbf{0},\mathbf{e}_1}| = 2$. By Lemmas 4.9 and 2.10(b), we will consider subgroups of $G_\mathbf{0}$ of index $v = 3$. Set K to be the group in $\mathcal{C}_1$ generated by $\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 3 \\ 4 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Then $G_\mathbf{0}$ has only one subgroup H of index 3. Hence $H \trianglelefteq G_\mathbf{0}$ and there is no 2-design as in Lemma 2.10(b) admitting a group $G \leqslant \mathrm{AGL}(2,5)$ as an automorphism group with $G_\mathbf{0} \in \mathcal{C}_1$.

When $i = 2$, we have $|K| = 120$ and $|G_{\mathbf{0},\mathbf{e}_1}| = 5$. By Lemma 4.9 this case cannot occur.

When $i = 3$, we have $|K| = 96$ and $|G_{\mathbf{0},\mathbf{e}_1}| = 4$. By Lemmas 4.9 and 2.10(b), we need to consider subgroups of $G_\mathbf{0}$ of index $v = 3$. Choose K to be the group in $\mathcal{C}_3$ generated by $\begin{bmatrix}1 & 1\\1 & 4\end{bmatrix}$, $\begin{bmatrix}2 & 4\\3 & 4\end{bmatrix}$, $\begin{bmatrix}0 & 3\\2 & 0\end{bmatrix}$, $\begin{bmatrix}4 & 0\\0 & 1\end{bmatrix}$ and $\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$. The subgroups of $G_{\mathbf{0}}$ of index 3 form a conjugacy class of length 3 (thus these groups are self-normalizing in $G_{\mathbf{0}}$). Let H be a subgroup of $G_{\mathbf{0}}$ of index 3. Since $|H| = 32$ and $|V^\sharp| = 24$, $H_{\mathbf{z}} \neq \{1\}$ for any $\mathbf{z} \in V^\sharp$. On the other hand, if $|H_{\mathbf{z}}| = 2$ for any $\mathbf{z} \in V^\sharp$, then $16 = |H|/2$ divides $|V^\sharp| = 24$, a contradiction. Hence there exists $\mathbf{x} \in V^\sharp$ such that $H_{\mathbf{x}} = G_{\mathbf{0},\mathbf{x}}$, and thus $R := \mathbf{x}^H$ is an imprimitive block of $G_{\mathbf{0}}$ on $V^\sharp$ ([10, Theorem 1.5A]). In addition, computing by Magma shows that H has two orbits on $V^\sharp$. Thus $\Omega := (\mathbf{0}, L)^G$ is a feasible orbit on the flags of the 2-$(25, 9, \lambda)$ design $\mathcal{D} := (V, L^G)$, where $L := R \cup \{\mathbf{0}\}$.

If $\lambda = 1$, then G_L is 2-transitive on L and $|G_L| = |L| \cdot |H|$ is a divisor of $|G|$, which is a contradiction. Therefore, $\lambda = |R| + 1 = 9$.

Let $\Sigma := R^{G_{\mathbf{0}}} = \{R = R_1, R_2, R_3\}$ and $L_\ell := R_\ell \cup \{\mathbf{0}\}$, $\ell = 1, 2, 3$. Suppose that $\Psi = ((\mathbf{0}, M), (\mathbf{x}, N))^G$ is a G-orbit on $\mathrm{F}(\mathcal{D}, \Omega)$, where $M \setminus \{\mathbf{0}\} = R_2$ and $N \setminus \{\mathbf{x}\} = R_j + \mathbf{x}$ for some $j > 1$. Then Ψ is self-paired if and only if there exists $\eta \in G$ interchanging $(\mathbf{0}, M)$ and $(\mathbf{x}, N)$. Hence $\eta = \delta\widehat{\mathbf{x}}$, where $\widehat{\mathbf{x}}$ is the translation induced by $\mathbf{x}$ and $\delta \in G_{\mathbf{0}}$ is such that $\mathbf{x}^\delta = -\mathbf{x}$ and δ interchanges R_2 and R_j. Thus $\delta \in \gamma G_{\mathbf{0},\mathbf{x}}$, where $\gamma = -\mathrm{id}_V$, and the action of δ on $\Sigma \setminus \{R\}$ has a cycle $(R_2\ R_j)$, possibly with $R_2 = R_j$. Since γ stabilises each element of Σ (by the proof of Lemma 4.9, L_ℓ is the union of some 1-subspaces of V, $\ell = 1, 2, 3$), we just need $\gamma\delta$ ($\in G_{\mathbf{0},\mathbf{x}}$) to have a cycle $(R_2\ R_j)$ on $\Sigma \setminus \{R\}$. Therefore, $\Psi = ((\mathbf{0}, M), (\mathbf{x}, N))^G$ is self-paired if and only if there exists an element of $G_{\mathbf{0},\mathbf{x}}$ which has a cycle $(R_2\ R_j)$ on $\Sigma \setminus \{R\}$. Since $G_{\mathbf{0},\mathbf{x}}$ acts nontrivially on $\Sigma \setminus \{R\}$, every orbit of G on $\mathrm{F}(\mathcal{D}, \Omega)$ is self-paired. Let Ψ be such a G-orbit. Then in the G-flag graph $\Gamma = \Gamma(\mathcal{D}, \Omega, \Psi)$, $(\mathbf{0}, L_2)$ is adjacent to $(\mathbf{x}, L_j + \mathbf{x})$ and $(\mathbf{0}, L_3)$ is adjacent to $(\mathbf{x}, L_n + \mathbf{x})$, where $\{j, n\} = \{2, 3\}$, and $\Gamma[\Omega(\mathbf{0}), \Omega(\mathbf{x})] \cong 2 \cdot K_2$.

Case 2: $p = 7$. There is only one conjugacy class $\mathcal{C}$ of subgroups of $\mathrm{GL}(2, 7)$ such that every $K \in \mathcal{C}$ is almost satisfactory. We have $|K| = 144$ and $|G_{\mathbf{0},\mathbf{e}_1}| = 3$. By Lemmas 4.9 and 2.10(b), it suffices to consider subgroups of $G_{\mathbf{0}}$ of index $v = 4$. Choose K to be the group in $\mathcal{C}$ generated by $\begin{bmatrix}5 & 5\\4 & 2\end{bmatrix}$, $\begin{bmatrix}2 & 4\\1 & 6\end{bmatrix}$, $\begin{bmatrix}2 & 1\\2 & 5\end{bmatrix}$, $\begin{bmatrix}1 & 3\\4 & 6\end{bmatrix}$ and $\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}$. The subgroups of $G_{\mathbf{0}}$ of index 4 form a conjugacy class of length 4. Let H be a subgroup of $G_{\mathbf{0}}$ of index 4. Then H is not semiregular on $V^\sharp$ and there exists $\mathbf{x} \in V^\sharp$ such that $H_{\mathbf{x}} \neq \{1\}$. Therefore $H_{\mathbf{x}} = G_{\mathbf{0},\mathbf{x}}$ and $R := \mathbf{x}^H$ is an imprimitive block of $G_{\mathbf{0}}$ on $V^\sharp$. Computing by Magma shows that H has two orbits on $V^\sharp$. Thus $\Omega := (\mathbf{0}, L)^G$ is a feasible G-orbit on the flags of the 2-$(49, 13, \lambda)$ design $\mathcal{D} := (V, L^G)$, where $L := R \cup \{\mathbf{0}\}$.

If $\lambda = 1$, then G_L is 2-transitive on L and $|G_L| = |L| \cdot |H|$ is a divisor of $|G|$, which is a contradiction. Therefore, $\lambda = |R| + 1 = 13$.

Let $\Sigma := R^{G_{\mathbf{0}}} = \{R = R_1, R_2, R_3, R_4\}$ and $L_\ell := R_\ell \cup \{\mathbf{0}\}$, $\ell = 1, 2, 3, 4$. Suppose that $\Psi = ((\mathbf{0}, M), (\mathbf{x}, N))^G$ is a G-orbit on $\mathrm{F}(\mathcal{D}, \Omega)$, where $M \setminus \{\mathbf{0}\} = R_2$ and $N \setminus \{\mathbf{x}\} = R_j + \mathbf{x}$ for some $j > 1$. Similar to case 1 above, we see that Ψ is self-paired if and only if there exists an element of $G_{\mathbf{0},\mathbf{x}}$ that has a cycle $(R_2\ R_j)$ on $\Sigma \setminus \{R\}$. Since the cycle decomposition of each nonidentity element of $G_{\mathbf{0},\mathbf{x}}$ on $\Sigma \setminus \{R\}$ is a 3-cycle, Ψ is self-paired if and only if $R_j = R_2$. In this case, in the corresponding G-flag graph $\Gamma = \Gamma(\mathcal{D}, \Omega, \Psi)$, $(\mathbf{0}, L_i)$ is adjacent to $(\mathbf{x}, L_i + \mathbf{x})$, $i = 2, 3, 4$, and $\Gamma[\Omega(\mathbf{0}), \Omega(\mathbf{x})] \cong 3 \cdot K_2$.

Case 3: $p = 11$. There are two conjugacy classes of subgroups of $\mathrm{GL}(2, 11)$, denoted by $\mathcal{C}_1$ and $\mathcal{C}_2$, such that every $K \in \mathcal{C}_i$ ($1 \leqslant i \leqslant 2$) is almost satisfactory.

When $i = 1$, we have $|K| = 240$ and $|G_{\mathbf{0},\mathbf{e}_1}| = 2$. By Lemmas 4.9 and 2.10(b), it suffices to consider subgroups of $G_{\mathbf{0}}$ of index $v = 3$. Choose K to be the group in $\mathcal{C}_1$ generated by $\begin{bmatrix} 8 & 0 \\ 6 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 6 \\ 9 & 2 \end{bmatrix}$, $\begin{bmatrix} 4 & 3 \\ 10 & 7 \end{bmatrix}$, $\begin{bmatrix} 5 & 5 \\ 3 & 6 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. The subgroups of $G_{\mathbf{0}}$ of index 3 form a conjugacy class of length 3. Let H be a subgroup of $G_{\mathbf{0}}$ of index 3. Then there exists $\mathbf{x} \in V^\sharp$ such that $H_{\mathbf{x}} = G_{\mathbf{0},\mathbf{x}}$, and thus $R := \mathbf{x}^H$ is an imprimitive block of $G_{\mathbf{0}}$ on $V^\sharp$. Computing by MAGMA shows that H has two orbits on $V^\sharp$. Hence $\Omega := (\mathbf{0}, L)^G$ is a feasible G-orbit on the flags of the 2-$(121, 41, \lambda)$ design $\mathcal{D} := (V, L^G)$ by Lemma 2.9, where $L := R \cup \{\mathbf{0}\}$. Similar to case 1 above, we have $\lambda = |R| + 1 = 41$ and each G-orbit on $\mathrm{F}(\mathcal{D}, \Omega)$ is self-paired. Let Ψ be such a G-orbit, and let $\Sigma := R^{G_{\mathbf{0}}} = \{R = R_1, R_2, R_3\}$ and $L_\ell := R_\ell \cup \{\mathbf{0}\}$, $\ell = 1, 2, 3$. Then in $\Gamma = \Gamma(\mathcal{D}, \Omega, \Psi)$, $(\mathbf{0}, L_2)$ is adjacent to $(\mathbf{x}, L_j + \mathbf{x})$ and $(\mathbf{0}, L_3)$ is adjacent to $(\mathbf{x}, L_n + \mathbf{x})$, where $\{j, n\} = \{2, 3\}$, and $\Gamma[\Omega(\mathbf{0}), \Omega(\mathbf{x})] \cong 2 \cdot K_2$.

When $i = 2$, we have $|K| = 600$ and $|G_{\mathbf{0},\mathbf{e}_1}| = 5$. By Lemmas 4.9 and 2.10(b), it suffices to consider subgroups of $G_{\mathbf{0}}$ of index $v = 6$. Choose K to be the group in $\mathcal{C}_2$ generated by $\begin{bmatrix} 6 & 1 \\ 4 & 5 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 2 & 8 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. The subgroups of $G_{\mathbf{0}}$ of index 6 form a conjugacy class of length 6 (thus these groups are self-normalizing in $G_{\mathbf{0}}$). Let H be a subgroup of $G_{\mathbf{0}}$ of index 6. Then H is not semiregular on $V^\sharp$ and there exists $\mathbf{x} \in V^\sharp$ such that $H_{\mathbf{x}} = G_{\mathbf{0},\mathbf{x}}$. Hence $R := \mathbf{x}^H$ is an imprimitive block of $G_{\mathbf{0}}$ on $V^\sharp$. Computing by MAGMA shows that H has two orbits on $V^\sharp$. Thus $\Omega := (\mathbf{0}, L)^G$ is a feasible G-orbit on the flags of the 2-$(121, 21, \lambda)$ design $\mathcal{D} := (V, L^G)$, where $L := R \cup \{\mathbf{0}\}$. Similar to case 2 above, we have $\lambda = |R| + 1 = 21$.

Let $\Sigma := R^{G_{\mathbf{0}}} = \{R = R_1, R_2, R_3, R_4, R_5, R_6\}$ and denote $L_i := R_i \cup \{\mathbf{0}\}$, $i = 1, 2, \ldots, 6$. Let $\Psi = ((\mathbf{0}, M), (\mathbf{x}, N))^G$ be a G-orbit on $\mathrm{F}(\mathcal{D}, \Omega)$, where $M \setminus \{\mathbf{0}\} = R_2$ and $N \setminus \{\mathbf{x}\} = R_j + \mathbf{x}$ for some $j > 1$. Similar to case 1 above, Ψ is self-paired if and only if there is an element of $G_{\mathbf{0},\mathbf{x}}$ that has a cycle $(R_2\ R_j)$ on $\Sigma \setminus \{R\}$. Since the cycle decomposition of each nonidentity element of $G_{\mathbf{0},\mathbf{x}}$ on $\Sigma \setminus \{R\}$ is a 5-cycle, Ψ is self-paired if and only if $R_2 = R_j$. In this case, $(\mathbf{0}, L_i)$ is adjacent to $(\mathbf{x}, L_i + \mathbf{x})$ in the corresponding G-flag graph $\Gamma = \Gamma(\mathcal{D}, \Omega, \Psi)$, $i = 2, 3, 4, 5, 6$, and $\Gamma[\Omega(\mathbf{0}), \Omega(\mathbf{x})] \cong 5 \cdot K_2$.

Case 4: $p = 19$. There is only one conjugacy class $\mathcal{C}$ of subgroups of $\mathrm{GL}(2, 19)$, such that every $K \in \mathcal{C}$ is almost satisfactory. We have $|K| = 1080$ and $|G_{\mathbf{0},\mathbf{e}_1}| = 3$. By Lemmas 4.9 and 2.10(b), it suffices to consider subgroups of $G_{\mathbf{0}}$ of index $v = 4$. Choose K to be the group in $\mathcal{C}$ generated by $\begin{bmatrix} 5 & 2 \\ 14 & 14 \end{bmatrix}$, $\begin{bmatrix} 9 & 11 \\ 3 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Then K has no subgroup of index 4, and thus there is no 2-design as in Lemma 2.10(b) admitting $G \leqslant \mathrm{AGL}(2, 19)$ as a group of automorphisms with $G_{\mathbf{0}} \in \mathcal{C}$.

Case 5: $p = 23$. There is no subgroup K of $\mathrm{GL}(2, 23)$ that is almost satisfactory. Hence this case cannot occur.

Case 6: $p = 29$. There is only one conjugacy class $\mathcal{C}$ of subgroups of $\mathrm{GL}(2, 29)$, such that every $K \in \mathcal{C}$ is almost satisfactory. We have $|K| = 1680$ and $|G_{\mathbf{0},\mathbf{e}_1}| = 2$. By Lemmas

4.9 and 2.10(b), it suffices to consider subgroups of $G_{\mathbf{0}}$ of index $v = 3$. Choose K to be the group in $\mathcal{C}$ generated by $\begin{bmatrix} 27 & 15 \\ 10 & 2 \end{bmatrix}$, $\begin{bmatrix} 3 & 12 \\ 8 & 5 \end{bmatrix}$, $\begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}$ and $\begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$. Then $G_{\mathbf{0}}$ has no subgroup of index 3, and so this case cannot occur.

Case 7: $p = 59$. There is no subgroup of $\mathrm{GL}(2, 59)$ that is almost satisfactory. Hence this case cannot occur.

4.10 $d = 4$, $p = 3$, and $G_{\mathbf{0}} \trianglerighteq \mathrm{SL}(2,5)$ or $G_{\mathbf{0}} \trianglerighteq E$, where E is an extraspecial group of order 32

In this case $V = \mathbb{F}_3^4$ and we set $V^\sharp := V \setminus \{\mathbf{0}\}$.

Case 1: $G_{\mathbf{0}} \trianglerighteq \mathrm{SL}(2,5)$. Suppose that P is an imprimitive block of $G_{\mathbf{0}}$ on $V^\sharp$ with $|P| \geqslant 2$ and $v := |V^\sharp|/|P| \geqslant 3$, such that $G_{\mathbf{0},\mathbf{x}}$ is transitive on $P^{G_{\mathbf{0}}} \setminus \{P\}$ for some $\mathbf{x} \in P$. Then $v \mid (3^4 - 1) = 80$ and $v - 1$ is a divisor of $|G_{\mathbf{0},\mathbf{x}}|$.

Using Magma we find that there are four conjugacy classes of subgroups of $\mathrm{GL}(4, 3)$, denoted by $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$, such that if $K \in \mathcal{C}_i$ then K is transitive but not regular on $V^\sharp$ and K contains a normal subgroup isomorphic to $\mathrm{SL}(2, 5)$. Let $K \in \mathcal{C}_i$ and $G := TK$. Then G is 2-transitive on V and $G_{\mathbf{0}} = K$. Similar to Section 4.9, it suffices to consider one representative group K in $\mathcal{C}_i$.

When $i = 1$, we have $|K| = 240$ and $|G_{\mathbf{0},\mathbf{x}}| = 3$. By Lemma 2.10(b), we need to consider subgroups of $G_{\mathbf{0}}$ of index $v = 4$. Since $G_{\mathbf{0}}$ has no subgroup of index 4, this case cannot occur.

When $i = 2$ or 3, we have $|K| = 480$ and $|G_{\mathbf{0},\mathbf{x}}| = 6$. By Lemma 2.10(b), we need to consider subgroups of $G_{\mathbf{0}}$ of index $v = 4$. Since $G_{\mathbf{0}}$ has only one subgroup H of index 4, we have $H \trianglelefteq G_{\mathbf{0}}$ and thus there is no 2-design as in Lemma 2.10(b) admitting $G \leqslant \mathrm{AGL}(4, 3)$ as a group of automorphisms with $G_{\mathbf{0}} \in \mathcal{C}_2$ or $G_{\mathbf{0}} \in \mathcal{C}_3$.

When $i = 4$, we have $|K| = 960$ and $|G_{\mathbf{0},\mathbf{x}}| = 12$. By Lemma 2.10(b), we need to consider subgroups of $G_{\mathbf{0}}$ of index $v = 4$ or 5. Magma shows that there are three conjugacy classes of subgroups of $G_{\mathbf{0}}$, each consisting of subgroups of $G_{\mathbf{0}}$ of order 240 and none of such subgroups is self-normalizing in $G_{\mathbf{0}}$. The subgroups of $G_{\mathbf{0}}$ of index 5 form a conjugacy class of length 5. Let H be such a subgroup of $G_{\mathbf{0}}$. By Magma H has two orbits on $V^\sharp$, which have lengths 32 and 48, respectively. Hence there is no 2-design as in Lemma 2.10(b) admitting $G \leqslant \mathrm{AGL}(4, 3)$ as a group of automorphisms with $G_{\mathbf{0}} \in \mathcal{C}_4$.

Case 2: $G_{\mathbf{0}} \trianglerighteq E$, where E is an extraspecial group of order 32. In this case $G_{\mathbf{0}}$ has a normal subgroup $J = \langle \gamma \rangle$ of order 2 which is the center of E. Thus γ is central in $G_{\mathbf{0}}$. Since $G_{\mathbf{0}}$ acts irreducibly on V, we have $\gamma = -\mathrm{id}_V$. Hence $G_{\mathbf{0}}$ contains $-\mathrm{id}_V$.

Since $G_{\mathbf{0}}$ is transitive on $V^\sharp$, E is 1/2-transitive on $V^\sharp$ and is not semiregular. By the proof of Theorem 19.6 in [22, p.237], if $E \leqslant D \trianglelefteq G_{\mathbf{0}}$ and D is a 2-group, then D is not semiregular on $V^\sharp$ and D must be in category (iv) there. Thus $|D| = 32$ and $D = E$. It follows that E is the maximal normal 2-subgroup of $G_{\mathbf{0}}$. Moreover, by the proof of Theorem 19.6 in [22, p.237], $V = U \oplus W$, where U and W are subspaces of dimension 2 over $\mathbb{F}_3$, and $\mathbf{x}^E = \mathbf{y}^E = (U \cup W) \setminus \{\mathbf{0}\}$ for any $\mathbf{x} \in U^\sharp$ and $\mathbf{y} \in W^\sharp$, where we set $Y^\sharp := Y \setminus \{\mathbf{0}\}$ for every subspace Y of V.

Fix an element $\mathbf{x}$ of $U^\sharp$ from now on. Then $P := \mathbf{x}^E$ is an imprimitive block of $G_\mathbf{0}$ on $V^\sharp$. Denote $\Lambda := P^{G_\mathbf{0}} = \{P_1 = P, P_2, P_3, P_4, P_5\}$.

Lemma 4.10. *The kernel of the action of $G_\mathbf{0}$ on Λ is equal to E.*

Proof. Let K be the kernel of the action of $G_\mathbf{0}$ on Λ. Then $E \leqslant K \trianglelefteq G_\mathbf{0}$. We aim to prove $K = E$.

By Frattini's argument, we have $G_{\mathbf{0},P} = G_{\mathbf{0},\mathbf{x}}E$, and thus $K = K \cap G_{\mathbf{0},P} = K \cap (G_{\mathbf{0},\mathbf{x}}E) = E(K \cap G_{\mathbf{0},\mathbf{x}}) = EK_\mathbf{x}$. Since E is a maximal normal 2-subgroup of $G_\mathbf{0}$, it suffices to show that K is a 2-group. Suppose otherwise. Then there exists some $\varphi \in K_\mathbf{x} \setminus E$ of odd order. Let ψ_i be a fixed element of $G_\mathbf{0}$ such that $P_i = P^{\psi_i} = (U^{\psi_i} \cup W^{\psi_i}) \setminus \{\mathbf{0}\}$, $i = 1, 2, \ldots, 5$. We choose ψ_1 to be id_V, and denote $U_i := U^{\psi_i}$ and $W_i := W^{\psi_i}$. Then, for any $\psi \in K$, since $U^\psi = U^\psi \cap (U \cup W) = (U^\psi \cap U) \cup (U^\psi \cap W)$, we have $U^\psi \cap U \subseteq U^\psi \cap W$ or $U^\psi \cap W \subseteq U^\psi \cap U$, and thus $U^\psi = U$ or W. Similarly, we have $U_i^\psi = U_i$ or W_i, $i = 2, 3, 4, 5$.

Suppose that φ stabilises U_i, $i = 1, 2, 3, 4, 5$. For each $i = 2, 3, 4, 5$, let $\mathbf{x} = \mathbf{a}_i + \mathbf{b}_i$, where $\mathbf{a}_i \in U_i^\sharp$, $\mathbf{b}_i \in W_i^\sharp$ (see Figure 2). Then $\mathbf{a}_i^\varphi = \mathbf{a}_i$ and $\mathbf{b}_i^\varphi = \mathbf{b}_i$, since $\mathbf{a}_i + \mathbf{b}_i = \mathbf{x} = \mathbf{x}^\varphi = \mathbf{a}_i^\varphi + \mathbf{b}_i^\varphi$ and U_i and W_i direct sum. If $i, \ell \in \{2, 3, 4, 5\}$ with $i \neq \ell$, then $\mathbf{a}_\ell \notin \langle \mathbf{a}_i, \mathbf{b}_i \rangle$ (for otherwise $\mathbf{a}_\ell = \mathbf{a}_i - \mathbf{b}_i$ or $\mathbf{a}_\ell = -\mathbf{a}_i + \mathbf{b}_i$ as $P_\ell \cap P_1 = P_\ell \cap P_i = \emptyset$, implying $\mathbf{b}_\ell (= \mathbf{x} - \mathbf{a}_\ell) = -\mathbf{b}_i$ or $-\mathbf{a}_i$, a contradiction). Hence $\mathbf{b}_3, \mathbf{a}_4, \mathbf{b}_4 \in \langle \mathbf{a}_2, \mathbf{b}_2, \mathbf{a}_3 \rangle$ as $\varphi \neq \mathrm{id}_V$. For each $j \in \{3, 4\}$, let $\mathbf{a}_2 = \mathbf{t}_j + \mathbf{w}_j$, where $\mathbf{t}_j \in U_j^\sharp$ and $\mathbf{w}_j \in W_j^\sharp$. If $\mathbf{t}_j \notin \langle \mathbf{a}_j \rangle$ and $\mathbf{w}_j \notin \langle \mathbf{b}_j \rangle$, then $U_j = \langle \mathbf{a}_j, \mathbf{t}_j \rangle$ and $W_j = \langle \mathbf{b}_j, \mathbf{w}_j \rangle$. As φ fixes $\mathbf{a}_2$, it fixes $\mathbf{t}_j$ and $\mathbf{w}_j$, and thus $\varphi = \mathrm{id}_V$, a contradiction. Hence $\mathbf{t}_j \in \langle \mathbf{a}_j \rangle$ or $\mathbf{w}_j \in \langle \mathbf{b}_j \rangle$, and $U_j \subseteq \langle \mathbf{a}_2, \mathbf{b}_2, \mathbf{a}_3 \rangle$ or $W_j \subseteq \langle \mathbf{a}_2, \mathbf{b}_2, \mathbf{a}_3 \rangle$. Since U_j and W_j are of dimension 2, $P_3 \cap P_4 \neq \emptyset$, a contradiction.

Therefore, φ interchanges U_i and W_i for some i with $2 \leqslant i \leqslant 5$ and $|\varphi|$ can not be odd, a contradiction. $\square$

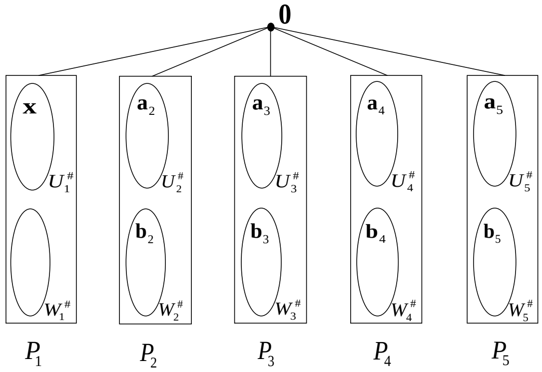

Figure 2

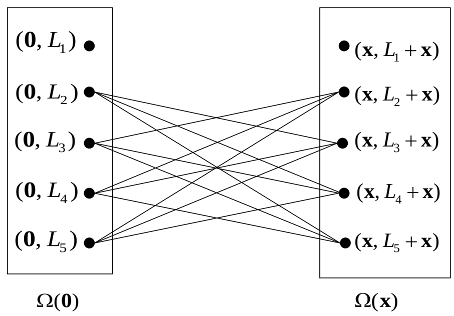

Figure 3

By Lemma 4.10, $G_\mathbf{0}/E$ can be embedded into S_5, and $G_\mathbf{0}$ is transitive on $V^\sharp$ if and only if $G_\mathbf{0}$ contains an element of order 5. Hence $G_\mathbf{0}/E \cong C_5$, D_{10}, $\mathrm{AGL}(1, 5)$, A_5 or S_5, and $|G_\mathbf{0}| = 160, 320, 640, 1920$ or 3840.

In what follows suppose that Q is an imprimitive block of $G_\mathbf{0}$ on $V^\sharp$ containing $\mathbf{x}$ with $|Q| \geqslant 2$ and $v := |V^\sharp|/|Q| \geqslant 3$ such that $G_{\mathbf{0},\mathbf{x}}$ is transitive on $\Sigma \setminus \{Q\}$, where

$\Sigma := Q^{G_{\mathbf{0}}} = \{Q_1 = Q, Q_2, \ldots, Q_v\}$. Let $L_i := Q_i \cup \{\mathbf{0}\}$, $i = 1, 2, \ldots, v$. Set $\mathcal{D} := (V, L^G)$, $\Omega := (\mathbf{0}, L)^G$ and $H := G_{\mathbf{0},Q}$, where $L = L_1$. Then $\mathcal{D}$ is a 2-$(81, |L|, \lambda)$ design.

Since $v \mid (3^4 - 1) = 80$, we have $v = 4, 5, 8, 10, 16, 20$ or 40. Since we want $\lambda > 1$, by Lemma 2.12 we have $|Q| = 80/v \geqslant v$ and thus $v = 4, 5$ or 8.

(i) If $v = 8$, then since $G_{\mathbf{0},\mathbf{x}}$ is transitive on $\Sigma \setminus \{Q\}$, $v - 1 = 7$ divides $|G_{\mathbf{0},\mathbf{x}}|$ and so divides $|\mathrm{GL}(4, 3)| = 80 \cdot 78 \cdot 72 \cdot 54$, a contradiction.

(ii) If $v = 4$, then since $v - 1 = 3$ divides $|G_{\mathbf{0},\mathbf{x}}|$, we have $G_{\mathbf{0}}/E \cong A_5$ or S_5. Consider the induced (faithful) action of $G_{\mathbf{0}}/E$ on Λ. $G_{\mathbf{0}}/E$ is 2-transitive on Λ, and since A_5 and S_5 have no subgroup of index 4, $E \not\leqslant H$ and HE/E is normal in $G_{\mathbf{0}}/E$. Thus HE/E is transitive on Λ. Moreover, since $G_{\mathbf{0},P} = G_{\mathbf{0},\mathbf{x}}E \leqslant HE$, we have $HE = G_{\mathbf{0}}$. Let J be the core of H in $G_{\mathbf{0}}$. Then J is exactly the kernel of the action of $G_{\mathbf{0}}$ on Σ and $G_{\mathbf{0}}/J$ is 2-transitive on Σ. Thus $G_{\mathbf{0}}/J \cong A_4$ or S_4 and 12 divides $|G_{\mathbf{0}}|/|J|$. On the other hand, since JE/E is normal in $G_{\mathbf{0}}/E$, $JE \neq E$ and JE/E is nonsolvable (otherwise $G_{\mathbf{0}}/E$ is solvable), JE/E is transitive on Λ and hence by [28, Theorem 11.7] JE/E is 2-transitive on Λ, which implies that $JE/E \cong A_5$ or S_5. Now we have 60 divides $|J|$ and 12 divides $|G_{\mathbf{0}}|/|J|$, which is a contradiction. Hence there is no 2-design as in Lemma 2.10(b) if $v = 4$.

(iii) If $v = 5$, then $|G_{\mathbf{0}} : H| = v = 5$. Since $\gcd(|E|, 5) = 1$, we have $E \leqslant H$, $Q = P = \mathbf{x}^E$ and $\Sigma = \Lambda$. Moreover, since $G_{\mathbf{0},P} = G_{\mathbf{0},\mathbf{x}}E$, $G_{\mathbf{0},\mathbf{x}}$ is transitive on $\Sigma \setminus \{P\}$ if and only if $G_{\mathbf{0},P}$ is transitive on $\Sigma \setminus \{P\}$, that is, if and only if $G_{\mathbf{0}}$ is 2-transitive on Σ. Therefore, Ω is feasible if and only if $G_{\mathbf{0}}/E \cong \mathrm{AGL}(1, 5)$, A_5 or S_5.

If $\lambda = 1$, then G_L is 2-transitive on L and $|G_L| = |L| \cdot |G_{\mathbf{0},L}| = 17 \cdot |H|$. But $|G_L|$ is a divisor of $|G| = |V| \cdot |G_{\mathbf{0}}| = 81 \cdot |G_{\mathbf{0}}|$, a contradiction. Hence $\lambda > 1$ and so $\lambda = |L| = 17$ by Lemma 2.8.

Let $\Psi = ((\mathbf{0}, M), (\mathbf{x}, N))^G$ be a G-orbit on $\mathrm{F}(\mathcal{D}, \Omega)$, where $M \setminus \{\mathbf{0}\} = P_2$, $N \setminus \{\mathbf{x}\} = P_j + \mathbf{x}$ for some $j > 1$. Similar to the discussion in case 1 (when $i = 3$) in Section 4.9, Ψ is self-paired if and only if there exists an element of $G_{\mathbf{0},\mathbf{x}}$ that has a cycle $(P_2\ P_j)$ on $\Sigma \setminus \{P\}$. We have

$$G_{\mathbf{0},P_1} = G_{\mathbf{0},\mathbf{x}}E, \text{ and } G_{\mathbf{0},P_1,P_j} = (G_{\mathbf{0},\mathbf{x}}E) \cap G_{\mathbf{0},P_j} = G_{\mathbf{0},\mathbf{x},P_j}E \text{ for } j > 1. \tag{17}$$

First assume that $G_{\mathbf{0}}/E \cong \mathrm{AGL}(1, 5)$. Then by (17) $G_{\mathbf{0},\mathbf{x}}$ induces a regular permutation group which is cyclic of order 4 on $\Sigma \setminus \{P\}$. Let $\varphi \in G_{\mathbf{0},\mathbf{x}}$ have a cycle decomposition $(P_2\ P_i\ P_\ell\ P_n)$ on $\Sigma \setminus \{P\}$, where $\{i, \ell, n\} = \{3, 4, 5\}$. Then $\Psi = ((\mathbf{0}, M), (\mathbf{x}, N))^G$ is self-paired if and only if $j = 2$ or $j = \ell$. Since $(\mathbf{x}, N)^{G_{\mathbf{0},\mathbf{x},P_2}} = \{(\mathbf{x}, N)\}$, we have $\Gamma[\Omega(\mathbf{0}), \Omega(\mathbf{x})] \cong 4 \cdot K_2$ for $\Gamma = \Gamma(\mathcal{D}, \Omega, \Psi)$.

Next assume that $G_{\mathbf{0}}/E \cong A_5$ or S_5. Then by (17), for any $n \in \{2, 3, 4, 5\}$ there is an element of $G_{\mathbf{0},\mathbf{x}}$ whose cycle decomposition on $\Sigma \setminus \{P\}$ is $(P_2\ P_n)(P_i\ P_\ell)$, where $i, \ell \neq 1, 2, n$. Thus each G-orbit on $\mathrm{F}(\mathcal{D}, \Omega)$ is self-paired.

If $P_j = P_2$, then in $\Gamma = \Gamma(\mathcal{D}, \Omega, \Psi)$, $(\mathbf{0}, L_i)$ is adjacent to $(\mathbf{x}, L_i + \mathbf{x})$, $i = 2, 3, 4, 5$, and $\Gamma[\Omega(\mathbf{0}), \Omega(\mathbf{x})] \cong 4 \cdot K_2$ since $(\mathbf{x}, N)^{G_{\mathbf{0},\mathbf{x},P_2}} = \{(\mathbf{x}, N)\}$.

If $P_j \neq P_2$, then by (17) we have $(\mathbf{x}, N)^{G_{\mathbf{0},\mathbf{x},P_2}} = \{(\mathbf{x}, L_e + \mathbf{x}) : e = 3, 4, 5\}$ and the edges of $\Gamma(\mathcal{D}, \Omega, \Psi)$ between $\Omega(\mathbf{0})$ and $\Omega(\mathbf{x})$ are as shown in Figure 3.

We have completed the proof of Theorem B.

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Appendix: Sample Magma codes

The following Magma codes are for Case 1 in Section 4.9. For other values of p and d in Sections 4.9 and 4.10, the Magma codes are similar.

```
d:=2; p:=5; G:=GL(d,p);
V:=VectorSpace(G); V; u:=V![1,0]; u;

L:=Subgroups(G:OrderMultipleOf:=p^d-1);
L:=[a`subgroup:a in L|#Orbits(a`subgroup) eq 2];
L:=[a:a in L|#a ne p^d-1];
L1:=[a:a in L|#[b:b in NormalSubgroups(a:OrderEqual:=120)|IsIsomorphic
    (b`subgroup,SL(2,5)) eq true]+#[b:b in NormalSubgroups
    (a:OrderEqual:=24)|IsIsomorphic(b`subgroup,SL(2,3)) eq true] gt 0];
L2:=[a:a in L1|IsCyclic(stabilizer(a,u)) eq true];
n:=#L2;
for i in [1..n] do #L2[i];
end for;

G0:=L2[1];
H:=Subgroups(G0:OrderEqual:=16); #H;
H[1]`length;

G0:=L2[3];
H:=Subgroups(G0:OrderEqual:=32); #H;
H[1]`length;
#Orbits(H[1]`subgroup);
```