# Lower Bounds for Cover-Free Families

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**Abstract**

Let $\mathcal{F}$ be a set of blocks of a $t$-set $X$. A pair $(X, \mathcal{F})$ is called an $(w, r)$-cover-free family ($(w, r)-$CFF) provided that, the intersection of any $w$ blocks in $\mathcal{F}$ is not contained in the union of any other $r$ blocks in $\mathcal{F}$.

We give new asymptotic lower bounds for the number of minimum points $t$ in a $(w, r)$-CFF when $w \leqslant r = |\mathcal{F}|^\epsilon$ for some constant $\epsilon \geqslant 1/2$.

**Keywords:** Cover-Free Family, Lower Bound.

## 1 Introduction

Let $\mathcal{F}$ be a set of blocks (subsets) of a $t$-set $X$. A pair $(X, \mathcal{F})$ is called a $(w, r)$-cover-free family ($(w, r)-$CFF) provided that, for any $w$ blocks $A_1, A_2, \ldots, A_w \in \mathcal{F}$ and any other $r$ blocks $B_1, B_2, \ldots, B_r \in \mathcal{F}$ we have

$$\bigcap_{i=1}^{w} A_i \not\subseteq \bigcup_{j=1}^{r} B_j.$$

Since using De Morgan, a $(w, r)-$CFF can be turned into $(r, w)-$CFF, throughout the paper we assume that $w \leqslant r$. Cover-free families were first introduced in 1964 by Kautz and Singleton [5].

Let $N(n, (w, r))$ denote the minimum number of points $|X|$ in any $(w, r)$-CFF having $|\mathcal{F}| = n$ blocks. The best known lower bound for $N(n, (1, r))$ is [2, 4, 7]

$$N(n, (1, r)) = \Omega\left(\frac{r^2}{\log r} \log n\right) \tag{1}$$

when $r \leqslant \sqrt{n}$, and, $\Omega(n)$ when $r > \sqrt{n}$. The constant of the $\Omega()$ is asymptotically 1/2, 1/4 and 1/8, respectively. Stinson et. al, [8], proved that

$$N(n, (w, r)) \geqslant N(n-1, (w-1, r)) + N(n-1, (w, r-1)). \tag{2}$$

They then use it with (1) to prove two bounds. The first bound is

$$N(n,(w,r)) \geqslant \Omega\left(\frac{\binom{w+r}{w}(w+r)}{\log\binom{w+r}{w}}\log n\right) \tag{3}$$

when $r \leqslant \sqrt{n}$, [8, 6], and

$$N(n,(w,r)) \geqslant \Omega\left(\frac{\binom{w+r}{w}}{\log(w+r)}\log n\right) \tag{4}$$

for any $r \leqslant n$, [8]. To the best of our knowledge (4) is the best bound known when $\sqrt{n} \leqslant r \leqslant n$. D'yachkov et. al. breakthrough result, [3], implies that for $r \leqslant \sqrt{n}$ and $r, n \to \infty$

$$N(n,(w,r)) = \Theta\left(\frac{\binom{w+r}{w}(w+r)}{\log\binom{w+r}{w}}\log n\right) \tag{5}$$

and for $r \geqslant \sqrt{n}$ and $r, n \to \infty$

$$N(n,(w,r)) \leqslant O\left(\frac{r}{w}\cdot\frac{\binom{w+r}{w}}{\log(w+r)}\log n\right). \tag{6}$$

In this paper we give a new lower bound for $(w,r)$-CFF when $r > \sqrt{n}$. We combine the two techniques used in [8, 6] and [1] to give the following asymptotic lower bound.

**Theorem 1.** *For any* $2 \leqslant k \leqslant w < r \leqslant n/2$ *we have*

$$N(n,(w,r)) \geqslant \frac{k^k k!}{2(k+1)^{2k}}\frac{r^{w+1}}{(w+1)!\ln^k r} = \Omega\left(\frac{\sqrt{k}}{e^k}\cdot\frac{r^{w+1}}{(w+1)!\ln^{k+1} r}\log n\right)$$

*for*

$$(n+k-1-w)^{\frac{k-1}{k}} \leqslant r \leqslant (n+k-w)^{\frac{k}{k+1}}$$

*and*

$$N(n,(w,r)) = \Theta\left(\binom{n}{w}\right)$$

*for*

$$r = \Omega\left((n\log n)^{\frac{w}{w+1}}\right).$$

Our bound is

$$\Theta\left(\frac{\sqrt{k}\cdot r}{w(e\ln r)^k}\right)$$

times greater than the previous bound in (4). In particular, when $k$ is constant, our lower bound improves the bound in (4) to

$$N(n,(w,r)) \geqslant \Omega\left(\frac{r}{w\log^k r}\cdot\frac{\binom{w+r}{w}}{\log(w+r)}\log n\right). \tag{7}$$

A slightly better bound can be achieved when

$$(n+k-w)^{\frac{k}{k+1}} \leqslant r \leqslant (n+k-w)^{\frac{k}{k+1}}\ln^{1/(k+1)} n.$$

For example, let $w=4$. Table 1 compares our results with the previous results (asymptotic values).

| $r$ | Previous Lower Bounds (3), (4) | Upper Bound [3] | Our Lower Bound |
|---|---|---|---|
| $r \leqslant n^{1/2}$ | $r^5\frac{\log n}{\log r}$ | $r^5\frac{\log n}{\log r}$ | —— |
| $n^{1/2} \leqslant r \leqslant n^{2/3}$ | $r^4\frac{\log n}{\log r}$ | $r^5\frac{\log n}{\log r}$ | $r^5\frac{\log n}{\log^3 r}$ |
| $n^{2/3} \leqslant r \leqslant n^{3/4}$ | $r^4\frac{\log n}{\log r}$ | $r^5\frac{\log n}{\log r}$ | $r^5\frac{\log n}{\log^4 r}$ |
| $n^{3/4} \leqslant r \leqslant n^{4/5}$ | $r^4\frac{\log n}{\log r}$ | $r^5\frac{\log n}{\log r}$ | $r^5\frac{\log n}{\log^5 r}$ |
| $n > r \geqslant (n\log n)^{4/5}$ | $r^4$ | $n^4$ | $n^4$ |

Table 1: Results for $w=4$.

## 2 First Lower Bound

In this section, we prove

**Lemma 2.** *Let $w \leqslant r \leqslant n/2$. If*

$$r = \Omega\left((n\log n)^{\frac{w}{w+1}}\right)$$

*then*

$$N(n,(w,r)) = \Theta\left(\binom{n}{w}\right). \tag{8}$$

*Otherwise,*

$$N(n,(w,r)) \geqslant \Omega\left(\left(\frac{r}{(w+1)\ln r}\right)^{w+1}\log n\right). \tag{9}$$

Lemma 2 follows from the following.

**Lemma 3.** *Let $\epsilon < 1$ be any constant. For $w \leqslant r \leqslant n/2$ we have*

$$N(n,(w,r)) \geqslant \min\left((1-\epsilon)\frac{w^w}{(w+1)^{2w+1}} \cdot \frac{r^{w+1}}{\ln^w r}\ ,\ \epsilon\binom{n}{w}\right). \tag{10}$$

*Proof.* Let $(X, \mathcal{F})$ be an optimal $(w,r)$-CFF. Let $\mathcal{F} = \{F_1, \ldots, F_n\}$,

$$|X| = N = N(n,(w,r))$$

and assume without loss of generality that $X = [N] := \{1, \ldots, N\}$. Define $v^{(i)} \in \{0,1\}^n$, $i = 1, \ldots, N$ where $v^{(i)}_j = 1$ if and only if $i \in F_j$. Let $V = \{v^{(i)} | i = 1, \ldots, N\}$. Let $V_0$ be the set of $v^{(i)}$ of weight $wt(v^{(i)})$ (i.e., $\sum_j v^{(i)}_j$) equal to $w$. Let

$$m = \frac{(w+1)^2 n \ln r}{wr}$$

and consider the two sets $V_1 = \{v^{(i)} \mid w < wt(v^{(i)}) < m\}$ and $V_2 = \{v^{(i)} \mid wt(v^{(i)}) \geqslant m\}$. Obviously, $V = V_0 \cup V_1 \cup V_2$ is a partition of $V$. Suppose by contradiction that

$$|V_0| \leqslant \epsilon\binom{n}{w}$$

and

$$\max(|V_1|, |V_2|) \leqslant (1-\epsilon)\frac{w^w}{(w+1)^{2w+1}} \cdot \frac{r^{w+1}}{\ln^w r}.$$

Consider $W = \{(j_1, \ldots, j_w) \mid 1 \leqslant j_1 < \cdots < j_w \leqslant n\}$ and $W' \subset W$ the set of all $(j_1, \ldots, j_w)$ where no $v^{(i)} \in V_0$, $i = 1, \ldots, N$, satisfies $v^{(i)}_{j_1} = \cdots = v^{(i)}_{j_w} = 1$. Obviously,

$$|W'| = \binom{n}{w} - |V_0| \geqslant (1-\epsilon)\binom{n}{w}.$$

Fix an element $v \in V_1$ and randomly and uniformly choose $j = (j_1, \ldots, j_w) \in W'$. We have

$$\Pr_{j \in W'}[v_{j_1} = \cdots = v_{j_w} = 1] \quad \leqslant \quad \frac{\binom{wt(v)}{w}}{|W'|} \leqslant \frac{\binom{m}{w}}{(1-\epsilon)\binom{n}{w}}.$$

Therefore, the expectation of the number of $v \in V_1$ for which $v_{j_1} = \cdots = v_{j_w} = 1$ is at most

$$\begin{aligned}
\frac{\binom{m}{w}|V_1|}{(1-\epsilon)\binom{n}{w}} \quad &\leqslant \quad \frac{1}{1-\epsilon}\left(\frac{m}{n}\right)^w |V_1| \\
&\leqslant \quad \frac{1}{1-\epsilon}\frac{(w+1)^{2w}\ln^w r}{w^w r^w} \cdot (1-\epsilon)\frac{w^w}{(w+1)^{2w+1}} \cdot \frac{r^{w+1}}{\ln^w r} \\
&= \quad \frac{r}{w+1}.
\end{aligned}$$

Therefore, there is $j' = (j'_1, \dots, j'_w) \in W'$ such that the number of $v \in V_1$ that satisfies $v_{j'_1} = \cdots = v_{j'_w} = 1$ is $r_1 \leqslant r/(w+1)$. Since the weight of every $v \in V_1$ is greater than $w$, we can choose $r_1$ new entries $j''_1, \dots, j''_{r_1} \notin \{j'_1, \dots, j'_w\}$ such that for every $v \in V_1$ where $v_{j'_1} = \cdots = v_{j'_w} = 1$ there is $j''_\ell$ such that $v_{j''_\ell} = 1$.

Now randomly and uniformly choose

$$r_2 := \left\lceil \frac{wr}{w+1} \right\rceil$$

distinct $k_1, \dots, k_{r_2} \in [n]$. Let $A$ be the event that $\{k_1, \dots, k_{r_2}\} \cap \{j'_1, \dots, j'_w\} \neq \emptyset$. The probability that $A$ does not happen is

$$\frac{\binom{n-w}{r_2}}{\binom{n}{r_2}} \geqslant \frac{\binom{n-w}{r_2}}{2^w \binom{n-w}{r_2}} = \frac{1}{2^w}.$$

Then

$$\begin{aligned}
\Pr[A \vee (\exists v \in V_2)\ v_{k_1} = \cdots = v_{k_{r_2}} = 0] &\leqslant 1 - \frac{1}{2^w} + |V_2| \frac{\binom{n-m}{r_2}}{\binom{n}{r_2}} \\
&\leqslant 1 - \frac{1}{2^w} + |V_2| \left(\frac{n-m}{n}\right)^{r_2} \\
&\leqslant 1 - \frac{1}{2^w} + |V_2| e^{-\frac{mr_2}{n}}
\end{aligned}$$

and

$$\begin{aligned}
|V_2| e^{-\frac{mr_2}{n}} &\leqslant (1-\epsilon) \frac{w^w}{(w+1)^{2w+1}} \cdot \frac{r^{w+1}}{\ln^w r} \cdot e^{-\frac{(w+1)^2 \ln r}{wr} r_2} \\
&\leqslant (1-\epsilon) \frac{w^w}{(w+1)^{2w+1}} \cdot \frac{r^{w+1}}{\ln^w r} \cdot e^{-(w+1)\ln r} \\
&= (1-\epsilon) \frac{w^w}{(w+1)^{2w+1}} \cdot \frac{1}{\ln^w r} \\
&< \frac{1}{2^w}.
\end{aligned}$$

Therefore,

$$\Pr[A \vee (\exists v \in V_2)\ v_{k_1} = \cdots = v_{k_{r_2}} = 0] < 1.$$

Therefore, there is $\{k_1, \dots, k_{r_2}\}$ such that $\{k_1, \dots, k_{r_2}\} \cap \{j'_1, \dots, j'_w\} = \emptyset$ and for every $v \in V_2$ there is $k_\ell \in \{k_1, \dots, k_{r_2}\}$ where $v_{k_\ell} = 1$.

Now it is easy to see that there is no $v \in V$ where $v_{j'_1} = \cdots = v_{j'_w} = 1$, $v_{j''_1} = \cdots = v_{j''_{r_1}} = 0$ and $v_{k_1} = \cdots = v_{k_{r_2}} = 0$. This implies that

$$\bigcap_{i=1}^{w} F_{j'_i} \subseteq \bigcup_{i=1}^{r_1} F_{j''_i} \cup \bigcup_{i=1}^{r_2} F_{k_i},$$

which is a contradiction. □

## 3 The Second Bound

In this section we prove Theorem 1.

**Lemma 4.** *For any* $2 \leqslant k \leqslant w \leqslant r \leqslant n/2$ *and*

$$2 \leqslant r \leqslant (n+k-w)^{\frac{k}{k+1}}$$

$$N(n,(w,r)) \geqslant \frac{k^k k!}{2(k+1)^{2k}} \frac{r^{w+1}}{(w+1)! \ln^k r} = \Omega\left(\frac{r^{w+1}}{(w+1)! \ln^k r}\right).$$

*Proof.* We prove the lemma by induction on $w$.

From Lemma 3 the lemma holds for $w = k$. Now assume the bound holds for some $w$ and every $r$ that satisfies $r \leqslant (n+k-w)^{\frac{k}{k+1}}$. We now prove the bound for $w+1$ and $r \leqslant (n+k-w-1)^{\frac{k}{k+1}}$. We have

$$\begin{aligned}
N(n,(w+1,r)) &\geqslant N(n-1,(w,r))) + N(n-1,(w+1,r-1)) && (11)\\
&\geqslant \sum_{j=1}^{r} N(n-r+j-1,(w,j)) && (12)\\
&\geqslant N(n-r,(w,1)) + \\
&\qquad \sum_{j=2}^{r} \frac{k^k k!}{2(k+1)^{2k}} \frac{j^{w+1}}{(w+1)! \ln^k j} && (13)\\
&\geqslant \frac{k^k k!}{2(k+1)^{2k}(w+1)! \ln^k r} \sum_{j=1}^{r} j^{w+1}\\
&\geqslant \frac{k^k k!}{2(k+1)^{2k}(w+1)! \ln^k r} \int_0^r x^{w+1}\, \mathrm{d}x\\
&\geqslant \frac{k^k k!}{2(k+1)^{2k}} \frac{r^{w+2}}{(w+2)! \ln^k r}.
\end{aligned}$$

Here, inequality (11) comes from [8]. Inequality (12) follows from the fact that $N(n-r+1,(w+1,1)) \geqslant N(n-r,(w,1))$. Inequality (13) follows from the induction hypothesis since

$$\begin{aligned}
j &= r-(r-j)\\
&\leqslant (n+k-w-1)^{\frac{k}{k+1}} - (r-j)\\
&\leqslant (n+k-w-1-(r-j))^{\frac{k}{k+1}}\\
&= ((n-r+j-1)+k-w)^{\frac{k}{k+1}}.
\end{aligned}$$

□

**Lemma 5.** *Let* $w \leqslant r \leqslant n/2$*. If*

$$r = \Omega\left((n \log n)^{\frac{w}{w+1}}\right)$$

*then*

$$N(n,(w,r)) = \Theta\left(\binom{n}{w}\right).$$

*Proof.* Let $r > c(n\log n)^{\frac{w}{w+1}}$ for large enough constant $c > 2e$ and $c' = c^{(w+1)/w}$. Since

$$\begin{aligned}
\frac{1}{2}\frac{w^w}{(w+1)^{2w+1}}\cdot\frac{r^{w+1}}{\ln^w r} &\geqslant \frac{1}{2}\frac{w^w}{(w+1)^{2w+1}}\cdot\frac{c^{w+1}n^w\log^w n}{\left(\frac{w}{w+1}\right)^w\ln^w(c'n\log n)}\\
&\geqslant \frac{1}{2}\frac{1}{(w+1)\left(1+\frac{1}{w}\right)^w}\cdot\frac{c^{w+1}n^w\ln^w n}{w^w\ln^w(n^2)}\\
&\geqslant \frac{c}{2e}\frac{1}{(w+1)}\left(\frac{c}{2e}\right)^w\cdot\left(\frac{en}{w}\right)^w\\
&\geqslant \frac{c}{2e}\frac{1}{(w+1)}\left(\frac{c}{2e}\right)^w\cdot\binom{n}{w}\geqslant\binom{n}{w},
\end{aligned}$$

by Lemma 3, the result follows. □

## References


[1] N. Alon and V. Asodi. Learning a Hidden Subgraph. *SIAM J. Discrete Math.* 18(4). pp. 697–712 (2005).

[2] A. G. D'yachkov and V. V. Rykov. Bounds on the length of disjunctive codes. *Problemy Peredachi Informatsii*, 18(3), pp. 7–13 (1982).

[3] A. G. D'yachkov, I. V. Vorob'ev, N. A. Polyansky and V. Yu. Shchukin. Bounds on the rate of disjunctive codes. *Probl. Inf. Transm.* 50(1), pp. 27–56 (2014).

[4] Z. Füredi. On $r$-Cover-free Families. *J. Comb. Theory, Ser. A*, 73(1). pp. 172–173 (1996).

[5] W. H. Kautz and R. C. Singleton. Nonrandom binary superimposed codes, *IEEE Trans. Inform. Theory.* 10, pp. 363–377 (1964).

[6] X. Ma and R. Wei. On Bounds of Cover-Free Families. *Des. Codes Cryptogr.*, 32, pp. 303–321 (2004).

[7] M. Ruszinkó. On the Upper Bound of the Size of the $r$-Cover-Free Families. *J. Comb. Theory, Ser. A.* 66(2). pp. 302–310 (1994).

[8] D. R. Stinson, R. Wei and L. Zhu. Some New Bounds for Cover-Free Families. *J. Comb. Theory, Ser. A.* 90, pp. 224–234 (2000).