2-Walk-regular dihedrants from group divisible designs

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Abstract

In this note, we construct bipartite 2-walk-regular graphs with exactly 6 distinct eigenvalues as the point-block incidence graphs of group divisible designs with the dual property. For many of them, we show that they are 2-arc-transitive dihedrants. We note that some of these graphs are not described in Du et al. (2008), in which they classified the connected 2-arc transitive dihedrants.

Keywords: 2-walk-regular graphs; distance-regular graphs; association schemes; group divisible designs with the dual property; cyclic relative difference sets; 2-arc-transitive dihedrants

1 Introduction

For unexplained terminology, see next section. C. Dalfó et al. [7] showed the following result.

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Proposition 1. *(cf. [7, Proposition 3.4, 3.5]) Let s, d be positive integers. Let Γ be a connected s-walk-regular graph with diameter $D \geqslant s$ and exactly $d+1$ distinct eigenvalues. Then the following two results hold:*

i) If $d \leqslant s+1$, then Γ is distance-regular;

ii) If $d \leqslant s+2$ and Γ is bipartite, then Γ is distance-regular.

In this note, we will construct infinitely many bipartite 2-walk-regular graphs with exactly 6 distinct eigenvalues and diameter $D = 4$, thus showing that Statement (ii) of Proposition 1 is not true for $d = 5$ and $s = 2$. We will construct these graphs as the point-block incidence graphs of certain group divisible designs with the dual property. We will show that infinitely many of these graphs are 2-arc transitive dihedrants, and, en passant, provide a new description of 2-arc transitive graphs found by Du et al. [9]. Note that, although most of the graphs we describe may not be new, the fact that many of them are 2-arc-transitive dihedrants seems to be new, as they give counter examples to a result of Du et al. [8, Theorem 1.2] in which they classified the connected 2-arc transitive dihedrants. The classical examples $\Gamma(d, q)$ ($d \geqslant 2$ and q a prime power), as described in Section 4, are not mentioned in Du et al. [8] for the case $d \geqslant 3$ and q any prime power, and also for the case $d = 2$ and q a power of two.

2 Preliminaries

All the graphs considered in this note are finite, undirected and simple. The reader is referred to [5, 4] for more information. Let $\Gamma := (V, E)$ be a connected graph with vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma)$. Denote $x \sim y$ if the vertices $x, y \in V$ are adjacent. The *distance* $d_\Gamma(x, y)$ between two vertices $x, y \in V$ is the length of a shortest path connecting x and y in Γ. If the graph Γ is clear from the context, then we simply use $d(x, y)$. The maximum distance between two vertices in Γ is the *diameter* $D = D(\Gamma)$. We use $\Gamma_i(x)$ for the set of vertices at distance i from x and denote $k_i(x) = |\Gamma_i(x)|$. For the sake of simplicity, we write $\Gamma(x) = \Gamma_1(x)$ and $k(x) = k_1(x)$. The *valency* of x is the number $|\Gamma(x)|$ of vertices adjacent to it. A graph is *regular* with valency k if the valency of each of its vertices is k.

The *distance-i matrix* $A_i = A(\Gamma_i)$ is the matrix whose rows and columns are indexed by the vertices of Γ and the (x, y)-entry is 1 whenever $d(x, y) = i$ and 0 otherwise. The *adjacency matrix* A of Γ equals A_1.

Let Γ be a graph with diameter D and let x, y be vertices of Γ at distance i ($0 \leqslant i \leqslant D$). Then the number of vertices which are at distance j from x and h from y is denoted by $p^i_{jh}(x, y)$ and is called an *intersection number* of Γ. Note that $p^i_{jh}(x, y) = |\Gamma_j(x) \cap \Gamma_h(y)|$. And we consider the numbers $c_i(x, y) = p^i_{i-1,1}(x, y)$, $a_i(x, y) = p^i_{i1}(x, y)$, $b_i(x, y) = p^i_{i+1,1}(x, y)$. Note that $k(y) = c_i(x, y) + a_i(x, y) + b_i(x, y)$ holds for all $0 \leqslant i \leqslant D$. The intersection numbers $p^i_{jh}(x, y)$ ($0 \leqslant i, j, h \leqslant D$) are called *well-defined* if the numbers do not depend on the choice of x and y but only on i, i.e., $p^i_{jh}(x, y) = p^i_{jh}(z, w)$ if $d(x, y) = d(z, w) = i$. A connected graph Γ with diameter D is called *distance-regular* if these numbers $c_i(x,y)$, $a_i(x,y)$ and $b_i(x,y)$ are well-defined. If this is the case, then these numbers are denoted simply by a_i, b_i and c_i for $0 \leqslant i \leqslant D$.

A graph Γ is called *t-walk-regular* if the number of walks of every given length ℓ between two vertices $x, y \in V$ depends only on the distance between them, provided that $d(x,y) \leqslant t$ (where it is implicitly assumed that the diameter of Γ is at least t). If a graph Γ is t-walk-regular, then for any two vertices x, y at distance i, the numbers $c_i = c_i(x,y), a_i = a_i(x,y)$ and $b_i = b_i(x,y)$ are well-defined for $0 \leqslant i \leqslant t$ (see [7, Proposition 3.15]) and the numbers $p^i_{jh} = p^i_{jh}(x,y)$ are also well-defined for $0 \leqslant i,j,h \leqslant t$ (see [6, Proposition 1]). And for a vertex x of a t-walk-regular graph Γ, the relation $k_{i-1}(x)b_{i-1} = k_i(x)c_i$ shows that $k_i = k_i(x)$ are well-defined for $0 \leqslant i \leqslant t$, where $k_i(x)$ can be considered as $p^0_{ii}(x,x)$. Note that $k_{t+1} = k_{t+1}(x)$ is also well-defined if the diameter D of Γ is equal to $t+1$. A D-walk-regular graph is a distance-regular graph, where D is the diameter of the graph.

Let Γ be a graph. The *eigenvalues* of Γ are the eigenvalues of its adjacency matrix A. We use $\{\theta_0 > \cdots > \theta_d\}$ for the set of distinct eigenvalues of Γ. If Γ has diameter D, then since $I, A, \ldots, A^D$ are linearly independent, it follows that $d \geqslant D$. The *multiplicity* of an eigenvalue θ is denoted by $m(\theta)$.

Let Γ be a graph. Let $\Pi = \{P_1, P_2, \ldots, P_t\}$ be a partition of the vertex set of Γ where t is a positive integer. We say Π is an *equitable partition* if there exist non-negative integers q_{ij} $(1 \leqslant i,j \leqslant t)$ such that any vertex in P_i has exactly q_{ij} neighbors in P_j. The $(t \times t)$-matrix $Q = (q_{ij})_{1 \leqslant i,j \leqslant t}$ is called the *quotient matrix* of Π. If Π is equitable, the *distribution diagram* of Γ with respect to Π is the diagram in which we present each P_i by a balloon such that the balloon representing P_i is joined by a line segment to the balloon representing P_j if $q_{ij} > 0$ and we will write the number q_{ij} just above the line segment close to the balloon representing P_i. We write $p_i := |P_i|$ and q_{ii} inside and below the balloon representing P_i, respectively. If $q_{ii} = 0$, we write $'-'$ instead of 0.

Let Γ be a graph and let x be a vertex of Γ. Then the *walk partition* $W(x)$ of the vertex x is the partition $\{\{x\}, P_1, \ldots, P_n\}$ of $V(\Gamma)$, such that two vertices y, z are in the same part if and only if for any ℓ, the numbers of walks of length ℓ between x, y and x, z are the same. A similar definition was introduced by Fiol [12]. For those graphs discussed in later sections, the walk partition is always equitable.

For example we consider a distance-regular graph Γ with diameter D. And the distribution diagram with respect to the walk partition $W(x)$ of any vertex x is shown in Figure 1 (for distance-regular graph, the walk partition $W(x)$ is the same as the partition according to the distance from the vertex x).

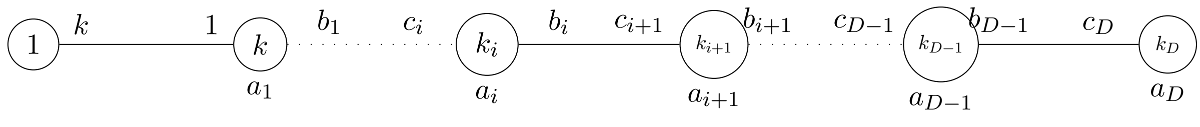

Figure 1

The action of a group G on a set V is *regular*, if it is transitive and no non-identity element of G fixes a point of V.

Let Γ be a graph and G be a group of automorphisms of Γ. The *quotient graph* Γ/G has as vertices the G-orbits on the vertices of Γ, as edges the G-orbits on the edges of Γ, and a vertex $\bar{v}$ is incident with an edge $\bar{e}$ in Γ/G if and only if some element of $\bar{v}$ is incident with some element of $\bar{e}$ in Γ, where $v \in V(\Gamma)$, $e \in E(\Gamma)$, $\bar{v}$ and $\bar{e}$ are the orbits of v and e respectively.

Let G be a group with identity 1 and let Q be a subset of $G^* := G - \{1\}$ closed under taking inverses. Then the *Cayley graph* $\mathrm{Cay}(G, Q)$ is the undirected graph with vertex set G and edge set $E(\mathrm{Cay}(G, Q)) = \{\{g, h\} \mid g^{-1}h \in Q\}$. It is known that $\mathrm{Cay}(G, Q)$ is vertex-transitive and it is connected if and only if Q generates G.

The *dihedral group* of order $2n$ is the group $D_{2n} = \langle a, b \mid a^n = 1 = b^2, bab = a^{-1}\rangle$. Let Q be a subset of D_{2n}^* closed under taking inverses. The graph $Cay(D_{2n}, Q)$ is called a *dihedrant* and is denoted by $\mathrm{Dih}(2n, S, T)$ where $Q = \{a^i \mid i \in S\} \cup \{a^j b \mid j \in T\}$.

Let G be a finite group of order mn and let N be a normal subgroup of G of order n. A k-element subset D of G is called an (m, n, k, λ)-*relative difference set* in G relative to N if every element in $G \setminus N$ has exactly λ representations $r_1 r_2^{-1}$ (or $r_1 - r_2$ if G is additive) with $r_1, r_2 \in D$, and no non-identity element in N has such a representation. When $n = 1$, we simply call D an (m, k, λ)-*difference set*. A difference set or a relative difference set is called *cyclic* if the group G is cyclic. Note that any cyclic relative difference set or cyclic difference set can be seen as a relative difference set in Z_{mn} or a difference set in Z_m, respectively.

An *incidence structure* $\mathcal{I} = (\mathcal{P}, \mathcal{B}, I)$ consists of a set $\mathcal{P}$ of *points*, a set $\mathcal{B}$ of *blocks* (disjoint from $\mathcal{P}$), and a relation $I \subseteq \mathcal{P} \times \mathcal{B}$ called *incidence*. If $(p, B) \in I$, then we say the point p and the block B are *incident*. We usually consider the blocks B as subsets of $\mathcal{P}$. If $\mathcal{I} = (\mathcal{P}, \mathcal{B}, I)$ is an incidence structure, then its *dual incidence structure* is given by $\mathcal{I}^* = (\mathcal{B}, \mathcal{P}, I^*)$, where $I^* = \{(B, p) \mid (p, B) \in I\}$. The *point-block incidence graph* $\Gamma(\mathcal{I})$ of an incidence structure $\mathcal{I}$ is the graph with vertex set $\mathcal{P} \cup \mathcal{B}$, where two vertices are adjacent if and only if they are incident. Note that the point-block incidence graph of an incidence structure is a bipartite graph.

A *group divisible design* $\mathcal{D} = (\mathcal{P}, \mathcal{G}, \mathcal{B})$ with parameters $(n, m; k; \lambda_1, \lambda_2)$, denoted by $GDD(n, m; k; \lambda_1, \lambda_2)$, consists of a set $\mathcal{P}$ of *points*, a partition $\mathcal{G}$ of $\mathcal{P}$ into m sets of size n, each set being called a *group*, and a collection $\mathcal{B}$ of k-subsets of $\mathcal{P}$, called *blocks*, such that each pair of points from the same group occurs in exactly λ_1 blocks and each pair of points from different groups occurs in exactly λ_2 blocks. The triple $\mathcal{I} = (\mathcal{P}, \mathcal{B}, I)$ of a group divisible design is an incidence structure (with the natural incidence relation I) and we consider the dual incidence structure $\mathcal{I}^* = (\mathcal{B}, \mathcal{P}, I^*)$. If there exists a partition $\mathcal{G}'$ of $\mathcal{B}$ such that the triple $(\mathcal{B}, \mathcal{G}', \mathcal{P})$ is a $GDD(n, m; k; \lambda_1, \lambda_2)$, then we say that the $\mathcal{D}$ is a *group divisible design with the dual property* with parameters $(n, m; k; \lambda_1, \lambda_2)$ and we denote such a design by $GDDDP(n, m; k; \lambda_1, \lambda_2)$.

Let X be a finite set and $\mathbb{C}^{X \times X}$ the set of complex matrices with rows and columns indexed by X. Let $\mathcal{R} = \{R_0, R_1, \ldots, R_n\}$ be a set of non-empty subsets of $X \times X$, where R_i $(0 \leqslant i \leqslant n)$ is called a *relation*. For each i, the *relation graph* $\Gamma_i^{\mathcal{R}} := (X, R_i)$ with respect to the relation R_i is the (directed, in general) graph with vertex set X and edge set R_i. Let F_i be the adjacency matrix of the graph $\Gamma_i^{\mathcal{R}}$. The pair $(X, \mathcal{R})$ is an *association* *scheme with* n *classes* if

i) $F_0 = I$, the identity matrix,

ii) $\sum_{i=0}^{n} F_i = J$, the all-ones matrix,

iii) $F_i^t \in \{F_0, F_1, \ldots, F_n\}$ for $0 \leqslant i \leqslant n$,

iv) $F_i F_j$ is a linear combination of $F_0, F_1, \ldots, F_n$ for $0 \leqslant i, j \leqslant n$.

The vector space $\mathbf{A}$ spanned by $\{F_0, F_1, \ldots, F_n\}$ is the *Bose-Mesner algebra* of $(X, \mathcal{R})$.

We say that $(X, \mathcal{R})$ is *commutative* if $\mathbf{A}$ is commutative, and that $(X, \mathcal{R})$ is *symmetric* if the F_i $(0 \leqslant i \leqslant n)$ are symmetric matrices. A symmetric association scheme is commutative. We only consider symmetric association scheme in this note.

Let $(X, \mathcal{R})$ be a symmetric association scheme with n classes. Then $\mathbb{C}^X$ can be decomposed as a direct sum of common eigenspaces V_i $(0 \leqslant i \leqslant n)$ of the Bose-Mesner algebra $\mathbf{A}$. Let E_i be the orthogonal projection onto the common eigenspace V_i, where we always set $E_0 = |X|^{-1} J$ (J is the all-ones matrix). Then $\{E_0, E_1, \ldots, E_n\}$ forms a basis of the primitive idempotents of $\mathbf{A}$, i.e. $E_i E_j = \delta_{ij} E_j$ $(0 \leqslant i, j \leqslant n)$, $\sum_{i=0}^{n} E_i = I$. We call the change-of-base matrices P and Q the *first and second eigenmatrices* of the association scheme $(X, \mathcal{R})$, where P and Q are defined as follows:

$$F_i = \sum_{j=0}^{n} P_{ji} E_j, \quad E_i = \frac{1}{|X|} \sum_{j=0}^{n} Q_{ji} F_j \quad (0 \leqslant i \leqslant n).$$

Let $(X, \mathcal{R})$ be a symmetric association scheme with n classes. Then the relation graph $\Gamma_i^{\mathcal{R}}$ $(1 \leqslant i \leqslant n)$ is an undirected graph. Choose a vertex x of the relation graph $\Gamma_i^{\mathcal{R}}$, then the partition $\{P_0, P_1, \ldots, P_n\}$ of $V(\Gamma_i^{\mathcal{R}})$ is equitable, where $P_j := \{y \mid (x, y) \in R_j\}$ $(0 \leqslant j \leqslant n)$. The *distribution diagram* of the symmetric association scheme $(X, \mathcal{R})$ with respect to the relation R_i is the distribution diagram of the relation graph $\Gamma_i^{\mathcal{R}}$ with respect to the equitable partition $\{P_0, P_1, \ldots, P_n\}$.

3 Group divisible designs with the dual property

In this section, we will construct bipartite 2-walk-regular graphs with diameter 4 having exactly 6 distinct eigenvalues, as the point-block incidence graphs of certain group divisible designs with the dual property.

Theorem 2. *Let $\mathcal{D}$ be a GDDDP$(n, m; k; 0, \lambda_2)$ with $n, m \geqslant 2$, $\lambda_2 \geqslant 1$, $k > n\lambda_2$ and let $\Gamma := \Gamma(\mathcal{D})$ be the point-block incidence graph of $\mathcal{D}$. Then Γ is a relation graph, say with respect to a relation R of a symmetric association scheme $(X, \mathcal{R})$ with 5 classes, such that the distribution diagram of $(X, \mathcal{R})$ with respect to the relation R is as in Figure 2, where $k_4 = n - 1$, $c_2 = \lambda_2$ and $b'_2 = (n-1)\lambda_2$. In particular, Γ is a bipartite 2-walk-regular graph with diameter 4 and exactly 6 distinct eigenvalues.*

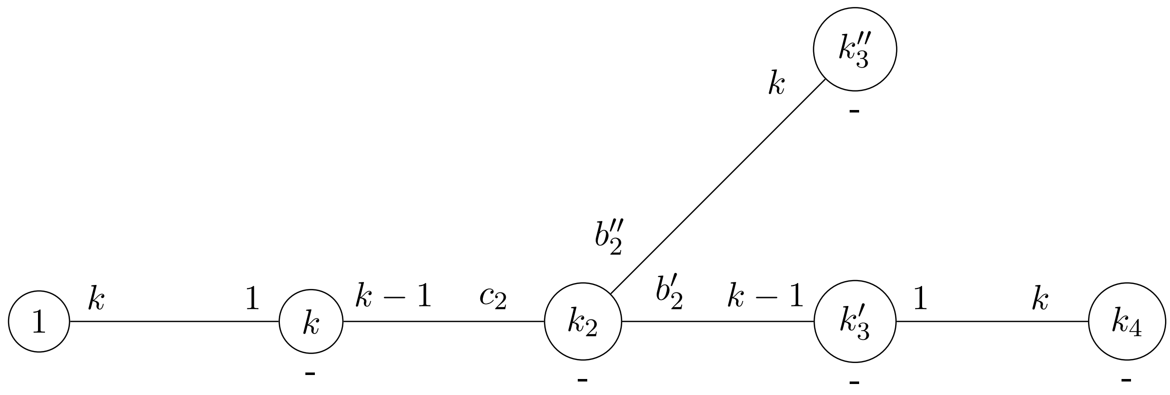

Figure 2

Proof. Note that Γ is a bipartite graph with valency k and diameter 4, as Γ is an incidence graph, each block contains k points and $\lambda_1 = 0$, $\lambda_2 \geqslant 1$. Let x be a vertex of Γ. By the dual property, we may assume without loss of generality that x is a point of $\mathcal{D}$, thus $\{x\} \cup \Gamma_2(x) \cup \Gamma_4(x)$ is the set of points of $\mathcal{D}$. As $\lambda_1 = 0$ and $\lambda_2 \geqslant 1$, we see that two vertices are in the same group if and only if they are at distance 4. It follows that the group of $\mathcal{D}$ that contains the vertex x is $\{x\} \cup \Gamma_4(x)$, i.e., $\Gamma_4(x)$ consists of $n-1$ vertices and they are mutually at distance 4, and $c_2(x, y) = \lambda_2$ holds for any vertex $y \in \Gamma_2(x)$. Now let $\Gamma'_3(x)$ be the set of vertices at distance 3 from x with a neighbor in $\Gamma_4(x)$ and $\Gamma''_3(x) := \Gamma_3(x) \setminus \Gamma'_3(x)$. As $\lambda_1 = 0$, we see that $b_3(x, y) = 1$ holds for any vertex $y \in \Gamma'_3(x)$. Choose a vertex $y \in \Gamma_2(x)$, then $\Gamma(y) \cap \Gamma'_3(x) = \cup_{z \in \Gamma_4(x)}(\Gamma(y) \cap \Gamma(z))$. Since those vertices in $\Gamma_4(x)$ are mutually at distance 4 from each other, we see that $|\Gamma(y) \cap \Gamma'_3(x)| = (n-1)\lambda_2$. It follows that the partition $\Pi = \{\{x\}, \Gamma(x), \Gamma_2(x), \Gamma'_3(x), \Gamma''_3(x), \Gamma_4(x)\}$ is an equitable partition of $V(\Gamma)$ with distribution diagram as in Figure 2 with $k_4 = n - 1$, $c_2 = \lambda_2$ and $b'_2 = (n-1)\lambda_2$. And $k > n\lambda_2$ implies that $k''_3 > 0$.

Now define the matrix B_3 by $(B_3)_{xy} = 1$ if $y \in \Gamma'_3(x)$ and 0 otherwise, where x and y are any two vertices of Γ. Note that for any pair of vertices x and y with $y \in \Gamma_3(x)$, the number of walks of length 3 between x and y equals $c_3(x, y)c_2 = c_3(y, x)c_2$, and $c_3(x, y) \neq k$ if and only if $y \in \Gamma'_3(x)$, which in turn implies that B_3 is symmetric. Let $C_3 = A_3 - B_3$, where A_i is the distance-i matrix of Γ for $i = 0, 1, 2, 3, 4$. It is straightforward to check that the set of matrices $\{A_0 = I, A_1, A_2, B_3, C_3, A_4\}$ satisfies the axioms of a symmetric association scheme. That Γ is 2-walk-regular follows from the fact that A_2 is a relation matrix of the association scheme. As Γ is the relation graph of a 5-class association scheme, it follows that Γ has at most 6 distinct eigenvalues. The fact that it has at least 6 eigenvalues follows from Proposition 1. This shows the theorem. □

Remark 3. i) The first and second eigenmatrices of the corresponding association scheme, where $b''_2 = k - (k_4 + 1)c_2$ are as follows:

$$P = \begin{pmatrix} 1 & k & \frac{(k-1)k}{c_2} & kk_4 & \frac{(k-1)b_2''}{c_2} & k_4 \\ 1 & -k & \frac{(k-1)k}{c_2} & -kk_4 & -\frac{(k-1)b_2''}{c_2} & k_4 \\ 1 & \sqrt{k} & 0 & -\sqrt{k} & 0 & -1 \\ 1 & -\sqrt{k} & 0 & \sqrt{k} & 0 & -1 \\ 1 & \sqrt{b_2''} & -(k_4+1) & k_4\sqrt{b_2''} & -(k_4+1)\sqrt{b_2''} & k_4 \\ 1 & -\sqrt{b_2''} & -(k_4+1) & -k_4\sqrt{b_2''} & (k_4+1)\sqrt{b_2''} & k_4 \end{pmatrix},$$

$$Q = \begin{pmatrix} 1 & 1 & \frac{(k^2-b_2'')k_4}{k-b_2''} & \frac{(k^2-b_2'')k_4}{k-b_2''} & \frac{(k-1)k}{k-b_2''} & \frac{(k-1)k}{k-b_2''} \\ 1 & -1 & \frac{(k^2-b_2'')k_4}{\sqrt{k}(k-b_2'')} & -\frac{(k^2-b_2'')k_4}{\sqrt{k}(k-b_2'')} & \frac{\sqrt{b_2''}(k-1)}{k-b_2''} & -\frac{\sqrt{b_2''}(k-1)}{k-b_2''} \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & -1 & -\frac{k^2-b_2''}{\sqrt{k}(k-b_2'')} & \frac{k^2-b_2''}{\sqrt{k}(k-b_2'')} & \frac{\sqrt{b_2''}(k-1)}{k-b_2''} & -\frac{\sqrt{b_2''}(k-1)}{k-b_2''} \\ 1 & -1 & 0 & 0 & -\frac{k}{\sqrt{b_2''}} & \frac{k}{\sqrt{b_2''}} \\ 1 & 1 & -\frac{k^2-b_2''}{k-b_2''} & -\frac{k^2-b_2''}{k-b_2''} & \frac{(k-1)k}{k-b_2''} & \frac{(k-1)k}{k-b_2''} \end{pmatrix}.$$

ii) It is easy to see that if $(X, \mathcal{R})$ is a symmetric association scheme such that the distribution diagram of $(X, \mathcal{R})$ with respect to a relation R is equal to the diagram in Figure 2, then $(X, \mathcal{R})$ comes from a GDDDP$(n, m; k; 0, \lambda_2)$ with $n = k_4 + 1$, $\lambda_2 = c_2$, as described in the above theorem.

iii) If the point-block incidence matrix of a GDDDP is symmetric with zeroes on the diagonal, it corresponds exactly to the divisible design graph as defined by Haemers et al. [14].

4 Classical Examples

In this section we discuss classical examples of group divisible designs with the dual property and in Proposition 4 we show that the point-block incidence graphs of these examples are 2-arc transitive dihedrants. Some of them were already found by Bose [3], for more information see [11].

We first introduce classical examples of group divisible designs with the dual property. Let $d \geqslant 2$ be an integer and let q be a prime power. Let V be a vector space of dimension d over $GF(q)$ (the finite field with q elements). We define the set of non-zero vectors in V as the point set $\mathcal{P}$ and the set of affine hyperplanes in V as the block set $\mathcal{B}$, i.e., $\mathcal{P} = \{x \in V \mid x \neq 0\}$ and $\mathcal{B} = \{x + H \mid H \text{ is a hyperplane in V and } x \notin H\}$.

We make a partition $\mathcal{G}$ of $\mathcal{P}$ such that the collinear non-zero vectors in V belong to the same group in $\mathcal{G}$. Note that each group in $\mathcal{G}$ has size $q-1$. Then $(\mathcal{P}, \mathcal{G}, \mathcal{B})$ is a $GDD(n, m; k; 0, \lambda_2)$, where $n = q - 1$, $m = \frac{q^d-1}{q-1}$ is the number of projective points in V, $k = q^{d-1}$ is the number of affine hyperplanes containing a given non-zero vector, and $\lambda_2 = q^{d-2}$ is the number of affine hyperplanes containing two given non-zero and non-collinear vectors.

Now we look at the dual incidence structure $\mathcal{I}^* = (\mathcal{B}, \mathcal{P}, I^*)$ of $\mathcal{I} = (\mathcal{P}, \mathcal{B}, I)$. We make a partition $\mathcal{G}'$ of $\mathcal{B}$ such that the parallel affine hyperplanes belong to the same group in $\mathcal{G}'$. Then $(\mathcal{B}, \mathcal{G}', \mathcal{P})$ becomes a $GDD(n, m; k; 0, \lambda_2)$, where $n = q - 1$, $m = \frac{q^d-1}{q-1}$ is the number of $d-1$ dimensional subspaces in V, $k = q^{d-1}$ is the number of non-zero vectors in an affine hyperplane, and $\lambda_2 = q^{d-2}$ is the number of non-zero vectors in the intersection of two given non-parallel affine hyperplanes.

This shows that $\mathcal{D}(d, q) := (\mathcal{P}, \mathcal{G}, \mathcal{B})$ is a $GDDDP(n, m; k; 0, \lambda_2)$, where n, m, k, λ_2 are given above. We denote $\Gamma(d, q) := \Gamma(\mathcal{D}(d, q))$ as the point-block incidence graph of $\mathcal{D}(d, q)$. It is clear that the general linear group $\mathrm{GL}(d, q)$ acts as a group of automorphism of the graph $\Gamma(d, q)$.

Now we show that the point-block incidence graphs $\Gamma(d, q)$ of classical group divisible designs with the dual property $\mathcal{D}(d, q)$ are 2-arc transitive dihedrants.

Proposition 4. *For all integers $d \geqslant 2$ and prime powers q, the point-block incidence graph $\Gamma(d, q)$ of the group divisible design with the dual property $\mathcal{D}(d, q)$ is a 2-arc transitive dihedrant.*

Proof. Let z be a primitive element of $\mathrm{GF}^*(q^d)$ and define the map $\tau_z : \mathrm{GF}^*(q) \to \mathrm{GF}^*(q)$ by $\tau_z(x) = zx$ for $x \in \mathrm{GF}^*(q^d)$. The map τ_z has order $n := q^d - 1$. We can identify the map τ_z as a linear map $A_z \in \mathrm{GL}(d, q)$ by identifying the field $\mathrm{GF}(q^d)$ with the vector space $\mathrm{GF}(q)^d$. Note that the group $\langle A_z \rangle$ is the well-known Singer-Zyklus subgroup of $\mathrm{GL}(d, q)$. It is clear that A_z is an automorphism of the graph $\Gamma(d, q)$. For any non-zero vector $y \in \mathrm{GF}(q)^d$, define $H_y := \{x \in \mathrm{GF}(q)^d \mid x^t y = 1\}$. A_z maps H_y to $H_{y'}$, where $y' = (A_z^t)^{-1} y$. Now let $u_0, u_1, \ldots, u_n$ and $v_0, v_1, \ldots, v_n$ be two orderings of the non-zero vectors of $\mathrm{GF}(q)^d$, such that A_z maps u_i to u_{i+1} and H_{v_i} to $H_{v_{i+1}}$ $(0 \leqslant i \leqslant n)$ (where we take the indices module n). Define the map $\phi : \mathcal{P} \cup \mathcal{B} \to \mathcal{P} \cup \mathcal{B}$ by $\phi(u_i) = H_{v_{n-i}}$ and $\phi(H_{v_i}) = u_{n-i}$ $(0 \leqslant i \leqslant n)$, where $\mathcal{P}$ is the point set of the design $\mathcal{D}(d, q)$, consisting of the non-zero vectors in $\mathrm{GF}(q)^d$, and $\mathcal{B}$ is the block set of the design $\mathcal{D}(d, q)$, consisting of the affine hyperplanes in $\mathrm{GF}(q)^d$. Then we see that $u_i \overset{\phi}{\longmapsto} H_{v_{n-i}} \overset{A_z}{\longmapsto} H_{v_{n-i+1}} \overset{\phi}{\longmapsto} u_{i-1}$, i.e., $\phi A_z \phi = A_z^{-1}$. Note that ϕ has order 2 and A_z has order n. We see that the group generated by ϕ and A_z is the dihedral group D_{2n} and it acts regularly on the vertex set of $\Gamma(d, q)$. This show that $\Gamma(d, q)$ is a dihedrant by [13, Lemma 3.7.2].

Now we show that $\Gamma(d, q)$ is 2-arc transitive. As $\Gamma(d, q)$ is vertex-transitive, we only need to show it is transitive on 2-arcs xHy, where x, y are non-zero vectors in $\mathrm{GF}(q)^d$ and H is an affine hyperplane (For a 2-arc $H_1 x H_2$, where H_1 and H_2 are affine hyperplanes and x is a non-zero vector, we may consider $\phi(H_1 x H_2)$). Note that xHy is a 2-arc if and only if x, y are linearly independent non-zero vectors and H is an affine hyperplane containing x, y. Let $x'H'y'$ be a 2-arc, with x', y' non-zero vectors and H' an affine hyperplane. Then there exists an element $\sigma \in \mathrm{GL}(d, q)$ that maps simultaneously x to x', y to y' and H to H'.

This shows that $\Gamma(d, q)$ is a 2-arc transitive dihedrant. □

Remark 5. The graphs $\Gamma(d, q)$ can also be described in a pure group theoretical way as a bi-coset graph (see Du and Xu [10]). Take $G = \mathrm{GL}(d, q)$. Let R be the set of matrices in G whose first row equals $(1, 0, 0, \ldots, 0)$, and let L be the set of matrices in G whose first column equals $(1, 0, 0, \ldots, 0)^t$. Note that R and L are subgroups of G. Then $\Gamma(d, q)$ is isomorphic to the bi-coset graph $\mathbf{B}(G, L, R; RL)$, which is bipartite with color classes $\{Lg \mid g \in G\}$ and $\{Rg \mid g \in G\}$, where Lg_1 is adjacent to Rg_2 if and only if $g_2 g_1^{-1} \in RL$.

Now we consider some quotient graphs of $\Gamma(d, q)$, which are also 2-arc transitive dihedrants. Consider the group $Z := \{\alpha I_d \mid \alpha \in \mathrm{GF}^*(q)\} \leqslant GL(d, q)$. Let n be a divisor of $q-1$. As Z is a cyclic group of order $q-1$, it contains a cyclic subgroup C of order $(q-1)/n$. Using a similar method as in Proposition 4, we may see that quotient graph $\Gamma(d, q, n) := \Gamma(d, q)/C$ is a 2-arc transitive dihedrant and when $n = q-1$, $\Gamma(d, q, n)$ is the same as $\Gamma(d, q)$. The distribution diagram of $\Gamma(d, q, n)$ with respect to the walk partition $W(x)$ of any vertex x is shown in Figure 2 with $k = q^{d-1}$, $c_2 = q^{d-2}(q-1)/n$ and $k_4 = n-1$.

5 Cyclic relative difference sets

In this section, we give another viewpoint on the examples of the last section and give a construction for dihedrants from cyclic difference sets.

Proposition 6. *Let D be a cyclic (m, n, k, λ)-relative difference set with $m, n \geqslant 2$. Then the dihedrant Dih$(2nm, \emptyset, D)$ is the point-block incidence graph of a GDDDP$(n, m; k; 0, \lambda)$. In particular, Dih$(2nm, \emptyset, D)$ is a connected 2-walk-regular graph.*

Proof. By definition, the dihedrant $\mathrm{Dih}(2nm, \emptyset, D)$ is bipartite. By direct verification, we see that the distribution diagram with respect to the walk partition of any vertex is as in Figure 2 with $k_4 = n-1$ and $c_2 = \lambda$. It follows that it is the point-block incidence graph of a $\mathrm{GDDDP}(n, m; k; 0, \lambda)$. Then the dihedrant $\mathrm{Dih}(2nm, \emptyset, D)$ is 2-walk-regular follows from Theorem 2. □

The graphs $\Gamma(d, q, n)$ as considered in the last section arise from cyclic $(\frac{q^d-1}{q-1}, n, q^{d-1}, \frac{q^{d-2}(q-1)}{n})$-relative difference sets. And Arasu et al. [2, 1] gave constructions for cyclic $(\frac{q^d-1}{q-1}, n, q^{d-1}, \frac{q^{d-2}(q-1)}{n})$-relative difference sets for q a prime power, where n is a divisor of $q-1$ when q is odd or d is even, and n is a divisor of $2(q-1)$ when q is even and d is odd. Arasu et al. [2, Theorem 1.2] showed that for a prime power q, cyclic relative difference sets with parameters $(\frac{q^d-1}{q-1}, n, q^{d-1}, \frac{q^{d-2}(q-1)}{n})$ exist if and only if the above restrictions are satisfied.

Note that those dihedrants in Section 4 are 2-arc transitive. But the dihedrants constructed from general cyclic relative difference are not always 2-arc transitive. We give two examples below.

The 2-arc transitive dihedrant generated by the cyclic $(7, 2, 4, 1)$-relative difference set $\{0, 1, 9, 11\}$ in Z_{14} relative to $\{0, 7\}$ has the distribution diagram with respect to the walk partition of any vertex as in Figure 3. It is the graph $C4[28, 3]$ in [15].

The cyclic $(13, 2, 9, 3)$-relative difference set $\{0, 9, 11, 15, 18, 19, 20, 23, 25\}$ in Z_{26} relative to $\{0, 13\}$ generates a dihedrant, which is not edge transitive. That graph and the

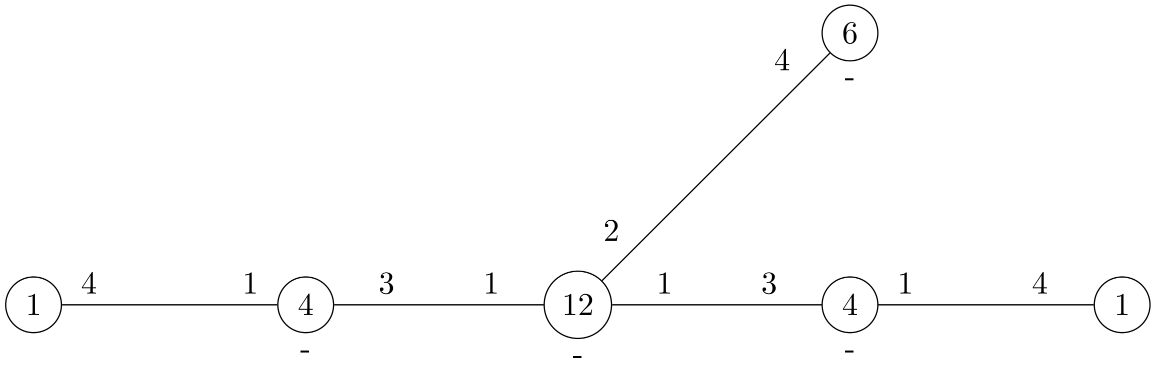

Figure 3

2-arc transitive graph $\Gamma(3,3)$ have the same distribution diagram with respect to the walk partition of any vertex as in Figure 4.

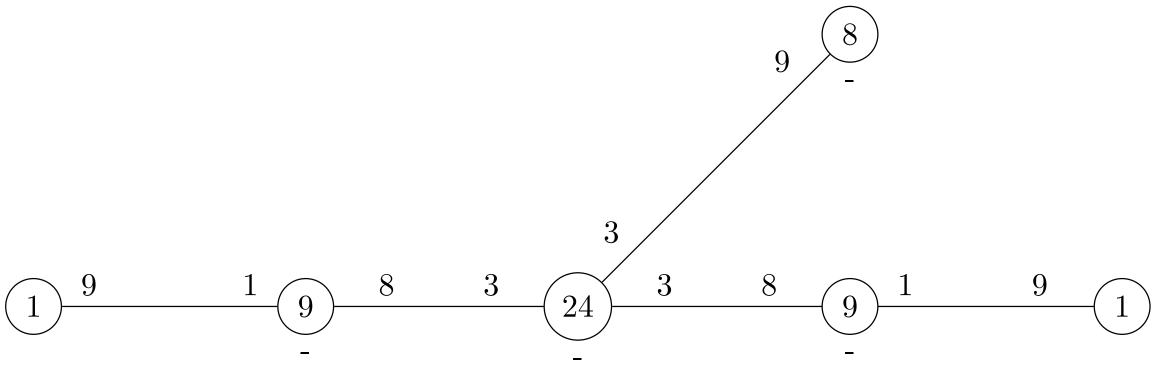

Figure 4

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