

On decomposing graphs of large minimum degree into locally irregular subgraphs

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Abstract

A *locally irregular graph* is a graph whose adjacent vertices have distinct degrees. We say that a graph G can be decomposed into k locally irregular subgraphs if its edge set may be partitioned into k subsets each of which induces a locally irregular subgraph in G . It has been conjectured that apart from the family of exceptions which admit no such decompositions, i.e., odd paths, odd cycles and a special class of graphs of maximum degree 3, every connected graph can be decomposed into 3 locally irregular subgraphs. Using a combination of a probabilistic approach and some known theorems on degree constrained subgraphs of a given graph, we prove this to hold for graphs of minimum degree at least 10^{10} . This problem is strongly related to edge colourings distinguishing neighbours by the pallets of their incident colours and to the 1-2-3 Conjecture. In particular, the contribution of this paper constitutes a strengthening of a result of Addario-Berry, Aldred, Dalal and Reed [J. Combin. Theory Ser. B 94 (2005) 237-244].

Keywords: locally irregular graph; graph decomposition; edge set partition; 1-2-3 Conjecture

1 Introduction

All graphs considered are simple and finite. We follow [6] for the notations and terminology not defined here. A *locally irregular graph* is a graph where every vertex has degree distinct from the degrees of all of its neighbours. In other words, it is a graph

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in which adjacent vertices have distinct degrees. Motivated by a few well known problems in edge colourings and labellings, we investigate a (non-evidently) related concept of decompositions of graphs into locally irregular subgraphs. More precisely, we say that a graph $G = (V, E)$ can be *decomposed into k locally irregular subgraphs* if its edge set can be partitioned into k subsets each of which induces a locally irregular subgraph, i.e., $E = E_1 \cup E_2 \cup \dots \cup E_k$ with $E_i \cap E_j = \emptyset$ for $i \neq j$ and $H_i := (V, E_i)$ is locally irregular for $i = 1, 2, \dots, k$. Naturally, instead of decomposing the graph G , we may paint its edges with k colours, say $1, 2, \dots, k$, so that every colour class induces a locally irregular subgraph in G . Such a colouring is called a *locally irregular k -edge colouring* of G . Equivalently it is just an edge colouring such that if an edge $uv \in E$ has colour $i \in \{1, 2, \dots, k\}$ assigned to it, then the number of edges coloured with i incident with u must be distinct from the number of edges coloured with i incident with v . As a weakening of this property, we may require u and v to differ in frequencies of any of the colours from $\{1, 2, \dots, k\}$, not specifically the colour i (which is assigned to uv). In other words, we want adjacent vertices to have distinct multisets of their incident colours under a colouring $c : E \rightarrow \{1, 2, \dots, k\}$. We call such a colouring a *neighbour multiset distinguishing k -edge colouring*. This variant of edge colourings has in fact already been investigated by Addario-Berry et al. in [2]. They proved that 4 colours are always sufficient to construct such a colouring for every graph containing no isolated edges, and provided the following improvement.

Theorem 1. *There exists a neighbour multiset distinguishing 3-edge colouring of every graph G of minimum degree at least 10^3 .*

Their research was motivated by the so called 1-2-3 Conjecture due to Karoński, Łuczak and Thomason [8], yet another concept introducing ‘local irregularity’ into a graph. Let $c : E \rightarrow \{1, 2, \dots, k\}$ be an edge colouring of G with positive integers. For every vertex v we denote by $s_c(v) := \sum_{u \in N(v)} c(uv)$ the sum of its incident colours and call it the *weighted degree* of v . We say that c is a *neighbour sum distinguishing k -edge colouring* of G if $s_c(u) \neq s_c(v)$ for all adjacent vertices u, v in G . Equivalently, instead of assigning integers from $\{1, 2, \dots, k\}$ to the edges, one might strive to multiply them the corresponding numbers of times in order to create a *locally irregular multigraph* of G , i.e., a multigraph in which adjacent vertices have distinct degrees. In [8] Karoński et al. posed the following elegant problem.

Conjecture 2 (1-2-3 Conjecture). There exists a neighbour sum distinguishing 3-edge colouring of every graph G containing no isolated edges.

Thus far it is known that a neighbour sum distinguishing 5-edge colouring exists for every graph without isolated edges, see [7]. In fact our interest in locally irregular graphs originated from the 1-2-3 Conjecture via the following easy observation from [5].

Observation 3. *If G is a regular graph, then there exists a neighbour sum distinguishing 2-edge colouring of G if and only if it can be decomposed into 2 locally irregular subgraphs.*

It is worth noting in this context that asymptotically almost surely a random d -regular graph can be decomposed into 2 locally irregular subgraphs for every constant $d > d_0$, where d_0 is a large constant, see [5]. In the same paper the authors investigate a special family \mathfrak{T} of graphs of maximum degree (at most) 3 whose every member can be constructed from a triangle by repeatedly performing the following operation: choose a triangle with a vertex of degree 2 in our constructed graph and append to this vertex either a hanging path of even length or a hanging path of odd length with a triangle glued to its other end. They posed the following conjecture.

Conjecture 4. Every connected graph G which does not belong to \mathfrak{T} and is not an odd length path nor an odd length cycle can be decomposed into 3 locally irregular subgraphs.

The graphs excluded in the conjecture above were also proven to be the only connected graphs which do not admit decompositions into any number of locally irregular subgraphs. This conjecture was verified in [5] for some classes of graph, e.g., complete graph, complete bipartite graphs, trees, cartesian products of graphs with the desired property (hypercubes for instance), and for regular graphs with large degrees.

The main result of this paper is the following strengthening of Theorem 1, which confirms Conjecture 4 for graphs of sufficiently large minimum degree.

Theorem 5. *Every graph G with minimum degree at least 10^{10} can be decomposed into three locally irregular subgraphs.*

Its proof combines a probabilistic approach with some known theorems on degree constrained subgraphs.

To exemplify the fact that the two graph invariants representing the minimum numbers of colors necessary to create a neighbour multiset distinguishing edge colouring and a locally irregular edge colouring, resp., are indeed distinct, let us consider a graph constructed as follows. Take a single edge, say uv , and append two hanging paths of length 2 to the vertex u and another two hanging paths of length 2 to the vertex v . It is easy to see that there exist multiset distinguishing 2-edge colourings of this graph, but no locally irregular 2-edge colouring. This example may also be easily generalized by substituting the paths of length 2 with any other even paths.

In the following, given two graphs $H_1 = (V_1, E_1)$, $H_2 = (V_2, E_2)$, usually subgraphs of a host graph G , by $H_1 \cup H_2$ we shall mean the graph $(V_1 \cup V_2, E_1 \cup E_2)$. Moreover, we shall write $H_2 \subset H_1$ if $V_2 \subset V_1$ and $E_2 \subset E_1$, and in case of $H_2 \subset H_1$, we shall also write $H_1 - E(H_2)$ to denote the graph obtained from H_1 by removing the edges of H_2 . Given a subset E' of edges of a graph $G = (V, E)$, the *graph induced by E'* shall be understood as $G' := (V, E')$.

2 Tools

We shall use the Lovász Local Lemma and the Chernoff Bound, classical tools of the probabilistic method, see e.g. [4] and [9], respectively.

Theorem 6 (The Local Lemma; General Case). *Let \mathcal{A} be a finite family of (typically bad) events in any probability space and let $D = (\mathcal{A}, E)$ be a directed graph such that every event $A \in \mathcal{A}$ is mutually independent of all the events $\{B : (A, B) \notin E\}$. Suppose that there are real numbers x_A ($A \in \mathcal{A}$) such that for every $A \in \mathcal{A}$, $0 \leq x_A < 1$ and*

$$\Pr(A) \leq x_A \prod_{B \leftarrow A} (1 - x_B). \quad (1)$$

Then $\Pr(\bigcap_{A \in \mathcal{A}} \bar{A}) > 0$.

Here $B \leftarrow A$ (or $A \rightarrow B$) means that there is an arc from A to B in D , the so called *dependency digraph*.

Theorem 7 (Chernoff Bound). *For any $0 \leq t \leq np$:*

$$\Pr(|\text{BIN}(n, p) - np| > t) < 2e^{-\frac{t^2}{3np}},$$

where $\text{BIN}(n, p)$ is the sum of n independent variables, each equal to 1 with probability p and 0 otherwise.

For the deterministic part of our proof we shall in turn use a consequence of the following theorem from [1] (see also [2, 3] for similar degree theorems and their applications).

Theorem 8. *Suppose that for some graph $G = (V, E)$ we have chosen, for every vertex v , two integers:*

$$a_v^- \in \left[\frac{d(v)}{3} - 1, \frac{d(v)}{2} \right], \quad a_v^+ \in \left[\frac{d(v)}{2} - 1, \frac{2d(v)}{3} \right].$$

Then there exists a spanning subgraph H of G such that for every $v \in V$:

$$d_H(v) \in \{a_v^-, a_v^- + 1, a_v^+, a_v^+ + 1\}.$$

Corollary 9. *Suppose that for some graph $G = (V, E)$ with minimum degree at least 12 we have chosen, for every vertex v , an integer $\lambda_v \geq 2$ with $6\lambda_v \leq d(v)$. Then for every assignment*

$$t : V \rightarrow \mathbb{Z},$$

there exists a spanning subgraph H of G such that $d_H(v) \in [\frac{d(v)}{3}, \frac{2d(v)}{3}]$ and $d_H(v) \equiv t(v) \pmod{\lambda_v}$ or $d_H(v) \equiv t(v) + 1 \pmod{\lambda_v}$ for each $v \in V$.

Proof. For every vertex $v \in V$ we have:

$$\left\lfloor \frac{d(v)}{2} \right\rfloor - \left\lfloor \frac{d(v)}{3} \right\rfloor + 1 \geq \frac{d(v) - 1}{2} - \frac{d(v)}{3} + 1 > \frac{d(v)}{6} \geq \lambda_v.$$

Since both sides of the inequality above are integers, then in fact:

$$\left\lfloor \frac{d(v)}{2} \right\rfloor - \left\lfloor \frac{d(v)}{3} \right\rfloor \geq \lambda_v.$$

Analogously,

$$\left\lfloor \frac{2d(v)}{3} \right\rfloor - \left\lfloor \frac{d(v)}{2} \right\rfloor + 1 \geq \frac{2d(v) - 2}{3} - \frac{d(v)}{2} + 1 > \frac{d(v)}{6} \geq \lambda_v,$$

hence,

$$\left\lfloor \frac{2d(v)}{3} \right\rfloor - \left\lfloor \frac{d(v)}{2} \right\rfloor \geq \lambda_v.$$

Thus the sets of integers

$$\left\{ \left\lfloor \frac{d(v)}{3} \right\rfloor + 1, \dots, \left\lfloor \frac{d(v)}{2} \right\rfloor \right\} \quad \text{and} \quad \left\{ \left\lfloor \frac{d(v)}{2} \right\rfloor, \dots, \left\lfloor \frac{2d(v)}{3} \right\rfloor - 1 \right\}$$

both contain all remainders modulo λ_v . The thesis follows then by Theorem 8 (it is sufficient to choose a_v^-, a_v^+ from these sets, resp., so that $a_v^-, a_v^+ \equiv t(v) \pmod{\lambda_v}$). \square

3 Proof of Theorem 5

3.1 Notions

Let $G = (V, E)$ be a graph of minimum degree at least 10^{10} . In the following by $d(v)$ we shall mean the degree of a vertex v in G , and we shall write $d(v)^p$ for short instead of $(d(v))^p$. Let us denote the auxiliary ‘optimizing’ constant

$$\beta := 2^{\frac{1}{0.38}},$$

where $\beta \approx 6.2$ ($6.19 < \beta < 6.2$), and set

$$\alpha(v) := \lceil \log_\beta d(v) \rceil$$

for every $v \in V$. In order to apply Corollary 9, we shall also need two auxiliary vertex labellings, say c_1 and c_2 , with certain regular features. Thus for every vertex v let us first randomly and independently choose one value in $\{0, 1, \dots, 2^{\alpha(v)} - 1\}$, each with equal probability, and denote it by $c_1(v)$. Then let us independently repeat our drawing, i.e., again for every $v \in V$ randomly and independently we choose one value in $\{0, 1, \dots, 2^{\alpha(v)} - 1\}$, each with equal probability, and denote it by $c_2(v)$.

By our construction below it shall be clear that every edge whose one end has the degree at least β times bigger than the other will be ‘safe’ from any potential conflicts between its end-vertices. Some of the remaining edges shall require extra attention though, and shall thus be called ‘risky’. We distinguish three kinds of these, i.e., we say that an edge uv with $d(v)/\beta < d(u) < \beta d(v)$ (hence $\alpha(u)$ and $\alpha(v)$ differ by at most 1) is:

- *risky of type 1* if $2^{\alpha(u)}c_1(u) \equiv 2^{\alpha(v)}c_1(v) \pmod{\min\{4^{\alpha(u)}, 4^{\alpha(v)}\}}$;
- *risky of type 2* if $2^{\alpha(u)}c_2(u) \equiv 2^{\alpha(v)}c_2(v) \pmod{\min\{4^{\alpha(u)}, 4^{\alpha(v)}\}}$;

- *risky of type 3* if

$$|d(u) - 3 \cdot 2^{\alpha(u)}(c_1(u) + c_2(u)) - d(v) + 3 \cdot 2^{\alpha(v)}(c_1(v) + c_2(v))| < \min\{3 \cdot 2^{\alpha(u)}, 3 \cdot 2^{\alpha(v)}\} \pmod{\min\{3 \cdot 4^{\alpha(u)}, 3 \cdot 4^{\alpha(v)}\}},$$

where, given integers b and k with $b \in \{1, \dots, k\}$, by writing $|a| < b \pmod{k}$ we mean that a is an integer which is congruent to one of the following: $-b + 1, -b + 2, \dots, b - 1$ modulo k . Denote the sets of risky edges of types 1, 2 and 3 by R_1, R_2 and R_3 , respectively. For each $v \in V$, let us also denote:

$$\begin{aligned} A(v) &:= \{u \in N_G(v) : uv \in R_1\}, \\ B(v) &:= \{u \in N_G(v) : uv \in R_2\}, \\ C(v) &:= \{u \in N_G(v) : uv \in R_3\}, \\ F(v) &:= B(v) \cap C(v) = \{u \in N_G(v) : uv \in R_2 \cap R_3\}. \end{aligned}$$

3.2 Probabilistic Lemma

Claim 10. *With positive probability, for every vertex $v \in V$:*

$$|A(v)|, |B(v)|, |C(v)| \leq 8d(v)^{0.62} \quad \text{and} \quad (2)$$

$$|F(v)| \leq 12d(v)^{0.24}. \quad (3)$$

Proof. For every $v \in V$, let X_v, Y_v, Z_v, T_v be the random variables of the cardinalities of the sets $A(v), B(v), C(v), F(v)$, resp., and let A_v, B_v, C_v, F_v denote the events that $X_v > 8d(v)^{0.62}, Y_v > 8d(v)^{0.62}, Z_v > 8d(v)^{0.62}$ and $T_v > 12d(v)^{0.24}$, respectively. Consider a vertex $v \in V$, and let u be any of its neighbours with $d(u) \in (d(v)/\beta, \beta d(v))$. Note that $\alpha(u) \in \{\alpha(v) - 1, \alpha(v), \alpha(v) + 1\}$. Then for arbitrarily fixed $c_1^*, c_2^* \in \{0, 1, \dots, 2^{\alpha(v)} - 1\}$ and $c_3^* \in \{0, 1, \dots, 2^{\alpha(u)} - 1\}$, in all cases:

$$\begin{aligned} \Pr(u \in A(v) | c_1(v) = c_1^*) &\leq \frac{1}{2^{\alpha(v)-1}} \leq \frac{1}{2^{\log_\beta d(v)-1}} \\ &= \frac{2}{d(v)^{\frac{1}{\log_2 \beta}}} = \frac{2}{d(v)^{0.38}}. \end{aligned} \quad (4)$$

To see the first inequality above, note that for any fixed c_1^* , in the case when $\alpha(u) \leq \alpha(v)$, $u \in A(v)$ (i.e., $2^{\alpha(u)}c_1(u) \equiv 2^{\alpha(v)}c_1^* \pmod{4^{\alpha(u)}}$) for exactly one out of $2^{\alpha(u)} \geq 2^{\alpha(v)-1}$ admissible choices for $c_1(u)$, while in the case when $\alpha(u) = \alpha(v) + 1$, $u \in A(v)$ (i.e., $2^{\alpha(v)} \cdot 2c_1(u) \equiv 2^{\alpha(v)}c_1^* \pmod{4^{\alpha(v)}}$) either for 0 or for 4 (dependent on the parity of c_1^*) out of $2^{\alpha(u)} = 2^{\alpha(v)+1}$ admissible choices for $c_1(u)$. Analogously:

$$\begin{aligned} \Pr(u \in B(v) | c_2(v) = c_2^*) &\leq \frac{1}{2^{\alpha(v)-1}} \leq \frac{1}{2^{\log_\beta d(v)-1}} \\ &= \frac{2}{d(v)^{\frac{1}{\log_2 \beta}}} = \frac{2}{d(v)^{0.38}}, \end{aligned} \quad (5)$$

$$\Pr(u \in C(v) | c_1(v) = c_1^* \wedge c_2(v) = c_2^* \wedge c_2(u) = c_3^*) \leq \frac{2}{2^{\alpha(v)-1}} \leq \frac{4}{d(v)^{\frac{1}{\log_2 \beta}}} = \frac{4}{d(v)^{.38}}. \quad (6)$$

To see the first inequality in (6), it is in turn sufficient to notice that for any fixed integer d^* (in particular $d^* = d(u) - 3 \cdot 2^{\alpha(u)} c_3^* - d(v) + 3 \cdot 2^{\alpha(v)} (c_1^* + c_2^*)$), in the case when $\alpha(u) \leq \alpha(v)$, we have $|d^* - 3 \cdot 2^{\alpha(u)} c_1(u)| < 3 \cdot 2^{\alpha(u)} \pmod{3 \cdot 4^{\alpha(u)}}$ for at most 2 admissible choices for $c_1(u)$ (cf. the definition of $C(v)$), while in the case when $\alpha(u) = \alpha(v) + 1$, we have $|d^* - 3 \cdot 2^{\alpha(v)} \cdot 2c_1(u)| < 3 \cdot 2^{\alpha(v)} \pmod{3 \cdot 4^{\alpha(v)}}$ for at most 4 admissible choices for $c_1(u)$.

Hence, by (6) and the law of total probability:

$$\Pr(u \in C(v) | c_1(v) = c_1^* \wedge c_2(v) = c_2^*) \leq \frac{4}{d(v)^{0.38}}. \quad (7)$$

Finally, since the choices for c_1 and c_2 are independent, by (5) and (6),

$$\begin{aligned} \Pr(u \in F(v) | c_1(v) = c_1^* \wedge c_2(v) = c_2^*) &= \Pr(u \in C(v) | c_1(v) = c_1^* \wedge c_2(v) = c_2^* \wedge u \in B(v)) \\ &\quad \times \Pr(u \in B(v) | c_1(v) = c_1^* \wedge c_2(v) = c_2^*) \\ &\leq \frac{4}{d(v)^{0.38}} \cdot \frac{2}{d(v)^{0.38}} = \frac{8}{d(v)^{0.76}}. \end{aligned} \quad (8)$$

Consequently, since all choices are independent and $2/d(v)^{0.38} \leq 4/d(v)^{0.38}$, by (4) and the Chernoff Bound we obtain:

$$\begin{aligned} \Pr(A_v | c_1(v) = c_1^*) &= \Pr(X_v > 8d(v)^{0.62} | c_1(v) = c_1^*) \\ &\leq \Pr\left(\text{BIN}\left(d(v), \frac{4}{d(v)^{0.38}}\right) > 8d(v)^{0.62}\right) \\ &\leq \Pr\left(\left|\text{BIN}\left(d(v), \frac{4}{d(v)^{0.38}}\right) - 4d(v)^{0.62}\right| > 4d(v)^{0.62}\right) \\ &< 2e^{-\frac{4d(v)^{0.62}}{3}}. \end{aligned}$$

By the law of total probability we thus obtain that:

$$\Pr(A_v) < 2e^{-\frac{4d(v)^{0.62}}{3}}. \quad (9)$$

Analogously, by (5) and (7),

$$\Pr(B_v) < 2e^{-\frac{4d(v)^{0.62}}{3}} \quad \text{and} \quad \Pr(C_v) < 2e^{-\frac{4d(v)^{0.62}}{3}}. \quad (10)$$

Finally, again by the Chernoff Bound and (8):

$$\begin{aligned} \Pr(F_v | c_1(v) = c_1^* \wedge c_2(v) = c_2^*) &= \Pr(T_v > 12d(v)^{0.24} | c_1(v) = c_1^* \wedge c_2(v) = c_2^*) \\ &\leq \Pr\left(\text{BIN}\left(d(v), \frac{8}{d(v)^{0.76}}\right) > 12d(v)^{0.24}\right) \\ &\leq \Pr\left(\left|\text{BIN}\left(d(v), \frac{8}{d(v)^{0.76}}\right) - 8d(v)^{0.24}\right| > 4d(v)^{0.24}\right) \\ &< 2e^{-\frac{2d(v)^{0.24}}{3}}, \end{aligned}$$

and hence, by the law of total probability,

$$\Pr(F_v) < 2e^{-\frac{2d(v)^{0.24}}{3}}. \quad (11)$$

Note that for every vertex $v \in V$, the events A_v, B_v, C_v and F_v depend only on the random choices for v and its adjacent vertices u with $d(v)/\beta < d(u) < \beta d(v)$. Thus each of these events (corresponding to the vertex v) is mutually independent of all events except (possibly) for these corresponding to the vertex v itself, those corresponding to the neighbours v' of v with $d(v)/\beta < d(v') < \beta d(v)$ and those corresponding to the neighbours v'' of such v' for which $d(v')/\beta < d(v'') < \beta d(v')$. In order to construct a dependency digraph D necessary to apply Theorem 6, from each of the events A_v, B_v, C_v, F_v , we draw arrows pointing at all other events corresponding to the vertices w ($w = v$ or $w = v'$ or $w = v''$) with the properties described above, for $v \in V$. For any event L corresponding to a vertex v of degree d in G (i.e., $L = A_v, L = B_v, L = C_v$ or $L = F_v$), we then set

$$x_L = \frac{1}{1 + d^3}. \quad (12)$$

By our construction, for every such L ,

$$d_D^+(L) \leq 3 + 4d + 4d(\lfloor \beta d \rfloor - 1) = 3 + 4d\lfloor \beta d \rfloor, \quad (13)$$

where $d_D^+(L)$ is the out-degree of L in D . Moreover, if $L \rightarrow Q$ in D , then Q is an event corresponding to a vertex w with

$$\frac{d}{\beta^2} < d(w) < \beta^2 d. \quad (14)$$

By (12), (13) and (14), since $\frac{x}{1+x} > e^{-\frac{1}{x}}$ for $x > 0$, we thus obtain:

$$\begin{aligned} x_L \prod_{Q \leftarrow L} (1 - x_Q) &= \left[x_L \cdot \frac{1}{1 - x_L} \right] \cdot \left[(1 - x_L) \prod_{Q \leftarrow L} (1 - x_Q) \right] \\ &> \left[\frac{1}{1 + d^3} \cdot \frac{1}{1 - \frac{1}{1 + d^3}} \right] \left[\left(1 - \frac{1}{1 + \left(\frac{d}{\beta^2}\right)^3} \right)^{1 + (3 + 4d\lfloor \beta d \rfloor)} \right] \\ &\geq \frac{1}{d^3} \left(\frac{\left(\frac{d}{\beta^2}\right)^3}{1 + \left(\frac{d}{\beta^2}\right)^3} \right)^{25d^2} > \frac{1}{d^3} e^{-\frac{1}{\left(\frac{d}{\beta^2}\right)^3} 25d^2} \\ &= \left(2 \frac{1}{2d^3} \right) \left(e^{-\frac{25 \cdot \beta^6}{d}} \right) > \left(2e^{-\frac{d^{0.24}}{3}} \right) \left(e^{-\frac{d^{0.24}}{3}} \right) = 2e^{-\frac{2d^{0.24}}{3}}, \end{aligned} \quad (15)$$

where $e^{-\frac{25 \cdot \beta^6}{d}} \geq e^{-\frac{d^{0.24}}{3}}$ for $d \geq (3 \cdot 25 \cdot \beta^6)^{\frac{1}{1.24}} \approx 221,460$, while $\frac{1}{2d^3} > e^{-\frac{d^{0.24}}{3}}$ is equivalent to the inequality

$$f(d) := \frac{d^{0.24}}{3} - \ln(2d^3) > 0,$$

which holds e.g. for $d \geq 10^{10}$, since $f(10^{10}) \approx 14 > 0$ and

$$f'(d) = \frac{0.08}{d^{0.76}} - \frac{3}{d} > 0$$

for $d > \left(\frac{3}{0.08}\right)^{\frac{1}{0.24}} \approx 3,617,959$.

By (9), (10), (11), (15) and the Local Lemma we thus obtain that

$$\Pr \left(\bigcap_{v \in V} \overline{A}_v \cap \overline{B}_v \cap \overline{C}_v \cap \overline{F}_v \right) > 0. \quad \square$$

3.3 Construction

Suppose then that we have chosen the labellings c_1 and c_2 so that (2) and (3) hold for every $v \in V$. We shall use twice the fact that

$$\frac{d}{3} - 16d^{0.62} > 72d^{0.76} \quad (16)$$

for $d \geq 10^{10}$. Indeed, if $d > 16^{\frac{1}{0.14}} \approx 398,893,555$, then the left hand side of inequality (16) is greater than $\frac{d}{3} - d^{0.76}$, what in turn is greater than the right hand side of inequality (16) if $d > (3 \cdot 73)^{\frac{1}{0.24}} \approx 5,647,425,084$.

Let us temporarily remove from G all risky edges of type 1 and denote the graph obtained by G' . By (2) and (16), for every vertex $v \in V$,

$$\begin{aligned} d_{G'}(v) &\geq d(v) - 8d(v)^{0.62} \geq 72 \cdot d(v)^{0.76} \\ &= 72 \cdot 4^{\log_\beta d(v)} \geq 6 \cdot 3 \cdot 4^{\alpha(v)}. \end{aligned} \quad (17)$$

By (17) and Corollary 9, we may thus find a subgraph H_1 of G' such that $d_{H_1}(v)$ has one of the two remainders modulo $3 \cdot 4^{\alpha(v)}$, namely:

$$d_{H_1}(v) \equiv 3 \cdot 2^{\alpha(v)} c_1(v), 3 \cdot 2^{\alpha(v)} c_1(v) + 1 \pmod{3 \cdot 4^{\alpha(v)}} \quad (18)$$

for every $v \in V$, and (by (17)):

$$d_{H_1}(v) \in \left[\frac{d_{G'}(v)}{3}, \frac{2d_{G'}(v)}{3} \right] \subset \left[\frac{d(v) - 8d(v)^{0.62}}{3}, \frac{2d(v)}{3} \right]. \quad (19)$$

We paint the edges of H_1 with colour 1, and claim that H_1 is locally irregular, what shall be exhibited in the next subsection.

Denote by G_1 the graph obtained from G by removing all (already painted) edges of H_1 . Let us (again temporarily) remove from G_1 all risky edges $e \in R_2 \cup R_3$ of types 2 and 3, and denote the graph obtained by G'' . By (2), (16) and (19), for every vertex $v \in V$,

$$\begin{aligned} d_{G''}(v) &\geq \frac{d(v)}{3} - 16d(v)^{0.62} \geq 72 \cdot d(v)^{0.76} \\ &= 72 \cdot 4^{\log_\beta d(v)} \geq 6 \cdot 3 \cdot 4^{\alpha(v)}. \end{aligned} \quad (20)$$

Let C be the subgraph induced by the edges of G_1 which belong to R_3 , hence C and G'' are edge-disjoint. For every $v \in V$, denote by

$$c_v := d_C(v) = |C(v) \cap N_{G_1}(v)| \tag{21}$$

the number of risky edges of type 3 incident with v in G_1 .

Subsequently consider the subgraph F induced by these edges of G_1 which belong to $R_2 \cap R_3$. Note that $F \subset C$. By (3), for every vertex $v \in V$,

$$d_F(v) \leq 12d(v)^{0.24} < \frac{1}{2}d(v)^{0.38},$$

where the second inequality holds since $d(v) > 24^{\frac{1}{0.14}} \approx 7,221,904,256$, and thus

$$d_F(v) < \frac{1}{2}d(v)^{\frac{1}{\log_2 \beta}} = 2^{\log_\beta d(v)-1} \leq 2^{\alpha(v)-1}.$$

Since both sides of the inequality above are integers, we in fact obtain that $d_F(v) \leq 2^{\alpha(v)-1} - 1$. Hence, we may greedily find a *proper* vertex colouring

$$h : V \rightarrow \{0, 1, \dots, 2^{\lceil \log_\beta \Delta(G) \rceil - 1} - 1\}$$

of F so that

$$h(v) \leq 2^{\alpha(v)-1} - 1 \tag{22}$$

for every $v \in V$.

By (20) and Corollary 9, we then may find a subgraph H'' of G'' such that

$$\begin{aligned} d_{H''}(v) &\equiv 3 \cdot 2^{\alpha(v)} c_2(v) + 3h(v) - c_v, \\ &3 \cdot 2^{\alpha(v)} c_2(v) + 3h(v) - c_v + 1 \pmod{3 \cdot 4^{\alpha(v)}} \end{aligned} \tag{23}$$

for every $v \in V$, and (by (2))

$$\begin{aligned} d_{H''}(v) &\in \left[\frac{d_{G''}(v)}{3}, \frac{2d_{G''}(v)}{3} \right] \\ &\subset \left[\frac{d(v) - d_{H_1}(v) - 16d(v)^{0.62}}{3}, \frac{2(d(v) - d_{H_1}(v))}{3} \right]. \end{aligned} \tag{24}$$

Then we denote $H_2 := H'' \cup C$, $H_3 := G - E(H_1 \cup H_2)$, and colour the edges of the first of these graphs with 2, while the edges of the second one with 3. Note that since $H_3 = G_1 - E(H'' \cup C)$, then H_3 contains no risky edges of type 3.

3.4 Validity

Consider an edge $uv \in E(H_1)$. By our construction, $uv \notin R_1$. Condition (18) implies that either $d_{H_1}(u)$ or $d_{H_1}(u) - 1$ must be a multiple of $\min\{3 \cdot 2^{\alpha(u)}, 3 \cdot 2^{\alpha(v)}\}$, and similarly, either $d_{H_1}(v)$ or $d_{H_1}(v) - 1$ is a multiple of the same quantity. If $d(v)/\beta < d(u) < \beta d(v)$,

then these multiples cannot be the same, since otherwise we would have $3 \cdot 2^{\alpha(u)}c_1(u) + k_u \cdot 3 \cdot 4^{\alpha(u)} = 3 \cdot 2^{\alpha(v)}c_1(v) + k_v \cdot 3 \cdot 4^{\alpha(v)}$ for some integers k_u, k_v , and hence uv would have to be a risky edge of type 1. Therefore, $d_{H_1}(u) \neq d_{H_1}(v)$ in this case. The same holds also for the (remaining) edges with a greater spread between the degrees of the end-vertices. We shall exhibit this explicitly for all such edges at the end of the proof.

Since H'' and C are edge-disjoint, by (21) and (23),

$$\begin{aligned} d_{H_2}(v) &\equiv 3 \cdot 2^{\alpha(v)}c_2(v) + 3h(v), \\ &3 \cdot 2^{\alpha(v)}c_2(v) + 3h(v) + 1 \pmod{3 \cdot 4^{\alpha(v)}} \end{aligned} \quad (25)$$

for every $v \in V$. Consider an edge $uv \in E(H_2)$ with $d(v)/\beta < d(u) < \beta d(v)$. Then either $d_{H_2}(u) - 3h(u)$ or $d_{H_2}(u) - 3h(u) - 1$ must be a multiple of $\min\{3 \cdot 2^{\alpha(u)}, 3 \cdot 2^{\alpha(v)}\} \geq \max\{3 \cdot 2^{\alpha(u)-1}, 3 \cdot 2^{\alpha(v)-1}\}$, and similarly, either $d_{H_2}(v) - 3h(v)$ or $d_{H_2}(v) - 3h(v) - 1$ is a multiple of the same quantity. If $uv \notin R_2$, then analogously as above these multiples cannot be the same, hence as by (22), $0 \leq 3h(u), 3h(v) \leq \max\{3 \cdot 2^{\alpha(u)-1}, 3 \cdot 2^{\alpha(v)-1}\} - 3$, we obtain that $d_{H_2}(u) \neq d_{H_2}(v)$. Otherwise, by our construction, $uv \in R_2 \cap R_3$ is an edge of F , and hence $d_{H_2}(u) \neq d_{H_2}(v)$ by the ‘properness’ of h , since $0 \leq 3h(u), 3h(v) \leq \max\{3 \cdot 2^{\alpha(u)-1}, 3 \cdot 2^{\alpha(v)-1}\} - 3$.

By (18) and (25),

$$\begin{aligned} d_{H_3}(v) &\equiv d(v) - 3 \cdot 2^{\alpha(v)}(c_1(v) + c_2(v)) - 3h(v), \\ &d(v) - 3 \cdot 2^{\alpha(v)}(c_1(v) + c_2(v)) - 3h(v) - 1, \\ &d(v) - 3 \cdot 2^{\alpha(v)}(c_1(v) + c_2(v)) - 3h(v) - 2 \pmod{3 \cdot 4^{\alpha(v)}} \end{aligned} \quad (26)$$

for every $v \in V$. Then, similarly as above, $d_{H_3}(u) \neq d_{H_3}(v)$ for every edge uv of H_3 with $d(v)/\beta < d(u) < \beta d(v)$, since then $uv \notin R_3$ by our construction.

As for the remaining edges of H_1 , H_2 and H_3 , let us first note that for every vertex $v \in V$, by (19),

$$d_{H_1}(v) \in \left[\frac{1}{3}d(v) - \frac{8}{3}d(v)^{0.62}, \frac{2}{3}d(v) \right], \quad (27)$$

hence, by (2), (24), (27) and our construction,

$$d_{H_2}(v) \in \left[\frac{1}{3}d(v) - \frac{1}{3}d_{H_1}(v) - \frac{16}{3}d(v)^{0.62}, \frac{2}{3}d(v) - \frac{2}{3}d_{H_1}(v) + 8d(v)^{0.62} \right] \quad (28)$$

$$\begin{aligned} &\subset \left[\frac{1}{3}d(v) - \frac{1}{3} \cdot \frac{2}{3}d(v) - \frac{16}{3}d(v)^{0.62}, \right. \\ &\quad \left. \frac{2}{3}d(v) - \frac{2}{3} \left(\frac{1}{3}d(v) - \frac{8}{3}d(v)^{0.62} \right) + 8d(v)^{0.62} \right] \\ &= \left[\frac{1}{9}d(v) - \frac{16}{3}d(v)^{0.62}, \frac{4}{9}d(v) + \frac{88}{9}d(v)^{0.62} \right], \end{aligned} \quad (29)$$

and thus, by (27) and (28),

$$\begin{aligned}
d_{H_3}(v) &\in \left[d(v) - d_{H_1}(v) - \left(\frac{2}{3}d(v) - \frac{2}{3}d_{H_1}(v) + 8d(v)^{0.62} \right), \right. \\
&\quad \left. d(v) - d_{H_1}(v) - \left(\frac{1}{3}d(v) - \frac{1}{3}d_{H_1}(v) - \frac{16}{3}d(v)^{0.62} \right) \right] \\
&= \left[\frac{1}{3}d(v) - \frac{1}{3}d_{H_1}(v) - 8d(v)^{0.62}, \frac{2}{3}d(v) - \frac{2}{3}d_{H_1}(v) + \frac{16}{3}d(v)^{0.62} \right] \\
&\subset \left[\frac{1}{3}d(v) - \frac{1}{3} \cdot \frac{2}{3}d(v) - 8d(v)^{0.62}, \right. \\
&\quad \left. \frac{2}{3}d(v) - \frac{2}{3} \left(\frac{1}{3}d(v) - \frac{8}{3}d(v)^{0.62} \right) + \frac{16}{3}d(v)^{0.62} \right] \\
&= \left[\frac{1}{9}d(v) - 8d(v)^{0.62}, \frac{4}{9}d(v) + \frac{64}{9}d(v)^{0.62} \right]. \tag{30}
\end{aligned}$$

Consequently, since $\frac{4}{9}d + \frac{88}{9}d^{0.62} \leq \frac{2}{3}d$ for $d \geq 44^{\frac{1}{0.38}} \approx 21,129$, by (27), (29) and (30),

$$d_{H_1}(v), d_{H_2}(v), d_{H_3}(v) \in \left[\frac{1}{9}d(v) - 8d(v)^{0.62}, \frac{2}{3}d(v) \right],$$

and thus

$$d_{H_1}(v), d_{H_2}(v), d_{H_3}(v) \in \left[\frac{4}{37}d(v), \frac{2}{3}d(v) \right],$$

since $\frac{1}{9}d - 8d^{0.62} \geq \frac{4}{37}d$ for $d \geq (8 \cdot 333)^{\frac{1}{0.38}} \approx 1,034,102,857$. Since $\frac{2}{3}/\frac{4}{37} < 6.17 < \beta$, this guarantees that if (without the loss of generality) $d(u) \geq \beta d(v)$ for some edge $uv \in E$, then $d_{H_1}(u) \neq d_{H_1}(v)$, $d_{H_2}(u) \neq d_{H_2}(v)$ and $d_{H_3}(u) \neq d_{H_3}(v)$. All subgraphs H_1 , H_2 and H_3 are thus locally irregular. \square

4 Concluding remarks

Note that Conjecture 4 still remains open. It would be interesting to settle it at least for bipartite graphs.

Problem 11. Can every connected bipartite graph which is not an odd length path be decomposed into three locally irregular subgraphs?

Moreover, thus far it is not even known if any finite number of locally irregular subgraphs admitted is sufficient in general.

Problem 12. Does there exist a constant K such that every connected graph which does not belong to \mathfrak{T} and is not an odd length path nor an odd length cycle can be decomposed into (at most) K locally irregular subgraphs?

This is not known in the case of bipartite graphs either.

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