Embedding factorizations for 3-uniform hypergraphs II: r-factorizations into s-factorizations

Amin Bahmanian
Department of Mathematics
Illinois State University
Normal, Illinois, U.S.A.

Mike Newman *
Department of Mathematics and Statistics
University of Ottawa
Ottawa, Ontario, Canada
mnewman@uottawa.ca

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Abstract

Motivated by a 40-year-old problem due to Peter Cameron on extending partial parallelisms, we provide necessary and sufficient conditions under which one can extend an r-factorization of a complete 3-uniform hypergraph on m vertices, K_m^3, to an s-factorization of K_n^3. This generalizes an existing result of Baranyai and Brouwer–where they proved it for the case $r = s = 1$.

Keywords: factorizations; embedding; detachments; amalgamations; hypergraphs; edge-colorings

1 Introduction

Let V be a given finite set of cardinality n; the elements of V will be called points. We denote the set of all h-subsets of V by $\binom{V}{h}$. A *parallelism* of $\binom{V}{h}$ is a partition of $\binom{V}{h}$ whose classes are themselves partitions of V; the classes are called *parallel classes*. Note that a parallelism satisfies the usual Euclidean axiom for parallels: for every point $v \in V$ and for each h-subset U of V, there is exactly one h-subset U' which is parallel to U (that is, contained in the same parallel class as U) and contains V. Obviously, a parallelism can exist only if h is a divisor of n. It was conjectured by Sylvester that this condition is sufficient as well, and Baranyai proved this conjecture [5]. The direction of research in similar subjects such as Steiner triple systems and Latin squares for which general existence theorems have been proved suggests the following problem.

Question 1. (Cameron [7, Question 1.2]) Under what conditions can partial parallelisms be extended to parallelisms?

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There are different possible interpretations based on the precise notion of "partial" and "extend". To formulate this problem precisely, let us first introduce some basic terminology.

Let K_n^h denote the *complete h-uniform* hypergraph on n vertices which is a hypergraph on n vertices whose edges are all h-subset of the vertex set. An *r-factorization* of a hypergraph is a partition (coloring) of the edges into r-regular spanning sub-hypergraphs. The following formulation of Problem 1 was investigated by the first author and Rodger in [4]:

Question 2. Under what conditions can arbitrary edge-colorings of K_m^h be extended to r-factorizations of K_n^h?

In this direction, it has been proven that

Theorem 3. (Bahmanian, Rodger [4, Theorem 3.1]) *Suppose that $n \geqslant (2+\sqrt{2})m$. Then a q-edge-coloring of $\mathcal{F} = K_m^3$ can be extended to an r-factorization of K_n^3 if and only if*

(i) $3 \mid rn$,

(ii) $r \mid \binom{n-1}{2}$,

(iii) $q \leqslant \binom{n-1}{2}/r$, *and*

(iv) $d_{\mathcal{F}(j)}(v) \leqslant r$ *for each* $v \in V(\mathcal{F})$ *and* $1 \leqslant j \leqslant q$.

Here $d_{\mathcal{F}(j)}(v)$ is the degree of vertex v in the sub-hypergraph of $\mathcal{F}$ induced by color j.

In this paper, we investigate the following formulation of Cameron's problem which is a special case of Problem 2: We are given K_n^h which has K_m^h as a sub-hypergraph, and the edges of K_m^h have been colored so that the degree of each vertex within each color class is r (so that we have an r-factorization of K_m^h). Can we color the remaining edges of K_n^h so as to achieve an s-factorization of K_n^h?

Question 4. Under what conditions can an r-factorization of K_m^h be extended to an s-factorization of K_n^h?

Baranyai and Brouwer [6] conjectured that a 1-factorization of K_m^h can be extended to a 1-factorization of K_n^h if and only if h divides m, n, and $n \geqslant 2m$. They proved this for $h = 2, 3$, and for arbitrary h when n is sufficiently large. This conjecture of Baranyai and Brouwer was beautifully settled by Häggkvist and Hellgren [9].

Theorem 5. (Häggkvist, Hellgren [9, Theorem 2]) *Let $n = qt$ and $m = pt$, where $p \leqslant q/2$. Suppose that we are given a coloring of a subgraph K_m^t, using $\binom{m-1}{t-1}$ colors. Then this coloring can be extended to a coloring of K_n^t using $\binom{n-1}{t-1}$ colors.*

In an attempt to generalize this result and extend Theorem 3 for larger values of h, in an earlier paper we showed that

Theorem 6. (Bahmanian, Newman [3, Theorem 1.7]) *If* $\gcd(m,n,h)=\gcd(n,h)$*, then an* r*-factorization of* K_m^h *can be extended to an* r*-factorization of* K_n^h *if and only if*

(G1) $h \mid rm$, $h \mid rn$;

(G2) $r \mid \binom{m-1}{h-1}$, $r \mid \binom{n-1}{h-1}$;

(G3) $n \geqslant 2m$.

In this paper, we completely solve Problem 4 for $h=3$, which can be seen as an improvement of Theorem 3 for the case when the arbitrary edge-coloring of K_m^h is replaced by a regular edge-coloring (see Theorem 16). Studying embedding factorization of graphs dates back to over 40 years ago, see for example the classical paper by Cruse [8], and its extensions by Andersen and Hilton [1]. For results concerning embedding connected factorization of graphs we refer the reader to [10, 11, 12].

This paper is organized as follows. In Section 2, we discuss the necessary conditions. In Section 3, we give the prerequisites, and in Section 4, we prove our main result.

2 General Necessary Conditions

Throughout this paper we assume that $m,n,r,s,h \in \mathbb{N}$. Moreover, in order to avoid trivial cases we assume that

$$h \geqslant 2, \text{ and } n > m > h. \tag{1}$$

Lemma 7. *If an* r*-factorization of* K_m^h *can be embedded into an* s*-factorization of* K_n^h*, then*

(N1) $h \mid rm$, $h \mid sn$;

(N2) $r \mid \binom{m-1}{h-1}$, $s \mid \binom{n-1}{h-1}$;

(N3) $1 \leqslant s/r \leqslant \binom{n-1}{h-1}/\binom{m-1}{h-1}$;

(N4) $n \geqslant \frac{h}{h-1}m$ *if* $1 < s/r < \binom{n-1}{h-1}/\binom{m-1}{h-1}$;

(N5) $n \geqslant 2m$ *if* $s=r$.

Proof. Suppose that an r-factorization of K_m^h can be embedded into an s-factorization of K_n^h. The degree sum of each r-factor in an r-factorization of K_m^h is rm, which must be divisible by the size of each edge, h. On the other hand the degree of each vertex in K_m^h is $\binom{m-1}{h-1}$ which must be divisible by r. A similar argument shows that $h \mid sn$, and $s \mid \binom{n-1}{h-1}$. This proves (N1) and (N2).

Let $q=\binom{m-1}{h-1}/r, k=\binom{n-1}{h-1}/s$. One can think of an r-factorization of K_m^h as a q-edge-coloring in which each color class induces an r-factor. So we are extending a q-edge-coloring of K_m^h to a k-edge-coloring of K_n^h by extending each r-factor in K_m^h to an s-factor in K_n^h, thus $s \geqslant r$ and $k \geqslant q$. In other words, $1 \leqslant s/r \leqslant \binom{n-1}{h-1}/\binom{m-1}{h-1}$. This proves (N3).

For convenience, let us refer to the vertices in $V(K_n^h)\backslash V(K_m^h)$ as the new vertices, the edges in $E(K_n^h)\backslash E(K_m^h)$ as the new edges, and the colors in $\{q+1,\dots,k\}$ as new colors if $k > q$.

Let e_j be the number of edges of color j in K_m^h for $1 \leqslant j \leqslant k$. In an s-factorization of K_n^h, each of the $n-m$ new vertices is adjacent with exactly s edges of each color class, therefore all the $n-m$ new vertices are adjacent with at most $s(n-m)$ edges of each color class. Since in an s-factorization of K_n^h the number of hyperedges of each color class is sn/h, for $1 \leqslant j \leqslant k$ we have

$$s(n-m)+e_j \geqslant sn/h.$$

If $1 < s/r < \binom{n-1}{h-1}/\binom{m-1}{h-1}$ (or $s > r$ and $k > q$), then since $e_j = 0$ for $q+1 \leqslant j \leqslant k$, we have $s(n-m) \geqslant sn/h$ which proves (N4).

If $s/r = 1$ (or $s = r$), fix a color $j \in \{1,\dots,q\}$. Since $r = s$, there is no edge colored j between $V(K_m^h)$ and the new vertices. Therefore, in order to to form an s-factor in K_n^h, there must be $r(n-m)/h$ edges colored j in K_{n-m}^h (the subgraph induced by the new vertices). But the total number of edges in K_{n-m}^h is $\binom{n-m}{h}$. Therefore

$$\binom{n-m}{h} \geqslant \frac{\binom{m-1}{h-1}}{r}\frac{r(n-m)}{h}.$$

Thus $\frac{h}{n-m}\binom{n-m}{h} \geqslant \binom{m-1}{h-1}$ which implies $\binom{n-m-1}{h-1} \geqslant \binom{m-1}{h-1}$, and so $n-m-1 \geqslant m-1$, and so $n \geqslant 2m$. This proves (N5) and the proof is complete. □

Remark 8. Note that if $1 = s/r = \binom{n-1}{h-1}/\binom{m-1}{h-1}$, then $n = m$ which is a trivial case.

3 Fair Detachments of Hypergraphs

If $x, y \in \mathbb{R}$, by $x \approx y$ we mean that $\lfloor y \rfloor \leqslant x \leqslant \lceil y \rceil$. For the purpose of this paper, a *hypergraph* $\mathcal{G}$ is a pair $(V(\mathcal{G}), E(\mathcal{G}))$ where $V(\mathcal{G})$ is a finite set called the *vertex* set, $E(\mathcal{G})$ is the *edge* multiset, where every edge is itself a multi-subset of $V(\mathcal{G})$. This means that not only can an edge occur multiple times in $E(\mathcal{G})$, but also each vertex can have multiple occurrences within an edge. By an edge of the form $\{u_1^{m_1}, u_2^{m_2}, \dots, u_r^{m_r}\}$, we mean an edge in which vertex u_i occurs m_i times for $1 \leqslant i \leqslant r$. The total number of occurrences of a vertex v among all edges of $E(\mathcal{G})$ is called the *degree*, $d_{\mathcal{G}}(v)$ of v in $\mathcal{G}$. The *multiplicity* of an edge e in $\mathcal{G}$, written $m_{\mathcal{G}}(e)$, is the number of repetitions of e in $E(\mathcal{G})$ (note that $E(\mathcal{G})$ is a multiset, so an edge may appear multiple times). If $\{u_1^{m_1}, u_2^{m_2}, \dots, u_r^{m_r}\}$ is an edge in $\mathcal{G}$, then we abbreviate $m_{\mathcal{G}}(\{u_1^{m_1}, u_2^{m_2}, \dots, u_r^{m_r}\})$ to $m_{\mathcal{G}}(u_1^{m_1}, u_2^{m_2}, \dots, u_r^{m_r})$. If $U_1, \dots, U_r$ are multi-subsets of $V(\mathcal{G})$, then $m_{\mathcal{G}}(U_1, \dots, U_r)$ means $m_{\mathcal{G}}(\bigcup_{i=1}^r U_i)$, where the union of U_is is the usual union of multisets. Whenever it is not ambiguous, we drop the subscripts; for example we write $d(v)$ and $m(e)$ instead of $d_{\mathcal{G}}(v)$ and $m_{\mathcal{G}}(e)$, respectively.

For a positive integer h, $\mathcal{G}$ is said to be *h-uniform* if $|e| = h$ for each $e \in E$. For a positive integer r, an r-factor in a hypergraph $\mathcal{G}$ is a spanning r-regular sub-hypergraph, and an *r-factorization* is a partition of the edge set of $\mathcal{G}$ into r-factors. The hypergraph

$K_n^h := (V, \binom{V}{h})$ with $|V| = n$ (by $\binom{V}{h}$ we mean the collection of all h-subsets of V) is called a *complete* h-uniform hypergraph. A k*-edge-coloring* of $\mathcal{G}$ is a mapping $f : V(\mathcal{G}) \to C$ (often the set of colors C is $\{1, \dots, k\}$) and color class j of $\mathcal{G}$, written $\mathcal{G}(j)$, is the sub-hypergraph of $\mathcal{G}$ induced by the edges of color j.

Let $\mathcal{G}$ be a hypergraph, let U be some finite set, and let $\Psi : V(\mathcal{G}) \to U$ be a surjective mapping. The map Ψ extends naturally to $E(\mathcal{G})$. For $A \in E(\mathcal{G})$ we define $\Psi(A) = \{\Psi(x) : x \in A\}$. Note that Ψ need not be injective, and A may be a multiset. Then we define the hypergraph $\mathcal{F}$ by taking $V(\mathcal{F}) = U$ and $E(\mathcal{F}) = \{\Psi(A) : A \in E(\mathcal{G})\}$. We say that $\mathcal{F}$ is an *amalgamation* of $\mathcal{G}$, and that $\mathcal{G}$ is a *detachment* of $\mathcal{F}$. Associated with Ψ is a function g defined by $g(u) = |\Psi^{-1}(u)|$; to be more specific we will say that $\mathcal{G}$ is a g-detachment of $\mathcal{F}$. Then $\mathcal{G}$ has $\sum_{u \in V(\mathcal{F})} g(u)$ vertices. Note that Ψ induces a bijection between the edges of $\mathcal{F}$ and the edges of $\mathcal{G}$, and that this bijection preserves the size of an edge. We adopt the convention that it preserves the color also, so that if we amalgamate or detach an edge-colored hypergraph the amalgamation or detachment preserves the same coloring on the edges. We make explicit a straightforward observation: Given $\mathcal{G}$, $V(\mathcal{F})$ and Ψ the amalgamation is uniquely determined, but given $\mathcal{F}$, $V(\mathcal{G})$ and Ψ the detachment is in general far from uniquely determined.

We need the following special case of a general result in [2].

Theorem 9. (Bahmanian [2, Theorem 4.1]) *Let $\mathcal{F}$ be a k-edge-colored hypergraph and let $g : V(\mathcal{F}) \to \mathbb{N}$. Then there exists a g-detachment $\mathcal{G}$ (possibly with multiple edges) of $\mathcal{F}$ whose edges are all sets, with amalgamation function $\Psi : V(\mathcal{G}) \to V(\mathcal{F})$, g being the number function associated with Ψ, such that:*

(F1) *$d_{\mathcal{G}(j)}(v) \approx d_{\mathcal{F}(j)}(u)/g(u)$ for each $u \in V(\mathcal{F})$, each $v \in \Psi^{-1}(u)$ and $1 \leqslant j \leqslant k$;*

(F2) *$m_{\mathcal{G}}(U_1, \dots, U_r) \approx m_{\mathcal{F}}(u_1^{m_1}, \dots, u_r^{m_r})/\Pi_{i=1}^{r}\binom{g(u_i)}{m_i}$ for distinct $u_1, \dots, u_r \in V(\mathcal{F})$ and $U_i \subset \Psi^{-1}(u_i)$ with $|U_i| = m_i \leqslant g(u_i)$ for $1 \leqslant i \leqslant r$.*

An immediate consequence of Theorem 9 is the following that will be most useful throughout this paper.

Corollary 10. *Let $\mathcal{F}$ be a hypergraph with vertex set $\{u, v\}$ such that $m(u^i, v^{h-i}) = \binom{m}{i}\binom{n-m}{h-i}$ for $0 \leqslant i \leqslant h-1$. Then an r-factorization of K_m^h can be embedded into an s-factorization of K_n^h if and only if we can color the edges of $\mathcal{F}$ with k colors so that*

$$d_j(u) = \begin{cases} m(s-r) & \text{for } 1 \leqslant j \leqslant q, \\ sm & \text{for } q+1 \leqslant j \leqslant k, \ \text{if } k > q, \end{cases} \tag{2}$$

$$d_j(v) = s(n-m) \quad \text{for } 1 \leqslant j \leqslant k. \tag{3}$$

where $q = \binom{m-1}{h-1}/r, k = \binom{n-1}{h-1}/s$, and $q, k \in \mathbb{N}$.

Proof. First, suppose an r-factorization of K_m^h can be embedded into an s-factorization of K_n^h. By Lemma 7, q, k both are integers. By removing the edges of K_m^h from K_n^h, amalgamating those m vertices in K_n^h that belong to K_m^h into a single vertex u, and the remaining $n-m$ vertices of K_n^h into a vertex v, we obtain the hypergraph $\mathcal{F}$. The k-edge-coloring of K_n^h (in which each color class is an s-factor) induces a k-edge-coloring in $\mathcal{F}$ that satisfies (2) and (3).

Conversely, suppose that an r-factorization of K_m^h is given, and the edges of $\mathcal{F}$ are colored with k colors so that (2) and (3) are satisfied. We show that we can embed the given r-factorization of K_m^h into as s-factorization of K_n^h. Let $g : V(\mathcal{F}) \to \mathbb{N}$ with $g(u) = m$, $g(v) = n - m$. By Theorem 9, there exists a g-detachment $\mathcal{G}$ of $\mathcal{F}$ such that:

(a) By (F1), for each $w \in \Psi^{-1}(u)$

$$d_{\mathcal{G}(j)}(w) \approx d_j(u)/g(u) = \begin{cases} m(s-r)/m = s-r & \text{for } 1 \leqslant j \leqslant q, \\ sm/m = s & \text{for } q+1 \leqslant j \leqslant k, \text{ if } k > q, \end{cases}$$

and for each $w \in \Psi^{-1}(v)$,

$$d_{\mathcal{G}(j)}(w) \approx d_j(v)/g(v) = s(n-m)/(n-m) = s \text{ for } 1 \leqslant j \leqslant k.$$

(b) By (F2), $m_{\mathcal{G}}(U,V) \approx \frac{m(u^i,v^{h-i})}{\binom{g(u)}{i}\binom{g(v)}{h-i}} = \frac{\binom{m}{i}\binom{n-m}{h-i}}{\binom{m}{i}\binom{n-m}{h-i}} = 1$ for $U \subset \Psi^{-1}(u), V \subset \Psi^{-1}(v)$ with $|U| = i, |V| = h-i$, for $0 \leqslant i \leqslant h-1$.

Let us assume that $V(K_m^h) = \Psi^{-1}(u)$, and think of the given r-factorization of K_m^h as a q-edge-coloring of K_m^h so that each color class induces an r-factor. Let $\mathcal{H}$ be a hypergraph whose vertex set is $V(\mathcal{G})$, whose edges are $E(K_m^h) \cup E(\mathcal{G})$, and its edges are colored according to the colors of edges of K_m^h and $\mathcal{G}$. Obviously, $\mathcal{H}$ contains an r-factorization of K_m^h. Moreover, the definition of $\mathcal{H}$ together with (a) and (b) respectively implies that $d_{\mathcal{H}(j)}(x) = s$ for $1 \leqslant j \leqslant k$, and $\mathcal{H} \cong K_n^h$. This completes the proof. □

4 The Main Result

In order to prove our main result, let us first review the obvious necessary conditions.

Lemma 11. *If an r-factorization of K_m^3 can be embedded into an s-factorization of K_n^3, then*

(C1) $3 \mid rm$, $3 \mid sn$;

(C2) $r \mid \binom{m-1}{2}$, $s \mid \binom{n-1}{2}$;

(C3) $1 \leqslant s/r \leqslant \binom{n-1}{2}/\binom{m-1}{2}$;

(C4) $n \geqslant 3m/2$ *if* $1 < s/r < \binom{n-1}{2}/\binom{m-1}{2}$;

(C5) $n \geqslant 2m$ *if* $s = r$;

(C6) $sm\binom{n-m}{2} \geqslant \binom{n-1}{2}$ *if* $m(s-r)$ *is odd and* $s/r = \binom{n-1}{2}/\binom{m-1}{2}$.

Proof. Taking $h = 3$ in Lemma 7 proves (C1)–(C5). To prove (C6), suppose $m(s-r)$ is odd and $s/r = \binom{n-1}{2}/\binom{m-1}{2}$. If by contrary, $m\binom{n-m}{2} < \binom{n-1}{2}/s$, and if $\mathcal{F}$ is the hypergraph described in Corollary 10, then there exists a color j for which $m_j(u, v^2) = 0$. Therefore, $m(s-r) = d_j(u) = 2m_j(u^2, v)$, contradicting the fact that $m(s-r)$ is odd. □

For the rest of this section, we assume that (C1)–(C6) are satisfied, and that

$$q := \binom{m-1}{2}/r, k := \binom{n-1}{2}/s.$$

Remark 12. A similar argument shows that it is necessary that

$$m\binom{n-m}{2} \geqslant \begin{cases} k & \text{if } m, s \text{ are odd and } r \text{ is even,} \\ q & \text{if } m, r \text{ are odd and } s \text{ is even,} \\ k-q & \text{if } m, r, s \text{ are odd.} \end{cases}$$

However, in Lemma 15 we will show that in most cases, (C4) implies this general necessary condition.

In order to prove that (C1)–(C6) are also sufficient for an r-factorization of K_m^3 to be embedded into an s-factorization of K_n^3, we need to prove a few elementary results.

Lemma 13.

(a) $m[\binom{n-1}{2} - \binom{m-1}{2}] = 2(n-m)\binom{m}{2} + m\binom{n-m}{2}$

(b) $(n-m)\binom{m}{2} = m\binom{n-1}{2} - \binom{n}{3} - 2\binom{m}{3} + \binom{n-m}{3}$

Proof. Let $\mathcal{F}$ be a hypergraph with vertex set $\{u, v\}$ such that $m(u^i, v^{3-i}) = \binom{m}{i}\binom{n-m}{3-i}$ for $0 \leqslant i \leqslant 2$. Counting the degree of u in two different ways proves (a). Using part (a), we have the following that proves (b).

$$\begin{aligned} m\binom{n-1}{2} - \binom{n}{3} - 2\binom{m}{3} + \binom{n-m}{3} &= m\binom{m-1}{2} + 2(n-m)\binom{m}{2} + m\binom{n-m}{2} - 2\binom{m}{3} \\ &\quad - \left[\binom{m}{3} + \binom{n-m}{3} + (n-m)\binom{m}{2} + m\binom{n-m}{2}\right] \\ &\quad -2\binom{m}{3} + \binom{n-m}{3} \\ &= (n-m)\binom{m}{2}. \end{aligned}$$ □

Lemma 14.

$$(n-m)\binom{m}{2} \geqslant \begin{cases} q(sm - sn/3 - 2rm/3) + (k-q)(sm - sn/3) & \\ (k-q)(rm - rn/3) & \textit{if } r = s. \end{cases} \tag{4}$$

Proof. To prove the first inequality, we have

$$\begin{aligned} q(sm - sn/3 - 2rm/3) + (k-q)(sm - sn/3) &= k(sm - sn/3) - q(2rm/3) \\ &= (m - n/3)\binom{n-1}{2} - (2m/3)\binom{m-1}{2} \\ &= m\binom{n-1}{2} - \binom{n}{3} - 2\binom{m}{3} \\ &< (n-m)\binom{m}{2}, \end{aligned}$$

where the last inequality is true by Lemma 13(b).

If $r = s$, then by (C5) $n \geqslant 2m$, and the following proves the second inequality.

$$\begin{aligned} (n-m)\binom{m}{2} &\geqslant (k-q)(rm - rn/3) \\ &= (m - n/3)[\binom{n-1}{2} - \binom{m-1}{2}] \iff \\ 3(n-m)m(m-1) &\geqslant (3m-n)[(n-1)(n-2) - (m-1)(m-2)] \\ &= (3m-n)(n-m)(n+m-3) \iff \\ (n-m)[3m(m-1) - (3m-n)(n+m-3)] &\geqslant 0 \iff \\ (n-3)(n-m)(n-2m) &\geqslant 0. \end{aligned}$$

□

Lemma 15.

$$(n-m)\binom{m}{2} \leqslant q\lfloor m(s-r)/2 \rfloor + (k-q)\lfloor ms/2 \rfloor \tag{5}$$

Proof. Let $\alpha = m(s-r)/2 - \lfloor m(s-r)/2 \rfloor, \beta = sm/2 - \lfloor sm/2 \rfloor$. Note that $\alpha, \beta \in \{0, 1/2\}$, and

$$\begin{aligned} 2q\lfloor m(s-r)/2 \rfloor + 2(k-q)\lfloor ms/2 \rfloor &= 2q[m(s-r)/2 - \alpha] + 2(k-q)(ms/2 - \beta) \\ &= kms - qmr/ - 2\alpha q - 2\beta(k-q) \\ &= m\binom{n-1}{2} - m\binom{m-1}{2} - 2\alpha q - 2\beta(k-q) \\ &\overset{\text{lem. 13}(a)}{=} 2(n-m)\binom{m}{2} + m\binom{n-m}{2} - 2\alpha q - 2\beta(k-q). \end{aligned}$$

Therefore, (5) is equivalent to

$$m\binom{n-m}{2} \geqslant 2\alpha q + 2\beta(k-q). \tag{6}$$

If $k > q$, there are four cases to consider.

(a) $\alpha = \beta = 0$: In this case m is even or r, s are even, and so (6) is equivalent to $m\binom{n-m}{2} \geqslant 0$ which is trivial.

(b) $\alpha = 0, \beta = 1/2$: In this case m, s, r are odd, and so (6) is equivalent to $m\binom{n-m}{2} \geqslant k-q$, and we have

$$\begin{aligned} m\binom{n-m}{2} &\geqslant \frac{\binom{n-1}{2}}{s} - \frac{\binom{m-1}{2}}{r} \iff \\ rsm\binom{n-m}{2} &\geqslant r\binom{n-1}{2} - s\binom{m-1}{2} \\ &= r[\binom{m-1}{2} + \binom{n-m}{2} + (n-m)(m-1)] - s\binom{m-1}{2} \iff \\ r(n-m)(m-1) &\leqslant r\binom{n-m}{2}(sm-1) + (s-r)\binom{m-1}{2}. \end{aligned} \quad (7)$$

By (1) $m \geqslant 4$, but m is odd, and so $m \geqslant 5$, which implies that $n-m \geqslant 3$. Therefore $\binom{n-m}{2} \geqslant n-m$, which proves (7).

(c) $\alpha = \beta = 1/2$: In this case m, s are odd and r is even, and so (6) is equivalent to $m\binom{n-m}{2} \geqslant k$, and we have

$$\begin{aligned} sm\binom{n-m}{2} &\geqslant \binom{n-1}{2} \\ &= \binom{m-1}{2} + \binom{n-m}{2} + (n-m)(m-1) \iff \\ (sm-1)\binom{n-m}{2} &\geqslant \binom{m-1}{2} + (n-m)(m-1) \iff \\ (sm-1)(n-m)(n-m-1) &\geqslant (m-1)(m-2) + 2(n-m)(m-1) \\ &= (m-1)(2n-m-2). \end{aligned}$$

Since m is odd, by (1) $m \geqslant 5$, and we have $m^2-4m+1 \geqslant 0$ or $(m+1)(m-3) \geqslant 2(m-2)$. But m is odd and so by (C4) $n-m \geqslant \frac{m+1}{2}$ which implies $2(n-m)(n-m-2) \geqslant \frac{m+1}{2}(\frac{m+1}{2}-2) \geqslant m-2$ and so $2(n-m)^2 - 4(n-m) \geqslant m-2$. Thus,

$$2(n-m)(n-m-1) = 2(n-m)^2 - 2(n-m) \geqslant 2n-m-2.$$

Since r is even, and s is odd, we have $s > r \geqslant 2$. Therefore

$$(sm-1)(n-m)(n-m-1) > 2(m-1)(n-m)(n-m-1) \geqslant (m-1)(2n-m-2).$$

(d) $\alpha = 1/2, \beta = 0$: In this case m, r are odd and s is even, and thus (6) is equivalent to $m\binom{n-m}{2} \geqslant q$. So we need to show that $rm\binom{n-m}{2} \geqslant \binom{m-1}{2}$ or equivalently, $rm(n-m)(n-m-1) \geqslant (m-1)(m-2)$. Since $m^2-4m+7 \geqslant 0$, we have $(m+1)(m-1) \geqslant 4m-8$, so $\frac{m+1}{2}(\frac{m+1}{2}-1) \geqslant m-2$, and since $r \geqslant 1$ and for m odd by (C4), $n \geqslant m + \frac{m+1}{2}$, we have

$$\begin{aligned} rm(n-m)(n-m-1) &> (m-1)(n-m)(n-m-1) \\ &\geqslant (m-1)\frac{m+1}{2}(\frac{m+1}{2}-1) \geqslant (m-1)(m-2). \end{aligned}$$

If $k = q$, there are two cases to consider.

(a) If $m(s-r)$ is even, then (6) is equivalent to $m\binom{n-m}{2} \geqslant 0$ which is trivial.

(b) If $m(s-r)$ is odd, then (6) is equivalent to $m\binom{n-m}{2} \geqslant q$ which is true by (C6). $\square$

Case $r = s$ of the following result is proved using a different method by the authors in [3].

Theorem 16. *An r-factorization of K_m^3 can be embedded into an s-factorization of K_n^3 if and only if*

(C1) $3 \mid rm$, $3 \mid sn$;

(C2) $r \mid \binom{m-1}{2}$, $s \mid \binom{n-1}{2}$;

(C3) $1 \leqslant s/r \leqslant \binom{n-1}{2}/\binom{m-1}{2}$;

(C4) $n \geqslant 3m/2$ *if* $1 < s/r < \binom{n-1}{2}/\binom{m-1}{2}$;

(C5) $n \geqslant 2m$ *if* $s = r$;

(C6) $sm\binom{n-m}{2} \geqslant \binom{n-1}{2}$ *if* $m(s-r)$ *is odd and* $s/r = \binom{n-1}{2}/\binom{m-1}{2}$.

Proof. The necessity is obvious by Lemma 11. To prove the sufficiency, let $\mathcal{F}$ be a hypergraph with vertex set $\{u, v\}$ such that $m(u^i, v^{3-i}) = \binom{m}{i}\binom{n-m}{3-i}$ for $i = 0, 1, 2$. By Corollary 10, it is enough to find a k-edge-coloring of $\mathcal{F}$ such that (2) and (3) are satisfied. In what follows, we find such a coloring. Observe that in any k-edge-coloring of $\mathcal{F}$, for $1 \leqslant j \leqslant k$ we have

$$\begin{aligned} d_j(u) &= 2m_j(u^2, v) + m_j(u, v^2), \text{ and} \\ d_j(v) &= 2m_j(u, v^2) + m_j(u^2, v) + 3m_j(v^3). \end{aligned} \tag{8}$$

There are two cases to consider.

Case 1. $s > r$: We color the edges of the form $\{u^2, v\}$ so that

$$\begin{aligned} sm - sn/3 - 2rm/3 &\leqslant m_j(u^2, v) \leqslant m(s-r)/2 \quad \text{for } 1 \leqslant j \leqslant q, \\ sm - sn/3 &\leqslant m_j(u^2, v) \leqslant ms/2 \quad \text{for } q+1 \leqslant j \leqslant k, \text{ if } k > q. \end{aligned} \tag{9}$$

In order to show that such a coloring is possible, first note that $ms/2 \geqslant sm - sn/3$ is equivalent to $n \geqslant 3m/2$, which is true if $k > q$ (by (C4)). Moreover, $m(s-r)/2 \geqslant sm - sn/3 - 2rm/3$ is equivalent to $n \geqslant \frac{m}{2}(3 - r/s)$ which is true by (1), and the following sequence of equivalences.

$$\begin{aligned} n \geqslant \frac{m}{2}(3 - r/s) = \frac{m}{2}[3 - \binom{m-1}{2}/\binom{n-1}{2}] &\iff \\ 2n \geqslant 3m - \frac{m(m-1)(m-2)}{(n-1)(n-2)} &\iff \\ 2n(n-1)(n-2) \geqslant 3m(n-1)(n-2) - m(m-1)(m-2) &\iff \\ 2(n-m)(n-m-1)(2n+m-4) \geqslant 0. & \end{aligned}$$

Therefore, it is enough to show that

$$q(sm-sn/3-2rm/3)+(k-q)(sm-sn/3)\leqslant m(u^2,v)\leqslant q\lfloor m(s-r)/2\rfloor+(k-q)\lfloor ms/2\rfloor,$$

which is true by Lemmas 14 and 15.

Then, we color the edges of the form $\{u,v^2\}$ so that

$$m_j(u,v^2)=\begin{cases} m(s-r)-2m_j(u^2,v) & \text{for } 1\leqslant j\leqslant q,\\ sm-2m_j(u^2,v) & \text{for } q+1\leqslant j\leqslant k, \text{ if } k>q.\end{cases}$$

This is possible, because by (9) $m_j(u,v^2)\geqslant 0$ for $1\leqslant j\leqslant k$, and

$$\begin{aligned}\sum_{j=1}^{k} m_j(u,v^2) &= qm(s-r)+sm(k-q)-2\sum_{j=1}^{k} m_j(u^2,v)\\ &= ksm-qrm-2(n-m)\binom{m}{2}\\ &= m\binom{n-1}{2}-m\binom{m-1}{2}-2(n-m)\binom{m}{2}\\ &\overset{\text{lem. }13(a)}{=} m\binom{n-m}{2}=m(u,v^2).\end{aligned}$$

Finally, we color the edges of the form $\{v^3\}$ so that

$$m_j(v^3)=\begin{cases} sn/3-sm+m_j(u^2,v)+2rm/3 & \text{for } 1\leqslant j\leqslant q,\\ sn/3-sm+m_j(u^2,v) & \text{for } q+1\leqslant j\leqslant k, \text{ if } k>q.\end{cases}$$

This coloring is possible, because by (C1) $m_j(v^3)$ is an integer for $1\leqslant j\leqslant k$, by (9) $m_j(v^3)\geqslant 0$ for $1\leqslant j\leqslant k$, and

$$\begin{aligned}\sum_{j=1}^{k} m_j(v^3) &= q(sn/3-sm+2rm/3)+(k-q)(sn/3-sm)+\sum_{j=1}^{k} m_j(u^2,v)\\ &= (n-m)\binom{m}{2}+skn/3-skm+2qrm/3\\ &= (n-m)\binom{m}{2}+n\binom{n-1}{2}/3-m\binom{n-1}{2}+2m\binom{m-1}{2}/3\\ &= (n-m)\binom{m}{2}+\binom{n}{3}-m\binom{n-1}{2}+2\binom{m}{3}\\ &\overset{\text{lem. }13(b)}{=} \binom{n-m}{3}=m(v^3).\end{aligned}$$

Using (8), we verify that the described edge-coloring satisfies (2) and (3).

$$d_j(u)=\begin{cases} 2m_j(u^2,v)+m(s-r)-2m_j(u^2,v)=m(s-r) & \text{for } 1\leqslant j\leqslant q,\\ 2m_j(u^2,v)+sm-2m_j(u^2,v)=sm & \text{for } q+1\leqslant j\leqslant k, \text{ if } k>q.\end{cases}$$

For $1 \leqslant j \leqslant q$,

$d_j(v) = 3(sn/3 - sm + m_j(u^2, v) + 2rm/3) + 2(sm - rm - 2m_j(u^2, v)) + m_j(u^2, v) = s(n-m)$,

and for $q+1 \leqslant j \leqslant k$, if $k > q$

$$d_j(v) = 3(sn/3 - sm + m_j(u^2, v)) + 2(sm - 2m_j(u^2, v)) + m_j(u^2, v) = s(n-m).$$

Case 2. $\boldsymbol{r = s}$**:** We color the edges of the form $\{u^2, v\}$ so that

$$\begin{aligned} m_j(u^2, v) = &\ 0 && \text{for } 1 \leqslant j \leqslant q, \\ rm - rn/3 \leqslant m_j(u^2, v) \leqslant &\ rm/2 && \text{for } q+1 \leqslant j \leqslant k. \end{aligned} \tag{10}$$

In order to show that such a coloring is possible, first note that $rm/2 \geqslant rm - rn/3$ is equivalent to $n \geqslant 3m/2$, which is true by (C5). Therefore, it is enough to show that

$$(k-q)(rm - rn/3) \leqslant m(u^2, v) \leqslant (k-q)\lfloor rm/2 \rfloor,$$

which is true by Lemmas 14 and 15.

Then, we color the edges of the form $\{u, v^2\}$ so that

$$m_j(u, v^2) = \begin{cases} 0 & \text{for } 1 \leqslant j \leqslant q, \\ rm - 2m_j(u^2, v) & \text{for } q+1 \leqslant j \leqslant k. \end{cases}$$

This is possible, because by (10) $m_j(u, v^2) \geqslant 0$ for $1 \leqslant j \leqslant k$, and

$$\begin{aligned} \sum_{j=1}^{k} m_j(u, v^2) &= rm(k-q) - 2\sum_{j=q+1}^{k} m_j(u^2, v) \\ &= m\binom{n-1}{2} - m\binom{m-1}{2} - 2(n-m)\binom{m}{2} \\ &\overset{\text{lem. } 13(a)}{=} m\binom{n-m}{2} = m(u, v^2). \end{aligned}$$

Finally, we color the edges of the form $\{v^3\}$ so that

$$m_j(v^3) = \begin{cases} r(n-m)/3 & \text{for } 1 \leqslant j \leqslant q, \\ rn/3 - rm + m_j(u^2, v) & \text{for } q+1 \leqslant j \leqslant k. \end{cases}$$

This coloring is possible, because by (C1) $m_j(v^3)$ is an integer for $1 \leqslant j \leqslant k$, by (10) $m_j(v^3) \geqslant 0$ for $1 \leqslant j \leqslant k$, and

$$\begin{aligned} \sum_{j=1}^{k} m_j(v^3) &= \sum_{j=1}^{q} m_j(v^3) + \sum_{j=q+1}^{k} m_j(v^3) \\ &= qr(n-m)/3 + (k-q)(rn/3 - rm) + m(u^2, v) \\ &= (n-m)\binom{m}{2} + 2qrm/3 + krn/3 - krm \\ &= (n-m)\binom{m}{2} + 2\binom{m}{3} + \binom{n}{3} - m\binom{n-1}{2} \\ &\overset{\text{lem. } 13(b)}{=} \binom{n-m}{3} = m(v^3). \end{aligned}$$

Using (8), we verify that the described edge-coloring satisfies (2) and (3).

$$d_j(u) = \begin{cases} 0 & \text{for } 1 \leqslant j \leqslant q, \\ 2m_j(u^2, v) + rm - 2m_j(u^2, v) = rm & \text{for } q+1 \leqslant j \leqslant k. \end{cases}$$

For $1 \leqslant j \leqslant q$,

$$d_j(v) = 3r(n-m)/3 = r(n-m),$$

and for $q+1 \leqslant j \leqslant k$,

$$d_j(v) = 3(rn/3 - rm + m_j(u^2, v)) + 2(rm - 2m_j(u^2, v)) + m_j(u^2, v) = r(n-m).$$

Applying Corollary 10 to $\mathcal{F}$, completes the proof. □

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