Combinatorics meets potential theory

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Abstract

Using potential theoretic techniques, we show how it is possible to determine the dominant asymptotics for the number of walks of length n, restricted to the positive quadrant and taking unit steps in a "balanced" set Γ. The approach is illustrated through an example of inhomogeneous space walk. This walk takes its steps in $\{\leftarrow, \uparrow, \rightarrow, \downarrow\}$ or $\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}$, depending on the parity of the coordinates of its positions. The exponential growth of our model is $(4\phi)^n$, where $\phi = \frac{1+\sqrt{5}}{2}$ denotes the Golden ratio, while the subexponential growth is like $1/n$. As an application of our approach we prove the non-D-finiteness in two dimensions of the length generating functions corresponding to nonsingular small step sets with an infinite group and zero-drift.

Keywords: Lattice path enumeration, analytic combinatorics in several variables, discrete potential theory, discrete harmonic functions.

1 Introduction

Counting lattice walks in a fixed region $\mathcal{R} \subset \mathbb{Z}^2$ is one of the most fundamental topics in enumerative combinatorics. In recent years, the case of walks confined to the first quadrant $\mathcal{Q} = \{(x, y) \in \mathbb{Z}^2;\ x > 0,\ y > 0\}$ has been a subject of several important works (see [3, 4, 5, 10, 11, 12, 17, 26, 27, 32, 33, 34]).

Given a set Γ of allowed steps, the basic enumerative question is to determine the number of walks confined to $\mathcal{Q}$, starting from $(x, y) \in \mathcal{Q}$, having length $n = 1, 2, \ldots$

and taking steps in Γ only. A walk in which the set of allowable steps Γ is contained in $\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}$ is said to have small steps.

The enumeration of small steps walks restricted to the positive quadrant has focused primarily on the associated generating functions. In contrast with the corresponding 1-dimensional problem, where the generating functions are always algebraic [2, 9, 14], in the 2-dimensional case, depending on the choice of the set Γ, the generating functions may be algebraic, D-finite and some times not even D-finite [8, 10, 12].

Three fundamental works should be mentioned. Firstly, the seminal work of Bousquet-Mélou and Mishna [11] (see also [33]) initiating a complete classification of small steps walks in $\mathcal{Q}$ and showing that the nature of the generating function is correlated to the finiteness of a certain group of 2-dimensional transformations associated with Γ. Bousquet-Mélou and Mishna proved that among the 256 possible small-step walks, there are exactly 79 different models (up to symmetries) and 23 have a finite group. Among the 23, they proved that 22 models have D-finite generating functions (the 23rd was shown algebraic in [4]). Secondly, the Kurkova and Raschel result [27] proving a conjecture of Bousquet-Mélou and Mishna about the non-D-finiteness of the trivariate generating function which takes into account the length of the walk and the position of its endpoint for the 51 nonsingular models among the 56 with infinite groups. The remaining 5 models were shown to have non-D-finite length generating functions by Mishna and Rechnitzer [34] and Melczer and Mishna [31]. Thirdly, the Bostan *et al.* result [5] establishing the non-D-finiteness of excursions generating functions for the same 51 nonsingular walks (the nature of the length generating function for all quadrant walks is still unknown in those 51 cases).

After the important progress made on understanding small steps quadrant walks, efforts are now deployed to move one dimension higher (small steps walks in octants [6]) and to address walks with big jumps (see [12, 18]).

A natural question about quadrant walks that has not received any special attention despite its interest provided the first motivation for this paper. What does happen if we let the set where the walk takes its steps depend on its positions?

At first glance, it might seem difficult to investigate inhomogeneous space walks because of the great difficulty of analysis of their generating functions [30]. A further major difficulty that appears in the non-homogeneous case is that paths are no longer equally probable and it is no longer possible to use the counting formula relating the enumeration problem to the survival probability in the quadrant to count them (see [13]).

Our aim is to offer an alternative approach allowing us to investigate quadrant walks. To explain the underlying principle we choose to present this approach through an example. However, our method has a more general scope and applies in higher dimension and for models with longer steps. It relies on a systematic use of tools and techniques from discrete potential theory developed in [24, 35, 36]. It assumes a centering condition and requires the construction of an appropriate harmonic function whose existence is difficult to establish in full generality. Nevertheless, if we restrict ourselves to the homogeneous case, the method applies quite generally and offer an alternative to the probabilistic approach. This allows us to answer some open questions in the subject. We show in particular in

§5, that for the three models with nonsingular small step set, zero-drift and infinite group (among the 51 nonsingular models considered in [5]) the length generating function is not D-finite.

2 Model description and main result

We consider walks that starts at $(x, y) \in \mathcal{Q}$ and take their steps $w = w_1, w_2, \dots$ in variable multi-sets $\Gamma \subset \{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}$ which depend on the position of $W_j = \sum_{i=1}^{j} w_i$. We denote by $\mathcal{Q}(x, y)$ the set of all walks that stay in $\mathcal{Q}$ and by $\mathcal{Q}_n(x, y) \subset \mathcal{Q}(x, y)$ the subset of walks of length n. The main quantity we investigate is $\nu_n(x, y) = |\mathcal{Q}_n(x, y)|$.

We shall denote by $\nu_n(x, y; x', y')$ the number of walks $\in \mathcal{Q}_n(x, y)$ ending at $(x', y') \in \mathcal{Q}$. The sequence $(\nu_n(x, y))_{n\in\mathbb{N}}$ satisfies a multivariate recurrence with non-constant coefficients:

$$\begin{aligned}
\nu_{n+1}(x, y) &= \sum_{(x',y')\in\mathcal{Q}} \nu_{n+1}(x, y; x', y') \\
&= \sum_{(x',y')\in\mathcal{Q}} \sum_{(h,k)\in\Gamma(x,y)} \nu_n(x + h, y + k; x', y') \\
&= \sum_{(h,k)\in\Gamma(x,y)} \sum_{(x',y')\in\mathcal{Q}} \nu_n(x + h, y + k; x', y') \\
&= \sum_{(h,k)\in\Gamma(x,y)} \nu_n(x + h, y + k).
\end{aligned}$$

Taking into account the obvious relations, $\nu_0(x, y) = 1$, $(x, y) \in \mathcal{Q}$ and $\nu_n(x, y) = 0$ if $x < 0$ or $y < 0$, we obtain that $\nu_n(x, y)$ is a solution of the following equation,

$$\nu_{n+1}(x, y) = \sum_{(h,k)\in\Gamma(x,y)} \nu_n(x + h, y + k), \quad \text{for all } (n; (x, y)) \in \mathbb{N} \times \mathcal{Q} \tag{1}$$

with:

$$\begin{cases} \nu_n(x, y) = 0 & \text{if} \quad (x, y) \in \partial\mathcal{Q}, \\ \nu_0(x, y) = 1 & \text{for all} \quad (x, y) \in \mathcal{Q}, \end{cases} \tag{2}$$

Equation (1) combined with the boundary conditions (2) uniquely determines the sequence $\nu_n(x, y)$ (see Figure 1 which gives the number of walks starting from $(1, 1)$ and ending at (x', y'), for $(x', y') \in [\![1, 14]\!] \times [\![1, 14]\!]$).

A general principle in combinatorics consists in introducing, for a fixed $(x, y) \in \mathcal{Q}$, the trivariate generating power series of the sequence $\nu_n(x, y; x', y')$

$$F_{x,y}(t; \xi, \eta) = \sum_{n\geqslant 0} \left(\sum_{(x',y')\in\mathcal{Q}} \nu_n(x, y; x', y') \xi^{x'} \eta^{y'} \right) t^n. \tag{3}$$

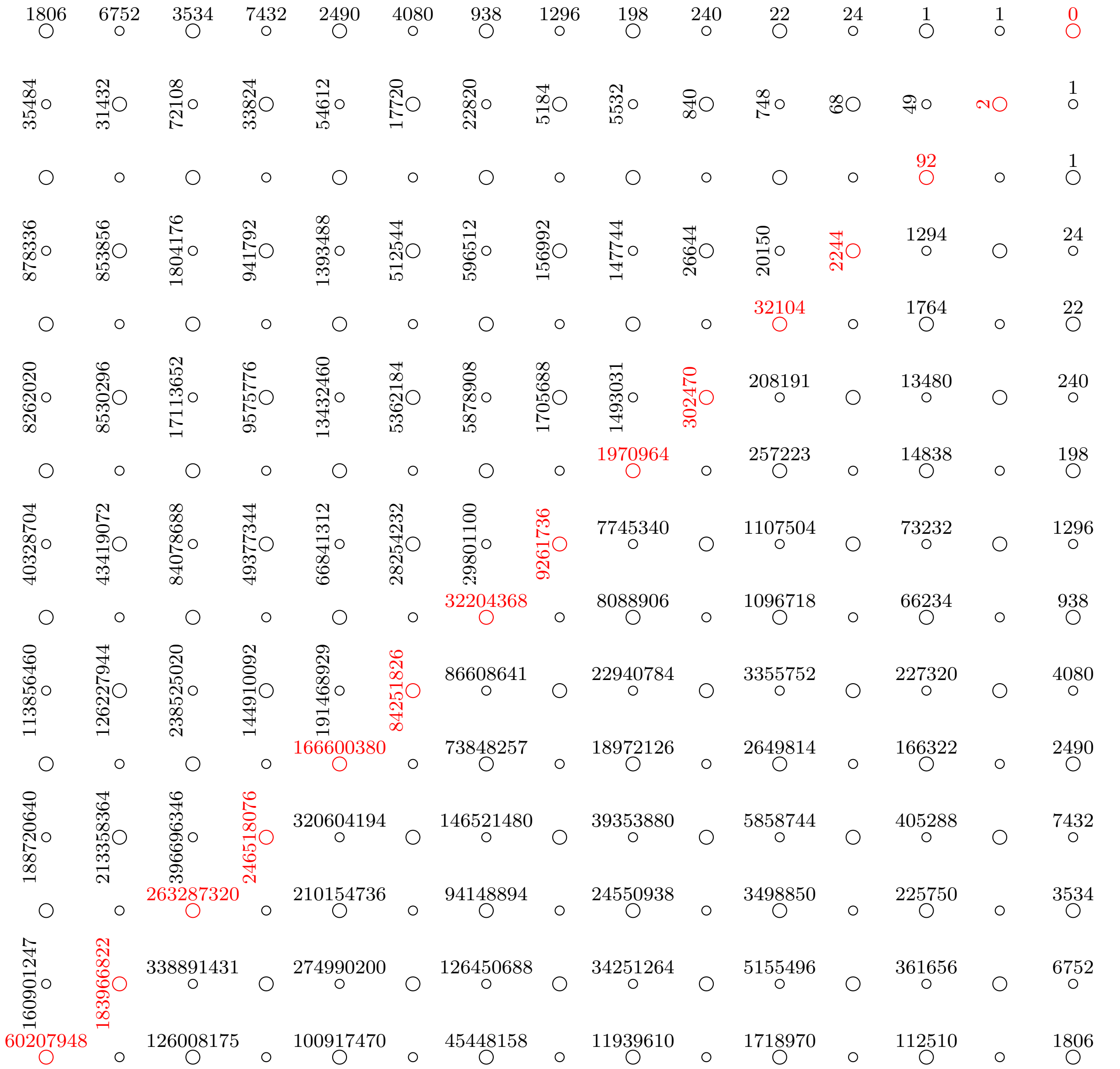

Figure 1: The number of 14-steps paths starting at $(1,1)$ and ending at (x',y'), for $(x',y') \in [\![1,14]\!] \times [\![1,14]\!]$.

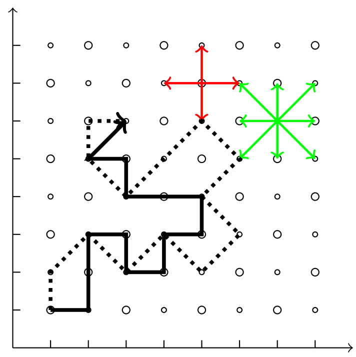

Figure 2: Two golden paths among the 238525020 14-steps quadrant paths, starting from $(1,1)$ and ending at $(3,6)$. The path corresponding to the solid line is much more likely than the dotted one (it has probability 2^{-36} vs 2^{-40} for the dotted path).

We use the notation $F_{x,y}$ to highlight the dependence of the generating function on the starting point (x,y). It is the will to understand this dependence that explains our formulation of the counting problem.

As $\nu_n(x,y;x',y')=0$ as soon as $|x-x'|>n$ or $|y-y'|>n$, the inner sum in (3) is finite, and $F_{x,y}(t;\xi,\eta)$ can be identified with a power series in t with coefficients in $\mathbb{Q}[\xi,\eta]$. Choosing $\xi=\eta=1$ yields a power series whose coefficients count the quadrant walks with prescribed number of steps. In the classical theory the growth of the coefficients of the generating functions is related to the location and nature of their singularities [19, 20]. The generating function methodology, however, seems difficult to implement in the case of spatially inhomogeneous walks, the recurrences in (1) being built with "variable" coefficients.

In order to make a first step towards understanding spatially inhomogeneous walks we investigate a new model of walks, taking their steps $w=w_1,w_2,\dots w_j,\dots$ alternately in one of the two multi- set

$$\Gamma_{\text{even}}=\{\leftarrow,\uparrow,\rightarrow,\downarrow\}\subset\Gamma_{\text{odd}}=\{\swarrow,\leftarrow,\nwarrow,\uparrow,\nearrow,\rightarrow,\searrow,\downarrow\}$$

according to the following rule: if $W_j=\sum_{i=1}^{j}w_i=(x,x+2k)$, $k\in\mathbb{Z}$, then the step w_{j+1} is taken from Γ_{even} and it is taken from Γ_{odd} otherwise (see Figure 2). We call them *golden walks*.

It is well known that the number of n-steps walks on the square lattice, with $\leftarrow,\uparrow,\rightarrow$ and $\downarrow$ steps (this is the so-called Pólya walk) that start from the origin and stay in the first quadrant grows like $4^n/n$. On the other hand, the number of n-steps quadrant walks taking their steps in the full set $\{\swarrow,\leftarrow,\nwarrow,\uparrow,\nearrow,\rightarrow,\searrow,\downarrow\}$ (the so-called King walk) grows like $8^n/n$. In our model the Pólya walk is modified so that it is allowed to move from the points whose coordinates are of opposite parity in the four additional directions $\swarrow,\nwarrow,\nearrow$ and $\searrow$. How does the asymptotics of walks of length n, restricted to the positive quadrant, change?

The answer is given by

Theorem 1. *There exist C, $c > 0$ such that*

$$\frac{(4\phi)^n\, xy}{Cn} \leqslant \nu_n(x,y) \leqslant C\frac{(4\phi)^n\, xy}{n} \tag{4}$$

for all $(x,y) \in \mathcal{Q}$ satisfying x, $y \leqslant c\sqrt{n}$, $n = 1,2,\dots$ and where $\phi = \frac{1+\sqrt{5}}{2}$ denotes the Golden ratio.

The following remarks may be helpful in placing the above theorem in the right perspective.

(i) As we are interested mainly by the asymptotic behavior of $\nu_n(x,y)$, $n \to \infty$, the condition x, $y \leqslant c\sqrt{n}$ is not really restrictive.

(ii) The factor $(4\phi)^n$ that appears in (4) is related to the number of unrestricted paths of length n (see §3.2. below). The same factor appears in the estimates of the number of n-steps paths confined to the half-space $\mathcal{H} = \{(x,y) \in \mathbb{Z}^2,\ y \geqslant 0\}$ and starting at $(x,y) \in \mathcal{H}$. Note this number $\nu_n^{\mathcal{H}}(x,y)$. Our method allows to show that

$$\frac{(4\phi)^n\, y}{C\sqrt{n}} \leqslant \nu_n^{\mathcal{H}}(x,y) \leqslant C\frac{(4\phi)^n\, y}{\sqrt{n}}, \quad y \leqslant c\sqrt{n}. \tag{5}$$

(iii) It is important to clarify the connection between the functions $u(x,y) = xy$, $u^{\mathcal{H}}(x,y) = y$ and the factors $1/n$ and $1/\sqrt{n}$ that appear in the lower and upper bounds of (4) and (5). It will be shown in §3.1 below that the function u satisfies a discrete heat type equation (see Eq. (9) below). Moreover, it is positive and vanishes on $\partial\mathcal{Q}$. The same kind of verification is easily done with $u^{\mathcal{H}}$ which vanishes on $\partial\mathcal{H}$. The existence and uniqueness of discrete harmonic functions was recently established, in a related context, for spatially inhomogeneous random walks on orthants in $\mathbb{Z}^d$ [7]. The same reasoning used in [7], based on an analog of Theorem 3 below, allows to show in the case here considered the uniqueness (up to multiplicative constants) of time-independent positive solutions of Eq. (9) vanishing on $\partial\mathcal{Q}$. In terms of the function u (resp. $u^{\mathcal{H}}$) the factor $1/n$ (resp. $1/\sqrt{n}$) can be interpreted as

$$\frac{1}{u(\sqrt{n},\sqrt{n})}, \quad \left(\text{resp. } \frac{1}{u^{\mathcal{H}}(\sqrt{n},\sqrt{n})}\right)$$

and the quotient xy/n (resp. y/n) as

$$\frac{u(x,y)}{u(\sqrt{n},\sqrt{n})}, \quad \left(\text{resp. } \frac{u^{\mathcal{H}}(x,y)}{u^{\mathcal{H}}(\sqrt{n},\sqrt{n})}\right).$$

What counts in the choice of the point $(\sqrt{n},\sqrt{n})$ is that it is at a distance $\approx \sqrt{n}$ from the boundary of $\mathcal{Q}$.

(iv) Due to the previous considerations, it is natural to divide $\nu_n(x,y)$ by $(4\phi)^n$ in (4) and try to find a similar interpretation for the resulting quotient in terms of a positive solution of (9) normalized by its value at a point located at a distance $\approx \sqrt{n}$ from $\partial\mathcal{Q}$. This can be achieved by considering unrestricted paths.

3 The heat type equation and the unrestricted paths

Let $\mathcal{N}_n(x,y)$ denote the number of unconstrained walks starting at (x,y) and of length $n \in \mathbb{N}$. We set $\mathcal{N}_n(x,y) = 1$ for $n = 0$ and denote by $\mathbf{1}_{\Gamma(x,y)}$ the characteristic function of the set $\Gamma(x,y)$. $\mathcal{N}_n(x,y)$ satisfies the equation

$$\mathcal{N}_{n+1}(x,y) = \sum_{(h,k)\in\mathbb{Z}^2} \mathbf{1}_{\Gamma(x,y)}(h,k)\mathcal{N}_n(x+h,y+k), \quad \text{for all } (n;(x,y)) \in \mathbb{N}\times\mathbb{Z}^2$$

which implies the relation

$$\sum_{(h,k)\in\Gamma(x,y)} \frac{\mathcal{N}_n(x+h,y+k)}{\mathcal{N}_{n+1}(x,y)} = 1, \quad \text{for all } (n;(x,y)) \in \mathbb{N}\times\mathbb{Z}^2. \tag{6}$$

Let

$$\mathcal{U}(n;(x,y)) = \frac{\nu_n(x,y)}{\mathcal{N}_n(x,y)}, \quad (n;(x,y)) \in \mathbb{N}\times\mathcal{Q}. \tag{7}$$

Dividing (1) by $\mathcal{N}_{n+1}(x,y)$ we see that $\mathcal{U}$ satisfies

$$\mathcal{U}(n+1;(x,y)) = \sum_{(h,k)\in\mathbb{Z}^2} \mathbf{1}_{\Gamma(x,y)}(h,k)\frac{\mathcal{N}_n(x+h,y+k)}{\mathcal{N}_{n+1}(x,y)}\mathcal{U}(n;(x+h,y+k)).$$

Setting

$$\pi\left(n;(x,y),(h,k)\right) = \mathbf{1}_{\Gamma(x,y)}(h,k)\frac{\mathcal{N}_n(x+h,y+k)}{\mathcal{N}_{n+1}(x,y)}; \tag{8}$$

and using (6) we can rewrite this equation

$$\mathcal{U}(n+1;(x,y)) - \mathcal{U}(n;(x,y)) = \sum_{(h,k)} \pi\left(n;(x,y),(h,k)\right)\Big(\mathcal{U}(n;(x+h,y+k)) - \mathcal{U}(n;(x,y))\Big).$$

Introducing the discrete time and space derivatives

$$\begin{aligned} \partial_n\mathcal{U}(n;(x,y)) &= \mathcal{U}(n+1;(x,y)) - \mathcal{U}(n;(x,y)) \\ \nabla_{(h,k)}\mathcal{U}(n;(x,y)) &= \mathcal{U}(n;(x+h,y+k)) - \mathcal{U}(n;(x,y)), \end{aligned}$$

we see that $\mathcal{U}(n;(x,y))$ satisfies the following discrete heat type equation:

$$\partial_n\mathcal{U}(n;(x,y)) = \sum_{(h,k)\in\mathbb{Z}^2} \pi\left(n;(x,y),(h,k)\right)\nabla_{(h,k)}\mathcal{U}(n;(x,y)). \tag{9}$$

3.1 The harmonic function u.

Let us show that the function $u : \mathcal{Q} \to \mathbb{R}$, $(x, y) \to u(x, y) = xy$ satisfies equation (9) in the case of golden walks. As u is independent of n the left-hand side is zero. To compute the right-hand side we first observe that $u(x + h, y + k) - u(x, y) = xk + yh + hk$. This implies that

$$\sum_{(h,k)\in\mathbb{Z}^2} \pi\left(n; (x, y), (h, k)\right) \nabla_{(h,k)} u(x, y) = x \sum_{(h,k)\in\mathbb{Z}^2} \pi\left(n; (x, y), (h, k)\right) k$$
$$+ y \sum_{(h,k)\in\mathbb{Z}^2} \pi\left(n; (x, y), (h, k)\right) h + \sum_{(h,k)\in\mathbb{Z}^2} \pi\left(n; (x, y), (h, k)\right) hk.$$

This shows that for u to be solution of (9) requires (see (8))

$$\begin{aligned} \sum_{(h,k)\in\Gamma(x,y)} \mathcal{N}_n(x + h, y + k)h &= \sum_{(h,k)\in\Gamma(x,y)} \mathcal{N}_n(x + h, y + k)k \\ &= \sum_{(h,k)\in\Gamma(x,y)} \pi\left(n; (x, y), (h, k)\right) hk = 0. \end{aligned} \tag{10}$$

3.2 Counting the unrestricted paths.

Let $\mathcal{N}_n(x, y)$ denote the number of unrestricted walks starting at (x, y) and of length $n \in \mathbb{N}$. We have

$$\mathcal{N}_n(x, y) = \mathcal{N}_n^o(x, y) + \mathcal{N}_n^e(x, y),$$

where $\mathcal{N}_n^o(x, y)$ (resp. $\mathcal{N}_n^e(x, y)$) denote the number of walks ending on odd sites, i.e. sites (x', y') such that $y' - x' \equiv 1 \bmod(2)$ (resp. on even sites). Since every odd site is accessible from its eight neighbors, we have

$$\mathcal{N}_{n+1}^o(x, y) = 4\mathcal{N}_n^o(x, y) + 4\mathcal{N}_n^e(x, y). \tag{11}$$

In contrast, even sites are accessible only from their four odd neighbors. This implies that

$$\mathcal{N}_{n+1}^e(x, y) = 4\mathcal{N}_n^o(x, y) \tag{12}$$

Adding (11) and (12) we obtain

$$\mathcal{N}_{n+1}(x, y) = 4(\mathcal{N}_n^o(x, y) + \mathcal{N}_n^e(x, y)) + 4\mathcal{N}_n^o(x, y) = 4\mathcal{N}_n(x, y) + 16\mathcal{N}_{n-1}(x, y).$$

Setting $\mathcal{F}_n(x, y) = 4^{-n}\mathcal{N}_n(x, y)$ we obtain

$$\mathcal{F}_{n+1}(x, y) = \mathcal{F}_n(x, y) + \mathcal{F}_{n-1}(x, y)$$

which shows that $(\mathcal{F}_n(x, y))$ is a Fibonacci sequence. Using the initial conditions

$$\begin{cases} \mathcal{F}_0(x, y) = 1, \quad \mathcal{F}_1(x, y) = 1 \quad \text{if} \quad y - x \equiv 0 \bmod(2) \\ \mathcal{F}_0(x, y) = 1, \quad \mathcal{F}_1(x, y) = 2 \quad \text{if} \quad y - x \equiv 1 \bmod(2) \end{cases}$$

we deduce the following formulas:

$$\mathcal{N}_n(x,y) = 4^n F_{n+1} = \frac{4^n}{\sqrt{5}}\left(\phi^{n+1} - \overline{\phi}^{n+1}\right) \text{ if } y-x \equiv 0 \bmod(2) \tag{13}$$

$$\mathcal{N}_n(x,y) = 4^n F_{n+2} = \frac{4^n}{\sqrt{5}}\left(\phi^{n+2} - \overline{\phi}^{n+2}\right) \text{ if } y-x \equiv 1 \bmod(2) \tag{14}$$

where $\overline{\phi} = \frac{1-\sqrt{5}}{2}$ and F_n denotes the n^{th} Fibonacci number; the second equality in (13) and (14) results from the Binet's Fibonacci number formula.

From (13) and (14) it follows immediately that $\mathcal{N}_n(x+h, y+k) = \mathcal{N}_n(x-h, y+k)$, $\mathcal{N}_n(x+h, y+k) = \mathcal{N}_n(x+h, y-k)$ for each step $(h,k) \in \Gamma(x,y)$. Combining with the symmetries of $\Gamma(x,y)$, we deduce (10).

On the other hand, using (8), (13) and (14) we deduce that

$$\pi\left(n; (x,y), (h,k)\right) = \frac{1}{4}\mathbf{1}_{\Gamma_{\text{even}}}(h,k),$$

in the case where $y-x \equiv 0 \bmod(2)$, and that

$$\pi\left(n; (x,y), (h,k)\right) = \frac{F_{n+1}}{4F_{n+3}}\mathbf{1}_{\Gamma_{\text{even}}}(h,k) + \frac{F_{n+2}}{4F_{n+3}}\mathbf{1}_{\Gamma_{\text{odd}}\setminus\Gamma_{\text{even}}}(h,k)$$

in the case where $y-x \equiv 1 \bmod(2)$. We will retain from the previous calculations, the following facts:

$$\frac{1}{C}(4\phi)^n \leqslant \mathcal{N}_n(x,y) \leqslant C(4\phi)^n, \quad \text{uniformly in } (x,y) \in \mathbb{Z}^2 \tag{15}$$

and

$$\pi\left(n; (x,y), (h,k)\right) \geqslant \alpha \mathbf{1}_{\Gamma_{\text{even}}}(h,k), \quad \text{uniformly in } (x,y);\ (h,k) \in \mathbb{Z}^2 \tag{16}$$

for appropriate C, $\alpha > 0$.

3.3 Towards the proof of Theorem 1

It follows from (15) that

$$\frac{\nu_n(x,y)}{C\mathcal{N}_n(x,y)} \leqslant \frac{\nu_n(x,y)}{(4\phi)^n} \leqslant C\frac{\nu_n(x,y)}{\mathcal{N}_n(x,y)}.$$

Using (7) and the obvious inequality $\mathcal{U}(n;(x,y)) \leqslant 1$; we see then that in order to establish the upper bound of Theorem 1, it is sufficient to show that

$$\frac{\mathcal{U}(n;(x,y))}{\mathcal{U}(n;(\sqrt{n},\sqrt{n}))} \leqslant C\frac{u(x,y)}{u(\sqrt{n},\sqrt{n})}. \tag{17}$$

For the lower estimate we will need to reverse inequality (17) and a lower bound

$$\mathcal{U}(n;(\sqrt{n},\sqrt{n})) \geqslant c, \quad n \geqslant C, \tag{18}$$

for appropriate c, $C > 0$. It is discrete potential theory that will allow us to establish (17), its reverse and the lower bound (18).

4 Discrete potential theory and Proof of Theorem 1

Ideas from potential theory, in particular, harmonic functions, maximum principle, Harnack inequalities and their parabolic and boundary variants have strong connections with our problem. In fact general concepts of potential theory, have been borrowed since a long time by discrete probability theory [15, 39]. This proved very useful and successful for random walk analysis [28, 29]. While it is beyond the scope of this paper to explain in detail the related concepts, we will try to explain the main tools that are relevant to our problem, i.e. Harnack inequalities. For this we need to introduce some notation.

Two points in $\mathbb{Z}^2$ will be said to be adjacent if the distance between them is unity. A subset $A \subset \mathbb{Z}^2$ of cardinality $|A| \geqslant 2$ will be called connected if for any two points of A there is a path consisting of segments of unit length connecting them in such a manner that the end points of these segments are all in A. A set of points is a domain if it is connected. The symbol A will be used in the following to denote a domain of $\mathbb{Z}^2$.

Given a map $\Gamma : (s,t) \in \mathbb{Z}^2 \to \Gamma(s,t) \subset \mathbb{Z}^2$, we define the boundary $\partial_\Gamma A$ of A (with respect to Γ) by

$$\partial_\Gamma A = \{(x',y') \in A^c, \quad (x',y') = (x+h, y+k), \quad \text{for some } (x,y) \in A \text{ and } (h,k) \in \Gamma(x,y)\}.$$

We shall assume that Γ satisfies:

$$\text{(19)} \qquad \text{There exists } C > 0, \quad |\Gamma(x,y)| \leqslant C, \quad \{\leftarrow, \uparrow, \rightarrow, \downarrow\} \subset \Gamma(x,y), \quad (x,y) \in \mathbb{Z}^2.$$

The closure of A will be denoted by $\overline{A}$ and defined by

$$\overline{A} = A \cup \partial_\Gamma A.$$

For a cylindrical subset $B = A \times \{a \leqslant n \leqslant b\} \subset \mathbb{Z}^2 \times \mathbb{Z}$ where $a < b \in \mathbb{Z}$ we define the lateral boundary and the parabolic boundary of B by

$$\partial_l B = \bigcup_{a<n<b} \partial_\Gamma A \times \{n\}, \qquad \partial_p B = \partial_l B \cup \left(\overline{A} \times \{a\}\right),$$

and we let

$$\overline{B} = B \cup \partial_p B.$$

Let Γ satisfying (19) and $\Pi : \mathbb{N} \times \mathbb{Z}^2 \times \mathbb{Z}^2 \to [0,1]$ be such that

$$\text{(20)} \qquad \Pi(n; (x,y); (h,k)) = 0, \text{if } (h,k) \notin \Gamma(x,y)$$

and

$$\text{(21)} \qquad \sum_{(h,k)\in\Gamma(x,y)} \Pi(n; (x,y); (h,k)) = 1, \quad n \in \mathbb{N}, \quad (x,y) \in \mathbb{Z}^2,$$

$$\text{(22)} \qquad \sum_{(h,k)\in\Gamma(x,y)} \Pi(n; (x,y); (h,k))h = \sum_{(h,k)\in\Gamma(x,y)} \Pi(n; (x,y); (h,k))k = 0, \quad n \in \mathbb{N}, \ (x,y) \in \mathbb{Z}^2,$$

there exists $\alpha > 0$ such that

$$\Pi(n; (x, y); (h, k)) \geqslant \alpha; \qquad n \in \mathbb{N}, \qquad (x, y) \in \mathbb{Z}^2,\ (h, k) \in \{\leftarrow, \uparrow, \rightarrow, \downarrow\}. \tag{23}$$

We shall also assume that Π is strongly aperiodic. This means that, given any $(x, y) \in \mathbb{Z}^2$, there exists some integer $n_0 = n_0(x, y)$ such that $\Pi(n; (0, 0); (x, y))) > 0$ for all $n \geqslant n_0$.

Let $v : \overline{B} \longrightarrow \mathbb{R}$, where $B = A \times \{a \leqslant n \leqslant b\} \subset \mathbb{Z}^2 \times \mathbb{Z}$. We say that v is Π-caloric in B if

$$v((x, y); n + 1) = \sum_{(h,k)\in\Gamma(x,y)} \Pi(n; (x, y); (h, k))v((x + h, y + k); n),$$

for all $((x, y); n) \in A \times \{a \leqslant n < b\}$.

The concept of a Π-caloric function generalizes the notion of discrete harmonic function. In particular, caloric functions satisfy an adapted version of the maximum principle. More precisely, a caloric function on a finite set B attains its maximum in $\overline{B}$ on the lateral boundary $\partial_l B$. Another basic property of harmonic functions is Harnack principle, which states that a positive harmonic function on a ball of radius R is roughly constant on the ball with the same center and radius $R/2$. Harnack principle generalizes for nonnegative Π-caloric as follows.

Theorem 2. *(Refer to [25, 24]) Let Γ and Π satisfy* (19)-(23)*. Then there exists a constant $C = C(\alpha, \Gamma) > 0$ such that, for any $((x, y); s) \in \mathbb{Z}^2 \times \mathbb{Z}$, $R \geqslant 1$ and any nonnegative Π-caloric function $v : \overline{B_{2R}(x, y)} \times \{s - 4R^2 \leqslant n \leqslant s\} \longrightarrow \mathbb{R}$, we have*

$$\begin{aligned} &\max\left\{v((x', y'); n); \quad (x', y') \in B_R(x, y), \quad s - 3R^2 \leqslant n \leqslant s - 2R^2\right\} \\ &\qquad \leqslant C \min\left\{v((x', y'); n); \quad (x', y') \in B_R(x, y), \ \ s - R^2 \leqslant n \leqslant s\right\}, \end{aligned} \tag{24}$$

where $B_R(x, y)$ denotes the discrete ball in $\mathbb{Z}^2$ centered on (x, y) and of radius R.

Kuo and Trudinger proved their Harnack principle in the setting of implicit difference schemes. To see how the explicit case, which corresponds to our Theorem 2, can be obtained by their method one can proceed as follows. Start with [[24], §2] and replace it by [[25], §3] (rewritten in the case $\alpha = 1$, and taking into account all the simplifications implied by the assumptions (19)-(23)), then carry out the same steps as in [[24], §3] until Eq. (3.17) on page 409. This is sufficient because we can always assume R large enough and there is no need to remove the restriction contained in (3.17). This gives an analogue of Lemma 3.2 of [[24], §3] valid in the present context. A weak Harnack inequality can be derived by an adaptation of the procedure introduced by Krylov and Safanov [[23], §2] for the continuous case. Harnack principle follows then as a direct consequence of the weak Harnack inequality (see [24], §4).

In the proof of Theorem 1, together with Theorem 2 we need the following boundary variant of (24). In classical potential theory, the boundary Harnack principle describes the boundary behavior of positive harmonic functions vanishing on a portion of the boundary [1, 21]. It asserts that two positive harmonic functions vanishing on a portion of the boundary decay at the same rate. This principle generalizes to Π-caloric functions. The proof in [[35], §5.1, §5.2 and §5.3], given for nonnegative $\mathcal{L}$-caloric functions in cylindrical domains, is based only on the maximum and Harnack principles and can be readily extended to Π-caloric functions.

Theorem 3. *Let Γ and Π satisfying (19)-(23). Then, there exists a constant $K > 1$ such that for all $s \in \mathbb{N}$, $R > K$ and all couple of nonnegative Π-caloric functions*

$$v_1,\, v_2 : \overline{(\mathcal{Q} \cap B_{3KR}) \times \{s - 9K^2R^2 \leqslant n \leqslant s + 9K^2R^2\}} \subset \mathbb{Z}^2 \times \mathbb{Z} \longrightarrow \mathbb{R}$$

with $v_1 = 0$ on $(\partial_\Gamma \mathcal{Q} \cap B_{2KR}) \times \{s - 4K^2R^2 \leqslant n \leqslant s + 4K^2R^2\}$, we have

$$\max\left\{\frac{v_1((x,y);n)}{v_2((x,y);n)},\ ((x,y);n) \in (\mathcal{Q} \cap B_R)) \times \{s - 2R^2 \leqslant n \leqslant s\}\right\}$$
$$\leqslant C\frac{v_1((KR,KR);s+2K^2R^2)}{v_2((KR,KR);s-2K^2R^2)},$$

where B_R denotes the discrete ball $B_R(0,0)$ and $C = C(\alpha,\Gamma) > 0$.

We now have all the ingredients to prove Theorem 1. The kernel π defined by (8) satisfies all the conditions required in order to apply Theorems 2 and 3. Γ_{even}, Γ_{odd} satisfy in an obvious way (19). (21) is implied by the normalization condition (6). (22) is a consequence of (10) and (23) follows from (16).

Proof Theorem 1. Note that in the proof of Theorem 1 we can assume $n \geqslant C$ for a large constant $C > 0$. Otherwise (4) becomes evident because

$$1 \leqslant \nu_n(x,y) \leqslant 8^C,\quad \frac{1}{C} \leqslant \frac{xy}{n} \leqslant 1$$

if we assume $1 \leqslant x\,, y \leqslant \sqrt{n}$ and $n \leqslant C$.

Assuming n large enough, it becomes possible to apply Theorem 3. The crucial observation is that each of the functions $(n;(x,y)) \longrightarrow u(x,y)$ and $(n;(x,y)) \longrightarrow \mathcal{U}(n;(x,y))$, initially defined on $\mathbb{N} \times \overline{\mathcal{Q}}$, can be extended to all $\mathbb{Z} \times \overline{\mathcal{Q}}$. As for the first, this simply happens because it is independent of n; as for the second, the extension can be done by setting

$$\mathcal{U}(n;(x,y)) \equiv 1, \quad (n;(x,y)) \in (\mathbb{Z}_-) \times \overline{\mathcal{Q}}. \tag{25}$$

The second relation in the boundary conditions (2) satisfied by $\nu(n;(x,y))$ guarantees that the extension (25) is π-caloric in $\mathbb{Z} \times \mathcal{Q}$. The idea of such a construction is inspired by the proof of Lemma 4.1 in [38].

We now have at our disposal two positive π-caloric functions satisfying conditions of Theorem 3 that we will be able to compare. Setting $v_1((x,y);n) = \mathcal{U}(n;(x,y))$, $v_2((x,y);n) = u(x,y)$, $s = n$ and $R = \sqrt{n}/4K$, we deduce then that

$$\frac{\mathcal{U}\big(n;(x,y)\big)}{u\big(x,y\big)} \leqslant C\frac{\mathcal{U}\big(9n/8;(\sqrt{n}/4,\sqrt{n}/4)\big)}{u\big(\sqrt{n}/4,\sqrt{n}/4\big)}. \tag{26}$$

Now we observe that if $(x, y) \in \mathcal{Q}$ is such that $x,\, y \geqslant c\sqrt{n}$ then $\mathcal{U}(n; (x, y)) \geqslant c'$, for appropriate positive constants c, $c' > 0$. This follows from (25) and an application of Harnack inequality (24). This remark also allows to see that, in the same way, (18) is an immediate consequence of Harnack principle.

Using the fact that

$$c \leqslant \mathcal{U}\big(9n/8; (\sqrt{n}/4, \sqrt{n}/4)\big) \approx \mathcal{U}(n; (\sqrt{n}, \sqrt{n})) \leqslant 1$$

and

$$u(\sqrt{n}/4, \sqrt{n}/4) \approx u(\sqrt{n}, \sqrt{n})$$

which easily results from the application of Harnack principle (24) we deduce (17).

To reverse inequality (26) we repeat the above reasoning exchanging v_1 and v_2. This gives

$$\frac{u(x, y)}{\mathcal{U}\big(n; (x, y)\big)} \leqslant C \frac{u\big(\sqrt{n}/4, \sqrt{n}/4\big)}{\mathcal{U}\big(7n/8; (\sqrt{n}/4, \sqrt{n}/4)\big)}.$$

Using Harnack inequality and (18) we get the desired inequality. This completes the proof of Theorem 1. □

5 Extensions and excursions

Our potential theoretic strategy relies on two crucial points: the existence of a positive harmonic function u vanishing on the boundary of $\partial\mathcal{Q}$ and the centering assumption (22). Once achieved the construction of such a function, the machinery applies and gives, the following bounds:

$$\frac{\mathcal{N}_n(x, y)u(x, y)}{Cu\big(\sqrt{n}, \sqrt{n}\big)} \leqslant \nu_n(x, y) \leqslant \frac{C\mathcal{N}_n(x, y)u(x, y)}{u\big(\sqrt{n}, \sqrt{n}\big)}, \tag{27}$$

$(x, y) \in \mathcal{Q}$, $\;x,\, y \leqslant c\sqrt{n}$, $\;n = 1, 2, \ldots$, where $\mathcal{N}_n(x, y)$ denotes the number of unrestricted walks starting at (x, y) and having length n.

In the case of centered homogeneous small steps walks, the existence of u follows from the results of K. Raschel [37] since in this case the kernel $\pi(n; (x, y); (h, k))$ is given by

$$\pi(n; (x, y); (h, k)) = \frac{1}{|\Gamma|}$$

and the function u is the discrete harmonic function explicitly constructed in [37].

Using the results of [5] (see step sets with number 30, 40 and 42 in column "Tag" of Table 2 of [5]), we easily deduce the following.

Theorem 4. *Let Γ be any of the three nonsingular step sets associated with an infinite group and having zero-drift (i.e. $\Gamma = \{\nearrow, \uparrow, \leftarrow, \swarrow, \searrow\}$, $\{\swarrow, \nwarrow, \uparrow, \rightarrow, \searrow\}$ or $\{\nwarrow, \nearrow, \leftarrow, \searrow, \downarrow\}$). Then for any $(x, y) \in \mathcal{Q}$ the length generating function for walks that start from (x, y) and always stay in $\mathcal{Q}$ is not D-finite.*

The proof of Theorem 4 is based on the form of the function $u(x,y)$ constructed in [37]. It follows from [13], that in case of step set $\Gamma = \{\nearrow, \uparrow, \leftarrow, \swarrow, \searrow\}$, this function satisfies $u(\sqrt{n}, \sqrt{n}) \approx n^{\pi/2 \arccos\left(-\frac{1}{4}\right)} = n^{\pi/2\left(\pi - \arccos\left(\frac{1}{4}\right)\right)}$, $n \to \infty$, and $u(\sqrt{n}, \sqrt{n}) \approx n^{\pi/2 \arccos\left(\frac{1}{4}\right)}$, $n \to \infty$ in the case of step sets $\{\swarrow, \nwarrow, \uparrow, \rightarrow, \searrow\}$ or $\{\nwarrow, \nearrow, \leftarrow, \searrow, \downarrow\}$

On the other hand, it is easy to see that $\pi/2 \arccos(1/4)$ is irrational (see [41]). A consequence of the irrationality of $\pi/2 \arccos(1/4)$ is that the asymptotics obtained for $u(\sqrt{n}, \sqrt{n})$ are not compatible with the growth of coefficients of D-finite series with integer-valued exponentially bounded coefficients (see [5], Theorem 3).

Another interesting aspect of (27) is that it generalizes to models with longer steps and also in higher dimensions (we should then use [7] instead of [37] to deduce the existence of the harmonic function u). For instance, the asymptotics obtained by Melczer and Mishna in [32] for d-dimensional highly symmetric walks

$$s_n \approx |\mathcal{S}|^n n^{-d/2},$$

where s_n denotes the number of walks of length n taking steps in $\mathcal{S}$, starting at the origin, and never leaving the positive orthant, can be retrieved easily using (27). Indeed, one can readily verify that for d-dimensional highly symmetric walks the function

$$u(x_1, x_2, .., x_d) = x_1 x_2 \dots x_d, \quad (x_1, x_2, \dots, x_d) \in \mathbb{N}^{\star d}$$

is a positive harmonic function vanishing on the boundary of $\mathbb{N}^{\star d}$ and it is immediate that $u(\sqrt{n}, \sqrt{n}, \dots \sqrt{n})$ equals the polynomial factor $n^{d/2}$.

For symmetric walks in Weyl chambers one can use the harmonic functions constructed in [16, 22] (see also [40]).

Finally the upper bound in (27) can also be used to derive upper estimates for excursions. Let us take, for example, the highly symmetric walks considered in [32]. Let us use the same notation as in §2 and denote by $\nu_n(x_1, \dots, x_d; y_1, \dots, y_d)$ the number of walks of length n taking steps in $\mathcal{S}$, starting from $(x_1, \dots, x_d)$, ending at $(y_1, \dots, y_d)$, and never leaving the positive orthant. We have:

$$\begin{aligned} \frac{\nu_{3n}(x_1, \dots x_d; y_1, \dots y_d)}{|\mathcal{S}|^{3n}} = & \sum_{(z_1, \dots z_d) \in \mathbb{N}^{\star d}} \sum_{(z'_1, \dots z'_d) \in \mathbb{N}^{\star d}} \frac{\nu_n(x_1, \dots x_d; z_1, \dots z_d)}{|\mathcal{S}|^n} \\ & \times \frac{\nu_n(z_1, \dots z_d; z'_1, \dots z'_d)}{|\mathcal{S}|^n} \times \frac{\nu_n(z'_1, \dots z'_d; y_1, \dots y_d)}{|\mathcal{S}|^n} \end{aligned}$$

Using the well known upper bound random walk estimate ([39], Chapter II, Proposition 7.6)

$$\frac{\nu_n(z_1, \dots z_d; z'_1, \dots z'_d)}{|\mathcal{S}|^n} \leqslant C n^{-d/2}$$

and combining with (27) we deduce that

$$\nu_n(x_1, \dots x_d; y_1, \dots y_d) \leqslant C \frac{|\mathcal{S}|^n (x_1 \dots x_d)(y_1 \dots y_d)}{n^{3d/2}}.$$

It follows that the number of walks e_n of length n taking steps in $\mathcal{S}$, beginning and ending at the origin, and never leaving the positive orthant satisfies

$$e_n = \mathcal{O}\left(\frac{C|\mathcal{S}|^n}{n^{3d/2}}\right),$$

which gives an alternative proof of Theorem 7.2 of [32].

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