On snarks that are far from being 3-edge colorable

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Abstract

In this note we construct two infinite snark families which have high oddness and low circumference compared to the number of vertices. Using this construction, we also give a counterexample to a suggested strengthening of Fulkerson's conjecture by showing that the Petersen graph is not the only cyclically 4-edge connected cubic graph which require at least five perfect matchings to cover its edges. Furthermore the counterexample presented has the interesting property that no 2-factor can be part of a cycle double cover.

1   Introduction

A cubic graph is said to be *colorable* if it has a 3-edge coloring and *uncolorable* otherwise. A *snark* is an uncolorable cubic cyclically 4-edge connected graph. It it well known that an edge minimal counterexample (if such exists) to some classical conjectures in graph theory, such as the cycle double cover conjecture [15, 18], Tutte's 5-flow conjecture [19] and Fulkerson's conjecture [4], must reside in this family of graphs.

There are various ways of measuring how far a snark is from being colorable. One such measure which was introduced by Huck and Kochol [8] is the *oddness*. The oddness of a bridgeless cubic graph G is defined as the minimum number of odd components in any 2-factor in G and is denoted by $o(G)$. Another measure is the *resistance* of G, $r_3(G)$, which was introduced by Steffen [17] and is defined as the minimal number of edges that needs to be removed from G in order to obtain a 3-edge colorable graph, i.e. $r_3(G) := \min\{|M| : M \subset E(G) \text{ and } \chi'(G - M) = 3\}$. It is easy to see that $r_3(G) \leqslant o(G)$. It is also known that there exists families of snarks where these measures are arbitrary large and arbitrary far apart [17].

Although snarks can be arbitrary far from being colorable in the sense of oddness and resistance, this might not be the case when we consider other uncolorablilty measures.

The *perfect matching index* $\tau(G)$ of a cubic graph G was introduced in [3] and is defined as the minimum number of perfect matchings needed to cover $E(G)$. A famous conjecture by Fulkerson asserts that every bridgeless cubic graph has a double cover by six perfect matchings [4], and if true, would imply that $\tau(G) \leqslant 5$ for every bridgeless cubic graph G. Recently Mazzuoccolo showed that these two statements are in fact equivalent [12]. However it is not known if there exists a constant k such that $\tau(G) < k$ for all snarks G.

It is easy to see that $\tau(P) = 5$, where P is the Petersen graph. It has been proposed [13, 3] that a possible strengthening of Fulkerson's conjecture could be the assertion that the Petersen graph is in fact the only snark with this property and that all other snarks have $\tau = 4$.

Conjecture 1.1 ([13, 3]). *Let G be a cyclically 4-edge connected cubic graph. If $\tau(G) = 5$, then G is the Petersen graph.*

In this paper we present a counterexample to this conjecture. Furthermore we note that this counterexample has the interesting property that no 2-factor can be part of a cycle double cover. We also give simple constructions for two infinite families of snarks with high oddness and resistance compared to the number of vertices.

2   The construction

The following simple lemma is well known (see e.g. [9, 14, 17]) and very useful when studying edge colorability of cubic graphs.

Lemma 2.1 (Parity lemma). *Let $\phi : E(G) \to \{1, 2, 3\}$ be a 3-edge-coloring of a cubic graph G. Then for every edge cut M in G we have that $|\phi^{-1}(1) \cap M| \equiv |\phi^{-1}(2) \cap M| \equiv |\phi^{-1}(3) \cap M| \equiv |M|(\mod 2)$*

Following the notation from [10] we say that a *semiedge* is an edge which is incident to exactly one vertex or one vertex and another semiedge. In the latter case we simply identify it with a normal edge. A *multipole* M is a triple $M = (V, E, S)$ where $V = V(M)$ is the vertex set, $E = E(M)$ is the edge set and $S = S(M)$ is the set of semiedges. A multipole with k semiedges is called a *k-pole*.

Let B be the 4-pole obtained by removing two adjacent vertices from the Petersen graph (see Figure 1).

Lemma 2.2. *In every 3-edge coloring of the 4-pole B the semiedges a_1 and a_2 receive the same color.*

Proof. Assume that a_1 and a_2 have different colors in a 3-edge coloring ϕ of B. W.l.o.g. we may assume that $\phi(a_1) = 1$ and $\phi(a_2) = 2$. Now, by Lemma 2.1, one of b_1 and b_2 must have color 1 and the other color 2. W.l.o.g. we can assume that $\phi(b_1) = 1$ and $\phi(b_2) = 2$. Furthermore we can assume w.l.o.g. that $\phi(v_1 v_2) = 2$ and $\phi(v_2 v_3) = 3$. Then $\phi(v_2 v_6) = 3$, but since one of the edges $v_6 v_7$ and $v_5 v_6$ must have color 3, we have a contradiction. $\square$
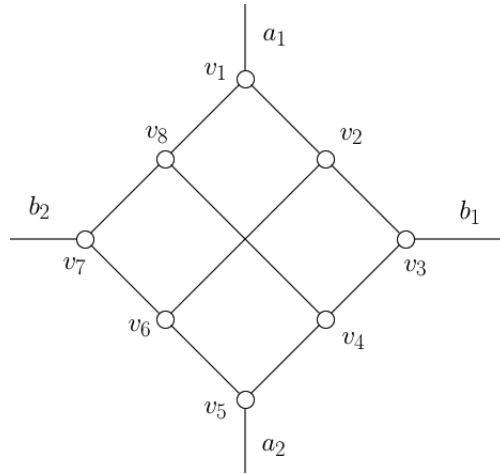
Figure 1: The 4-pole B is constructed from the Petersen graph by removing two adjacent vertices.
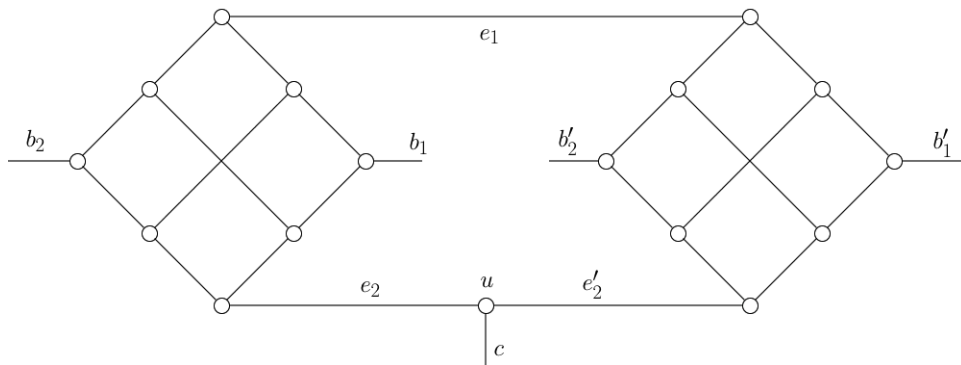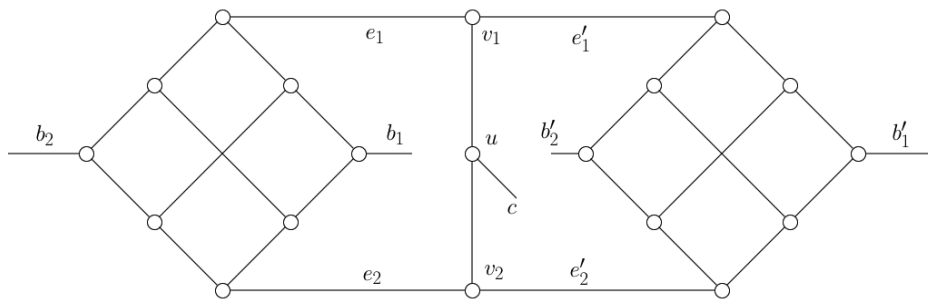

Figure 2: The 5-pole H_1.

Figure 3: The 5-pole H_2.

Now consider the 5-pole H_1 obtained the following way: Take two copies B_1 and B_2 of B and identify the semiedges b_1 from B_1 with b_1 from B_2 to get an edge e_1. Then add

a vertex u which is connected with the semiedges b_2 from both B_1 and B_2 and denote the corresponding new edges by e_2 and e_2' (see Figure 2). It is straightforward to verify that this 5-pole cannot be 3-edge colorable.

Lemma 2.3. *H_1 is not 3-edge colorable.*

Proof. Assume the opposite. Then by Lemma 2.2 e_1 must have the same color as e_2, but e_1 must also have the same color as e_2' which is impossible since e_2 and e_2' are incident edges. $\square$

The 5-pole H_2 is formed in a similar way by joining two copies of B with three vertices v_1, v_2 and u as shown in Figure 3.

Lemma 2.4. *H_2 is not 3-edge colorable.*

Proof. Assume the the graph has a 3-edge coloring. Then by Lemma 2.2 e_1 must have the same color as e_2 and e_1' must have the same color as e_2'. W.l.o.g. we may assume that e_1 and e_2 has color 1 and e_1' and e_2' has color 2. But then both $v_1 u$ and $v_2 u$ must have color 3, which is impossible. $\square$

We can now use that fact that any cubic graph which contains either H_1 or H_2 as a subgraph cannot be colorable to create two infinite families of snarks.

Construction 1.

Let G be any 2-edge connected cubic graph and let D be a 2-regular subgraph of G. Let $C = (v_1, v_2, \ldots, v_k)$ be a cycle in D. Now, form a graph by removing all edges $v_i v_{i+1}$ (indices are taken modulo k). Then add k copies $B_1, \ldots, B_k$ of the 4-pole B and denote the semiedge edges in B_i by a_l^i, b_l^i for $l \in \{1, 2\}$. Now, for all $i \in \{1, \ldots, k\}$, connect v_i with the semiedge b_1^i, the semiedges a_2^i with b_2^{i+1} and a_1^i with v_{i+1}. Repeat this process for every cycle in D and denote the resulting proper graph by G'.

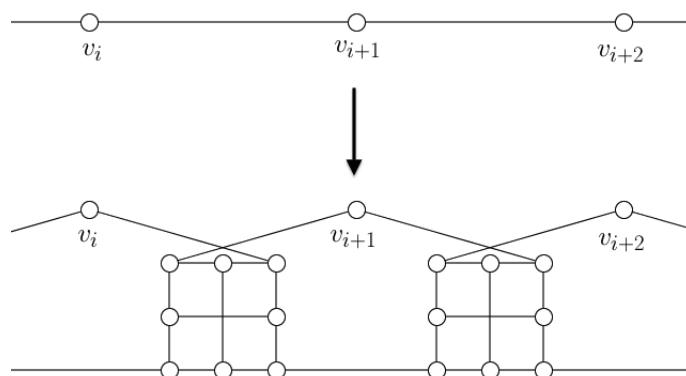We call G' a *semi blowup* of (G, D) and denote G' by SemiBlowup(G, D).

Figure 4: An illustration of Construction 1.

Construction 2.

The second construction is similar to the previous one. Let G be any 2-edge connected cubic graph and let D be a 2-regular subgraph of G. Furthermore let $C = (v_1, v_2, \dots, v_k)$ be a cycle in D and remove all edges $v_i v_{i+1}$, $i = 1, \dots, k$ and add k copies $B_1, \dots, B_k$ of B. Now add two additional vertices u_i and w_i. Then add the edges $\{v_i u_i, v_i w_i\}$ and form edges from the semiedges by connecting u_i with b_1^i, w_i with b_2^i, a_1^i with u_{i+1} and a_2^i with w_{i+1}. We continue this process for every cycle in D and call the resulting graph G''. We say that G'' a *blowup* of D and G'' is denoted by $\mathrm{Blowup}(G, D)$.
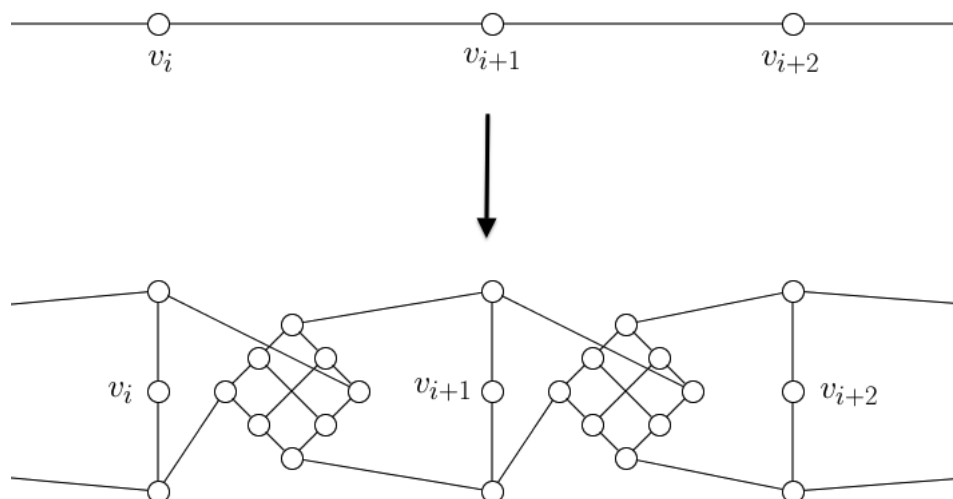

Figure 5: An illustration of construction 2.

Theorem 2.5. *Let G be a 3-edge connected cubic graph with a 2-regular subgraph D. Then $\mathrm{Blowup}(G, D)$ and $\mathrm{SemiBlowup}(G, D)$ are not 3-edge colorable. Furthermore if $D_1, \dots, D_l$ are the disjoint cycles of D, we have that*

$$o(\mathrm{Blowup}(G, D)) \geqslant r_3(\mathrm{Blowup}(G, D)) \geqslant \sum_{i=1}^{l} \left\lceil \frac{|V(D_i)|}{2} \right\rceil$$

and the same inequality holds for $\mathrm{SemiBlowup}(G, D)$.

Proof. Let $G_1 := \mathrm{SemiBlowup}(G, D)$ and $G_2 := \mathrm{Blowup}(G, D)$. Every pair of adjacent edges on a cycle in D gives a subgraph isomorphic to H_1 in G_1 and a subgraph isomorphic to H_2 in $\mathrm{SemiBlowup}(G, D)$. Since a cubic graph cannot be colorable if either H_1 or H_2 are subgraphs, we have that $r_3(G_1)$ and $r_3(G_2)$ is at least the number of edges that needs to be removed in order to make G_1 H_1-free or G_2 H_2-free. It is easy to see that this is at least $\sum_{i=1}^{l} \beta(D_i)$ where β is the vertex cover number of a graph (the minimum number of vertices needed to cover all the edges in a graph). Since an optimal vertex cover of an

even cycle is to choose every second vertex we have that $\beta(C_{2k}) = k$ and for odd cycles at least two adjacent vertices must be in the vertex cover, so $\beta(C_{2k+1}) = k + 1$. Hence

$$o(G_j) \geqslant r_3(G_j) \geqslant \sum_{i=1}^{l} \beta(D_i) = \sum_{i=1}^{l} \left\lceil \frac{|V(D_i)|}{2} \right\rceil$$

for $j = 1, 2$. $\hfill\square$

If G has no cyclic k-edge-cuts where $k \leqslant 3$, then both blowup and semiblowup will produce snarks. This also gives a simple construction of cyclically 4-edge connected snarks with rather high oddness compared to the number of vertices.

By considering the semiblowup of a hamiltonian cycle in a hamiltonian cubic graph, we get the following:

Corollary 2.6. *For every $k \in \mathbb{N}, k > 2$ there exists a snark G_k with $|V(G_k)| = 18k$ and $o(G_k) \geqslant k$.*

This can be improved slightly by instead use many short cycles of odd lengths.

Corollary 2.7. *For every $k \in \mathbb{N}$ there exists a snark G_k with $|V(G_k)| = 90k$ and $o(G_k) \geqslant 6k$.*

Proof. Given $k \in \mathbb{N}$, let F_k be the 2-factor formed from $2k$ copies of C_5. Then add $10k$ edges between the cycles in F in order to obtain a cubic graph G'_k. Now let $G_k :=$ SemiBlowup(G', F). Since $\beta(C_5) = 3$ we get $o(G_k) \geqslant 2k\beta(C_5) = 6k$. $\hfill\square$

In Corollary 2.7 the oddness of G_k grows linearly in the number of vertices. Obviously it is impossible to construct a family of snarks where the oddness grows superlinearly. Let

$$q_k := \max \left\{ \frac{k}{|V(G)|} : G \in \mathcal{G} \text{ and } o(G) \geqslant k \right\}$$

where $\mathcal{G}$ is the family of snarks. The Petersen graph is the smallest snark so $q_2 = \frac{1}{5}$. It is known that the smallest snark with oddness 4 has at least 38 vertices [1] and from Theorem 2.5 we can construct a snark with oddness 4 on 46 vertices (use the semiblowup construction on the cubic graph on 6 vertices which consists of a 5-cycle, a cord and a $K_{1,3}$). Hence, $\frac{2}{23} \leqslant q_4 \leqslant \frac{2}{19}$. From Corollary 2.7 we get $q_{6k} \geqslant \frac{1}{15}$ and from Corollary 2.6 we get $q_k \geqslant \frac{1}{18}$. We pose the following two problems:

Problem 1. Let k be a given even number. Determine q_k when $k > 2$.

Problem 2. What is the largest value c such that $q_k \geqslant c$ for all even k.

3 Perfect matching covers

Let C be a cycle of length 3 in K_4 and consider the graph $Blowup(K_4, C)$. By using a computer, it is easy to see that this graph has perfect matching index 5. Note that this graph was also observed in [1], as an example of a snark of minimum order for which the removal of any vertex yields a graph homeomorphic to a non 3-edge colorable cubic graph.

Observation 3.1. $\tau(Blowup(K_4, C)) = 5$.

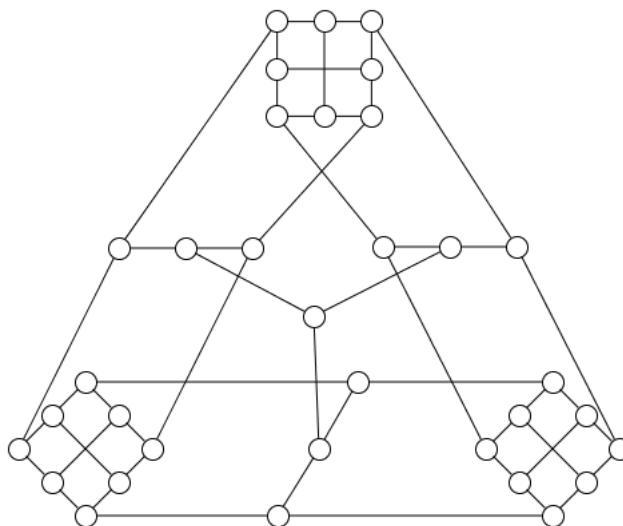Figure 6: An illustration of $Blowup(K_4, C)$.

It is possible to construct other snarks with this property. Another example is the blowup of a 4-cycle in the prism depicted in Figure 7. However we do not have any good characterization of the cubic graphs and 2-regular subgraphs for which the blowup-construction yields snarks with perfect matching index 5.

Problem 3. Is it possible to give a simple characterization of cubic graphs G with $\tau(G) = 5$?

Graphs constructed with the blowup-construction always contains cyclic 4-edge cuts, and other than the Petersen graph, we do not know of any cyclically 5-edge connected snarks G with $\tau(G) = 5$.

Problem 4. Are there any cyclically 5-edge connected snarks G, different from the Petersen graph, such that $\tau(G) = 5$?

4 Cycle double covers

We say that a *cycle double cover* (abbreviated CDC) of a graph is multiset of cycles such that every edge is covered by exactly two cycles. A *k-CDC* is a CDC where we can color
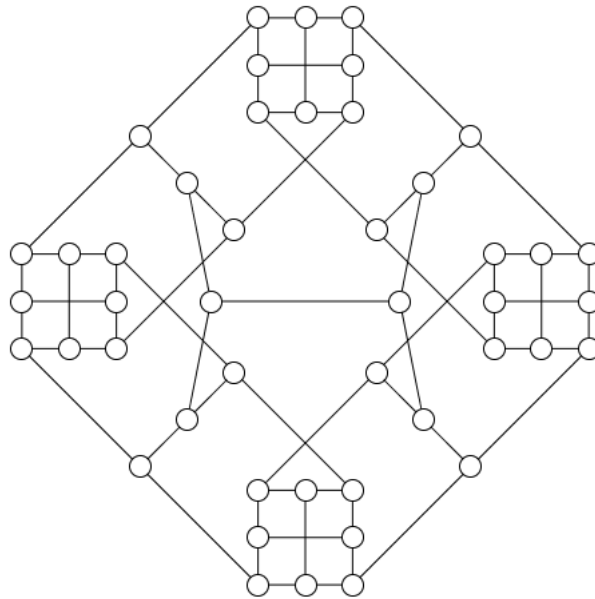
Figure 7: An illustration of Blowup($Prism, C_4$).

assign k colors to the cycles in such a way that if two cycles share an edge, they will always receive different colors. A famous conjecture by Seymour [15] and Szekeres [18] asserts that every bridgeless cubic graph has a CDC, and a conjecture by Celmis [2] strengthen this further by asserting that every such graph in fact has a 5-CDC. Celmins also observed a connection between the perfect matching index and the existence of a 5-CDC where one color class induces a 2-factor.

Theorem 4.1 (Celmins [2]). *Let G be a cubic graph. Then $\tau(G) \leqslant 4$ if and only if G has a 5-CDC where one color class induces a 2-factor in G.*

From Theorem 4.1 and Observation 3.1 we see that Blowup(K_4, C), where C is a 3-cycle in K_4, also has the following interesting property.

Corollary 4.2. *No 2-factor in Blowup(K_4, C) can be part of a 5-CDC.*

By a computer search we also make the following stronger observation.

Observation 4.3. *No 2-factor in Blowup(K_4, C) can be part of a CDC.*

4.1 Circumference

It is known that cubic graphs with oddness 4 or less have cycle double covers [5, 7]. It is also known that the cycle double cover conjecture holds for graphs with sufficiently long cycles. Given a graph G, let the *circumference* of G be the length of the longest cycle. We denote this by circ(G). In [6] it is shown that if a cubic graph G has circ(G) $\geqslant |V(G)| - 9$, then G has a cycle double cover. This result was further improved in [1] to $|V(G)| - 10$.

We have already seen that the snarks constructed in Section 2 can have arbitrary high oddness. We will now show that the same constructions also produces snarks with low circumference compared to the number of vertices.

Given a cubic graph G, let $\rho(G)$ denote the minimum number of vertices that needs to be removed from G in order to obtain a 3-edge colorable graph. It was shown in [16] that this number equals the resistance of the graph.

Theorem 4.4 (Steffen [16])**.** *Let G be a cubic graph. Then $\rho(G) = r_3(G)$.*

This theorem was later generalized in [11] to apply to all graphs with vertices of degee at most 3. A simple consequence of this theorem is a bound on the circumference.

Lemma 4.5. *Let G be a cubic graph. Then $circ(G) \leqslant |V(G)| - r_3(G) + 1$.*

Proof. Assume that G has a cycle C with $|V(C)| > |V(G)| - r_3(G) + 1$. Now consider the graph $G' = G - (V(G) - V(C))$. If $|V(C)|$ is even, then this graph is 3-edge colorable, since we can color the edges of C alternating between colors 1 and 2, and the cords of C with color 3. However this gives $\rho(G) < r_3(G) - 1$ contradicting Theorem 4.4. If, on the other hand, $|V(C)|$ is odd, then obviously $G' - v$ is 3-edge colorable for any vertex v in C. We now have $\rho(G) < r_3(G)$ which once again contradicts Theorem 4.4. $\square$

From Lemma 4.5 and Theorem 2.5 we can see that the graphs obtained from the blowup and semiblowup constructions cannot have too long cycles.

Corollary 4.6. *Given a cubic graph G and a 2-regular subgraph D with disjoint cycles $C_1, C_2, \ldots, C_k$, then*

$$circ(Blowup(G, D)) \leqslant |V(G)| - \sum_{i=1}^{k} \lceil \frac{|V(C_i)|}{2} \rceil + 1$$

and the same holds for $SemiBlowup(G, D)$.

5 Concluding remarks

There are other constructions of snarks that are far from 3-edge colorable that are similar to the blowup and semiblowup constructions using other k-poles than B (any k-pole which is either non-3-edge colorable or where the colors are forced as above works). See e.g. [10], [17] and [14] for more comprehensive studies of this.

It is somewhat unsatisfactory to require the assistance of a computer to show that the blowup of a 3-cycle in K_4 has $\tau = 5$. Given a good characterization of this property, it might be possible to construct an infinite family of such snarks.

References

[1] Gunnar Brinkmann, Jan Goedgebeur, Jonas Hägglund, and Klas Markström. Generation and properties of snarks. *J. Combin. Theory Ser. B*, 103(4):468–488, 2013.

[2] Uldis Alfred Celmins. *On cubic graphs that do not have an edge 3-colouring*. ProQuest LLC, Ann Arbor, MI, 1985. Thesis (Ph.D.)–University of Waterloo (Canada).

[3] Jean-Luc Fouquet and Jean-Marie Vanherpe. On the perfect matching index of bridgeless cubic graphs. `arXiv:0904.1296`.

[4] D. R. Fulkerson. Blocking and anti-blocking pairs of polyhedra. *Math. Programming*, 1:168–194, 1971.

[5] Roland Häggkvist and Sean McGuinness. Double covers of cubic graphs with oddness 4. *J. Combin. Theory Ser. B*, 93(2):251–277, 2005.

[6] Jonas Hägglund and Klas Markström. On stable cycles and cycle double covers of graphs with large circumference. *Discrete Mathematics*, 2011.

[7] Andreas Huck. On cycle-double covers of graphs of small oddness. *Discrete Math.*, 229(1-3):125–165, 2001. Combinatorics, graph theory, algorithms and applications.

[8] Andreas Huck and Martin Kochol. Five cycle double covers of some cubic graphs. *J. Combin. Theory Ser. B*, 64(1):119–125, 1995.

[9] Rufus Isaacs. Infinite families of nontrivial trivalent graphs which are not Tait colorable. *Amer. Math. Monthly*, 82:221–239, 1975.

[10] Martin Kochol. Snarks without small cycles. *J. Combin. Theory Ser. B*, 67(1):34–47, 1996.

[11] Martin Kochol. Three measures of edge-uncolorability. *Discrete Math.*, 311(1):106–108, 2011.

[12] G. Mazzuoccolo. The equivalence of two conjectures of Berge and Fulkerson. *J. Graph Theory*, 68(2):125–128, 2011.

[13] Giuseppe Mazzuoccolo. Covering a 3-graph with perfect matchings. `arXiv:1111.1871`.

[14] Roman Nedela and Martin Škoviera. Decompositions and reductions of snarks. *J. Graph Theory*, 22(3):253–279, 1996.

[15] P. D. Seymour. Sums of circuits. In *Graph theory and related topics (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1977)*, pages 341–355. Academic Press, New York, 1979.

[16] Eckhard Steffen. Classification and characterizations of snarks. *Discrete Math.*, 188(1-3):183–203, 1998.

[17] Eckhard Steffen. Measurements of edge-uncolorability. *Discrete Math.*, 280(1-3):191–214, 2004.

[18] G. Szekeres. Polyhedral decompositions of cubic graphs. *Bull. Austral. Math. Soc.*, 8:367–387, 1973.

[19] W. T. Tutte. A contribution to the theory of chromatic polynomials. *Canadian J. Math.*, 6:80–91, 1954.