On some conjectures concerning critical independent sets of a graph

Taylor Short[*]

Department of Mathematics
University of South Carolina
shorttm2@mailbox.sc.edu

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Abstract

Let G be a simple graph with vertex set $V(G)$. A set $S \subseteq V(G)$ is independent if no two vertices from S are adjacent. For $X \subseteq V(G)$, the difference of X is $d(X) = |X| - |N(X)|$ and an independent set A is critical if $d(A) = \max\{d(X) : X \subseteq V(G)$ is an independent set$\}$ (possibly $A = \emptyset$). Let nucleus(G) and diadem(G) be the intersection and union, respectively, of all maximum size critical independent sets in G. In this paper, we will give two new characterizations of König-Egerváry graphs involving nucleus(G) and diadem(G). We also prove a related lower bound for the independence number of a graph. This work answers several conjectures posed by Jarden, Levit, and Mandrescu.

Keywords: maximum independent set; maximum critical independent set; König-Egerváry graph; maximum matching; core; corona; ker; diadem; nucleus.

1 Introduction

In this paper G is a simple graph with vertex set $V(G)$, $|V(G)| = n$, and edge set $E(G)$. The set of neighbors of a vertex v is $N_G(v)$ or simply $N(v)$ if there is no possibility of ambiguity. If $X \subseteq V(G)$, then the set of neighbors of X is $N(X) = \cup_{u \in X} N(u)$, $G[X]$ is the subgraph induced by X, and X^c is the complement of the subset X. For sets $A, B \subseteq V(G)$, we use $A \setminus B$ to denote the vertices belonging to A but not B. For such disjoint A and B we let (A, B) denote the set of edges such that each edge is incident to both a vertex in A and a vertex in B.

A *matching* M is a set of pairwise non-incident edges of G. A matching of maximum cardinality is a *maximum matching* and $\mu(G)$ is the cardinality of such a maximum

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matching. For a set $A \subseteq V(G)$ and matching M, we say A is *saturated* by M if every vertex of A is incident to an edge in M. For two disjoint sets $A, B \subseteq V(G)$, we say there is a matching M of A into B if M is a matching of G such that every edge of M belongs to (A, B) and each vertex of A is saturated. An *M-alternating path* is a path that alternates between edges in M and those not in M. An *M-augmenting path* is an M-alternating path which begins and ends with vertices not saturated by M.

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent. An independent set of maximum cardinality is a *maximum independent set* and $\alpha(G)$ is the cardinality of such a maximum independent set. For a graph G, let $\Omega(G)$ denote the family of all its maximum independent sets, let

$$\mathrm{core}(G) = \bigcap\{S : S \in \Omega(G)\}, \quad \text{and} \quad \mathrm{corona}(G) = \bigcup\{S : S \in \Omega(G)\}.$$

See [1, 15] for background and properties of $\mathrm{core}(G)$ and $\mathrm{corona}(G)$.

For a graph G and a set $X \subseteq V(G)$, the *difference* of X is $d(X) = |X| - |N(X)|$ and the *critical difference* $d(G)$ is $\max\{d(X) : X \subseteq V(G)\}$. Zhang [24] showed that $\max\{d(X) : X \subseteq V(G)\} = \max\{d(S) : S \subseteq V(G) \text{ is an independent set}\}$. The set X is a *critical set* if $d(X) = d(G)$. The set $S \subseteq V(G)$ a *critical independent set* if S is both a critical set and independent. A critical independent set of maximum cardinality is called a *maximum critical independent set*. Note that for some graphs the empty set is the only critical independent set, for example odd cycles or complete graphs. See [2, 12, 13, 24] for more background and properties of critical independent sets.

Finding a maximum independent set is a well-known **NP**-hard problem. Zhang [24] first showed that a critical independent set can be found in polynomial time. Butenko and Trukhanov [2] showed that every critical independent set is contained in a maximum independent set, thereby directly connecting the problem of finding a critical independent set to that of finding a maximum independent set.

For a graph G the inequality $\alpha(G) + \mu(G) \leqslant n$ always holds. A graph G is a *König-Egerváry graph* if $\alpha(G) + \mu(G) = n$. According to the classical result of König [10] and Egerváry [4], all bipartite graphs are König-Egerváry graphs. There are non-bipartite graphs which are König-Egerváry as well, see Figure 2 for an example. We adopt the convention that the empty graph K_0, without vertices, is a König-Egerváry graph.

Deming [3] and Sterboul [22] were the first to give characterizations of König-Egerváry graphs. A matching M of a graph is *perfect* if every vertex of the graph is saturated by M. With respect to a matching M, a *blossom* is an odd cycle where half of one less than the number of edges in the cycle belong to M. The unique vertex of the blossom not saturated by M is called the *blossom tip*. A *blossom pair* is a pair of blossoms whose tips are joined by an M-alternating path with an odd number of edges that begins and ends with edges in M. Deming proved that if G is a graph with a perfect matching M, then G is a König-Egerváry graph if, and only if, G contains no blossom pair. Sterboul gave an equivalent characterization.

Gavril [7] introduced red/blue-split graphs, a generalization of König-Egerváry graphs and split graphs. A graph is a *red/blue-split graph* if its edges can be colored using red, blue, or both colors such that the vertices can be partitioned into a red and blue independent set (where red or blue independent set is an independent set in the graph made of red or blue edges, respectively). Gavril [6] also proved that given a maximum matching of a graph G, the problem of determining whether G is a König-Egerváry graph has complexity $O(n + |E(G)|)$.

Korach *et al.* [11] described red/blue-split graphs in terms of certain forbidden configurations. This led them to a characterization of König-Egerváry graphs in terms of certain forbidden subgraphs with respect to a maximum matching. Lovász [20] gave a characterization of König-Egerváry graphs having a perfect matching, in terms of certain forbidden subgraphs with respect to a particular perfect matching.

Larson and Pepper [14] gave a partial characterization of König-Egerváry graphs involving the annihilation number of a graph. For a graph G with degree sequence $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_n$, the *annihilation number* $a = a(G)$ is the largest index such that $\sum_{i=1}^{a} d_i \leqslant |E(G)|$. An *annihilating set* A is a subset of the vertices such that the sum of the degrees of the vertices in A does not exceed $|E(G)|$. We say that A is a *maximum annihilating set* if $|A| = a(G)$. Larson and Pepper proved that if G is a graph with $a(G) \geqslant \frac{n}{2}$, then $a(G) = \alpha(G)$ if, and only if, G is a König-Egerváry graph and every maximum independent set is also a maximum annihilating set.

Larson [13] also showed that König-Egerváry graphs are closely related to critical independent sets.

Theorem 1. [13] *A graph G is König-Egerváry if, and only if, every maximum independent set in G is critical.*

Theorem 2. [13] *For any graph G, there is a unique set $X \subseteq V(G)$ such that all of the following hold:*

(*i*) $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$,

(*ii*) $G[X]$ *is a König-Egerváry graph,*

(*iii*) *for every non-empty independent set S in $G[X^c]$, $|N(S)| > |S|$, and*

(*iv*) *for every maximum critical independent set I of G, $X = I \cup N(I)$.*

Larson [12] proved that a maximum critical independent set can be found in polynomial time. So the decomposition in Theorem 2 of a graph G into X and X^c is also computable in polynomial time. Figure 1 gives an example of this decomposition, where both the sets X and X^c are non-empty. Recall, for some graphs the empty set is the only critical independent set, so for such graphs the set X would be empty. If a graph G is a König-Egerváry graph, then the set X^c would be empty. We adopt the convention that if K_0 is empty graph, then $\alpha(K_0) = 0$.

In [9, 17] the following concepts were introduced: for a graph G,

$$\ker(G) = \bigcap\{S : S \text{ is a critical independent set in } G\},$$

$$\text{diadem}(G) = \bigcup\{S : S \text{ is a critical independent set in } G\}, \text{ and}$$

$$\text{nucleus}(G) = \bigcap\{S : S \text{ is a maximum critical independent set in } G\}.$$

However, the following result due to Larson allows us to use a more suitable definition for diadem(G).

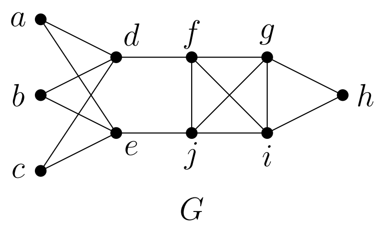

Figure 1: G has maximum critical independent set $I = \{a, b, c\}$. Theorem 2 gives that $X = \{a, b, c, d, e\}$ and $X^c = \{f, g, h, i, j\}$.

Theorem 3. [12] *Each critical independent set is contained in some maximum critical independent set.*

For the remainder of this paper we define

$$\operatorname{diadem}(G) = \bigcup\{S : S \text{ is a maximum critical independent set in } G\}.$$

Note that if G is a graph where the empty set is the only critical indepedent set (including the case $G = K_0$, the empty graph), then $\ker(G), \operatorname{diadem}(G)$, and $\operatorname{nucleus}(G)$ are all empty. See Figure 2 for examples of the sets $\ker(G)$, $\operatorname{diadem}(G)$, and $\operatorname{nucleus}(G)$.

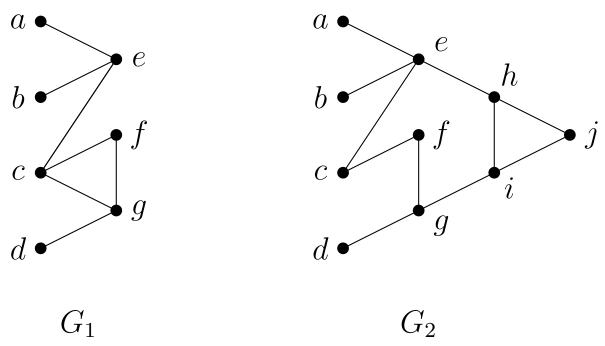

Figure 2: G_1 is a König-Egerváry graph with $\ker(G_1) = \{a, b\} \subsetneq \operatorname{core}(G_1) = \operatorname{nucleus}(G_1) = \{a, b, d\}$ and $\operatorname{diadem}(G_1) = \operatorname{corona}(G_1) = \{a, b, c, d, f\}$. G_2 is not a König-Egerváry graph and has $\ker(G_2) = \operatorname{core}(G_2) = \{a, b\} \subsetneq \operatorname{nucleus}(G_2) = \{a, b, d\}$ and $\operatorname{diadem}(G_2) = \{a, b, c, d, f\} \subsetneq \operatorname{corona}(G) = \{a, b, c, d, f, g, h, i, j\}$.

In [8, 9], the following necessary conditions for König-Egerváry graphs were given:

Theorem 4. [8] *If G is a König-Egerváry graph, then*
(i) $\operatorname{diadem}(G) = \operatorname{corona}(G)$, *and*
(ii) $|\ker(G)| + |\operatorname{diadem}(G)| \leqslant 2\alpha(G)$.

Theorem 5. [9] *If G is a König-Egerváry graph, then* $|\operatorname{nucleus}(G)| + |\operatorname{diadem}(G)| = 2\alpha(G)$.

In [8] it was conjectured that condition (i) of Theorem 4 is sufficient for König-Egerváry graphs and in [9] it was conjectured the necessary condition in Theorem 5 is also sufficient. The purpose of this paper is to affirm these conjectures by proving the following new characterizations of König-Egerváry graphs.

Theorem 6. *For a graph G, the following are equivalent:*
(i) *G is a König-Egerváry graph,*
(ii) $\operatorname{diadem}(G) = \operatorname{corona}(G)$, *and*
(iii) $|\operatorname{diadem}(G)| + |\operatorname{nucleus}(G)| = 2\alpha(G)$.

The paper [8] gives an upper bound for $\alpha(G)$ in terms of unions and intersections of maximum independent sets, proving

$$2\alpha(G) \leqslant |\operatorname{core}(G)| + |\operatorname{corona}(G)|$$

for any graph G. It is natural to ask whether a similar lower bound for $\alpha(G)$ can be formulated in terms of unions and intersections of critical independent sets. Jarden, Levit, and Mandrescu in [8] conjectured that for any graph G, the inequality $|\ker(G)| + |\operatorname{diadem}(G)| \leqslant 2\alpha(G)$ always holds. We will prove a slightly stronger statement. By Theorem 3 we see that $\ker(G) \subseteq \operatorname{nucleus}(G)$ holds implying that $|\ker(G)|+|\operatorname{diadem}(G)| \leqslant |\operatorname{nucleus}(G)| + |\operatorname{diadem}(G)|$. In section 4 we will prove the following statement, resolving the cited conjecture:

Theorem 7. *For any graph G,*

$$|\operatorname{nucleus}(G)| + |\operatorname{diadem}(G)| \leqslant 2\alpha(G).$$

It would be interesting to know whether the sets $\operatorname{nucleus}(G)$ and $\operatorname{diadem}(G)$, or their sizes, can be computed in polynomial time.

2 Some structural lemmas

Here we prove several crucial lemmas which will be needed in our proofs. Our results hinge upon the structure of the set X as described in Theorem 2.

Lemma 8. *Let I be a maximum critical independent set in G and set $X = I \cup N(I)$. Then* $\operatorname{diadem}(G) \cup N(\operatorname{diadem}(G)) = X$.

Proof. By Theorem 2 the set X is unique in G, that is, for any maximum critical independent set S, $X = S \cup N(S)$. Then $\operatorname{diadem}(G) \cup N(\operatorname{diadem}(G)) = X$ follows by definition. ☐

Lemma 9. *Let I be a maximum critical independent set in G and set $X = I \cup N(I)$. Then* $\operatorname{diadem}(G) \subseteq \operatorname{diadem}(G[X])$ *and* $\operatorname{nucleus}(G[X]) \subseteq \operatorname{nucleus}(G)$.

Proof. Let S be a maximum critical independent set in G. Using Theorem 2 we see that S is a maximum independent set in $G[X]$ and also $G[X]$ is a König-Egerváry graph. Then Theorem 1 gives that S must also be critical in $G[X]$, which implies that $\text{diadem}(G) \subseteq \text{diadem}(G[X])$.

Now let $v \in \text{nucleus}(G[X])$. Then v belongs to every maximum critical indepedent set in $G[X]$. As remarked above, since every maximum critical independent set in G is also a maximum critical independent set in $G[X]$, then v belongs to every maximum critical independent set in G. This shows that $v \in \text{nucleus}(G)$ and $\text{nucleus}(G[X]) \subseteq \text{nucleus}(G)$ follows. □

Lemma 10. *Suppose I is a non-empty maximum critical independent set in G, set $X = I \cup N(I)$, let $A = \text{nucleus}(G) \setminus \text{nucleus}(G[X])$, and let S be a maximum independent set in $G[X]$. For $S' \subseteq S \cap N(A)$, if there exists $A' \subseteq A$ such that $N(A') \cap S \subseteq S'$, then $|S'| \geqslant |A'|$.*

Proof. For $S' \subseteq S \cap N(A)$ suppose such an A' exists. For sake of contradiction, suppose that $|S'| < |A'|$. Since $A' \subseteq \text{nucleus}(G)$, then A' is an independent set. Also since $A' \subseteq \text{nucleus}(G) \subseteq \text{diadem}(G)$, by Lemma 8 we have $A' \subseteq X$. Furthermore, since $N(A') \cap S \subseteq S'$ then $A' \cup (S \setminus S')$ is an independent set in $G[X]$. Now by assumption $|S'| < |A'|$, so $A' \cup (S \setminus S')$ is an independent set in $G[X]$ larger than S, which cannot happen. Therefore we must have $|S'| \geqslant |A'|$ as desired. □

Lemma 11. *Let I be a maximum critical independent set in G and set $X = I \cup N(I)$. Then*

$$|\,\text{nucleus}(G)| + |\,\text{diadem}(G)| \leqslant |\,\text{nucleus}(G[X])| + |\,\text{diadem}(G[X])|.$$

Proof. First note that if the set X is empty, then by Lemma 8 both sides of the inequality are zero. So let us assume that X is non-empty. Now consider the set $A = \text{nucleus}(G) \setminus \text{nucleus}(G[X])$. If this independent set is empty, then $\text{nucleus}(G) = \text{nucleus}(G[X])$ and there is nothing to prove since $\text{diadem}(G) \subseteq \text{diadem}(G[X])$ holds by Lemma 9. If A is non-empty, for each $v \in A$ there is some maximum independent set S of $G[X]$ which doesn't contain v. Since S is a maximum independent set there exists $u \in N(v) \cap S$. Since $v \in \text{nucleus}(G)$, then u does not belong to any maximum critical independent set in G. Recall by Theorem 2 (*ii*) $G[X]$ is a König-Egerváry graph, so Theorem 1 gives that S is a maximum critical independent set in $G[X]$. It follows that $u \in \text{diadem}(G[X]) \setminus \text{diadem}(G)$, which shows each vertex in A is adjacent to at least one vertex in $\text{diadem}(G[X]) \setminus \text{diadem}(G)$.

Now we will show there is a maximum matching from A into $\text{diadem}(G[X]) \setminus \text{diadem}(G)$ with size $|A|$. For sake of contradiction, suppose such a matching M has less than $|A|$ edges. Then there exists some vertex $v \in A$ not saturated by M. By the above, v is adjacent to some vertex $u \in \text{diadem}(G[X]) \setminus \text{diadem}(G)$. Since M is maximum, u is matched to some vertex $w \in A$ under M. Now let S be a maximum independent set of $G[X]$ containing u. We now restrict ourselves to the subgraph induced by the edges $(A \cap N(S), S \cap N(A))$, noting this subgraph is bipartite since both $A \cap N(S)$ and $S \cap N(A)$ are independent. In this subgraph, consider the set $\mathcal{P}$ of all M-alternating paths starting with the edge vu. Note that all such paths must start with the vertices v, u, then w. Also, such paths must end at either a matched vertex in $A \cap N(S)$ or an unmatched vertex in $S \cap N(A)$.

We wish to show that there is some alternating path ending at an unmatched vertex in $S \cap N(A)$. For sake of contradiction, suppose all alternating paths end at a matched vertex in $A \cap N(S)$ and let $V(\mathcal{P})$ denote the union of all vertices belonging to such an alternating path. We aim to show this scenario contradicts Lemma 10. Now clearly we must have $N(V(\mathcal{P}) \cap A) \cap S \subseteq V(\mathcal{P}) \cap S$, else we could extend an alternating path to any vertex in $(N(V(\mathcal{P}) \cap A) \cap S) \setminus (V(\mathcal{P}) \cap S)$. Also, since all paths in $\mathcal{P}$ end at a matched vertex in $A \cap N(S)$, then every vertex of $V(\mathcal{P}) \cap S$ is matched under M, and such a situation should look as in Figure 3.

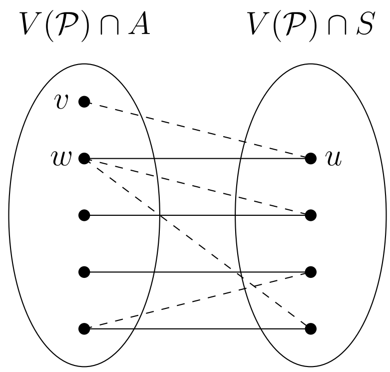

Figure 3: What the M-alternating paths could look like between $V(\mathcal{P}) \cap A$ and $V(\mathcal{P}) \cap S$, where solid lines represent matched edges in M and dotted lines represent the unmatched edges.

From this it follows that $|V(\mathcal{P}) \cap S| < |V(\mathcal{P}) \cap A|$. The previous statements exactly contradict Lemma 10, so there is some alternating path P ending at an unmatched vertex $x \in S \cap N(A)$. This means that P is an M-augmenting path. A well-known theorem in graph theory states that a matching is maximum in G if, and only if, there is no augmenting path [23]. So P being an M-augmenting path contradicts our assumption that M is a maximum matching.

Therefore there is a matching M from A into $\text{diadem}(G[X]) \setminus \text{diadem}(G)$. This matching implies that $|\,\text{nucleus}(G) \setminus \text{nucleus}(G[X])| \leqslant |\,\text{diadem}(G[X]) \setminus \text{diadem}(G)|$. Since both $\text{nucleus}(G[X]) \subseteq \text{nucleus}(G)$ and $\text{diadem}(G) \subseteq \text{diadem}(G[X])$ by Lemma 9, the lemma follows. $\square$

3 New characterizations of König-Egerváry graphs

Proof (of Theorem 6). First we prove $(ii) \Rightarrow (i)$. Suppose that $\text{diadem}(G) = \text{corona}(G)$ holds and let I be a maximum critical independent set with $X = I \cup N(I)$. We will use the decomposition in Theorem 2 to show that X^c must be empty and hence, $G = G[X]$

is a König-Egerváry graph. By Lemma 8 we have $\mathrm{corona}(G) = \mathrm{diadem}(G) \subseteq X$, in other words every maximum independent set in G is contained in X. This implies that $|I| = \alpha(G[X]) = \alpha(G)$. Now by Theorem 2 (i), $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$ showing that we must have $\alpha(G[X^c]) = 0$. Now clearly the result follows, since $\alpha(G[X^c]) = 0$ implies that X^c must be empty.

To prove $(iii) \Rightarrow (i)$, again we will use the decomposition in Theorem 2 to show that X^c must be empty and hence, G is a König-Egerváry graph. So suppose that $|\mathrm{diadem}(G)| + |\mathrm{nucleus}(G)| = 2\alpha(G)$ and let I be a maximum critical independent set in G with $X = I \cup N(I)$. Lemma 11 implies that

$$2\alpha(G) = |\mathrm{diadem}(G)| + |\mathrm{nucleus}(G)| \leqslant |\mathrm{diadem}(G[X])| + |\mathrm{nucleus}(G[X])|.$$

Theorem 2(ii) gives that $G[X]$ is König-Egerváry , so by Corollary 5 we have

$$|\mathrm{diadem}(G[X])| + |\mathrm{nucleus}(G[X])| = 2\alpha(G[X])$$

implying that $\alpha(G) \leqslant \alpha(G[X])$. It follows by Theorem 2(i) we must have $\alpha(G) = \alpha(G[X])$, so again we know that $\alpha(G[X^c]) = 0$ which finishes this part of the proof.

The implications $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ are given in Theorem 4 and in Theorem 5. □

4 A bound on $\alpha(G)$

Proof (of Theorem 7). Let I be a maximum critical independent set in G and $X = I \cup N(I)$. By Theorem 2 (ii), $G[X]$ is a König-Egerváry graph so by Theorem 5 we have

$$|\mathrm{nucleus}(G[X])| + |\mathrm{diadem}(G[X])| = 2\alpha(G[X]) \leqslant 2\alpha(G).$$

Now by Lemma 11 we must have

$$|\mathrm{nucleus}(G)| + |\mathrm{diadem}(G)| \leqslant |\mathrm{nucleus}(G[X])| + |\mathrm{diadem}(G[X])|$$

and the theorem follows. □

Combining Theorem 7 and the inequality $2\alpha(G) \leqslant |\mathrm{core}(G)| + |\mathrm{corona}(G)|$ proven in [8], the following corollary is immediate.

Corollary 12. *For any graph G,*

$$|\mathrm{nucleus}(G)| + |\mathrm{diadem}(G)| \leqslant 2\alpha(G) \leqslant |\mathrm{core}(G)| + |\mathrm{corona}(G)|.$$

These upper and lower bounds are quite interesting. The fact that every critical independent set is contained in a maximum independent set implies that $\mathrm{diadem}(G) \subseteq \mathrm{corona}(G)$ for all graphs G. However, the graph G_2 in Figure 2 has $\mathrm{core}(G_2) \subsetneq \mathrm{nucleus}(G_2)$ while the graph G in Figure 1 has $\mathrm{nucleus}(G) = \{a, b, c\} \subsetneq \mathrm{core}(G) = \{a, b, c, h\}$.

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