On well-covered, vertex decomposable and Cohen-Macaulay graphs

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Abstract

Let $G = (V, E)$ be a graph. If G is a König graph or if G is a graph without 3-cycles and 5-cycles, we prove that the following conditions are equivalent: Δ_G is pure shellable, R/I_Δ is Cohen-Macaulay, G is an unmixed vertex decomposable graph and G is well-covered with a perfect matching of König type $e_1, \ldots, e_g$ without 4-cycles with two e_i's. Furthermore, we study vertex decomposable and shellable (non-pure) properties in graphs without 3-cycles and 5-cycles. Finally, we give some properties and relations between critical, extendable and shedding vertices.

Keywords: Cohen-Macaulay, shellable, well-covered, unmixed, vertex decomposable, König, girth

1 Introduction

Let G be a simple graph (without loops and multiplies edges) whose vertex set is $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. The *edge ideal* of G, denoted by $I(G)$, is the ideal of R generated by all monomials $x_i x_j$ such that $\{x_i, x_j\} \in E(G)$. G is a *Cohen-Macaulay graph* if $R/I(G)$ is a Cohen-Macaulay

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ring (see [3], [20]). A subset F of $V(G)$ is a *stable set* or *independent set* if $e \nsubseteq F$ for each $e \in E(G)$. The cardinality of the maximum stable set is denoted by $\beta(G)$. G is called *well-covered* if every maximal stable set has the same cardinality. On the other hand, a subset D of $V(G)$ is a *vertex cover* of G if $D \cap e \neq \varnothing$ for every $e \in E(G)$. The number of vertices in a minimum vertex cover of G is called the *covering number* of G and it is denoted by $\tau(G)$. This number coincide with $\mathrm{ht}(I(G))$, the *height* of $I(G)$. If the minimal vertex covers have the same cardinality, then G is called an *unmixed* graph. Notice that, D is a vertex cover if and only if $V(G) \setminus D$ is a stable set. Hence, $\tau(G) = n - \beta(G)$ and G is well-covered if and only if G is unmixed. The *Stanley-Reisner complex* of $I(G)$, denoted by Δ_G, is the simplicial complex whose faces are the stable sets of G. Recall that a simplicial complex Δ is called *pure* if every facet has the same number of elements. Thus, Δ_G is pure if and only if G is well-covered.

Some properties of G, Δ_G and $I(G)$ allow an interaction between Commutative Algebra and Combinatorial Theory. Examples of these properties are: Cohen-Macaulayness, shellability, vertex decomposability and well-coveredness. These properties have been studied in ([3], [4], [6], [7], [11], [12], [13], [16], [17], [18], [20], [22]). In general, we have the following implications (see [3], [16], [20], [22])

$$\begin{array}{c}\text{Unmixed}\\ \text{vertex decomposable}\end{array} \Rightarrow \begin{array}{c}\text{Pure}\\ \text{shellable}\end{array} \Rightarrow \text{Cohen - Macaulay} \Rightarrow \text{Well - covered}$$

The equivalence between the Cohen-Macaulay property and the unmixed vertex decomposable property has been studied for some families of graphs: bipartite graphs (in [7] and [11]); very well-covered graphs (in [5] and [13]); graphs with girth at least 5, block-cactus (in [12]); and graphs without 4-cycles and 5-cycles (in [2]). For this paper, a cycle $C = (z_1, z_2, \ldots, z_n)$ can have chords (edges between non-consecutive vertices in C) in G. A cycle without chords is called an *induced cycle*.

If a bipartite graph is well-covered, pure shellable or Cohen-Macaulay, then it is König and has a perfect matching. The perfect matching is important because it allowed Hibi and Herzog to characterize Cohen-Macaulay bipartite graph (see [11]). Similarly, the existence of a perfect matching allows one to find a classification of well-covered bipartite graphs (see [15] and [19]). However, a 3-cycle and a 5-cycle are Cohen-Macaulay graphs, but they does not have a perfect matching. This is the motivation for the study of Cohen-Macaulay graphs without 3-cycles and 5-cycles. In particular, we are interested in knowing if these graphs have a perfect matching. In this paper we prove that it is affirmative.

The paper is organized as follow: in section 2 we give some properties and relations between critical, shedding and extendable vertices that we will use in the following sections. In section 3 we prove some results about well-covered graphs. In section 4 we prove the equivalences of unmixed vertex decomposable and Cohen-Macaulay properties for König graphs and graphs without 3-cycles and 5-cycles. We prove that theses properties are equivalent to the following condition: G is an unmixed König graph with a perfect matching $e_1, \ldots, e_g$ without 4-cycles with two e_i's. This result extends the criterion of

Herzog-Hibi for Cohen-Macaulay bipartite graphs, given in [11]. In [17] Van Tuyl proved that the vertex decomposable property, the shellable (non-pure) property and the sequentially Cohen-Macaulay property are equivalent in bipartite graphs. Furthermore, in [18] Van Tuyl and Villarreal give a criterion that characterize shellable bipartite graphs. These results and results obtained in section 4, motivate us to study the vertex decomposable property and the shellable (non-pure) property for graphs without 3-cycles and 5-cycles. In section 5, we prove that the neighborhood of a 2-connected block of G has a free vertex, if G is a bipartite shellable graph or if G is a vertex decomposable graph without 3-cycles and 5-cycles. Also, we prove that the criterion of Van Tuyl-Villarreal can be extended to vertex decomposable graphs without 3-cycles and 5-cycles and shellable graphs with girth at least 11. The equivalence between the shellable property and the vertex decomposable property for graphs without 3-cycles and 5-cycles is an open problem.

2 Critical, extendable and shedding vertices.

Let X be a subset of $V(G)$. The *subgraph induced* by X in G, denoted by $G[X]$ is the graph with vertex set X and whose edge set is $\{\{x,y\} \in E(G) \mid x,y \in X\}$. Furthermore, let $G \setminus X$ denote the induced subgraph $G[V(G) \setminus X]$. Now, if $v \in V(G)$, then the set of *neighbors* of v (in G) is denoted by $N_G(v)$ and its closed neighborhood is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v in G is $\deg_G(v) = |N_G(v)|$.

Definition 1. G is vertex decomposable if G is a totally disconnected graph or there is a vertex v such that

(a) $G \setminus v$ and $G \setminus N_G[v]$ are both vertex decomposable, and

(b) each stable set in $G \setminus N_G[v]$ is not a maximal stable set in $G \setminus v$.

A *shedding vertex* of G is any vertex which satisfies the condition (b). Equivalently, v is a shedding vertex if for every stable set S contained in $G \setminus N_G[v]$, there is some $x \in N_G(v)$ such that $S \cup \{x\}$ is stable.

Lemma 2. *If x is a vertex of G, then x is a shedding vertex if and only if $|N_G(x) \setminus N_G(S)| \geqslant 1$ for every stable set S of $G \setminus N_G[x]$.*

Proof. $\Rightarrow$) We take a stable set S of $G \setminus N_G[x]$. Since x is a shedding vertex, then there is a vertex $z \in N_G(x)$ such that $S \cup \{z\}$ is a stable set of $G \setminus x$. Thus, $z \notin N_G[S]$. Therefore, $|N_G(x) \setminus N_G(S)| \geqslant 1$.

$\Leftarrow$) We take a stable set S of $G \setminus N_G[x]$. Thus, there exists a vertex $z \in N_G(x) \setminus N_G(S)$. Since $z \in N_G(x)$, we have that $z \notin S$. Furthermore, $z \notin N_G(S)$, then $S \cup \{z\}$ is a stable set of $G \setminus x$. Consequently, S is not a maximal stable set of $G \setminus x$. Therefore, x is a shedding vertex. ☐

Definition 3. Let S be a stable set of G. If x is of degree zero in $G \setminus N_G[S]$, then x is called *isolated vertex* in $G \setminus N_G[S]$, or we say that S *isolates* to x.

By Lemma 2, we have that x is not a shedding vertex if and only if there exists a stable set S of $G \setminus N_G[x]$ such that $N_G(x) \subseteq N_G(S)$, i.e. x is an isolated vertex in $G \setminus N_G[S]$.

Corollary 4. *Let S be a stable set of G. If S isolates x in G, then x is not a shedding vertex in $G \setminus N_G[y]$ for all $y \in S$.*

Proof. Since S isolates x, then $\deg_{G\setminus N_G[S]}(x) = 0$ and in particular $x \in V(G \setminus N_G[S])$. Thus, $N_G(x) \subseteq N_G[S] \setminus S$. Hence, if $y \in S$ and $G' = G \setminus N_G[y]$, then $x \in V(G')$. Furthermore, since $S \cap N_G[x] = \varnothing$, then $S' = S \setminus y$ is a stable set in $G' \setminus N_{G'}[x]$. Now, since S isolates x, if $a \in N_{G'}(x)$, then there exists $s \in S$ such that $\{a, s\} \in E(G)$. But $a \in N_{G'}(x)$, then $a \notin N_G[y]$, consequently $s \in S'$ and $\{a, s\} \in E(G')$. This implies $|N_{G'}(x) \setminus N_{G'}(S')| = 0$. Therefore, by Lemma 2, x is not a shedding vertex in G'. □

Theorem 5. *If x is a shedding vertex of G, then one of the following conditions hold:*

(a) *There is $y \in N_G(x)$ such that $N_G[y] \subseteq N_G[x]$.*

(b) *x is in a 5-cycle with at most one chord.*

Proof. We take $N_G(x) = \{y_1, y_2, \ldots, y_k\}$. If G does not satisfy (a), then there is

$$\{z_1, \ldots, z_k\} \subseteq V(G) \setminus N_G[x]$$

such that $\{y_i, z_i\} \in E(G)$ for $i \in \{1, \ldots, k\}$. We denote by $L = \{z_1, \ldots, z_q\} = \{z_1, \ldots, z_k\}$ and suppose that $z_i \neq z_j$ for $1 \leqslant i < j \leqslant q$. By Lemma 2, if L is a stable set of G, then $|N_G(x) \setminus N_G(L)| \geqslant 1$. But $N_G(x) = \{y_1, \ldots, y_k\} \subseteq N_G(L)$, then L is not a stable set. Hence, $q \geqslant 2$ and there exist $z_{i_1}, z_{i_2} \in L$ such that $\{z_{i_1}, z_{i_2}\} \in E(G)$. Thus, there exist y_{j_1} and y_{j_2} such that $y_{j_1} \neq y_{j_2}$ and $\{y_{j_1}, z_{i_1}\}, \{y_{j_2}, z_{i_2}\} \in E(G)$. Furthermore, $\{z_i, y_{j_2}\}, \{z_{i_2}, y_{j_1}\}, \{z_{i_1}, x\}, \{z_{i_2}, x\} \notin E(G)$. Therefore, $(x, y_{j_1}, z_{i_1}, z_{i_2}, y_{j_2})$ is a 5-cycle of G with at most one chord. □

Definition 6. A vertex v is called simplicial if the induced subgraph $G[N_G(v)]$ is a complete graph (or clique). Equivalently, a simplicial vertex is a vertex that appears in exactly one clique.

Remark 7. If $v, w \in V(G)$ such that $N_G[v] \subseteq N_G[w]$, then w is a shedding vertex of G (see Lemma 6 in [22]). In particular, if v is a simplicial vertex, then any $w \in N_G(v)$ is a shedding vertex (see Corollary 7 in [22]).

Corollary 8. *Let G be graph without 4-cycles. If x is a shedding vertex of G, then x is in a 5-cycle or there exists a simplicial vertex z such that $\{x, z\} \in E(G)$ with $|N_G[z]| \leqslant 3$.*

Proof. By Theorem 5, if x is not in a 5-cycle, then there is $z \in N_G(x)$ such that $N_G[z] \subseteq N_G[x]$. If $\deg_G(z) = 1$, then z is a simplicial vertex. If $\deg_G(z) = 2$, then $N_G(z) = \{x, w\}$. Consequently, (z, x, w) is a 3-cycle since $N_G[z] \subseteq N_G[x]$. Thus, z is a simplicial vertex. Now, if $\deg_G(z) \geqslant 3$, then there are $w_1, w_2 \in N_G(z) \setminus x$. Since $N_G[z] \subseteq N_G[x]$, we have that (w_1, z, w_2, x) is a 4-cycle of G. This is a contradiction. Therefore, $|N_G[z]| \leqslant 3$ and z is a simplicial vertex. □

Remark 9. If G is a 5-cycle with $V(G) = \{x_1, x_2, x_3, x_4, x_5\}$, then each x_i is a shedding vertex.

Proof. We can assume that $i = 1$, then $\{x_3\}$ and $\{x_4\}$ are the stable sets in $G \setminus N_G[x_1]$. Furthermore, $\{x_3, x_5\}$ and $\{x_2, x_4\}$ are stable sets in $G \setminus x_1$. Hence, each stable set of $G \setminus N_G[x_1]$ is not a maximal stable set in $G \setminus x_1$. Therefore, x_1 is a shedding vertex. □

Definition 10. A vertex v of G is critical if $\tau(G \setminus v) < \tau(G)$. Furthermore, G is called a *vertex critical* graph if each vertex of G is critical.

Lemma 11. *If $\tau(G \setminus v) < \tau(G)$, then $\tau(G) = \tau(G \setminus v) + 1$. Moreover, v is a critical vertex if and only if $\beta(G) = \beta(G \setminus v)$.*

Proof. If t is a minimal vertex cover such that $|t| = \tau(G \setminus v)$, then $t \cup \{v\}$ is a vertex cover of G. Thus, $\tau(G) \leqslant |t \cup \{v\}| = \tau(G \setminus v) + 1$. Consequently, if $\tau(G) > \tau(G \setminus v)$, then $\tau(G) = \tau(G \setminus v) + 1$.

Now, we have that $\tau(G) + \beta(G) = |V(G)| = |V(G \setminus v)| + 1 = \tau(G \setminus v) + \beta(G \setminus v) + 1$. Hence, $\beta(G) = \beta(G \setminus v)$ if and only if $\tau(G) = \tau(G \setminus v) + 1$. Therefore, v is a critical vertex if and only if $\beta(G) = \beta(G \setminus v)$. □

Definition 12. A vertex v of G is called an *extendable* vertex if G and $G \setminus v$ are well-covered graphs with $\beta(G) = \beta(G \setminus v)$.

Note that if v is an extendable vertex, then every maximal stable set S of $G \setminus v$ contains a vertex of $N_G(v)$.

Corollary 13. *Let G be an unmixed graph and $x \in V(G)$. The following conditions are equivalent:*

(a) *x is an extendable vertex.*

(b) $|N_G(x) \setminus N_G(S)| \geqslant 1$ *for every stable set S of $G \setminus N_G[x]$.*

(c) *x is a shedding vertex.*

(d) *x is a critical vertex and $G \setminus x$ is unmixed.*

Proof. $(a) \Leftrightarrow (b)$ ([8], Lemma 2).

$(b) \Leftrightarrow (c)$ By Lemma 2.

$(a) \Leftrightarrow (d)$ Since G is unmixed, then by Lemma 11, x is extendable if and only if x is a critical vertex and $G \setminus x$ is unmixed. □

3 König and well-covered graphs

In this paper we denoted by Z_G the set of the isolated vertices of G, that is,

$$Z_G = \{x \in V(G) \mid \deg_G(x) = 0\}.$$

Definition 14. G is a *König* graph if $\tau(G) = \nu(G)$ where $\nu(G)$ is the maximum number of pairwise disjoint edges. A *perfect matching of König type* of G is a collection $e_1, \ldots, e_g$ of pairwise disjoint edges whose union is $V(G)$ and $g = \tau(G)$.

Proposition 15. *Let G be a König graph and $G' = G \setminus Z_G$. Then the following are equivalent:*

(a) *G is unmixed.*

(b) *G' is unmixed.*

(c) *If $V(G') \neq \varnothing$, then G' has a perfect matching $e_1, \ldots, e_g$ of König type such that for any two edges $f_1 \neq f_2$ and for two distinct vertices $x \in f_1$, $y \in f_2$ contained in some e_i, one has that $(f_1 \setminus x) \cup (f_2 \setminus y)$ is an edge.*

Proof. (a)⇔(b) Since $V(G) \setminus V(G') = Z_G$, then C is a vertex cover of G if and only if C is a vertex cover of G'. Therefore, G is unmixed if and only if G' is unmixed.

(b)⇔(c) By ([14], Lemma 2.3 and Proposition 2.9). □

Definition 16. A graph G is called *very well-covered* if it is well-covered without isolated vertices and $|V(G)| = 2\mathrm{ht}(I(G))$.

Lemma 17. *G is an unmixed König graph if and only if G is totally disconnected or $G' = G \setminus Z_G$ is very well-covered.*

Proof. ⇒) If G is not totally disconnected, then from Proposition 15, G' has a perfect matching $e_1, \ldots, e_g$ of König type. Hence, $|V(G')| = 2g = 2\tau(G') = 2\mathrm{ht}(I(G'))$. Furthermore, G' is unmixed, therefore G' is very well-covered.

⇐) If G is totally disconnected, then $\nu(G) = 0$ and $\tau(G) = 0$. Hence, G is an unmixed König graph. Now, if G is not totally disconnected, then G' is very well-covered. Consequently, by ([10], Corollary 3.7) G' has a perfect matching. Thus, $\nu(G') = |V(G')|/2 = \mathrm{ht}(G') = \tau(G')$. Hence, G' is König. Furthermore, $\nu(G) = \nu(G')$ and $\tau(G) = \tau(G')$, then G is König. Finally, since G' is unmixed, by Proposition 15, G is also unmixed. □

Definition 18. A subgraph H of G is called a c-minor (of G) if there exists a stable set S of G, such that $H = G \setminus N_G[S]$.

Remark 19. Each connected component of a graph G is a c-minor of G.

Remark 20. The unmixed property is closed under c-minors. That is, each c-minor of G has the same property (see [20]).

Definition 21. A vertex of degree one is called *leaf* or *free vertex*. Furthermore, an edge which is incident with a leaf is called *pendant*.

Lemma 22. *If G is an unmixed graph and $x \in V(G)$, then $N_G(x)$ does not contain two free vertices.*

Proof. We suppose that there exists $x \in V(G)$ such that $y_1, \dots, y_s$ are free vertices in $N_G(x)$. Hence, $G_1 = G \setminus N_G[y_1, \dots, y_s] = G \setminus \{x, y_1, \dots, y_s\}$ is unmixed. Now, we take a maximal stable set S of G_1. Thus, $|S| = \beta(G_1)$ since G_1 is unmixed. Consequently, $S_1 = S \cup \{y_1, \dots, y_s\}$ is a stable set in G. We take S_2 a maximal stable in G such that $x \in S_2$. Since G is unmixed, we have that $|S_2| \geqslant |S_1| = |S| + s$. Furthermore, $S_2 \setminus x$ is a stable set in G_1, then $|S_2| \leqslant \beta(G_1) + 1$. This implies $\beta(G_1) + 1 \geqslant |S| + s$. But, $|S| = \beta(G_1)$, therefore, $s \leqslant 1$. □

Definition 23. If $v, w \in V(G)$, then the distance $\mathrm{d}(u, v)$ between u and v in G is the length of the shortest path joining them, otherwise $d(u, v) = \infty$. Now, if $H \subseteq G$, then the distance from a vertex v to H is $d(v, H) = min\{d(v, u) \mid u \in V(H)\}$. Furthermore, if $W \subseteq V(G)$, then we define $d(v, W) = d(v, G[W])$ and $D_i(W) = \{v \in V(G) \mid d(v, W) = i\}$.

Proposition 24. *Let G be an unmixed connected graph without 3-cycles and 5-cycles. If C is a 7-cycle and H is a c-minor of G with $C \subseteq H$ such that C has three non-adjacent vertices of degree 2 in H, then C is a c-minor of G.*

Proof. We take a minimal c-minor H of G such that $C \subseteq H$ and C has three non-adjacent vertices of degree 2 in H. We can suppose that $C = (x, z_1, w_1, a, b, w_2, z_2)$ with $\deg_H(x) = \deg_H(w_1) = \deg_H(w_2) = 2$. If $\{z_1, b\} \in E(H)$, then (z_1, b, w_2, z_2, x) is a 5-cycle of G. Thus, $\{z_1, b\} \notin E(H)$, similarly $\{z_2, a\} \notin E(H)$. Furthermore, since G does not have 3-cycles, then $\{z_1, z_2\}, \{z_1, a\}, \{z_2, b\} \notin E(H)$. Hence, C is an induced cycle in H. On the other hand, if there exists $v \in V(H)$ such that $d(v, C) \geqslant 2$, then $H' = H \setminus N_G[v]$ is a c-minor of G and $C \subseteq H' \subset H$. This is a contradiction by the minimality of H. Therefore, $d(v, C) \leqslant 1$ for each $v \in V(H)$.

Now, if $\deg_H(b) \geqslant 3$, then there exists $c \in V(H) \setminus V(C)$ such that $\{b, c\} \in E(H)$. If $\{c, z_2\} \notin E(G)$ implies that $N_{H_1}(z_2)$ has two free vertices, w_2 and x, in $H_1 = H \setminus N_H[w_1, c]$, this is a contradiction by Lemma 22. Thus $\{c, z_2\} \in E(H)$. Furthermore, $\{a, c\}$, $\{z_1, c\} \notin E(H)$ since (a, b, c) and (z_1, w_1, a, b, c) are not cycles in G. Hence, if $\deg_H(c) \geqslant 3$, then there exists $d \in V(H) \setminus V(C)$ such that $\{c, d\} \in E(H)$. Also, $\{d, b\}, \{d, z_2\}, \{d, z_1\} \notin E(H)$ since (c, b, d), (z_2, d, c) and (z_1, x, z_2, c, d) are not cycles of G. But $d(d, C) \leqslant 1$, so $\{a, d\} \in E(H)$. Consequently, $N_{H_2}(z_1)$ has two free vertices, w_1 and x, in $H_2 = H \setminus N_H[d, w_2]$, a contradiction by Lemma 22, then $\deg_H(c) = 2$. This implies, $N_{H_3}(z_2)$ has two free vertices, w_2 and c, in $H_3 = H \setminus N_H[a]$. This is not possible, therefore $\deg_H(b) = 2$. Similarly, $\deg_H(a) = 2$.

Now, if $\deg_H(z_2) \geqslant 3$ we have that there exists $c' \in V(H) \setminus V(C)$ such that $\{c', z_2\} \in E(H)$. If there exists $d' \in V(H) \setminus V(C)$ such that $\{c', d'\} \in E(H)$, then $\{d', z_1\}$ or $\{d', z_2\} \in E(G)$, since $d(d', C) \leqslant 1$. But (c', d', z_2) and (x, z_2, c', d', z_1) are not cycles of H, thus, $N_H(c') \subseteq \{z_1, z_2\}$. Consequently, $N_{H_4}(z_2)$ has two free vertices, x and c', in $H_4 = H \setminus N_H[w_1]$, a contradiction. Hence $\deg_H(z_2) = 2$. Similarly, $\deg_H(z_1) = 2$. Furthermore, since H is minimal, then it is connected. Therefore, $H = C$ and C is a c-minor of G. □

4 König and Cohen-Macaulay graphs without 3-cycles and 5-cycles

Definition 25. A simplicial complex Δ is shellable if the facets (maximal faces) of Δ can be ordered $F_1, \ldots, F_s$ such that for all $1 \leqslant i < j \leqslant s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$. In this case, $F_1, \ldots, F_s$ is called a shelling of Δ. A graph G is called shellable if Δ_G is shellable. Furthermore, the facet set of Δ is denoted by $\mathcal{F}(\Delta)$.

Remark 26. The following properties: shellable, Cohen-Macaulay, sequentially Cohen-Macaulay and vertex decomposable are closed under c-minors (see [1], [20]).

Remark 27. If G is very well-covered with a perfect matching $e_1, \ldots, e_g$, then the following conditions are equivalent:

(1) G is Cohen-Macaulay.

(2) There are no 4-cycles with two e_i's.

Proof. By ([5], Theorem 3.4). □

Proposition 28. *Let G be a König graph where $G' = G \setminus Z_G$. Then the following properties are equivalent:*

(i) *G is unmixed vertex decomposable.*

(ii) *Δ_G is pure shellable.*

(iii) *$R/I(G)$ is Cohen-Macaulay.*

(iv) *$V(G') = \varnothing$ or G' is an unmixed graph with a perfect matching $e_1, \ldots, e_g$ of König type without 4-cycles with two e_i's.*

(v) *$V(G') = \varnothing$ or there exists a relabelling of the vertices $V(G') = \{x_1, \ldots, x_h, y_1, \ldots, y_h\}$ such that $\{x_1, y_1\}, \ldots, \{x_h, y_h\}$ is a perfect matching, $X = \{x_1, \ldots, x_h\}$ is a minimal vertex cover of G' and the following conditions holds:*

  (a) *If $a_i \in \{x_i, y_i\}$ and $\{a_i, x_j\}, \{y_j, x_k\} \in E(G')$, then $\{a_i, x_k\} \in E(G')$ for $i \neq j$ and $j \neq k$;*

  (b) *If $\{x_i, y_j\} \in E(G')$, then $i \leqslant j$.*

Proof. (i)⇔(ii)⇔(iii) In each case G is unmixed and König. Hence, by Lemma 17, G is totally disconnected or G' is very well-covered. If G is totally disconnected, then we obtain the equivalences. Now, if G' is very well-covered, then by ([13], Theorem 1.1) we obtain the equivalences.

(iv)⇒(iii) We can assume that $V(G') \neq \varnothing$. Thus, by Lemma 17, G' is very well-covered. Hence, by Remark 27 G' is Cohen-Macaulay. Therefore, G is Cohen-Macaulay.

(iii)⇒ (v) Since $R/I(G)$ is Cohen-Macaulay, then G is unmixed. Consequently, by Lemma 17, we can assume that G' is very well-covered. Hence, by ([13], Lemma 3.1), G' satisfies (v).

(v)⇒(iv) We can assume that $V(G') \neq \varnothing$. Since, $e_1 = \{x_1, y_1\}, \ldots, e_h = \{x_h, y_h\}$ is a perfect matching, then $\nu(G') = h$. Furthermore, X is a minimal vertex cover, so $\tau(G') = h$. Hence, $e_1, \ldots, e_h$ is a perfect matching of König type. Thus, from (a) and Proposition 15, G' is unmixed. On the other hand $\{y_1, \ldots, y_h\}$ is a stable set. Therefore, from (b), there are no 4-cycles with two e_i's. □

Corollary 29. *Let G be a connected König graph. If G is Cohen-Macaulay, then G is an isolated vertex or G has at least one free vertex.*

Proof. By Proposition 28, if G is not an isolated vertex, then G has a perfect matching $e_1 = \{x_1, y_1\}, \ldots, e_h = \{x_h, y_h\}$ where $\{x_1, \ldots, x_h\}$ is a minimal vertex cover. Thus, $\{y_1, \ldots, y_h\}$ is a maximal stable set. Furthermore, if $\{x_i, y_j\}$, then $i \leqslant j$. Hence, $N_G(y_1) = \{x_1\}$. Therefore, y_1 is a free vertex. □

Lemma 30. *Let G be an unmixed connected graph with a perfect matching $e_1, \ldots, e_g$ of König type without 4-cycles with two e_i's and $g \geqslant 2$. For each $z \in V(G)$ we have that:*

(a) *If $\deg_G(z) \geqslant 2$, then there exist $\{z, w_1\}, \{w_1, w_2\} \in E(G)$ such that $\deg_G(w_2) = 1$. Furthermore, $e_i = \{w_1, w_2\}$ for some $i \in \{1, \ldots, g\}$.*

(b) *If $\deg_G(z) = 1$, then there exist $\{z, w_1\}$,$\{w_1, w_2\}$,$\{w_2, w_3\} \in E(G)$ such that $\deg_G(w_3) = 1$. Moreover, $e_i = \{z, w_1\}$ and $e_j = \{w_2, w_3\}$ for some $i, j \in \{1, \ldots, g\}$.*

Proof. Since $e_1 = \{x_1, y_1\}, \ldots, e_g = \{x_g, y_g\}$ is a perfect matching of König type we can assume $D = \{x_1, \ldots, x_g\}$ is a minimal vertex cover. Thus, $F = \{y_1, \ldots, y_g\}$ is a maximal stable set. By Proposition 28, we can assume that if $\{x_i, y_j\} \in E(G)$, then $i \leqslant j$. Now, we take a vertex $z \in V(G)$.

(a) First, we suppose that $z = x_k$ and there is a vertex x_j in $N_G(x_k)$. If y_j is a free vertex, then we take $w_1 = x_j$ and $w_2 = y_j$, and $e_j = \{w_1, w_2\}$. Now, we can assume $N_G(y_j) \setminus x_j = \{x_{p_1}, \ldots, x_{p_r}\}$ with $p_1 < \cdots < p_r < j$. If y_{p_1} is not a free vertex, then there is a vertex x_p with $p < p_1$ such that $\{x_p, y_{p_1}\} \in E(G)$. Since G is unmixed, from Proposition 15, we obtain that $\{x_p, y_j\} = (\{x_p, y_{p_1}\} \setminus y_{p_1}) \cup (\{y_j, x_{p_1}\} \setminus x_{p_1}) \in E(G)$. But $p < p_1$, a contradiction since p_1 is minimal. Consequently, $\deg_G(y_{p_1}) = 1$. Also, from Proposition 15, we have that $\{x_k, x_{p_1}\} = (\{x_k, x_j\} \setminus x_j) \cup (\{x_{p_1}, y_j\} \setminus y_j) \in E(G)$. Hence, we take $w_1 = x_{p_1}$ and $w_2 = y_{p_1}$, and we have that $e_{p_1} = \{w_1, w_2\}$. Now, we assume that $z = x_k$ and $N_G(x_k) \setminus y_k = \{y_{j_1}, \ldots, y_{j_t}\}$ with $k < j_1 < \cdots < j_t$. We suppose that $\deg_G(x_{j_t}) \geqslant 2$. If there is a vertex y_r such that $\{x_{j_t}, y_r\} \in E(G)$, then $r > j_t$. Since G is unmixed, $\{x_k, y_r\} = (\{x_k, y_{j_t}\} \setminus y_{j_t}) \cup (\{y_r, x_{j_t}\} \setminus x_{j_t}) \in E(G)$, a contradiction since j_t is maximal. Thus, there exists a vertex x_p such that $\{x_{j_t}, x_p\} \in E(G)$. But, since G is unmixed, then $\{x_k, x_p\} = (\{x_k, y_{j_t}\} \setminus y_{j_t}) \cup (\{x_p, x_{j_t}\} \setminus x_{j_t}) \in E(G)$. This is a contradiction since $N_G(x_k) \setminus y_k = \{y_{j_1}, \ldots, y_{i_t}\}$. Consequently, $\deg_G(x_{j_t}) = 1$. Therefore, we take $w_1 = y_{j_t}$ and $w_2 = x_{j_t}$, with $e_{j_t} = \{w_1, w_2\}$.

Finally, we assume that $z = y_k$, since y_k is not a free vertex, then $N_G(y_k) \setminus x_k = \{x_{j_1}, \ldots, x_{j_r}\}$ with $j_1 < \cdots < j_r < k$. If y_{j_1} is not a free vertex, then there is a vertex x_q such that $\{x_q, y_{j_1}\} \in E(G)$ with $q < j_1$. This implies $\{x_q, y_k\} = (\{x_q, y_{j_1}\} \setminus y_{j_1}) \cup (\{x_{j_1}, y_k\} \setminus x_{j_1}) \in E(G)$. But $q < j_1$, a contradiction. Therefore, $\deg_G(y_{j_1}) = 1$ and we take $w_1 = x_{j_1}$ and $w_2 = y_{j_1}$. Hence, $e_{j_1} = \{w_1, w_2\}$.

(b) Since $e_1, \ldots, e_g$ is a perfect matching, then there exists $i \in \{1, \ldots, g\}$ such that $e_i = \{z, z'\}$. Since G is connected, z is a free vertex and $g \geqslant 2$, then $\deg_G(z') \geqslant 2$. Thus, by (a) there exist $w'_1, w'_2 \in V(G)$ such that $\{z', w'_1\}, \{w'_1, w'_2\} \in E(G)$ where $\deg_G(w'_2) = 1$ and $\{w'_1, w'_2\} = e_j$ for some $j \in \{1, \ldots, g\}$. Therefore, we take $w_1 = z'$, $w_2 = w'_1$, $w_3 = w'_2$. Consequently, $e_i = \{z, w_1\}$ and $e_j = \{w_2, w_3\}$. □

Remark 31. If C_n is a n-cycle, then C_n is vertex decomposable, shellable or sequentially Cohen-Macaulay if and only if $n = 3$ or 5 (see [9] and [22]). Furthermore, a chordal graph, which is a graph whose induced cycles are 3-cycles, is vertex decomposable (see Corollary 7 in [22]). In particular trees are vertex decomposable.

Theorem 32. *Let G be a graph without 3-cycles and 5-cycles. If $G_1, \ldots, G_k$ are the connected components of G, then the following conditions are equivalent:*

(a) *G is unmixed vertex decomposable.*

(b) *G is pure shellable.*

(c) *G is Cohen-Macaulay*

(d) *G is unmixed and if G_i is not an isolated vertex, then G_i has a perfect matching $e_1, \ldots, e_g$ of König type without 4-cycles with two e'_is.*

Proof. (a) ⇒ (b) ⇒ (c) (see [16], [20], [22]).

(d) ⇒ (a) Since each component G_i is König, then G is König. Therefore, from Proposition 28, G is unmixed vertex decomposable.

(c) ⇒ (d) Since G is Cohen-Macaulay, then G is unmixed. We proceed by induction on $|V(G)|$. We take $x \in V(G)$ such that $\deg_G(x)$ is minimal and we suppose that $N_G(x) = \{z_1, \ldots, z_r\}$. By Remark 26, $G' = G \setminus N_G[x]$ is a Cohen-Macaulay graph. We take $G'_1, \ldots, G'_s$, the connected components of G'. We can assume that $V(G'_i) = \{y_i\}$ for $i \in \{1, \ldots, s'\}$. Since $\deg_G(x)$ is minimal, this implies $\{y_i, z_j\} \in E(G)$ for all $i \in \{1, \ldots, s'\}$ and $j \in \{1, \ldots, r\}$. Since G does not contain 3-cycles, we have that $N_G(x)$ is a stable set. If $s' = s$, then the only maximal stable sets of G are $\{y_1, \dots, y_{s'}, x\}$ and $\{z_1, \dots, z_r\}$. Thus, G is a bipartite graph. So, G is König. Hence, by Proposition 28, G satisfies (d). Consequently, we can assume $s > s'$, implying that there is a component G'_i with an edge $e = \{w, w'\}$.

Now, we suppose that $r \geqslant 2$. Since $\deg_G(x)$ is minimal there exist $a, b \in V(G)$ such that $\{a, w\}, \{b, w'\} \in E(G)$. If $a = b$, then (a, w, w') is a 3-cycle in G. Hence, $a \neq b$. If $a, b \in N_G(x)$, then (x, a, w, w', b) is a 5-cycle in G. Thus, $|\{w, w', a, b\} \cap V(G'_i)| \geqslant 3$. By induction hypothesis, G' satisfies (d). So, G'_i has a perfect matching and $\tau(G'_i) \geqslant 2$. Furthermore, by Corollary 29, G'_i has a free vertex a'. Then, by Lemma 30 (b), there exist edges $\{a', w_1\}, \{w_1, w_2\}, \{w_2, b'\} \in E(G'_i)$ such that $\deg_{G'_i}(a') = \deg_{G'_i}(b') = 1$. By the minimality of $\deg_G(x)$ we have that a' and b' are adjacent with at least $r-1$ neighbor vertices of x. If $r \geqslant 3$, then there exists z_j such that $z_j \in N_G(a') \cap N_G(b')$. This implies that (a', w_1, w_2, b', z_j) is a 5-cycle of G. But G does not have 5-cycles, consequently, $r = 2$. We can assume that $\{a', z_1\}, \{b', z_2\} \in E(G)$, implying $C = (x, z_1, a', w_1, w_2, b', z_2)$ is a 7-cycle with $\deg_G(a') = \deg_G(b') = \deg_G(x) = 2$. Hence, by Proposition 24, C is a c-minor of G. Thus, by Remark 26, C is Cohen-Macaulay. This is a contradiction by Remark 31. Therefore, $\deg_G(x) = r \leqslant 1$.

If $r = 0$, then the result is clear. Now, if $r = 1$ we can assume that $G_1, \dots, G_k$ are the connected components of G and $z_1 \in V(G_1)$. Consequently, the connected components of $G \setminus N_G[x]$ are $F_1, \dots, F_l, G_2, \dots, G_k$ where $F_1, \dots, F_l$ are the connected components of $G_1 \setminus N_{G_1}[x]$. By induction hypothesis $G_2, \dots, G_k$ satisfy (d). If $F_j = \{d_j\}$, then $N_G(z_1)$ has two free vertices, d_j and x, a contradiction by Lemma 22. Hence, $|V(F_i)| \geqslant 2$ for $i \in \{1, \dots, l\}$. By induction hypothesis, we have that F_i has a perfect matching $M_i = \{e^i_1, \dots, e^i_{g_i}\}$ of König type. Thus, $\{e\} \cup (\bigcup_{i=1}^{l} M_i)$ is a perfect matching of G_1, where $e = \{x, z_1\}$. Also, $\{z_1\} \cup (\bigcup_{i=1}^{l} X_i)$ is a vertex cover of G_1, where X_i is a minimal vertex cover of F_i. Consequently, $\nu(G_1) \geqslant 1 + \sum_{i=1}^{l} |M_i| = 1 + \sum_{i=1}^{l} g_i = 1 + \sum_{i=1}^{l} |X_i| \geqslant \tau(G_1)$. This implies that G_1 is König. Furthermore, by Remark 26, we have that G_1 is Cohen-Macaulay. Therefore, by Proposition 28, G_1 satisfies (d). □

Corollary 33. *Let G be a connected graph without 3-cycles and 5-cycles. If G is Cohen-Macaulay, then G has at least one extendable vertex x adjacent to a free vertex.*

Proof. From Theorem 32, G is König. Thus, by Corollary 29 there exists a free vertex x. If $N_G(x) = \{y\}$, then by Remark 7, y is a shedding vertex. Therefore, from Corollary 13, y is an extendable vertex, since G is unmixed. □

Definition 34. G is called *whisker graph* if there exists an induced subgraph H of G such that $V(H) = \{x_1, \dots, x_s\}$, $V(G) = V(H) \cup \{y_1, \dots, y_s\}$ and $E(G) = E(H) \cup W(H)$ where $W(H) = \{\{x_1, y_1\}, \dots, \{x_s, y_s\}\}$. The edges of $W(H)$ are called whiskers and they form a perfect matching.

Definition 35. The *girth* of G is the length of the smallest cycle or infinite if G is a forest.

Corollary 36. *Let G be a connected graph of girth 6 or more. If G is not an isolated vertex, then the following conditions are equivalent:*

(i) *G is unmixed vertex decomposable.*

(ii) *Δ_G is pure shellable.*

(iii) *$R/I(G)$ is Cohen-Macaulay.*

(iv) *G is an unmixed König graph.*

(v) *G is very well-covered.*

(vi) *G is unmixed with $G \neq C_7$.*

(vii) *G is a whisker graph.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) (see [16], [20], [22]). (iii) $\Rightarrow$ (iv) G is unmixed and from Theorem 32, G is König. (iv) $\Rightarrow$ (v) From Lemma 17. (v) $\Rightarrow$ (vi) It is clear, since C_7 is not very well-covered.

(vi) $\Rightarrow$ (vii) By ([8], Corollary 5), the pendant edges $\{x_1, y_1\}, \dots, \{x_g, y_g\}$ of G form a perfect matching. Since $\{x_i, y_i\}$ is a pendant edge, we can assume that $\deg_G(y_i) = 1$ for each $1 \leqslant i \leqslant g$. We take $H = G[x_1, \dots, x_n]$. Therefore, G is a whisker graph with $W(H) = \{\{x_1, y_1\}, \dots, \{x_g, y_g\}\}$.

(vii) $\Rightarrow$ (i) By ([6], Theorem 4.4). □

5 Vertex decomposable and shellable properties in graphs without 3-cycles and 5-cycles

Definition 37. A 5-cycle C of G is called *basic* if C does not contain two adjacent vertices of degree three or more in G.

Lemma 38. *If G is a graph, then any vertex of degree at least 3 in a basic 5-cycle is a shedding vertex.*

Proof. Let $C = (x_1, x_2, x_3, x_4, x_5)$ be a basic 5-cycle. We suppose that $\deg_G(x_1) \geqslant 3$, since C is a basic 5-cycle, then $\deg_G(x_2) = \deg_G(x_5) = 2$. Also, we can assume that $\deg_G(x_3) = 2$. We take a stable set S of $G \setminus N_G[x_1]$. Since $\{x_3, x_4\} \in E(G)$, then $|S \cap \{x_3, x_4\}| \leqslant 1$. Hence, $x_3 \notin S$ or $x_4 \notin S$. Consequently, $S \cup \{x_2\}$ or $S \cup \{x_5\}$ is a stable set of $G \setminus x_1$. Therefore, x_1 is a shedding vertex. □

Remark 39. If G has a shedding vertex v where $G \setminus v$ and $G \setminus N_G[v]$ are shellable with shelling $F_1, \dots, F_k$ and $G_1, \dots, G_q$, respectively, then G is shellable with shelling $F_1, \dots, F_k, G_1 \cup \{v\}, \dots, G_q \cup \{v\}$ (see Lemma 6 in [21]).

Theorem 40. *Let G be a connected graph with a basic 5-cycle C. G is a shellable graph if and only if there is a shedding vertex $x \in V(C)$ such that $G \setminus x$ and $G \setminus N_G[x]$ are shellable graphs.*

Proof. $\Rightarrow$) We can suppose that $C = (x_1, x_2, x_3, x_4, x_5)$. If $G = C$, then G is shellable. By Remark 9, each vertex is a shedding vertex. Furthermore, $G \setminus x_1$ is a path with shelling $\{x_2, x_4\}, \{x_2, x_5\}, \{x_3, x_5\}$ and $G \setminus N_G[x_1]$ is an edge. Therefore, $G \setminus x_1$ and $G \setminus N_G[x_1]$ are shellable graphs. Now, we suppose $G \neq C$. We can assume that $\deg_G(x_1) \geqslant 3$. Since C is a basic 5-cycle, then $\deg_G(x_2) = \deg_G(x_5) = 2$. Also, we can suppose $\deg_G(x_3) = 2$ and $\deg_G(x_4) \geqslant 2$. By Lemma 38, x_1 is a shedding vertex. Furthermore by Remark 26, we have that $G \setminus N_G[x_1]$ is a shellable graph. Now, we will prove that $G_1 = G \setminus x_1$ is shellable. Since G is shellable and since shellability is closed under c-minors, then $G_2 = G \setminus N_G[x_2]$ is shellable. We assume that $F_1, \dots, F_r$ is a shelling of Δ_{G_2}. Also, $G_3 = G \setminus N_G[x_3, x_5]$ is shellable. We suppose that $H_1, H_2, \dots, H_k$ is shelling of Δ_{G_3}. We take $F \in \mathcal{F}(\Delta_{G_1})$. If $x_2 \in F$, then $F \setminus x_2 \in \mathcal{F}(\Delta_{G_2})$ and there exists F_i such that $F = F_i \cup \{x_2\}$. If $x_2 \notin F$, then $x_3 \in F$ and $x_4 \notin F$. Thus, $x_5 \in F$. Hence, $F \setminus \{x_3, x_5\} \in \mathcal{F}(\Delta_{G_3})$, then there exists H_j such that $F = H_j \cup \{x_3, x_5\}$. This implies, $\mathcal{F}(\Delta_{G_1}) = \{F_1 \cup \{x_2\}, \dots, F_r \cup \{x_2\}, H_1 \cup \{x_3, x_5\}, \dots, H_k \cup \{x_3, x_5\}\}$. Furthermore, $F_1 \cup \{x_2\}, \dots, F_r \cup \{x_2\}$ and $H_1 \cup \{x_3, x_5\}, \dots, H_k \cup \{x_3, x_5\}$ are shellings. Now, $x_3 \in (H_j \cup \{x_3, x_5\}) \setminus (F_i \cup \{x_2\})$ and H_j is a stable set of G without vertices of C. So, $H_j \cup \{x_2, x_5\}$ is a maximal stable set of G_1 since $N_G(x_2, x_5) = V(C)$ and $\{x_2, x_5\} \notin E(G)$. Consequently, $H_j \cup \{x_2, x_5\} = F_l \cup \{x_2\}$ for some $l \in \{1, \dots, r\}$ and $(H_j \cup \{x_3, x_5\}) \setminus (F_l \cup \{x_2\}) = \{x_3\}$. Therefore, G_1 is a shellable graph.

$\Leftarrow$) By Remark 39. □

Definition 41. A *cut vertex* of a graph is one whose removal increases the number of connected components. A *block* of a graph is a maximal subgraph without cut vertices. A connected graph without cut vertices with at least three vertices is called 2-connected graph.

In the following result P is a property closed under c-minors.

Theorem 42. *Let G be a graph without 3-cycles and 5-cycles with a 2-connected block B. If G satisfies the property P and B does not satisfy P, then there exists $x \in D_1(B)$ such that* $\deg_G(x) = 1$.

Proof. By contradiction, we assume that if $x \in D_1(B)$, then $|N_G(x)| > 1$. Thus, there exist $a, b \in N_G(x)$ with $a \neq b$. We can suppose that $a \in V(B)$. If $b \in V(B)$, then $G[\{x\} \cup V(B)]$ is 2-connected. But $B \subsetneq G[\{x\} \cup V(B)]$. This is a contradiction since B is a block. Consequently, $V(B) \cap N_G(x) = \{a\}$. Now, we suppose that $b \in D_1(B)$. Since there is no 3-cycle in G, then $a \notin N_G(b)$. Hence, there exists $c \in N_G(b) \cap V(B)$ such that $c \neq a$. This implies $G[\{x, b\} \cup V(B)]$ is 2-connected. But $B \subsetneq G[\{x, b\} \cup V(B)]$, a contradiction. Then $D_1(B) \cap N_G(x) = \varnothing$. Thus, $N_G(x) \cap (V(B) \cup D_1(B)) = \{a\}$ and $b \in D_2(B)$. Now, if $D_1(B) = \{x_1, \dots, x_r\}$, then there exists an a_i such that $V(B) \cap N_G(x_i) = \{a_i\}$. Also, there exists b_i such that $b_i \in N_G(x_i) \cap D_2(B)$. We can suppose that $L = \{b_1, \dots, b_s\} =$

$\{b_1, \ldots, b_r\}$ with $b_i \neq b_j$ for $1 \leqslant i < j \leqslant s$. We will prove that L is a stable set. Suppose that $\{b_i, b_j\} \in E(G)$, if $a_i = a_j$, then $(a_i, x_i, b_i, b_j, x_j, a_i)$ is a 5-cycle in G, this is a contradiction. Consequently $a_i \neq a_j$ and the induced subgraph $G[\{x_i, b_i, b_j, x_j\} \cup V(B)]$ is 2-connected. But B is a block, then $\{b_i, b_j\} \notin E(G)$. Therefore, L is a stable set. Furthermore, $G' = G \setminus N_G[L]$ is a c-minor of G, implying that G' satisfies the property P. Since $D_1(B) \subset N_G(L)$, we have that B is a connected component of G'. But, B does not satisfy P. This is a contradiction since each connected component of G is a c-minor. Therefore, there exists a free vertex in $D_1(B)$. □

Corollary 43. *Let G be a graph without 3-cycles and 5-cycles and B a 2-connected block. If G is shellable (unmixed, Cohen-Macaulay, sequentially Cohen-Macaulay or vertex decomposable) and B is not shellable (unmixed, Cohen - Macaulay, sequentially Cohen-Macaulay or vertex decomposable), then there exists $x \in D_1(B)$ such that $\deg_G(x) = 1$.*

Proof. From Remark 20, Remark 26 and Theorem 42. □

Corollary 44. *Let G be a bipartite graph and B a 2-connected block. If G is shellable, then there exists $x \in D_1(B)$ such that $\deg_G(x) = 1$.*

Proof. Since G is bipartite, then B is bipartite. If H is a shellable bipartite graph, then H has a free vertex (see [18], Lemma 2.8), and so H is not 2-connected. Hence, B is not shellable. Therefore, by Corollary 43, there exists $x \in D_1(B)$ such that $\deg_G(x) = 1$. □

Lemma 45. *Let G be a graph without 3-cycles and 5-cycles. If G is vertex decomposable, then G has a free vertex.*

Proof. Since G is vertex decomposable, then there is a shedding vertex x. Furthermore, there are no 5-cycles in G. Hence, by Theorem 5, there exists $y \in N_G(x)$ such that $N_G[y] \subseteq N_G[x]$. If $z \in N_G(y) \setminus x$, then (x, y, z) is a 3-cycle. This is a contradiction. Therefore, $N_G(y) = \{x\}$, implying that y is a free vertex. □

Theorem 46. *Let G be a graph without 3-cycles and 5-cycles. G is vertex decomposable if and only if there exists a free vertex x with $N_G(x) = \{y\}$ such that $G_1 = G \setminus N_G[x]$ and $G_2 = G \setminus N_G[y]$ are vertex decomposable.*

Proof. $\Rightarrow$) By Lemma 45 there exists a free vertex x. Furthermore, by Remark 26, G_1 and G_2 are vertex decomposable.

$\Leftarrow$) By Remark 7, y is a shedding vertex. Moreover, $G \setminus y = G_1 \cup \{x\}$. Furthermore, since G_1 is vertex decomposable, then $G \setminus y$ is also it. Therefore, G is vertex decomposable, since G_2 is vertex decomposable. □

Corollary 47. *If G is a 2-connected graph without 3-cycles and 5-cycles, then G is not a vertex decomposable.*

Proof. Since G is 2-connected, then G does not have a free vertex. Therefore, by Lemma 45, G is not vertex decomposable. □

Theorem 48. *Let G be a vertex decomposable graph without 3-cycles and 5-cycles. If B is a 2-connected block of G, then $D_1(B)$ has a free vertex.*

Proof. By Corollary 47, B is not vertex decomposable. Therefore, by Theorem 42, $D_1(B)$ has a free vertex. ☐

Definition 49. Let G_1, G_2 be graphs. If $K = G_1 \cap G_2$ is a complete graph with $|V(K)| = k$, then $G = G_1 \cup G_2$ is called the *k-clique-sum* (or clique-sum) of G_1 and G_2 in K.

Corollary 50. *If G is the 2-clique-sum of the cycles C_1 and C_2 with $|V(C_1)| = r_1 \leqslant r_2 = |V(C_2)|$, then G is vertex decomposable if and only if $r_1 = 3$ or $r_1 = r_2 = 5$.*

Proof. $\Leftarrow$) First, we suppose that $r_1 = 3$. Consequently, we can assume $C_1 = (x_1, x_2, x_3)$ and $x_2, x_3 \in V(C_1) \cap V(C_2)$. Thus, x_1 is a simplicial vertex. Hence, by Remark 7, x_2 is a shedding vertex. Furthermore, $G \setminus x_2$ and $G \setminus N_G[x_2]$ are trees. Consequently, by Remark 31, $G \setminus x_2$ and $G \setminus N_G[x_2]$ are vertex decomposable graphs. Therefore, G is vertex decomposable.

Now, we assume that $r_1 = r_2 = 5$ with $C_1 = (x_1, x_2, x_3, x_4, x_5)$ and $C_2 = (y_1, x_2, x_3, y_4, y_5)$. We take a stable set S in $G \setminus N_G[x_5]$. If $x_2 \in S$, then $S \cup \{x_4\}$ is a stable set in $G_1 = G \setminus x_5$. If $x_2 \notin S$, then $S \cup \{x_1\}$ is a stable set in G_1. Consequently, by Lemma 2, x_5 is a shedding vertex. Since x_2 is a neighbor of a free vertex in G_1, then x_2 is a shedding vertex in G_1. Furthermore, since $G_1 \setminus x_2$ and $G_1 \setminus N_{G_1}[x_2]$ are forests, then they are vertex decomposable graphs, by Remark 31. Thus, G_1 is vertex decomposable. Snce $G \setminus N_G[x_5] = C_2$, it is vertex decomposable by Remark 31. Therefore, G is vertex decomposable.

$\Rightarrow$) By Corollary 47, we have that $\{r_1, r_2\} \cap \{3, 5\} \neq \varnothing$. We suppose $r_1 \neq 3$. So $r_1 = 5$ or $r_2 = 5$. Consequently, we can assume that $\{C_1, C_2\} = \{C, C'\}$ where $C = (x_1, x_2, x_3, x_4, x_5)$ and $x_2, x_3 \in V(C) \cap V(C')$. Thus, $G \setminus N_G[x_5] = C'$ is vertex decomposable. Hence, from Remark 31, $|V(C')| \in \{3, 5\}$. But $r_1 \neq 3$, then $|V(C')| = 5$ and $r_1 = r_2 = 5$. Therefore, $r_1 = 3$ or $r_1 = r_2 = 5$. ☐

Lemma 51. *Let G be a 2-connected graph with girth at least 11. Then G is not shellable.*

Proof. Since G is 2-connected, then G is not a forest. Consequently, if r is the girth of G, then there exists a cycle $C = (x_1, x_2, \dots, x_r)$. If $G = C$, then G is not shellable. Hence, $G \neq C$ implying $D_1(C) \neq \varnothing$. We take $y \in D_1(C)$, without loss of generality we can assume that $\{x_1, y\} \in E(G)$. If $\{x_i, y\} \in E(G)$ for some $i \in \{2, \dots, r\}$, then we take the cycles $C_1 = (y, x_1, x_2, \dots, x_i)$ and $C_2 = (y, x_1, x_r, x_{r-1}, \dots, x_i)$. Thus, $|V(C_1)| = i + 1$ and $|V(C_2)| = r - i + 3$. Since r is the girth of G, then $i + 1 \geqslant r$ and $r - i + 3 \geqslant r$. Consequently, $3 \geqslant i$ implies $4 \geqslant r$. But $r \geqslant 11$, this is a contradiction. This implies that $|N_G(y) \cap V(C)| = 1$. Now, we suppose that there exist $y_1, y_2 \in D_1(C)$ such that $\{y_1, y_2\} \in E(G)$. We can assume that $\{x_1, y_1\}, \{x_i, y_2\} \in E(G)$. Since $r \geqslant 11$, then there are no 3-cycles in G. In particular, $x_1 \neq x_i$. Now, we take the cycles $C' = (y_1, x_1, \dots, x_i, y_2)$ and $C'' = (y_1, x_1, x_r, x_{r-1}, \dots, x_i, y_2)$. So, $|V(C')| = i + 2$ and $|V(C'')| = r - i + 4$. Since r is the girth, we have that $i + 2 \geqslant r$ and $r - i + 4 \geqslant r$. Hence, $4 \geqslant i$ and

$6 \geqslant r$, this is a contradiction. Then $D_1(C)$ is a stable set. Now, since G is 2-connected, then for each $y \in D_1(C)$ there exists $z \in N_G(y) \cap D_2(C)$. If there exist $z_1, z_2 \in D_2(C)$ such that $\{z_1, z_2\} \in E(G)$, then there exist $y_1, y_j \in D_1(C)$ such that $\{z_1, y_1\}, \{z_2, y_j\} \in E(G)$. Since there are no 3-cycles in G, we have that $y_1 \neq y_j$. We can assume that $\{x_1, y_1\}, \{x_i, y_j\} \in E(G)$. Since there are no 5-cycles, then $i \neq 1$. Consequently, there exist cycles $C_1' = (x_1, \ldots, x_i, y_j, z_2, z_1, y_1)$ and $C_2' = (x_i, \ldots, x_r, x_1, y_1, z_1, z_2, y_j)$. This implies $r \leqslant |V(C_1')| = i + 4$ and $r \leqslant |V(C_2')| = r - i + 6$. Hence, $i \leqslant 6$ and $r \leqslant 10$, this is a contradiction. Then $D_2(C)$ is a stable set. Furthermore, C is a connected component of $G \setminus N_G[D_2(C)]$. But C is not shellable, therefore G is not shellable, from Remark 26. □

Theorem 52. *If G has girth at least 11, then G is shellable if and only if there exists $x \in V(G)$ with $N_G(x) = \{y\}$ such that $G \setminus N_G[x]$ and $G \setminus N_G[y]$ are shellable.*

Proof. $\Leftarrow$) By ([18], Theorem 2.9).

$\Rightarrow$) By Remark 26, shellability is closed under c-minors. Consequently, it is only necessary to prove that G has a free vertex. If every block of G is an edge or a vertex, then G is a forest and there exists $x \in V(G)$ with $\deg_G(x) = 1$. Hence, we can assume that there exists a 2-connected block B of G. The girth of B is at least 11, since B is an induced subgraph of G. Thus, by Lemma 51, B is not shellable. Therefore, by Theorem 42, there exists $x \in D_1(B)$ such that $\deg_G(x) = 1$. □

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