Evaluations of Hecke algebra traces at Kazhdan-Lusztig basis elements

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Submitted: Feb 3, 2015; Accepted: Mar 28, 2016; Published: Apr 15, 2016
Mathematics Subject Classifications: 05E10, 20C08, 05E05

Abstract

For irreducible characters $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$, induced sign characters $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$, and induced trivial characters $\{\eta_q^\lambda \,|\, \lambda \vdash n\}$ of the Hecke algebra $H_n(q)$, and Kazhdan-Lusztig basis elements $C'_w(q)$ with w avoiding the patterns 3412 and 4231, we combinatorially interpret the polynomials $\chi_q^\lambda(q^{\ell(w)/2}C'_w(q))$, $\epsilon_q^\lambda(q^{\ell(w)/2}C'_w(q))$, and $\eta_q^\lambda(q^{\ell(w)/2}C'_w(q))$. This provides a new algebraic interpretation of chromatic quasisymmetric functions of Shareshian and Wachs, and a new combinatorial interpretation of special cases of results of Haiman. We prove similar results for other $H_n(q)$-traces, and confirm a formula conjectured by Haiman.

Keywords: Hecke algebra, character, trace, Kazhdan-Lusztig basis, chromatic quasisymmetric function, planar network, P-tableau, pattern avoidance

1 Introduction

The *symmetric group algebra* $\mathbb{Z}\mathfrak{S}_n$ and the *(Iwahori-) Hecke algebra* $H_n(q)$ have similar presentations as algebras over $\mathbb{Z}$ and $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ respectively, with multiplicative identity elements e and T_e, generators $s_1, \ldots, s_{n-1}$ and $T_{s_1}, \ldots, T_{s_{n-1}}$, and relations

$$\begin{array}{rclrcll} s_i^2 &=& e & T_{s_i}^2 &=& (q-1)T_{s_i} + qT_e & \text{for } i = 1, \ldots, n-1, \\ s_i s_j s_i &=& s_j s_i s_j & T_{s_i} T_{s_j} T_{s_i} &=& T_{s_j} T_{s_i} T_{s_j} & \text{for } |i-j| = 1, \\ s_i s_j &=& s_j s_i & T_{s_i} T_{s_j} &=& T_{s_j} T_{s_i} & \text{for } |i-j| \geqslant 2. \end{array}$$

Analogous to the *natural basis* $\mathfrak{S}_n$ of $\mathbb{Z}\mathfrak{S}_n$ is the *natural basis* $\{T_w \,|\, w \in \mathfrak{S}_n\}$ of $H_n(q)$, where we define $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$ whenever $s_{i_1} \cdots s_{i_\ell}$ is a reduced (short as possible) expression for w in $\mathfrak{S}_n$. We call ℓ the *length* of w and write $\ell = \ell(w)$. It is known that $\ell(w)$ is equal to $\textsc{inv}(w)$, the number of *inversions* in the one-line notation $w_1 \cdots w_n$ of w, i.e., the number of pairs (i,j) with $i < j$ and $w_i > w_j$. The specialization of $H_n(q)$ at $q^{\frac{1}{2}} = 1$ is isomorphic to $\mathbb{Z}\mathfrak{S}_n$.

In addition to the natural bases of $\mathbb{Z}\mathfrak{S}_n$ and $H_n(q)$, we have the (signless) *Kazhdan-Lusztig bases* [13] $\{C'_w(1) \,|\, w \in \mathfrak{S}_n\}$, $\{C'_w(q) \,|\, w \in \mathfrak{S}_n\}$, defined in terms of certain *Kazhdan-Lusztig polynomials* $\{P_{v,w}(q) \,|\, v, w \in \mathfrak{S}_n\}$ in $\mathbb{N}[q]$ by

$$C'_w(1) = \sum_{v \leqslant w} P_{v,w}(1) v, \qquad C'_w(q) = q_{e,w}^{-1} \sum_{v \leqslant w} P_{v,w}(q) T_v, \tag{1.1}$$

where $\leqslant$ denotes the Bruhat order on $\mathfrak{S}_n$ and we define $q_{v,w} = q^{\frac{\ell(w)-\ell(v)}{2}}$. We have the identity $P_{v,w}(q) = 1$ when w avoids the patterns 3412 and 4231, i.e., when no subword $w_{i_1} w_{i_2} w_{i_3} w_{i_4}$ of $w_1 \cdots w_n$ consists of letters which appear in the same relative order as 3412 or 4231 [17]. These particular permutations are of interest to algebraic geometers because they correspond to smooth Schubert varieties. (See [3, Ch. 4, Ch. 13].)

Representations of $\mathbb{Z}\mathfrak{S}_n$ and $H_n(q)$ are often studied in terms of linear maps called *characters*. (See [22, Ch. 1] for definitions.) The span of the $\mathfrak{S}_n$-characters is called the space of $\mathfrak{S}_n$*-class functions*, and has dimension equal to the number of integer partitions of n. Three well-studied bases are the irreducible characters $\{\chi^\lambda \,|\, \lambda \vdash n\}$, induced sign characters $\{\epsilon^\lambda \,|\, \lambda \vdash n\}$, and induced trivial characters $\{\eta^\lambda \,|\, \lambda \vdash n\}$, where $\lambda \vdash n$ denotes that λ is a partition of n. Letting $\mathfrak{S}_\lambda$ denote the Young subgroup of $\mathfrak{S}_n$ of type λ, we have

$$\epsilon^\lambda := \mathrm{sgn}\big\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}, \qquad \eta^\lambda := \mathrm{triv}\big\uparrow_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}.$$

The span of the $H_n(q)$-characters, called the space of $H_n(q)$*-traces*, has the same dimension and analogous character bases $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$, $\{\epsilon_q^\lambda \,|\, \lambda \vdash n\}$, $\{\eta_q^\lambda \,|\, \lambda \vdash n\}$, specializing at $q^{\frac{1}{2}} = 1$ to the $\mathfrak{S}_n$-character bases. Each of the two spaces has a fourth basis consisting of *monomial* class functions $\{\phi^\lambda \,|\, \lambda \vdash n\}$ or traces $\{\phi_q^\lambda \,|\, \lambda \vdash n\}$, and a fifth basis consisting of *power sum* class functions $\{\psi^\lambda \,|\, \lambda \vdash n\}$ or traces $\{\psi_q^\lambda \,|\, \lambda \vdash n\}$. These are defined via the inverse Kostka numbers $\{K_{\lambda,\mu}^{-1} \,|\, \lambda, \mu \vdash n\}$ and the numbers $\{L_{\lambda,\mu} \,|\, \lambda, \mu \vdash n\}$ of row-constant Young tableaux of shape λ and content μ by

$$\phi^\lambda := \sum_\mu K_{\lambda,\mu}^{-1} \chi^\mu, \quad \phi_q^\lambda := \sum_\mu K_{\lambda,\mu}^{-1} \chi_q^\mu, \qquad \psi^\lambda := \sum_\mu L_{\lambda,\mu} \phi^\mu, \quad \psi_q^\lambda := \sum_\mu L_{\lambda,\mu} \phi_q^\mu. \tag{1.2}$$

Each of these functions is not a character, but is a difference of two characters. In each space, the five bases are related to one another by the same transition matrices which relate the Schur $\{s_\lambda \,|\, \lambda \vdash n\}$, elementary $\{e_\lambda \,|\, \lambda \vdash n\}$, complete homogeneous $\{h_\lambda \,|\, \lambda \vdash n\}$, monomial $\{m_\lambda \,|\, \lambda \vdash n\}$, and power sum $\{p_\lambda \,|\, \lambda \vdash n\}$ bases of the space Λ_n of homogeneous degree n symmetric functions. It follows from the theory of symmetric functions that basis elements in each space are integer linear combinations of irreducible characters in that space. (For more information on the transition matrices and symmetric functions, see [2, Sec. 2], [29, Ch. 7], respectively.) A correspondence between these class functions and symmetric functions is given by the Frobenius *characteristic map*

$$\mathrm{ch}(\theta) := \frac{1}{n!} \sum_{w \in \mathfrak{S}_n} \theta(w) p_{\mathrm{ctype}(w)},$$

where ctype(w) is the cycle type of w. In particular, we have $\mathrm{ch}(\chi^\lambda) = s_\lambda$, $\mathrm{ch}(\epsilon^\lambda) = e_\lambda$, $\mathrm{ch}(\eta^\lambda) = h_\lambda$, $\mathrm{ch}(\phi^\lambda) = m_\lambda$, $\mathrm{ch}(\psi^\lambda) = p_\lambda$, which explains our names for the fourth and fifth bases. We naturally use the Greek ancestors ϵ, η of e, h in our class function notation, but we prefer not to use the ancestors σ, μ, π of s, m, p. Instead we follow the standard practices of using χ for irreducible characters, while reserving μ for integer partitions, and π for path families in planar networks. We follow [11], [32] in using ϕ, and we use ψ because *psi* begins with p.

For any $\mathfrak{S}_n$-class function θ belonging to the bases above, and for any element z of the natural or Kazhdan-Lusztig basis of $\mathbb{Z}\mathfrak{S}_n$, we have $\theta(z) \in \mathbb{Z}$. This follows from the linearity of θ and the fact that $\chi^\lambda(w)$ can be expressed as the trace of an integer matrix. (See, e.g., [22, Sec. 2.3].) On the other hand, we do not in general have an elementary formula for the integer $\theta(z)$. This incomplete understanding of the $\mathfrak{S}_n$-class functions is unfortunate, since the functions encode much important information about $\mathfrak{S}_n$. For some class functions and basis elements, we may associate sets R, S to the pair (θ, z) to combinatorially interpret the integer $\theta(z)$ as $(-1)^{|S|}|R|$, or simply as $|R|$ if $\theta(z) \in \mathbb{N}$. We summarize results and open problems in the following table.

θ	$\theta(w) \in \mathbb{N}$?	interpretation of $\theta(w)$ as $(-1)^{\lvert S\rvert}\lvert R\rvert$?	$\theta(C'_w(1)) \in \mathbb{N}$?	interpretation of $\theta(C'_w(1))$ as $\lvert R\rvert$ for w avoiding 3412 and 4231?
η^λ	yes	yes	yes	yes
ϵ^λ	no	yes	yes	yes
χ^λ	no	open	yes	yes
ψ^λ	yes	yes	yes	yes
ϕ^λ	no	yes	conj. by Stembridge, Haiman	open

For the above combinatorial interpretations of $\theta(w)$, see [2]. The number $\chi^\lambda(w)$ may be computed by the well-known algorithm of Murnaghan and Nakayama (See, e.g., [29, Ch. 7].) but has no conjectured expression of the type stated above. Interpretations of $\theta(C'_w(1))$ are not known for general $w \in \mathfrak{S}_n$, but nonnegativity follows from work of Haiman [11] and Stembridge [31]. Interpretations of $\eta^\lambda(C'_w(1))$, $\epsilon^\lambda(C'_w(1))$, $\chi^\lambda(C'_w(1))$,

$\psi^\lambda(C'_w(1))$, for w avoiding the patterns 3412, 4231 follow via straightforward arguments from results of the fourth author [25, Thms. 4.3, 5.4] and others, notably Gasharov [7], Karlin-MacGregor [12], Lindström [18], Littlewood [19], Merris-Watkins [20], Stanley-Stembridge [30], [31]. These will be discussed in Section 4. There is no conjectured combinatorial interpretation of $\phi^\lambda(C'_w(1))$, even for w avoiding the patterns 3412 and 4231, but interpretations have been given for particular partitions λ by Stembridge [32], several of the authors [4], and Wolfgang [34]. The problem of interpreting $\theta(C'_w(1))$ when w does not avoid the patterns 3412 and 4231 is open.

Our understanding of $H_n(q)$-traces is even less complete. We know that irreducible $H_n(q)$-characters $\{\chi_q^\lambda \,|\, \lambda \vdash n\}$ satisfy $\chi_q^\lambda(T_w) \in \mathbb{Z}[q]$ for all $w \in \mathfrak{S}_n$, since $\chi_q^\lambda(T_w)$ can be expressed as the trace of a $\mathbb{Z}[q]$ matrix [13]. Thus for any element θ_q of the mentioned $H_n(q)$-trace bases and any element $z \in \mathrm{span}_{\mathbb{Z}[q]}\{T_w \,|\, w \in \mathfrak{S}_n\}$, we have $\theta_q(z) \in \mathbb{Z}[q]$ as well. For instance, elements of a modified Kazhdan-Lusztig basis $\{q_{e,w}C'_w(q) \,|\, w \in \mathfrak{S}_n\}$ belong to this span. On the other hand, we do not have a general elementary formula for the polynomial $\theta_q(z)$. This is unfortunate, since $H_n(q)$-characters are important in the study of $H_n(q)$ and quantum groups. In some cases, we may associate sequences $(S_k)_{k\geqslant 0}$, $(R_k)_{\geqslant 0}$ of sets to the pair (θ_q, z) to combinatorially interpret $\theta_q(z)$ as $\sum_k (-1)^{|S_k|}|R_k|q^k$, or simply as $\sum_k |R_k|q^k$ if $\theta_q(z) \in \mathbb{N}[q]$. We summarize results and open problems in the following table.

θ_q	$\theta_q(T_w) \in \mathbb{N}[q]$?	interpretation of $\theta_q(T_w)$ as $\sum_k (-1)^{\lvert S_k\rvert}\lvert R_k\rvert q^k$?	$\theta_q(q_{e,w}C'_w(q)) \in \mathbb{N}[q]$?	interpretation of $\theta_q(q_{e,w}C'_w(q))$ as $\sum_k \lvert R_k\rvert q^k$ for w avoiding 3412 and 4231?
η_q^λ	no	open	yes	stated in Section 5
ϵ_q^λ	no	open	yes	stated in Section 6
χ_q^λ	no	open	yes	stated in Section 8
ψ_q^λ	no	open	conj. by Haiman	stated in Section 9
ϕ_q^λ	no	open	conj. by Haiman	open

The polynomial $\chi_q^\lambda(T_w)$, and therefore all polynomials $\theta_q(T_w)$ above, may be computed via a q-extension of the Murnaghan-Nakayama algorithm, developed in [14], [15], [21], [33]. However, none of these has a conjectured expression of the type stated above. Interpretations of $\theta_q(q_{e,w}C'_w(q))$ are not known for general $w \in \mathfrak{S}_n$, but results concerning containment in $\mathbb{N}[q]$ follow principally from work of Haiman [11]. In all cases above, coefficients of $\theta_q(q_{e,w}C'_w(q))$ are symmetric about $q_{e,w}$. They are also unimodal for $\theta_q \in \{\eta_q^\lambda, \epsilon_q^\lambda, \chi_q^\lambda\}$ and conjectured to be so for $\theta_q \in \{\psi_q^\lambda, \phi_q^\lambda\}$ [11, Lem. 1.1, Conj. 2.1]. For w avoiding the patterns 3412 and 4231, formulas for $\eta_q^\lambda(q_{e,w}C'_w(q))$, $\epsilon_q^\lambda(q_{e,w}C'_w(q))$, $\chi_q^\lambda(q_{e,w}C'_w(q))$, and $\psi_q^\lambda(q_{e,w}C'_w(q))$ follow from work of Athanasiadis [1], Gasharov [7], Shareshian-Wachs [24], and the authors. There is no conjectured combinatorial interpretation of $\phi_q^\lambda(q_{e,w}C'_w(q))$, even for w avoiding the patterns 3412, 4231, although for particular partitions λ interpretations are given by the authors in Section 10. The problem of combinatorially interpreting $\theta_q(q_{e,w}C'_w(q))$ when w does not avoid the patterns 3412 and 4231 is open.

Another way to understand the evaluations $\theta(w)$ is to define a generating function $\mathrm{Imm}_\theta(x)$ for $\{\theta(w) \mid w \in \mathfrak{S}_n\}$ in the polynomial ring $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$. Similarly, we may define a generating function $\mathrm{Imm}_{\theta_q}(x)$ for $\{\theta_q(T_w) \mid w \in \mathfrak{S}_n\}$ in a certain noncommutative ring $\mathcal{A}_n(q)$. In some cases these generating functions have simple forms. We summarize known results in the following tables.

θ	simple expression for $\mathrm{Imm}_\theta(x)$?
η^λ	yes
ϵ^λ	yes
χ^λ	open
ψ^λ	yes
ϕ^λ	open

θ_q	simple expression for $\mathrm{Imm}_{\theta_q}(x)$?
η_q^λ	yes
ϵ_q^λ	yes
χ_q^λ	open
ψ_q^λ	open
ϕ_q^λ	open

Simple expressions for $\mathrm{Imm}_{\eta^\lambda}(x)$ and $\mathrm{Imm}_{\epsilon^\lambda}(x)$ are due to Littlewood [19] and Merris-Watkins [20], and a simple expression for $\mathrm{Imm}_{\psi^\lambda}(x)$ follows immediately from a standard definition of ψ. An expression for $\mathrm{Imm}_{\chi^\lambda}(x)$ as a coefficient of a generating function in two sets of variables was given by Goulden-Jackson [9]. There is no conjectured simple formula for $\mathrm{Imm}_{\phi^\lambda}(x)$, although a simple formula for particular partitions λ was stated by Stembridge [32]. Simple expressions for $\mathrm{Imm}_{\eta_q^\lambda}(x)$ and $\mathrm{Imm}_{\epsilon_q^\lambda}(x)$ are due to the fourth author and Konvalinka [16], as is a (less simple) expression for $\mathrm{Imm}_{\chi_q^\lambda}(x)$ as a coefficient in a generating function in two sets of variables.

In Section 2 we discuss generating functions $\mathrm{Imm}_\theta(x)$ for $\mathfrak{S}_n$-class functions θ and generating functions $\mathrm{Imm}_{\theta_q}(x)$ for $H_n(q)$-traces θ_q. These generating functions belong to the ring $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ and to a certain q-analog $\mathcal{A}_n(q)$ of $\mathbb{Z}[x_{1,1}, \ldots, x_{n,n}]$ known as the *quantum matrix bialgebra*. In Section 3 we relate these generating functions to structures called *zig-zag networks* [25], which serve as combinatorial interpretations for Kazhdan-Lusztig basis elements indexed by permutations avoiding the patterns 3412 and 4231. In Section 4 we introduce a partial order on paths in zig-zag networks, and show how this poset and results in the literature lead to combinatorial interpretations for the evaluations of $\mathfrak{S}_n$-class functions at the above Kazhdan-Lusztig basis elements of $\mathbb{Z}\mathfrak{S}_n$. For the remainder of the article, we concentrate on combinatorial interpretations of trace evaulations of the form $\theta_q(q_{e,w}C'_w(q))$ where w avoids the patterns 3412 and 4231. In Sections 5 – 6 we interpret $\eta_q^\lambda(q_{e,w}C'_w(q))$ and $\epsilon_q^\lambda(q_{e,w}C'_w(q))$. In Section 7 we recall the relationship between $\mathfrak{S}_n$-class functions and Stanley's chromatic symmetric functions, and prove that $H_n(q)$-traces are similarly related to the Shareshian-Wachs chromatic quasisymmetric functions. In Sections 8 – 9 we use results of Shareshian-Wachs and Athanasiadis to interpret $\chi_q^\lambda(q_{e,w}C'_w(q))$ and $\psi_q^\lambda(q_{e,w}C'_w(q))$. One of our interpretations proves a formula conjectured by Haiman [11, Conj. 4.1]. Finally, in Section 10 we state several results concerning $\phi_q^\lambda(q_{e,w}C'_w(q))$.

2 Generating functions for $\theta(w)$ and $\theta_q(T_w)$ when θ (θ_q) is fixed

For a fixed $\mathfrak{S}_n$-class function θ, we create a generating function for $\{\theta(w) \,|\, w \in \mathfrak{S}_n\}$ by writing a matrix of variables $x = (x_{i,j})_{i,j\in[n]}$ and defining

$$\mathrm{Imm}_\theta(x) := \sum_{w\in\mathfrak{S}_n} \theta(w) x_{1,w_1} \cdots x_{n,w_n} \;\in\; \mathbb{Z}[x] := \mathbb{Z}[x_{1,1}, \ldots, x_{n,n}],$$

where $w_1 \cdots w_n$ is the one-line notation of w. We call this polynomial the *θ-immanant.* The sign character ($w \mapsto (-1)^{\ell(w)}$) immanant and trivial character ($w \mapsto 1$) immanant are the determinant and permanent,

$$\det(x) = \sum_{w\in\mathfrak{S}_n} (-1)^{\ell(w)} x_{1,w_1} \cdots x_{n,w_n}, \qquad \mathrm{per}(x) = \sum_{w\in\mathfrak{S}_n} x_{1,w_1} \cdots x_{n,w_n}.$$

Simple formulas for the ϵ^λ-immanants and η^λ-immanants employ determinants and permanents of square submatrices $x_{I,J}$ of x,

$$x_{I,J} := (x_{i,j})_{i\in I, j\in J}, \qquad I, J \subset [n] := \{1, \ldots, n\}, \qquad |I| = |J|.$$

In particular, we have the Littlewood-Merris-Watkins identities [19], [20]

$$\mathrm{Imm}_{\epsilon^\lambda}(x) = \sum_{(I_1,\ldots,I_r)} \det(x_{I_1,I_1}) \cdots \det(x_{I_r,I_r}), \quad \mathrm{Imm}_{\eta^\lambda}(x) = \sum_{(I_1,\ldots,I_r)} \mathrm{per}(x_{I_1,I_1}) \cdots \mathrm{per}(x_{I_r,I_r}), \tag{2.1}$$

where $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ and the sums are over all sequences of pairwise disjoint subsets of $[n]$ satisfying $|I_j| = \lambda_j$. We will call such a sequence an *ordered set partition* of $[n]$ *of type* λ, and will sometimes write $I \vdash [n]$ and $\ell(\lambda) = r$.

A simple formula for the ψ^λ-immanant involves a sum over all permutations of cycle type λ. Specifically, we have

$$\mathrm{Imm}_{\psi^\lambda}(x) = z_\lambda \sum_{\substack{w \\ \mathrm{cyc}(w)=\lambda}} x_{1,w_1} \cdots x_{n,w_n}, \tag{2.2}$$

where z_λ is the product $1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n} \alpha_1! \cdots \alpha_n!$, and λ has α_i components (or *parts*) equal to i for $i = 1, \ldots, n$. No such simple formulas are known for the χ^λ-immanants or ϕ^λ-immanants in general, although Stembridge gave a formula [32, Thm. 2.8] for $\mathrm{Imm}_{\phi^\lambda}(x)$ when λ is the rectangular partition $(k)^r = (k, \ldots, k)$. In this case we have

$$\mathrm{Imm}_{\phi^{(k)^r}}(x) = \sum_{(I_1,\ldots,I_k)} \det(x_{I_1,I_2}) \det(x_{I_2,I_3}) \cdots \det(x_{I_k,I_1}), \tag{2.3}$$

where the sum is over all ordered set partitions of $[n] = [kr]$ of type $(r)^k$. (See [9] for an expression for $\mathrm{Imm}_{\chi^\lambda}(x)$ as a coefficient of a generating function in two sets of variables.)

For a fixed $H_n(q)$-trace θ_q, we create a generating function for $\{\theta_q(T_w) \,|\, w \in \mathfrak{S}_n\}$ as before, except that we interpret polynomials in $x = (x_{i,j})$ as elements of the *quantum* *matrix bialgebra* $\mathcal{A}_n(q)$, the noncommutative $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-algebra generated by n^2 variables $x_{1,1}, \ldots, x_{n,n}$, subject to the relations

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{i,\ell}, \qquad & x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k} \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{j,k} \qquad & x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})x_{i,\ell}x_{j,k}, \end{aligned} \tag{2.4}$$

for all indices $1 \leqslant i < j \leqslant n$ and $1 \leqslant k < \ell \leqslant n$. As a $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module, $\mathcal{A}_n(q)$ has a natural basis of monomials $x_{\ell_1,m_1} \cdots x_{\ell_r,m_r}$ in which index pairs appear in lexicographic order. The relations (2.4) allow one to express other monomials in terms of this natural basis.

As a generating function for $\{\theta_q(T_w) \,|\, w \in \mathfrak{S}_n\}$, we define

$$\mathrm{Imm}_{\theta_q}(x) := \sum_{w \in \mathfrak{S}_n} \theta_q(T_w) q_{e,w}^{-1} x_{1,w_1} \cdots x_{n,w_n}$$

in $\mathcal{A}_n(q)$, and call this the θ_q-*immanant*. The $H_n(q)$ sign character ($T_w \mapsto (-1)^{\ell(w)}$) immanant and trivial character ($T_w \mapsto q^{\ell(w)}$) immanant are called the *quantum determinant* and *quantum permanent*,

$$\det_q(x) = \sum_{w \in \mathfrak{S}_n} (-q^{\frac{-1}{2}})^{\ell(w)} x_{1,w_1} \cdots x_{n,w_n}, \qquad \mathrm{per}_q(x) = \sum_{w \in \mathfrak{S}_n} (q^{\frac{1}{2}})^{\ell(w)} x_{1,w_1} \cdots x_{n,w_n}.$$

Specializing $\mathcal{A}_n(q)$ at $q^{\frac{1}{2}} = 1$ gives the commutative polynomial ring $\mathbb{Z}[x]$, with elements $\det_q(x)$ and $\mathrm{per}_q(x)$ specializing to the classical determinant $\det(x)$ and permanent $\mathrm{per}(x)$. Simple formulas for the ϵ_q^λ-immanants and η_q^λ-immanants employ quantum determinants and quantum permanents of submatrices of x. In particular, the fourth author and Konvalinka [16, Thm. 5.4] proved quantum analogs of the Littlewood-Merris-Watkins identities (2.1),

$$\mathrm{Imm}_{\epsilon_q^\lambda}(x) = \sum_{(I_1,\ldots,I_r)} \det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r}), \quad \mathrm{Imm}_{\eta_q^\lambda}(x) = \sum_{(I_1,\ldots,I_r)} \mathrm{per}_q(x_{I_1,I_1}) \cdots \mathrm{per}_q(x_{I_r,I_r}), \tag{2.5}$$

where the sums are as in (2.1). See [16] for an expression for the χ_q^λ-immanant as a coefficient of a generating function in two sets of variables. No formulas are known for the ψ_q^λ- or ϕ_q^λ- immanants. It would be interesting to state a q-analog of (2.3) for rectangular partitions $\lambda = (k)^r$.

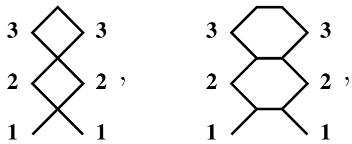
3 Planar networks and path matrices

Call a directed planar graph G a *planar network of order n* if it is acyclic and may be embedded in a disc with $2n$ boundary vertices labeled clockwise as *source* $1, \ldots,$ *source* n (with indegrees of 0) and *sink* $n, \ldots,$ *sink* 1 (with outdegrees of 0). In figures, we will draw sources on the left and sinks on the right, implicitly labeled $1, \ldots, n$ from bottom to top. Edges will be implicitly oriented from left to right. Given a planar network G, define the *path matrix* $B = B(G) = (b_{i,j})$ of G by

$$b_{i,j} = \text{ number of paths in } G \text{ from source } i \text{ to sink } j. \tag{3.1}$$

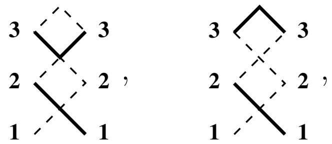
The path matrix of any planar network is totally nonnegative, i.e., each square submatrix has nonnegative determinant. Specifically, for sets $I = \{i_1, \dots, i_k\}$, $J = \{j_1, \dots, j_k\} \subset [n]$, and the corresponding submatrix $B_{I,J} := (b_{i,j})_{i \in I, j \in J}$, we have that $\det(B_{I,J})$ is equal to the number of families $(\pi_1, \dots, \pi_k)$ of mutually nonintersecting paths from sources $i_1, \dots, i_k$ (respectively) to sinks $j_1, \dots, j_k$ (respectively). This fact is known as *Lindström's Lemma* [18]. (See also [12].) We will call two planar networks G_1, G_2 *isomorphic* and will write $G_1 \cong G_2$ if $B(G_1) = B(G_2)$.

For example, consider two isomorphic planar networks, their common path matrix B and its submatrix $B_{23,13}$:



$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \qquad B_{23,13} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

We can interpret $\det(B_{23,13}) = 2$ as counting the two path families

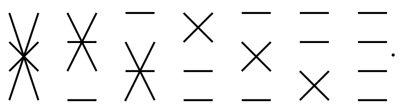


from sources $\{2, 3\}$ to sinks $\{1, 3\}$ in the first network.

An easy fact about planar networks is the following.

Observation 3.1. *Let G be a planar network of order n and assume that for some indices $i, i', j, j' \in [n]$, G contains a path π_i from source i to sink j and a path $\pi_{i'}$ from source i' to sink j'. If these two paths intersect, then G also contains a path from source i to sink j' and a path from source i' to sink j. If $i' < i$ and $j < j'$ then the paths π_i and $\pi_{i'}$ cross.*

For $[a, b]$ a subinterval of $[n]$, let $G_{[a,b]}$ be the planar network consisting of $a - 1$ horizontal edges from sources $1, \dots, a - 1$ to corresponding sinks, a "star" of $b - a + 1$ edges from sources $a, \dots, b$ to an intermediate vertex and $b - a + 1$ more edges from this vertex to sinks $a, \dots, b$, and $n - b$ more horizontal edges from sources $b + 1, \dots, n$ to corresponding sinks. For $n = 4$, there are seven such networks: $G_{[1,4]}$, $G_{[2,4]}$, $G_{[1,3]}$, $G_{[3,4]}$, $G_{[2,3]}$, $G_{[1,2]}$, $G_{[1,1]} = \cdots = G_{[4,4]}$, respectively,



Given planar networks G, H of order n, in which all sources have outdegree 1 and all sinks have indegree 1, define $G \circ H$ to be the concatenation of G and H, where for

$i = 1, \ldots, n$, sink i of G is dropped, source i of H is dropped, and the unique edge in G from vertex x to sink i and the unique edge in H from source i to vertex y are merged to form a single edge from x to y in $G \circ H$. Note that for star networks $G_{[c_1,d_1]}$, $G_{[c_2,d_2]}$ indexed by nonintersecting intervals, the two concatenations $G_{[c_1,d_1]} \circ G_{[c_2,d_2]}$ and $G_{[c_2,d_2]} \circ G_{[c_1,d_1]}$ are isomorphic.

We will be interested in concatenations

$$G = G_{[c_1,d_1]} \circ \cdots \circ G_{[c_t,d_t]} \tag{3.2}$$

such that

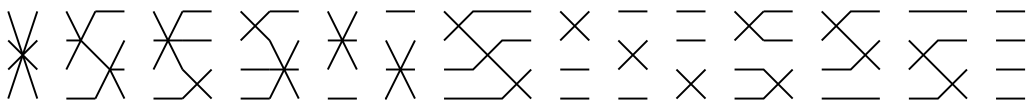
1. the sequence $([c_1, d_1], \ldots, [c_t, d_t])$ consists of t distinct, pairwise nonnesting intervals,
2. for $i < j < k$, if $[c_i, d_i] \cap [c_j, d_j] \neq \emptyset$ and $[c_j, d_j] \cap [c_k, d_k] \neq \emptyset$, then we have $c_i < c_j < c_k$ (and $d_i < d_j < d_k$) or $c_i > c_j > c_k$ (and $d_i > d_j > d_k$).

We define a relation $\lessdot$ on the set of intervals appearing in the concatenation (3.2) by declaring $[c_i, d_i] \lessdot [c_k, d_k]$ if

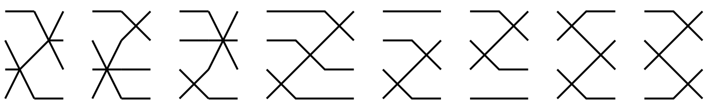
1. $i < k$,
2. $[c_i, d_i] \cap [c_k, d_k] \neq \emptyset$,
3. $[c_i, d_i] \cap [c_j, d_j] \cap [c_k, d_k] = \emptyset$ for $j = i + 1, \ldots, k - 1$.

This is the covering relation of a partial order $\preceq$.

For each planar network G of the form (3.2) satisfying the conditions following (3.2), we define a related planar network F by considering each covering pair $[c_i, d_i] \lessdot [c_j, d_j]$ with $|[c_i, d_i] \cap [c_j, d_j]| = k$, and deleting all but one of the k paths from the central vertex of $G_{[c_i,d_i]}$ to the central vertex of $G_{[c_j,d_j]}$. Following [25], we call the resulting network F a *zig-zag network*. In the special case that we have $c_1 > \cdots > c_t$, we call F a *descending star network*. The descending star networks (up to isomorphism) of order 4 are


. (3.3)

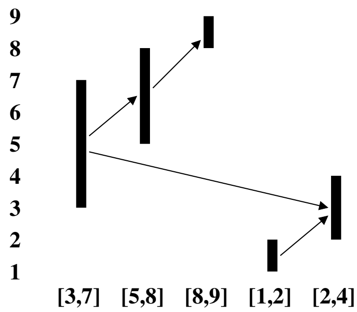
The zig-zag networks of order 4 which are not descending star networks are


. (3.4)

It is easy to see that if F is a zig-zag network of order n, then there is at most one interval in the concatenation (3.2) containing n, and this interval must be maximal or minimal (or both) in the partial order $\preceq$.

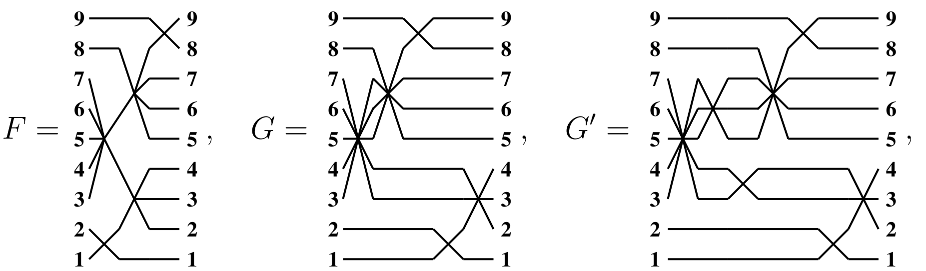
It was shown in [25, Thm. 3.5, Lem. 5.3] that zig-zag networks of order n correspond bijectively to 3412-avoiding, 4231-avoiding permutations in $\mathfrak{S}_n$. To summarize this bijection, we let F be the zig-zag network obtained from the concatenation G in (3.2), and construct another concatenation of star networks as follows. For $i = 1, \ldots, t-1$, if the interval $[c_i, d_i]$ is covered by $[c_j, d_j]$ in the order $\preceq$ and if $|[c_i, d_i] \cap [c_j, d_j]| > 1$, then insert $G_{[c_i,d_i]\cap[c_j,d_j]}$ immediately after $G_{[c_i,d_i]}$ in the current concatenation. (If $[c_i, d_i]$ is also covered by a second interval $[c_k, d_k]$, then $G_{[c_i,d_i]\cap[c_k,d_k]}$ may be inserted before or after $G_{[c_i,d_i]\cap[c_j,d_j]}$.) Call the resulting augmented network G'. Now visually follow paths from sources to sinks in G', passing "straight" through each star, to create a 3412-avoiding, 4231-avoiding permutation $w \in \mathfrak{S}_n$. See [25, Sec. 3] for a description of the inverse of this bijection, which maps w to G'. We will let F_w and G'_w denote the zig-zag network and augmented star network corresponding to a fixed 3412-avoiding, 4231-avoiding permutation w, and we will let $w(F)$ denote the 3412-avoiding, 4231-avoiding permutation corresponding to a fixed zig-zag network F.

For example, suppose that F is the zig-zag network obtained from the concatenation $G = G_{[3,7]} \circ G_{[5,8]} \circ G_{[8,9]} \circ G_{[1,2]} \circ G_{[2,4]}$ of star networks of order 9. Drawing the poset $\preceq$ on these intervals from left to right, we have



.

Since the only covering pairs which intersect at more than an endpoint are $[3,7] \prec\!\!\cdot\; [5,8]$ and $[3,7] \prec\!\!\cdot\; [2,4]$, we construct G' by inserting $G_{[3,7]\cap[5,8]} = G_{[5,7]}$ and $G_{[3,7]\cap[2,4]} = G_{[3,4]}$ after $G_{[3,7]}$. Thus we have $G' = G_{[3,7]} \circ G_{[5,7]} \circ G_{[3,4]} \circ G_{[5,8]} \circ G_{[8,9]} \circ G_{[1,2]} \circ G_{[2,4]}$, and we obtain the 3412-avoiding, 4231-avoiding permutation $w = w(F) = 419763258$:



$w = \begin{pmatrix} 123456789 \\ 419763258 \end{pmatrix}.$

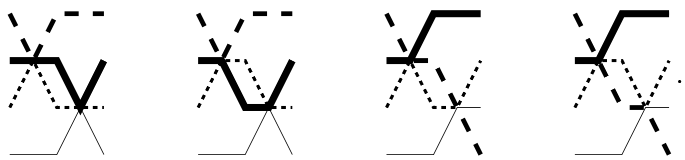
Note that we have $G_{[3,4]} \circ G_{[5,7]} \cong G_{[5,7]} \circ G_{[3,4]}$, since the intervals $[3,4]$, $[5,7]$ do not overlap. In general, we have the following.

Observation 3.2. *If $[c_{i_1}, d_{i_1}], \ldots, [c_{i_t}, d_{i_t}]$ is a linear extension of the poset $\preceq$ defined in terms of the concatenation (3.2), then we have*

$$G_{[c_1,d_1]} \circ \cdots \circ G_{[c_t,d_t]} \cong G_{[c_{i_1},d_{i_1}]} \circ \cdots \circ G_{[c_{i_t},d_{i_t}]},$$

and the corresponding zig-zag networks are isomorphic as well.

Call a sequence $\pi = (\pi_1, \dots, \pi_n)$ of source-to-sink paths in a planar network G of order n a *path family.* We will always assume that path π_i begins at source i. If for some $w \in \mathfrak{S}_n$ with one-line notation $w_1 \cdots w_n$, each component path π_i terminates at sink w_i, we will say that π has *type* w and we will write $\mathrm{type}(\pi) = w$. If the union of the paths of π is equal to G, we will say that π *covers* G. For example, the planar network $G_{[2,4]} \circ G_{[1,3]}$ can be covered by many different path families, including two of type e, and two of type 2341:



(3.5)

The result [25, Lem. 5.3] states that the path families covering a zig-zag network F_w correspond bijectively to elements of a principal order ideal in the Bruhat order:

Theorem 3.3. *Fix $v, w \in \mathfrak{S}_n$ with w avoiding the patterns* 3412 *and* 4231*. There is a unique path family of type v covering F_w if and only if we have $v \leqslant w$. Otherwise, there is no such path family.*

Thus w is the unique Bruhat-maximal permutation for which some path family of type w covers F_w. It follows also that there is exactly one path in F_w from any source i to the corresponding sink i, and at most one path from source i to sink $j \neq i$. Whether or not such a path exists may be determined by the intervals in the corresponding concatenation of star networks and the partial order $\preceq$.

Observation 3.4. *Let F be a zig-zag network which corresponds to the concatenation (3.2). There exists a path in F from source i to sink j if and only if we have one of the following.*

1. *$i, j \in [c_k, d_k]$ for some k.*
2. *$i \in [c_k, d_k]$, $j \in [c_\ell, d_\ell]$ and $[c_k, d_k] \prec [c_\ell, d_\ell]$ for some k, ℓ.*

It is easy to see that if F is a descending star network with sources i and j belonging to the same connected component, then the second condition in Observation 3.4 is equivalent to the inequality $i > j$. This fact allows us to refine Observation 3.1 slightly.

Lemma 3.5. *Let π_{ℓ_1}, π_{ℓ_2} be paths in a descending star network F from sources $\ell_1 < \ell_2$ to sinks m_1, m_2, respectively. Then the paths π_{ℓ_1} and π_{ℓ_2} intersect if and only if there exists a path in F from source ℓ_1 to sink m_2.*

Proof. Assume that F has order n and corresponds to the concatenation (3.2) of star networks. Let $x_1, \dots, x_t$ be the vertices in F corresponding to the central vertices of the t star networks in (3.2), and for any index $j \in [n]$, let $f(j)$ and $g(j)$ denote the indices of the first and last intervals, respectively, in $[c_1, d_1], \dots, [c_t, d_t]$ to contain j. Then for any indices $\ell, m \in [n]$, the unique path in F from source ℓ to sink m contains the vertices $x_{f(\ell)}, \dots, x_{g(m)}$.

It is easy to see that if the intersection of π_{ℓ_1} and π_{ℓ_2} is nonempty, then there is a path in F from source ℓ_1 to sink m_2. Now suppose that there is a path in F from source ℓ_1 to sink m_2. By Observation 3.4, we have $m_2 < \ell_1$ or we have an interval $[c_k, d_k]$ containing both ℓ_1 and m_2. Either case implies that we have $f(\ell_2) \leqslant f(\ell_1) \leqslant g(m_2)$, and the paths π_{ℓ_1} and π_{ℓ_2} share the vertex $x_{f(\ell_1)}$. □

The subset of zig-zag networks which are descending star networks can be characterized using pattern avoidance.

Theorem 3.6. *Let $v \in \mathfrak{S}_n$ avoid the patterns* 3412 *and* 4231*. Then v avoids the pattern* 312 *if and only if F_v is a descending star network.*

Proof. Let G be the concatenation (3.2) of star networks which leads to F_v, and define G' as in the example above.

Suppose first that F_v is not a descending star network. Then in the concatenation (3.2) there exists an index i such that the interval $[c_i, d_i]$ is minimal in the poset $\preceq$, and an index $j > i$ such that $c_i < c_j$ and $[c_i, d_i] \prec\!\!\cdot\; [c_j, d_j]$. Considering the relationship between G' and v, one sees that we have $v_{d_i} < c_j$ and that for some index $\ell \geqslant d_j$ we have $v_\ell = c_j$. On the other hand, since G' is constructed by inserting $G_{[c_j,d_i]}$ between $G_{[c_i,d_i]}$ and $G_{[c_j,d_j]}$, we have that $v_{c_i} \geqslant d_j$. Thus the indices $c_i < d_i < \ell$ satisfy $v_{d_i} < v_\ell < v_{c_i}$ and v does not avoid the pattern 312.

Now suppose that F_v is a descending star network. We claim that if F_v arises from the concatenation (3.2), then v avoids the pattern 312 and satisfies $v_{c_1} > \cdots > v_{d_1}$. By inspection of (3.3), this is true for descending star networks of orders $1-4$. Now assume this to be true for each descending star network F_w of order $1, \ldots, n-1$, and let F_v be a descending star network of order n. If in the first interval of (3.2) we have $d_1 < n$, then $v_n = n$ and $F_{v_1 \cdots v_{n-1}}$ is a descending star network of order $n-1$. Thus F_v has the claimed properties. If in the first interval of (3.2) we have $d_1 = n$, then consider the descending star network F_w arising from the concatenation $G_{[c_2,d_2]} \circ \cdots \circ G_{[c_t,d_t]}$. Then $v = s_{[c_1,d_1]} s_{[c_1,d_2]} w$, where $s_{[a,b]}$ is the unique Bruhat-maximal permutation for which a path family of this type covers the star network $G_{[a,b]}$. By the above argument, w avoids the pattern 312, and satisfies $w_{c_2} > \cdots > w_{d_2}$, and $w_i = i$ for $i = d_2 + 1, \ldots, n$. It follows that the letters in positions $c_1, \ldots, n$ of $s_{[c_1,d_2]} w$ form an increasing sequence, and that v satisfies $v_{c_1} > \cdots > v_n$. Thus the subword $v_1 \cdots v_{c_1 - 1} = w_1 \cdots w_{c_1-1}$ avoids the pattern 312, as does the subword $v_{c_1} \cdots v_n$. If v contains any subword $z_3 z_1 z_2$ that matches the pattern 312, then $z_3 z_1$ must be a subword of $v_1 \cdots v_{c_1-1}$ while z_2 is a letter of the decreasing word $v_{c_1} \cdots v_n$. But in this case, $z_3 z_1$ is a subword of $w_1 \cdots w_{c_1-1}$ while z_2 is a letter of $w_{c_1} \cdots w_n$, which contradicts our assumption that w avoids the pattern 312. □

It follows that there are $\frac{1}{n+1}\binom{2n}{n}$ descending star networks of order n, since there are this many 312-avoiding permutations in $\mathfrak{S}_n$, all of which avoid the patterns 3412 and 4231. Also related to pattern avoidance are the sizes of the stars in the concatenation (3.2).

Theorem 3.7. *Let $w \in \mathfrak{S}_n$ avoid the patterns* 3412 *and* 4231*. Then $w_1 \cdots w_n$ contains a decreasing subsequence of size k if and only if some interval in the concatenation (3.2) corresponding to F_w has cardinality at least k.*

Proof. Let $w_{i_1}, \ldots, w_{i_k}$ be a decreasing subsequence of w. Then there is a path in F_w from source i_j to sink w_{i_j} for $j = 1, \ldots, k$. These paths pairwise intersect, since $i_r < i_s$ if and only if $w_{i_r} > w_{i_s}$. Since F_w is acyclic, these paths must all intersect at a single vertex. Such a vertex must correspond to the central vertex of a star network indexed by an interval of cardinality at least k in the concatenation (3.2).

The converse is clearly true if we have $t = 1$ in (3.2). Suppose that the converse holds for each zig-zag network corresponding to a concatenation of $t - 1$ star networks. Let F_w correspond to a concatenation (3.2) of t star networks, and let F_v correspond to the concatenation $G_{[c_1,d_1]} \circ \cdots \circ G_{[c_{t-1},d_{t-1}]}$. Suppose that some interval $[c_i, d_i]$, $1 \leqslant i \leqslant t$, has cardinality at least k. If $i \leqslant t - 1$, then v contains a decreasing subsequence of size k. By [25, Cor. 3.7], there is a reduced expression for w which consists of a reduced expression for v, followed by some reduced expression $s_{i_1} \cdots s_{i_k}$ for the permutation $v^{-1}w$. It is well known that each permutation in the sequence $(v, vs_{i_1}, vs_{i_1}s_{i_2}, \ldots, w)$ preserves all inversions of the previous permutation and introduces one more. It follows that w also has a decreasing subsequence of length k. If $i = t$, then apply the above argument to w^{-1}, which corresponds to the concatenation $G_{[c_t,d_t]} \circ \cdots \circ G_{[c_1,d_1]}$. It is well known that w has a decreasing subsequence of length k if and only if w^{-1} does. □

We may use path matrices of zig-zag networks to evaluate $\mathfrak{S}_n$-class functions at Kazhdan-Lusztig basis elements $\{C'_w(1) \,|\, w$ avoids the patterns 3412 and 4231$\}$. Specifically, if B is the path matrix of F_w, then by [25, Sec. 4, Thm. 5.4] we have

$$\theta(C'_w(1)) = \mathrm{Imm}_\theta(B). \tag{3.6}$$

This fact is a crucial ingredient in the proofs of Theorems 4.6, 4.7, which interpret the evaluations of certain $\mathfrak{S}_n$-class functions at Kazhdan-Lusztig basis elements of $\mathbb{Z}\mathfrak{S}_n$.

In order to interpret the evaluations of $H_n(q)$-traces at Kazhdan-Lusztig basis elements of $H_n(q)$, we will prove a q-extension of Equation (3.6) in Proposition 3.8. Namely, we will show that path matrices can also be used to evaluate $H_n(q)$-traces at the (modified) Kazhdan-Lusztig basis elements $\{q_{e,w}C'_w(q) \,|\, w$ avoids the patterns 3412 and 4231$\}$. A bit of care is required though: the evaluation $\mathrm{Imm}_{\theta_q}(B)$ does not make sense because the substitution $x_{i,j} \mapsto b_{i,j}$ does not respect the relations (2.4) and therefore does not give a well-defined map from $\mathcal{A}_n(q)$ to $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$. Thus we define a $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-linear map for each $n \times n$ integer matrix B by

$$\begin{aligned} \sigma_B : \mathcal{A}_n(q) &\to \mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}] \\ x_{1,v_1} \cdots x_{n,v_n} &\mapsto q_{e,v} b_{1,v_1} \cdots b_{n,v_n}. \end{aligned}$$

Proposition 3.8. *Let θ_q be an $H_n(q)$-trace and let $w \in \mathfrak{S}_n$ avoid the patterns* 3412 *and* 4231*. Then the path matrix B of F_w satisfies*

$$\theta_q(q_{e,w}C'_w(q)) = \sigma_B(\mathrm{Imm}_{\theta_q}(x)).$$

Proof. The right-hand side is equal to

$$\sigma_B\Big(\sum_{v\in\mathfrak{S}_n}\theta_q(T_v)q_{e,v}^{-1}x_{1,v_1}\cdots x_{n,v_n}\Big)=\sum_{v\in\mathfrak{S}_n}\theta_q(T_v)b_{1,v_1}\cdots b_{n,v_n}.$$

By Theorem 3.3 ([25, Lem. 5.3]), the product $b_{1,v_1}\cdots b_{n,v_n}$ is 1 when $v\leqslant w$ and is 0 otherwise. Thus the above expression is equal to

$$\sum_{v\leqslant w}\theta_q(T_v)=\theta_q\Big(\sum_{v\leqslant w}T_v\Big).$$

Since w avoids the patterns 3412 and 4231, [17] implies that the Kazhdan-Lusztig polynomials $\{P_{v,w}(q)\,|\,v\leqslant w\}$ are identically 1. (See also [3, Ch. 6].) Comparing to (1.1), we see that the parenthesized sum is equal to $q_{e,w}C'_w(q)$. □

Stated from another point of view, Proposition 3.8 asserts that for w avoiding the patterns 3412 and 4231, the zig-zag network F_w combinatorially encodes the modified Kazhdan-Lusztig basis element $q_{e,w}C'_w(q)$ in the sense that

$$q_{e,w}C'_w(q)=\sum_{\pi}T_{\mathrm{type}(\pi)},$$

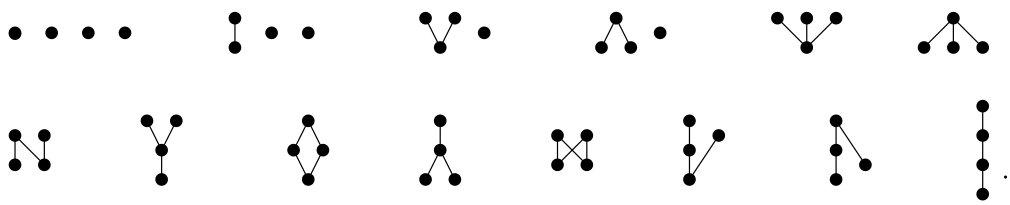
where the sum is over all path families which cover F_w.

4 Path posets, planar network tableaux and interpretation of $\mathfrak{S}_n$-class function evaluations

4.1 Path posets

In a planar network G of order n, the source-to-sink paths have a natural partial order $Q=Q(G)$. Given paths π_i, ρ_j, originating at sources i, j, respectively, we define $\pi_i<_Q\rho_j$ if $i<j$ and π_i and ρ_j do not intersect. Let $P(G)$ be the subposet of $Q(G)$ induced by paths whose source and sink indices are equal. For each zig-zag network F_w, the poset $P(F_w)$ has exactly n elements: there is exactly one path from source i to sink i, for $i=1,\dots,n$.

The posets $P(G)$ corresponding to the descending star networks G in (3.3) are



(4.1)

These are precisely the $(\mathbf{3}+\mathbf{1})$-free, $(\mathbf{2}+\mathbf{2})$-free posets on four elements, where $(\mathbf{a}+\mathbf{b})$-*free* means that no induced subposet is a disjoint union of an a-element chain and a b-element chain. Such posets are also called *unit interval orders*. It is known that unit interval orders may be naturally labeled by the numbers $[n]$ so that $i <_P j$ implies $i < j$ as integers and so that the conditions $i < j < k$ and i incomparable to k in P imply that $\{i, j, k\}$ is an antichain in P [5, Sec. 3]. (See [26, Prop. 2.4] for the algorithm.) It is known that there are $\frac{1}{n+1}\binom{2n}{n}$ unit interval orders on n elements [5, Sec. 4].

Theorem 4.1. *The posets* $\{P(F_v) \mid v \in \mathfrak{S}_n$ *avoids* $312\}$ *are precisely the unit interval orders on* n *elements.*

Proof. Let $\mathcal{P}$ and $\mathcal{U}$ be the two sets of posets in the theorem. We define a map $\zeta : \mathcal{U} \to \mathcal{P}$ as follows. By the above fact on incomparability, we may naturally label $P \in \mathcal{U}$ by $1, \ldots, n$ so that its maximal antichains are $[a_1, b_1], \ldots, [a_t, b_t]$, with $a_1 > \cdots > a_t$. Now define F to be the descending star network corresponding to the concatenation $G_{[a_1,b_1]} \circ \cdots \circ G_{[a_t,b_t]}$, and let $(\pi_1, \ldots, \pi_n)$ be the unique path family of type e that covers F. Let $\zeta(P)$ be the poset $P(F)$.

We claim that for each poset $P \in \mathcal{U}$, the map $i \mapsto \pi_i$ is an isomorphism of P and $\zeta(P)$. To see this, note that $i <_P j$ if and only if $i < j$ as integers and i, j belong to no common antichain in P, i.e., if and only if $i < j$ and i, j belong to no common interval $[a_k, b_k]$ defining the concatenation $G_{[a_1,b_1]} \circ \cdots \circ G_{[a_t,b_t]}$ of star networks. But this is true if and only if we have $\pi_i <_{\zeta(P)} \pi_j$. It follows that the $\frac{1}{n+1}\binom{2n}{n}$ images $\{\zeta(P) \mid P \in \mathcal{U}\}$ are pairwise nonisomorphic. On the other hand, by Theorem 3.6, we have $|\mathcal{P}| \leqslant \frac{1}{n+1}\binom{2n}{n}$. The claim follows. $\square$

While a star network is covered by a unique path family of type e, a concatenation of star networks need not be. Nevertheless, such a concatenation is covered by a unique noncrossing path family of type e: the concatenation of the unique path families of type e that cover the component star networks. Given a concatenation G of star networks, let $\hat{P}(G)$ be the n-element subposet of $P(G)$ induced by the paths in the unique noncrossing path family of type e which covers G. For example, the first two figures in (3.5) show the two path families of type e which cover $G_{[2,4]} \circ G_{[1,3]}$. Call these $(\pi_1, \pi_2, \pi_3, \pi_4)$ and $(\pi_1, \pi_2', \pi_3', \pi_4)$, respectively. Ordering all six of these source-i-to-sink-i paths, or only the mutually noncrossing paths $\{\pi_1, \pi_2, \pi_3, \pi_4\}$, we form the posets

$$P(G_{[2,4]} \circ G_{[1,3]}) = \begin{array}{c}\pi_4\\ |\\ \pi_1\end{array} \quad \pi_2 \;\; \pi_3 \;\; \pi_2' \;\; \pi_3', \qquad \hat{P}(G_{[2,4]} \circ G_{[1,3]}) = \begin{array}{c}\pi_4\\ |\\ \pi_1\end{array} \quad \pi_2 \;\; \pi_3,$$

respectively.

It is easy to show that for a concatenation G of star networks, the poset $\hat{P}(G)$ does not depend upon the ordering of the factors of G.

Proposition 4.2. *Let* $[a_1, b_1], \ldots, [a_t, b_t]$ *be subintervals of* $[n]$*. Then for any permutation* $u \in \mathfrak{S}_t$*, we have* $\hat{P}(G_{[a_1,b_1]} \circ \cdots \circ G_{[a_t,b_t]}) \cong \hat{P}(G_{[a_{u_1},b_{u_1}]} \circ \cdots \circ G_{[a_{u_t},b_{u_t}]})$.

Proof. Define $G = G_{[a_1,b_1]} \circ \cdots \circ G_{[a_t,b_t]}$, $H = G_{[a_{u_1},b_{u_1}]} \circ \cdots \circ G_{[a_{u_t},b_{u_t}]}$, and let $(\rho_1, \ldots, \rho_n)$, $(\tau_1, \ldots, \tau_n)$ be the unique noncrossing path families of type e covering G, H, respectively.

For $i = 1, \dots, t$, let $(\pi_1^{(i)}, \dots, \pi_n^{(i)})$ be the unique noncrossing path family of type e covering $G_{[a_i,b_i]}$. For $j, k \in [n]$, the definition of $\hat{P}(G)$ implies we have $\rho_j \leqslant_{\hat{P}(G)} \rho_k$ if and only if $\pi_j^{(i)} \leqslant_{\hat{P}(G_{[a_i,b_i]})} \pi_k^{(i)}$ for $i = 1, \dots, t$. But this condition is also equivalent to $\tau_j \leqslant_{\hat{P}(H)} \tau_k$. $\square$

It is also easy to show that the path poset of a zig-zag network F may be obtained directly from the concatenation of star networks which lead to F as in Section 3.

Proposition 4.3. *Let F be a zig-zag network constructed from a concatenation G of star networks as after (3.2). Then $P(F)$ is isomorphic to $\hat{P}(G)$.*

Proof. Let $(\rho_1, \cdots, \rho_n)$ and $(\tau_1, \dots, \tau_n)$ be the unique noncrossing path families of type e in G and F, respectively, and let $x_1, \dots, x_t$ be the vertices of F which correspond to the central vertices of the t star networks in (3.2). For $i < j$, we have $\rho_i <_{\hat{P}(G)} \rho_j$ if and only if ρ_i, ρ_j share none of the central vertices of the t star networks. But this condition holds if and only if τ_i, τ_j share none of the vertices $x_1, \dots, x_t$. $\square$

By Propositions 4.2 and 4.3, it is possible to have an isomorphism of path posets for nonisomorphic zig-zag networks. Define an equivalence relation $\sim$ on 3412-avoiding, 4231-avoiding permutations by

$$v \sim w \quad \text{if and only if} \quad P(F_v) \cong P(F_w). \tag{4.2}$$

For example, it is easy to see the equivalence of the four permutations corresponding to the seventh descending star network in (3.3) and the fourth, seventh, and eighth zig-zag networks in (3.4): in each case the path poset is isomorphic to the seventh unit interval order in (4.1). It is also easy to see that we have $w \sim w^{-1}$ for w avoiding the patterns 3412 and 4231: the networks F_w and $F_{w^{-1}}$ differ only by reflection in a vertical line.

Theorem 4.4. *Each equivalence class of the relation $\sim$ (4.2) contains exactly one representative which avoids the pattern* 312.

Proof. Fix $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231, let $G = G_{[c_1,d_1]} \circ \cdots \cdots \circ G_{[c_t,d_t]}$ be the concatenation of star networks which leads to the zig-zag network F_w as after (3.4), and let $u \in \mathfrak{S}_t$ be the unique permutation satisfying $c_{u_1} > \cdots > c_{u_t}$. Then the concatenation $G_{[c_{u_1},d_{u_1}]} \circ \cdots \cdots \circ G_{[c_{u_t},d_{u_t}]}$ leads to a descending star network F_v. By Theorem 3.6, v avoids the pattern 312, and by Propositions 4.2 and 4.3 we have $P(F_v) \cong P(F_w)$. By Theorem 4.1, v is the only 312-avoiding permutation in its equivalence class. $\square$

4.2 Planar network tableaux

To combinatorially interpret evaluations of $\mathfrak{S}_n$-class functions and $H_n(q)$-traces, we will repeatedly fill a (French) Young diagram with a path family $\pi = (\pi_1, \dots, \pi_n)$ covering a zig-zag network F_w, and will call the resulting structures *F_w-tableaux*, or more specifically *π-tableaux*. (See, e.g., [2, Sec. 2] for French notation.) If π has type v, then we will also say that each π-tableau has *type v*. Since π induces a subposet Q_π of the poset

$Q(F_w)$, π-tableaux form a special case of Gessel and Viennot's *P-tableaux* [8]: they are Q_π-tableaux.

Several properties which π-tableaux may posess can be defined for P-tableaux where P is an arbitrary poset. We say that a P-tableau U has *shape* λ for some partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ if it has λ_i cells in row i for all i. If U has λ_i cells in *column* i for all i, we say that U has shape $\lambda^\top$, where we define $\lambda^\top$ to be the partition whose ith part is equal to the number of cells in row i of U. Call an element $x \in P$ a *nontrivial record* in a row of U if it is greater in P than all entries appearing to its left in the same row, and if it is not the leftmost entry of its row.

- Call U *column-strict* (*row-strict*) if whenever elements x, y appear consecutively from bottom to top in a column (left to right in a row), then we have $x <_P y$.
- Call U *row-semistrict* if whenever elements x, y appear consecutively from left to right in a row, then we have $x <_P y$ or x incomparable to y in P.
- Call U *cyclically row-semistrict* if it is row-semistrict and the above condition also holds when x, y are the rightmost and leftmost (respectively) entries in a row.
- Call U *standard* if it is both column-strict and row-semistrict.
- Call U *record-free* if no row contains a nontrivial record.

Another property of π-tableaux depends upon each path being labeled by its source vertex. Call a row of U *left anchored* (*right anchored*) if its leftmost (rightmost) entry has the least source vertex of all paths in the row.

- Call U *left-anchored* (*right-anchored*) if each row is left-anchored (right-anchored).

More properties of π-tableaux depend upon the fact that each element of the poset Q_π has a source vertex and a (potentially different) sink vertex. Given a π-tableau U, let $L(U)$ and $R(U)$ denote the Young tableaux of integers obtained from U by replacing paths $\pi_1, \ldots, \pi_n$ with their corresponding source and sink indices, respectively.

- Call U *row-closed* if each row of $R(U)$ is a permutation of the corresponding row of $L(U)$.
- Call U *left row-strict* if $L(U)$ is row-strict as a $\mathbb{Z}$-tableau.
- Call U *cylindrical* if for each row of $L(U)$ containing indices $i_1, \ldots, i_k$ from left to right, the corresponding row of $R(U)$ contains $i_2, \ldots, i_k, i_1$ from left to right.

It will be convenient to let $\mathcal{T}(F_w, \lambda)$ be the set of F_w-tableaux of shape λ.

Lemma 4.5. *Let $v \in \mathfrak{S}_n$ avoid the pattern 312, and fix $\lambda \vdash n$. Within the set $\mathcal{T}(F_v, \lambda)$, tableaux which are row-closed and left row-strict correspond bijectively to tableaux which are row-semistrict of type e.*

Proof. Observe that for a row-closed, left row-strict tableau $V \in \mathcal{T}(F_v, \lambda)$, the tableau V_k is itself a row-closed, left row-strict tableau in $\mathcal{T}(F_v|_{I_k}, \lambda_k)$, where I_k is the set of indices appearing in $L(V_k)$. Similarly, for a row-semistrict tableau $V' \in \mathcal{T}(F_v, \lambda)$ of type e, the tableau V'_k is itself a row-semistrict tableau of type e in $\mathcal{T}(F_v|_{I_k}, \lambda_k)$. Letting $r = \lambda_k$, we have that $F_v|_{I_k}$ is a descending star network of the form F_u for some $u \in \mathfrak{S}_r$ avoiding the pattern 312. We therefore construct a bijection which preserves the set of path indices appearing in each row of a tableau, and we state it as a product of bijections on one-rowed tableaux.

Map each left row-strict tableau $U \in \mathcal{T}(F_u, (r))$ to a row-semistrict tableau U' of type e in $\mathcal{T}(F_u, (r))$ as follows. Let $\rho = (\rho_1, \ldots, \rho_r)$ be the unique path family of type e covering F_u. Let U contain the path family $\pi = (\pi_1, \ldots, \pi_r)$ from left to right, and define $w \in \mathfrak{S}_r$ to be the word of right indices of these paths. Write w^{-1} in cycle notation, with each cycle starting with its greatest element, and cycles ordered by increasing greatest elements. (See Cycle Structure subsection in [28, Sec. 1.3]). Drop the parentheses, and interpret the resulting string $x = x_1 \cdots x_r$ of letters as the one-line notation of an element of $\mathfrak{S}_r$. Then write the paths $\rho_{x_1}, \ldots, \rho_{x_r}$ from left to right in U'.

To see that U' is row-semistrict, assume that we have $\rho_{x_i} >_{P(F_u)} \rho_{x_{i+1}}$ for some i. Then there is no path from source x_{i+1} to sink x_i in F_u. If x_i and x_{i+1} belong to the same cycle of w^{-1}, then $w_{x_{i+1}} = x_i$ and there is a path from x_{i+1} to x_i in F_u, a contradiction. If x_i and x_{i+1} do not belong to the same cycle of w^{-1}, then $x_{i+1} > x_i$ as integers, contradicting the assumed inequality in $P(F_u)$.

To see that the map $U \mapsto U'$ is a bijection, we construct its inverse. Let V be a row-semistrict tableau of type e in $\mathcal{T}(F_u, (r))$, containing paths $\rho_{x_1}, \ldots, \rho_{x_r}$ from left to right. Define $w \in \mathfrak{S}_n$ to be the permutation whose cycle notation is given by

$$(x_1, \ldots, x_{i_1-1})(x_{i_1}, \ldots, x_{i_2-1}) \cdots (x_{i_k}, \ldots, x_r),$$

where $x_1, x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ are the records of the word $x_1 \cdots x_r$, i.e., $x_{i_j} = \max\{x_1, \ldots, x_{i_j}\}$. Then write $w^{-1} = w_1^{-1} \cdots w_r^{-1}$ in one-line notation and define V' to be the tableau in $\mathcal{T}(F_u, (r))$ whose ith entry is the unique path in F_u from source i to sink w_i^{-1}. It is clear that this map, if well defined, is inverse to the map $U \mapsto U'$, and therefore that the two are bijections. (See [28, Sec. 1.3]).

To see that the necessary paths exist in F_u, consider a cycle $(x_j, \ldots, x_{j+\ell})$ of w and the pairs $(i, w_i^{-1}) \in \{(x_{j+a+1}, x_{j+a}) \,|\, 1 < a \leqslant \ell\} \cup \{(x_{j+\ell}, x_j)\}$. Since V is row-semistrict, i.e., $\rho_{x_{j+a}} \not>_{P(F_u)} \rho_{x_{j+a+1}}$, any integer inequality $x_{j+a} > x_{j+a+1}$ implies that there are paths in F_u from sources x_{j+a} and x_{j+a+1} to (both) sinks x_{j+a} and x_{j+a+1}. In particular, there are paths in F_u from sources x_j and x_{j+1} to sinks x_j and x_{j+1}. Now assume that there are paths from source x_{j+s} to sink x_{j+s-1} and from source x_j to sink x_{j+s}, and consider the pair (x_{j+s+1}, x_{j+s}). If $x_{j+s+1} < x_{j+s}$, then by the above argument there is a path from source x_{j+s+1} to sink x_{j+s}. Since there are paths from sources x_j and x_{j+s+1} to sink x_{j+s}, the two sources must belong to the same connected component of F_u. By the comment following Observation 3.4, we also have a path from source x_j to sink x_{j+s+1}. If on the other hand $x_{j+s+1} < x_{j+s}$, then since x_j is the maximum index in its cycle we have $x_j > x_{j+s+1} > x_{j+s}$. Since there are paths in F_u from source x_j to sink x_{j+s} and from source x_{j+s+1} to sink x_{j+s+1}, Observation 3.1 implies that there are also a paths from source x_j to sink x_{j+s+1} and from source x_{j+s+1} to x_{j+s}. By induction, we have that for $a = 1, \ldots, \ell$, there is a path from source x_{j+a} to sink x_{j+a-1}, and that there is also a path from source x_j to sink $x_{j+\ell}$. $\square$

4.3 Interpretation of $\mathfrak{S}_n$-class function evaluations

The equivalence relation in (4.2) has applications in the enumeration of certain F-tableaux and in the evaluation of $\mathfrak{S}_n$-class functions.

Theorem 4.6. *Let $v, w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231 and satisfy $v \sim w$, and let X be a property of F-tableaux which depends only upon the poset $P(F)$ (rather than on $Q(F)$). Then F_v-tableaux and F_w-tableaux having property X are in bijective correspondence. Moreover, for any $\mathfrak{S}_n$-class function θ we have $\theta(C'_v(1)) = \theta(C'_w(1))$.*

Proof. Since the property X depends only upon the poset $P(F_v) \cong P(F_w)$, we have a bijection between the sets of F_v-tableaux and F_w-tableaux having property X.

Now apply Lindström's Lemma and (3.6) to the first Littlewood-Merris-Watkins identity in (2.1) to see that for all $\lambda \vdash n$, the evaluations $\epsilon^\lambda(C'_v(1))$ and $\epsilon^\lambda(C'_w(1))$ are equal to the numbers of column-strict F_v-tableaux and F_w-tableaux, respectively, of type e and shape λ. Since column-strictness of these tableaux depends only upon $P(F_v) \cong P(F_w)$, the above bijection implies that we have $\epsilon^\lambda(C'_v(1)) = \epsilon^\lambda(C'_w(1))$ for all $\lambda \vdash n$. Since $\{\epsilon^\lambda \,|\, \lambda \vdash n\}$ is a basis for the space of $\mathfrak{S}_n$-class functions, each class function θ satisfies $\theta(C'_v(1)) = \theta(C'_w(1))$. $\square$

For some $\mathfrak{S}_n$-class functions θ, and all 3412-avoiding, 4231-avoiding permutations w, we may combinatorially interpret $\theta(C'_w(1))$ in terms of a zig-zag network F_w as follows. (See [29, p. 288] for information on the majorization order, used in (v-a) below.)

Theorem 4.7. *Let $w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231, and fix $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$. Then we have the following.*

(i) $\epsilon^\lambda(C'_w(1)) = \#\{U \in \mathcal{T}(F_w, \lambda^\top) \,|\, U$ *column-strict of type* $e\}$.

(ii-a) $\eta^\lambda(C'_w(1)) = \#\{U \in \mathcal{T}(F_w, \lambda) \,|\, U$ *row-closed, left row-strict*$\}$.

(ii-b) $\eta^\lambda(C'_w(1)) = \#\{U \in \mathcal{T}(F_w, \lambda) \,|\, U$ *row-semistrict of type* $e\}$.

(iii) $\chi^\lambda(C'_w(1)) = \#\{U \in \mathcal{T}(F_w, \lambda) \,|\, U$ *standard of type* $e\}$.

(iv-a) $\psi^\lambda(C'_w(1)) = \#\{U \in \mathcal{T}(F_w, \lambda) \,|\, U$ *cylindrical*$\}$.

(iv-b) $\psi^\lambda(C'_w(1)) = \#\{U \in \mathcal{T}(F_w, \lambda) \,|\, U$ *cyclically row-semistrict of type* $e\}$.

(iv-c) $\psi^\lambda(C'_w(1)) = \#\{U \in \mathcal{T}(F_w, \lambda) \,|\, U$ *record-free, row-semistrict of type* $e\}$.

(iv-d) $\psi^\lambda(C'_w(1)) = \lambda_1 \cdots \lambda_r \cdot \#\{U \in \mathcal{T}(F_w, \lambda) \,|\, U$ *right-anchored, row-semistrict of type* $e\}$.

(v-a) *Suppose* $\lambda_1 \leqslant 2$. *We have* $\phi^\lambda(C'_w(1)) = \#\{U \in \mathcal{T}(F_w,\lambda) \,|\, U$ *column-strict of type* $e\}$ *if for all* μ *majorized by* λ *we have* $\mathcal{T}(F_w,\mu) = \emptyset$*; otherwise we have* $\phi^\lambda(C'_w(1)) = 0$.

(v-b) *For* $\lambda = (k)^r$, *we have* $\phi^\lambda(C'_w(1)) = \#\{U \in \mathcal{T}(F_w,\lambda) \,|\, U$ *column-strict, cylindrical*$\}$.

Proof. (i) See the proof of Theorem 4.6.

(ii-a) Apply the definition (3.1) of path matrix and (3.6) to the second Littlewood-Merris-Watkins identity in (2.1).

(ii-b) Since row-semistrictness in F_w-tableaux of type e is a property of the poset $P(F_w)$, we may apply Theorem 4.6 and Lemma 4.5 to the interpretation in (ii-a).

(iii) Applying (i) and Theorem 4.1 to Gasharov's [7, Thm. 2], we obtain the claimed interpretation for 312-avoiding permutations. (See also Section 7.) Since standardness of F_w-tableaux depends only upon $P(F_w)$, we may apply Theorem 4.6 to extend the result to 3412-avoiding, 4231-avoiding permutations as well.

(iv-a) Apply the definition (3.1) of path matrix to the identity (2.2).

(iv-b) Let $v \sim w$ avoid the pattern 312. We define a map from cylindrical tableaux in $\mathcal{T}(F_v,\lambda)$ to cyclically row-semistrict tableaux in $\mathcal{T}(F_v,\lambda)$ having type e as follows. For $U \in \mathcal{T}(F_v,\lambda)$ cylindrical with rows k of $L(U)$ and $R(U)$ containing indices $i_1,\ldots,i_{\lambda_k}$ and $i_2,\ldots,i_{\lambda_k},i_1$ (respectively) from left to right, and $\rho = (\rho_1,\ldots,\rho_n)$ the unique path family of type e covering F_v, create a cyclically row-semistrict tableau $U' \in \mathcal{T}(F_v,\lambda)$ by inserting $\rho_{i_1},\ldots,\rho_{i_{\lambda_k}}$ into row k, from right to left. This map is bijective since in the descending star network F_v, there exists a (unique) path from source i_j to sink i_m if and only if $i_j > i_m$ or ρ_{i_j} and ρ_{i_m} intersect. Thus the claimed interpretation holds for 312-avoiding permutations. Since cyclical row-semistrictness in F_w-tableaux of type e depends only upon $P(F_w)$, we may apply Theorem 4.6 to extend the result to 3412-avoiding, 4231-avoiding permutations as well.

(iv-c) Shareshian and Wachs [23, Sec. 4] have shown that for 312-avoiding permutations, this formula is equivalent to Stanley's [27, Thm. 2.6]. Since the claimed property of F_w-tableaux depends only upon $P(F_w)$, we may apply Theorem 4.6 to extend the result to 3412-avoiding, 4231-avoiding permutations as well. (See also, [1, Lem. 6].)

(iv-d) The number of tableaux in (iv-b) is equal to the cardinality of the subset that are right-anchored, times $\lambda_1 \cdots \lambda_r$. This subset is precisely the right-anchored row-semistrict F_w-tableaux of type e and shape λ. Alternatively, we may use the Shareshian-Wachs argument of (iv-c).

(v-a) This was first stated in [4], and will be proved in Theorem 10.3. A different interpretation was given in [34, Thm. 2.5.1].

(v-b) Apply Lindström's Lemma and (3.6) to Stembridge's identity (2.3). $\square$

Conspicuously absent from Theorem 4.7 is an interpretation of monomial class function evaluations of the form $\phi^\lambda(C'_w(1))$ which holds for all $\lambda \vdash n$. As we have mentioned in the first table of Section 1, these integers are conjectured to be nonnegative. The problem of interpreting them has been posed from different points of view by Haiman, Stanley and Stembridge [11, Conj. 2.1], [27, Conj. 5.1], [30, Conj. 5.5], [32, Conj. 2.1]. Any extension of the statements in Theorem 4.7 (v-a), (v-b) would be interesting.

Problem 4.8. For (special cases of) $w \in \mathfrak{S}_n$ and $\lambda \vdash n$, find a combinatorial proof that $\phi^\lambda(C'_w(1))$ is nonnegative.

4.4 Inversions in path tableaux

For $\lambda \vdash n$ and $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231, parts (i) – (iv-d) of Theorem 4.7 interpret $\epsilon^\lambda(C'_w(1))$, $\eta^\lambda(C'_w(1))$, $\chi^\lambda(C'_w(1))$, and $\psi^\lambda(C'_w(1))$ as cardinalities of certain sets of F_w-tableaux. Using these same sets of F_w-tableaux and variations of the permutation statistic $\textsc{inv}$, we show in Sections 5, 6, 8, 9 that $\epsilon_q^\lambda(q_{e,w}C'_w(q))$, $\eta_q^\lambda(q_{e,w}C'_w(q))$, $\chi_q^\lambda(q_{e,w}C'_w(q))$, and $\psi_q^\lambda(q_{e,w}C'_w(q))$ are generating functions for tableaux on which the statistics take the values $k = 0, 1, \ldots$.

Specifically, we adapt the permutation statistic $\textsc{inv}$ for use on path tableaux as follows. Let $\pi = (\pi_1, \ldots, \pi_n)$ be a path family of type v in some zig-zag network F, and let U be a π-tableau. Let (π_i, π_j) be a pair of intersecting paths in F such that π_i appears in a column of U to the left of the column containing π_j. Call (π_i, π_j) a *(left) inversion* in U if we have $i > j$ and a *right inversion* in U if we have $v_i > v_j$. Let $\textsc{inv}(U)$ denote the number of inversions in U, and let $\textsc{rinv}(U)$ denote the number of right inversions in U.

Sometimes we will compute inversions in a one-rowed tableau formed by concatenating all of the rows of a path tableau U. Let U_i be the ith row of U, and let $U_1 \circ \cdots \circ U_r$ and $U_r \circ \cdots \circ U_1$ be the F-tableaux of shape n consisting of the rows of U concatenated in increasing and decreasing order, respectively. We will also compute inversions in the *transpose* $U^\top$ of a path tableau U, whose rows are the columns of U. It is easy to see that inversions in these one-rowed and transposed tableaux are related by the identities

$$\begin{aligned} \textsc{inv}(U_1 \circ \cdots \circ U_r) &= \textsc{inv}(U_1) + \cdots + \textsc{inv}(U_r) + \textsc{inv}(U^\top), \\ \textsc{rinv}(U_1 \circ \cdots \circ U_r) &= \textsc{rinv}(U_1) + \cdots + \textsc{rinv}(U_r) + \textsc{rinv}(U^\top). \end{aligned} \tag{4.3}$$

5 Interpretation of $\eta_q^\lambda(q_{e,w}C'_w(q))$

Let $w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231, and let B be the path matrix of F_w. Using (2.5) and Proposition 3.8, we have

$$\eta_q^\lambda(q_{e,w}C'_w(q)) = \sigma_B(\mathrm{Imm}_{\eta_q^\lambda}(x)) = \sum_{(I_1,\ldots,I_r)} \sigma_B(\mathrm{per}_q(x_{I_1,I_1}) \cdots \mathrm{per}_q(x_{I_r,I_r})), \tag{5.1}$$

where the sum is over all ordered set partitions $(I_1, \ldots, I_r)$ of $[n]$ of type $\lambda = (\lambda_1, \ldots, \lambda_r)$. Let $\mathfrak{S}_\lambda$ denote the Young subgroup of $\mathfrak{S}_n$ generated by

$$\{s_1, \ldots, s_{n-1}\} \smallsetminus \{s_{\lambda_1}, s_{\lambda_1+\lambda_2}, s_{\lambda_1+\lambda_2+\lambda_3}, \ldots, s_{n-\lambda_r}\},$$

and let $\mathfrak{S}_\lambda^-$ be the set of Bruhat-minimal representatives of cosets of the form $\mathfrak{S}_\lambda u$, i.e., the elements $u \in \mathfrak{S}_n$ for which each of the subwords

$$u_1 \cdots u_{\lambda_1}, \qquad u_{\lambda_1+1} \cdots u_{\lambda_1+\lambda_2}, \qquad \ldots, \qquad u_{n-\lambda_r+1} \cdots u_n \tag{5.2}$$

is strictly increasing. It is clear that such elements correspond bijectively to the ordered set partitions $(I_1, \dots, I_r)$ in (5.1). Expanding the product of permanents, we obtain monomials of the form $q_{u,v}$ times

$$x^{u,v} := x_{u_1,v_1} \cdots x_{u_n,v_n},$$

where v is the concatenation, in order, of rearrangements of the r words (5.2). Thus v may be written as yu with $y \in \mathfrak{S}_\lambda$, or as uy with $y \in u^{-1}\mathfrak{S}_\lambda u$. Now the sum in (5.1) becomes

$$\sum_{u \in \mathfrak{S}_\lambda^-} \sum_{y \in u^{-1}\mathfrak{S}_\lambda u} \sigma_B(q_{u,uy} x^{u,uy}). \tag{5.3}$$

Let us therefore consider evaluations of the form $\sigma_B(q_{u,v} x^{u,v})$.

To combinatorially interpret these evaluations, let $\pi = (\pi_1, \dots, \pi_n)$ be a path family (of arbitrary type) which covers a zig-zag network F, and define $U(u, \pi)$ to be the π-tableau of shape (n) containing π in the order $\pi_{u_1}, \dots, \pi_{u_n}$. Clearly the left tableau of $U(u, \pi)$ is $u_1 \cdots u_n$. If the right tableau is $v_1 \cdots v_n$ then π has type $u^{-1}v$. If s_i is a left descent for u, then right inversions in $U(u, \pi)$ and $U(s_i u, \pi)$ are related as follows.

Proposition 5.1. *Fix $u, v \in \mathfrak{S}_n$, let F be a zig-zag network, and let $\pi = (\pi_1, \dots, \pi_n)$ be a path family of type $u^{-1}v$ which covers F. If $s_i u < u$ then we have*

$$\mathrm{RINV}(U(u,\pi)) = \begin{cases} \mathrm{RINV}(U(s_i u, \pi)) - 1 & \text{if } s_i v > v, \\ \mathrm{RINV}(U(s_i u, \pi)) & \text{if } s_i v < v \text{ and no path family of type } u^{-1} s_i v \\ & \text{covers } F, \\ \mathrm{RINV}(U(s_i u, \pi)) + 1 & \text{if } s_i v < v \text{ and some path family of type} \\ & u^{-1} s_i v \text{ covers } F. \end{cases}$$

Proof. The tableaux $U(u, \pi)$ and $U(s_i u, \pi)$ are identical except that π_{u_i} appears before $\pi_{u_{i+1}}$ in $U(u, \pi)$. Thus we have

$$\mathrm{RINV}(U(u,\pi)) = \begin{cases} \mathrm{RINV}(U(s_i u, \pi)) - 1 & \text{if } (\pi_{u_i}, \pi_{u_{i+1}}) \text{ is a right inversion in } U(s_i u, \pi) \\ & \text{but not in } U(u, \pi), \\ \mathrm{RINV}(U(s_i u, \pi)) + 1 & \text{if } (\pi_{u_i}, \pi_{u_{i+1}}) \text{ is a right inversion in } U(u, \pi) \\ & \text{but not in } U(s_i u, \pi), \\ \mathrm{RINV}(U(s_i u, \pi)) & \text{if } (\pi_{u_i}, \pi_{u_{i+1}}) \text{ is not a right inversion in} \\ & U(s_i u, \pi) \text{ or } U(u, \pi). \end{cases}$$

Since $s_i u < u$, we have $u_i > u_{i+1}$.

If $s_i v > v$, then we have $v_i < v_{i+1}$, and Observation 3.1 implies that the paths π_{u_i} and $\pi_{u_{i+1}}$ intersect, forming a right inversion in $U(s_i u, \pi)$ and not in $U(u, \pi)$.

If $s_i v < v$, then we have $v_i > v_{i+1}$, and the paths π_{u_i} and $\pi_{u_{i+1}}$ do not form a right inversion in $U(s_i u, \pi)$. Suppose that some path family $\pi' = (\pi'_1, \dots, \pi'_n)$ of type $u^{-1} s_i v$ covers F. Then the tableau $U(u, \pi')$ satisfies

$$L(U(u, \pi')) = (u_1, \dots, u_n), \qquad R(U(u, \pi')) = (v_1, \dots, v_{i-1}, v_{i+1}, v_i, v_{i+2}, \dots, v_n).$$

By the uniqueness of source-to-sink paths in zig-zag networks, this tableau is identical to the tableau $U(u,\pi)$ except for the paths π'_{u_i} and $\pi'_{u_{i+1}}$ in positions i and $i+1$, which terminate at sinks $v_{i+1} < v_i$, respectively. By Observation 3.1, the paths $\pi'_{u_i}, \pi'_{u_{i+1}}$ cross and the paths π_{u_i}, $\pi_{u_{i+1}}$ intersect, forming a right inversion in the tableau $U(u,\pi)$. On the other hand, suppose that no path family of type $u^{-1}s_iv$ covers F. Since π has type $u^{-1}v$, we can deduce that either there is no path in F from source u_i to sink v_{i+1} or there is no path from source u_{i+1} to sink v_i. By Observation 3.1, the paths π_{u_i}, $\pi_{u_{i+1}}$ do not intersect and therefore do not form a right inversion in $U(u,\pi)$. $\square$

Now we evaluate $\sigma_B(q_{u,v}x^{u,v})$, first in the case that $u=e$.

Proposition 5.2. *Let w in $\mathfrak{S}_n$ avoid the patterns 3412 and 4231, let B be the path matrix of F_w, and fix v in $\mathfrak{S}_n$. Then we have*

$$\sigma_B(q_{e,v}x^{e,v}) = \begin{cases} q^{\text{RINV}(U(e,\pi))} & \text{if there exists a path family } \pi \text{ of type } v \text{ which covers } F_w, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By definition we have

$$\sigma_B(q_{e,v}x^{e,v}) = q_{e,v}q_{e,v}b^{e,v} = q^{\ell(v)}b^{e,v}. \tag{5.4}$$

First assume that there exists a (unique) path family π of type v that covers F_w. Then we have $b^{e,v}=1$. In the tableau $U(e,\pi)$, paths appear in the order $(\pi_1,\ldots,\pi_n)$. Now observe that for each inversion in v, i.e., each pair (v_i,v_j) with $i<j$ and $v_i>v_j$, the paths π_i (from source i to sink v_i) and π_j (from source j to sink v_j) cross in F_w and therefore form a right inversion in $U(e,\pi)$. Conversely, for each noninversion (v_i,v_j) in v, the paths π_i and π_j do not form a right inversion in $U(e,\pi)$. Thus we have $\ell(v) = \text{RINV}(U(e,\pi))$, and the expression in (5.4) is equal to $q^{\text{RINV}(U(e,\pi))}$.

Now assume that there is no path family of type v which covers F_w. Then we have $b^{e,v}=0$ and the expressions in (5.4) are equal to 0. $\square$

More generally, we evaluate $\sigma_B(q_{u,v}x^{u,v})$ as follows.

Proposition 5.3. *Let w in $\mathfrak{S}_n$ avoid the patterns 3412 and 4231, let B be the path matrix of F_w, and fix u, v in $\mathfrak{S}_n$. Then we have*

$$\sigma_B(q_{u,v}x^{u,v}) = \begin{cases} q^{\text{RINV}(U(u,\pi))} & \text{if there exists a path family } \pi \text{ of type } u^{-1}v \text{ covering } F_w, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We use induction on the length of u. By Proposition 5.2, the claimed formula holds when u has length 0. Now assume that the formula holds when u has length $1,\ldots,k-1$, and consider u of length k. Choosing a left descent s_i of u, we may write

$$\begin{aligned} \sigma_B(q_{u,v}x^{u,v}) &= \begin{cases} \sigma_B(q_{u,v}x^{s_iu,s_iv}) & \text{if } s_iv > v, \\ \sigma_B(q_{u,v}x^{s_iu,s_iv} + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})q_{u,v}x^{s_iu,v}) & \text{if } s_iv < v, \end{cases} \\ &= \begin{cases} q^{-1}\sigma_B(q_{s_iu,s_iv}x^{s_iu,s_iv}) & \text{if } s_iv > v, \\ \sigma_B(q_{s_iu,s_iv}x^{s_iu,s_iv}) + (1-q^{-1})\sigma_B(q_{s_iu,v}x^{s_iu,v}) & \text{if } s_iv < v. \end{cases} \end{aligned}$$

Suppose first that we have $s_i v > v$. Then by induction we have

$$\sigma_B(q_{s_i u, s_i v} x^{s_i u, s_i v}) = \begin{cases} q^{\mathrm{RINV}(U(s_i u, \pi))} & \text{if there exists a path family } \pi \text{ of type } u^{-1}v \\ & \text{covering } F_w, \\ 0 & \text{otherwise,} \end{cases}$$

and Proposition 5.1 implies that the claim is true in this case.

Now suppose that we have $s_i v < v$ and consider path families of types $u^{-1}v$ and $u^{-1}s_i v$ which cover F_w. If there are no path families of types $u^{-1}v$ and $u^{-1}s_i v$ which cover F_w, then by induction we have

$$\sigma_B(q_{s_i u, s_i v} x^{s_i u, s_i v}) = \sigma_B(q_{s_i u, v} x^{s_i u, v}) = 0. \tag{5.5}$$

If there exists a path family $\pi = (\pi_1, \ldots, \pi_n)$ of type $u^{-1}v$ which covers F_w, but no path family of type $u^{-1}s_i v$ which covers F_w, then by induction and Proposition 5.1 we have

$$\begin{gathered} \sigma_B(q_{s_i u, s_i v} x^{s_i u, s_i v}) = q^{\mathrm{RINV}(U(s_i u, \pi))} = q^{\mathrm{RINV}(U(u, \pi))}, \\ \sigma_B(q_{s_i u, v} x^{s_i u, v}) = 0. \end{gathered} \tag{5.6}$$

If there exists a path family $\pi' = (\pi'_1, \ldots, \pi'_n)$ of type $u^{-1}s_i v$ which covers F_w, then the paths π'_{u_i}, $\pi'_{u_{i+1}}$ from sources $u_i > u_{i+1}$ to sinks $v_{i+1} < v_i$ (respectively) cross. It follows that there exists a path family $\pi = (\pi_1, \ldots, \pi_n)$ of type $u^{-1}v$ which covers F_w, and which agrees with π' except that the paths π_{u_i}, $\pi_{u_{i+1}}$ from sources $u_i > u_{i+1}$ to sinks $v_i > v_{i+1}$ (respectively) intersect but do not cross. By induction and the existence of π' and π we have

$$\begin{gathered} \sigma_B(q_{s_i u, s_i v} x^{s_i u, s_i v}) = q^{\mathrm{RINV}(U(s_i u, \pi))}, \\ \sigma_B(q_{s_i u, v} x^{s_i u, v}) = q^{\mathrm{RINV}(U(s_i u, \pi'))}. \end{gathered}$$

The tableaux $U(s_i u, \pi)$ and $U(s_i u, \pi')$ agree except in positions i and $i+1$, where paths $\pi'_{u_{i+1}}$ and π'_{u_i} form a right inversion, but $\pi_{u_{i+1}}$ and π_{u_i} do not. This fact and Proposition 5.1 imply that we have

$$\mathrm{RINV}(U(s_i u, \pi')) = \mathrm{RINV}(U(s_i u, \pi)) + 1 = \mathrm{RINV}(U(u, \pi)),$$

and

$$\begin{gathered} \sigma_B(q_{s_i u, s_i v} x^{s_i u, s_i v}) = q^{\mathrm{RINV}(U(u, \pi)) - 1}, \\ (1 - q^{-1})\sigma_B(q_{s_i u, v} x^{s_i u, v}) = q^{\mathrm{RINV}(U(u, \pi))} - q^{\mathrm{RINV}(U(u, \pi)) - 1}. \end{gathered} \tag{5.7}$$

Thus by Equations (5.5), (5.6), and (5.7) the claim is true when $s_i v < v$. □

Now we have the following q-analog of Theorem 4.7 (ii-a).

Theorem 5.4. *Let $w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231. Then for $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ we have*

$$\eta_q^\lambda(q_{e,w} C'_w(q)) = \sum_U q^{\mathrm{RINV}(U_1 \circ \cdots \circ U_r)},$$

where the sum is over all row-closed, left row-strict F_w-tableaux of shape λ.

Proof. Let B be the path matrix of F_w and let $(I_1, \ldots, I_r)$ be a set partition of $[n]$ of type λ. By (5.2) – (5.3), there is a permutation $u \in \mathfrak{S}_\lambda^-$ corresponding to $(I_1, \ldots, I_r)$ such that we have

$$\sigma_B(\mathrm{per}_q(x_{I_1,I_1}) \cdots \mathrm{per}_q(x_{I_r,I_r})) = \sum_{y \in u^{-1}\mathfrak{S}_\lambda u} \sigma_B(q_{u,uy} x^{u,uy}).$$

By Proposition 5.3, this is equal to

$$\sum_{(y,\pi)} q^{\mathrm{RINV}(U(u,\pi))}, \tag{5.8}$$

where the sum is over pairs (y, π) such that $y \in u^{-1}\mathfrak{S}_\lambda u$ and π is a path family of type y which covers F. If such a path family π exists for a given permutation y, it is necessarily unique. Thus as y varies over $u^{-1}\mathfrak{S}_\lambda u$ we have that $U(u, \pi)$ varies over all bijective path tableaux of shape (n) which satisfy

1. For $j = 1, \ldots, r$, the paths in positions $\lambda_1 + \cdots + \lambda_{j-1} + 1, \ldots,\ \lambda_1 + \cdots + \lambda_j$ are indexed by I_j, in increasing order.
2. The sequence of sink indices of these same paths are a rearrangement of I_j.

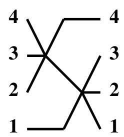
Thus the expression in (5.8) may be rewritten as

$$\sum_U q^{\mathrm{RINV}(U_1 \circ \cdots \circ U_r)}, \tag{5.9}$$

where this last sum is over all row-closed, left row-strict F_w-tableaux U of shape λ for which path indices of U_j are I_j for $j = 1, \ldots, r$. Summing over ordered set partitions and using (5.1), we have the desired result. $\square$

For example, consider the network

$$F_{3421} = \tag{5.10}$$



It is easy to verify that there are twenty row-closed, left row-strict F_{3421}-tableaux of shape 31. Four of these are

$$\begin{array}{|c|cc}\hline 4,4 & \multicolumn{2}{c}{} \\ \hline 1,2 & \multicolumn{1}{|c|}{2,1} & \multicolumn{1}{c|}{3,3} \\ \hline \end{array}\ ,\quad \begin{array}{|c|cc}\hline 4,4 & \multicolumn{2}{c}{} \\ \hline 1,3 & \multicolumn{1}{|c|}{2,1} & \multicolumn{1}{c|}{3,2} \\ \hline \end{array}\ ,\quad \begin{array}{|c|cc}\hline 3,3 & \multicolumn{2}{c}{} \\ \hline 1,2 & \multicolumn{1}{|c|}{2,4} & \multicolumn{1}{c|}{4,1} \\ \hline \end{array}\ ,\quad \begin{array}{|c|cc}\hline 2,2 & \multicolumn{2}{c}{} \\ \hline 1,3 & \multicolumn{1}{|c|}{3,4} & \multicolumn{1}{c|}{4,1} \\ \hline \end{array}\ ,$$

where i, j represents the unique path from source i to sink j. These tableaux U of shape 31 yield tableaux $U_1 \circ U_2$ of shape 4,

$$\begin{array}{|c|c|c|c|}\hline 1,2 & 2,1 & 3,3 & 4,4 \\ \hline \end{array}\ ,\quad \begin{array}{|c|c|c|c|}\hline 1,3 & 2,1 & 3,2 & 4,4 \\ \hline \end{array}\ ,\quad \begin{array}{|c|c|c|c|}\hline 1,2 & 2,4 & 4,1 & 3,3 \\ \hline \end{array}\ ,\quad \begin{array}{|c|c|c|c|}\hline 1,3 & 3,4 & 4,1 & 2,2 \\ \hline \end{array}\ ,$$

which have 1, 2, 3, and 4 right inversions, respectively. Together, the tableaux contribute $q + q^2 + q^3 + q^4$ to $\eta_q^{31}(q_{e,3421}C'_{3421}(q)) = 1 + 3q + 6q^2 + 6q^3 + 3q^4 + q^5$.

Expanding η_q^λ in terms of irreducible characters and Kostka numbers, $\eta_q^\lambda = \sum K_{\mu,\lambda}\chi_q^\mu$, and using Haiman's result [11, Lem 1.1], we have that the coefficients of $\eta_q^\lambda(q_{e,w}C'_w(q))$ are symmetric and unimodal about $q_{e,w}$ for all $w \in \mathfrak{S}_n$. In the case that w avoids the patterns 3412 and 4231, it would be interesting to explain this phenomenon combinatorially in terms of Theorem 5.4.

It would also be interesting to extend Theorem 5.4 to include a q-analog of Theorem 4.7 (ii-b). In particular, the identity

$$\eta_q^{(n)} = \sum_{\lambda \vdash n} \phi_q^\lambda$$

suggests that an answer to Problem 4.8 and its q-analog are related to a set partition of tableaux counted by $\eta^{(n)}$. It is not clear whether such a partition is more easily expressed in terms of left row-strict tableaux of shape (n), or row-semistrict tableaux of type e and shape (n).

Problem 5.5. Find a statistic STAT on F-tableaux such that we have

$$\eta_q^\lambda(q_{e,w}C'_w(q)) = \sum_U q^{\mathrm{STAT}(U)},$$

where the sum is over all row-semistrict F_w-tableaux of type e and shape λ.

As a consequence of Theorem 5.4, we obtain the following q-analog of Theorem 4.6.

Theorem 5.6. *Let $v, w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231 and satisfy $v \sim w$, and let (X, STAT) be a property of F-tableaux and a statistic on F-tableaux which depend only upon the poset $P(F)$. Then F_v-tableaux and F_w-tableaux having property X and satisfying $\mathrm{STAT}(U) = k$ are in bijective correspondence. Moreover, for each $H_n(q)$-trace θ_q we have*

$$\theta_q(q_{e,v}C'_v(q)) = \theta_q(q_{e,w}C'_w(q)).$$

Proof. Since the pair (X, STAT) depends only upon the poset $P(F_v) \cong P(F_w)$, we have a bijection between the sets of F_v-tableaux and F_w-tableaux having property X and satisfying $\mathrm{STAT}(U) = k$. In particular, we have $\eta_q^\lambda(q_{e,v}C'_v(q)) = \eta_q^\lambda(q_{e,w}C'_w(q))$, and since $\{\eta_q^\lambda \mid \lambda \vdash n\}$ is a basis of the space of $H_n(q)$-traces, we have $\theta_q(q_{e,v}C'_v(q)) = \theta_q(q_{e,w}C'_w(q))$ for all $H_n(q)$-traces θ_q as well. ☐

Let θ_q be an $H_n(q)$-trace. If the posets $P(F_v)$, $P(F_w)$ of two zig-zag networks are dual, rather than isomorphic, we still have $\theta_q(q_{e,v}C'_v(q)) = \theta_q(q_{e,w}C'_w(q))$. In this case v and w satisfy $v \sim w_0ww_0$ and $\ell(v) = \ell(w)$, where w_0 is the longest element of $\mathfrak{S}_n$. Since natural basis elements of $H_n(q)$ satisfy $T_{w_0}T_wT_{w_0}^{-1} = T_{w_0ww_0}$, and since any $H_n(q)$-trace θ_q satisfies $\theta_q(gh) = \theta_q(hg)$ for all $h, g \in H_n(q)$, we have that the equations

$$\theta_q(q_{e,w_0ww_0}C'_{w_0ww_0}(q)) = \theta_q(q_{e,w}T_{w_0}C'_w(q)T_{w_0}^{-1}) = \theta_q(q_{e,w}C'_w(q))$$

hold for all $w \in \mathfrak{S}_n$.

6 Interpretation of $\epsilon_q^\lambda(q_{e,w}C'_w(q))$

Let $w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231 and let B be the path matrix of F_w. Following the computations of Equations (5.1) – (5.3), we have

$$\begin{aligned}\epsilon_q^\lambda(q_{e,w}C'_w(q)) = \sigma_B(\mathrm{Imm}_{\epsilon_q^\lambda}(x)) &= \sum_{(I_1,\dots,I_r)} \sigma_B(\det_q(x_{I_1,I_1})\cdots\det_q(x_{I_r,I_r})) \\ &= \sum_{u\in\mathfrak{S}_\lambda^-}\sum_{y\in u^{-1}\mathfrak{S}_\lambda u} (-1)^{\ell(uy)-\ell(u)}\sigma_B(q_{u,uy}^{-1}x^{u,uy}),\end{aligned} \tag{6.1}$$

where the first sum is over all ordered set partitions $(I_1,\dots,I_r)$ of $[n]$ of type λ. Let us therefore consider evaluations of the form $\sigma_B(q_{u,v}^{-1}x^{u,v})$.

To combinatorially interpret these evaluations, let $\pi = (\pi_1,\dots,\pi_n)$ be a path family (of arbitrary type) which covers a zig-zag network F and define $U(u,\pi,\lambda)$ to be the π-tableau of shape λ containing π in the order $\pi_{u_1},\dots,\pi_{u_n}$. That is, $U(u,\pi,\lambda)_j$ contains the λ_j paths whose indices are

$$u_{\lambda_1+\cdots+\lambda_{j-1}+1},\dots,u_{\lambda_1+\cdots+\lambda_j}.$$

Clearly $L(U(u,\pi,\lambda))$ contains the numbers $u_1,\dots,u_{\lambda_1}$ in row 1, $u_{\lambda_1+1},\dots,u_{\lambda_1+\lambda_2}$ in row 2, etc. If $R(U(u,\pi,\lambda))$ contains the numbers $v_1,\dots,v_{\lambda_1}$ in row 1, $v_{\lambda_1+1},\dots,v_{\lambda_1+\lambda_2}$ in row 2, etc., then π has type $u^{-1}v$. In terms of this notation, our earlier tableau $U(u,\pi)$ defined after (5.3) is equal to $U(u,\pi,(n))$. If $u \in \mathfrak{S}_\lambda^-$ corresponds to $(I_1,\dots,I_r)$ as in (5.2) – (5.3), then the path indices in row j of $U(u,\pi,\lambda)$ are simply the elements of I_j, in increasing order.

Proposition 6.1. *Let $w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231, and let B be the path matrix of F_w. Fix $\lambda \vdash n$, $u \in \mathfrak{S}_\lambda^-$, and $y \in u^{-1}\mathfrak{S}_\lambda u$. Then we have*

$$\sigma_B(q_{u,uy}^{-1}x^{u,uy}) = \begin{cases} q^{\mathrm{RINV}(U(u,\pi,\lambda)^\top)} & \text{if there exists a path family } \pi \text{ of type } y \text{ covering } F_w, \\ 0 & \text{otherwise.}\end{cases}$$

Proof. By Proposition 5.3 we have

$$\begin{aligned}\sigma_B(q_{u,uy}^{-1}x^{u,uy}) &= q_{u,uy}^{-2}\sigma_B(q_{u,uy}x^{u,uy}) \\ &= \begin{cases} q^{\ell(u)-\ell(uy)}q^{\mathrm{RINV}(U(u,\pi,(n)))} & \text{if some path family of type } y \text{ covers } \hat{F}, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Letting $V = U(u,\pi,\lambda)$ and using (5.9), we may rewrite the above exponent of q as $\mathrm{RINV}(V_1\circ\cdots\circ V_r) + \ell(u) - \ell(uy)$. By (4.3), this is

$$\mathrm{RINV}(V_1) + \cdots + \mathrm{RINV}(V_r) + \mathrm{RINV}(V^\top) - (\ell(uy) - \ell(u)).$$

We claim that this expression reduces further to $\mathrm{RINV}(V^\top)$. To see this, recall that the tableaux $L(V_1)\circ\cdots\circ L(V_r)$ and $R(V_1)\circ\cdots\circ R(V_r)$ contain the one-line notations of u

and uy, respectively. Since u belongs to $\mathfrak{S}_\lambda^-$ and each tableau $R(V_j)$ is a permutation of the corresponding tableau $L(V_j)$, we have

$$\begin{aligned}\ell(u) &= \#\{(k,k') \,|\, k > k',\ k \text{ in an earlier row of } L(V) \text{ than } k'\} \\ &= \#\{(k,k') \,|\, k > k',\ k \text{ in an earlier row of } R(V) \text{ than } k'\}.\end{aligned}$$

$$\begin{aligned}\ell(uy) &= \sum_{j=1}^{r} \text{INV}(R(V_j)) + \#\{(k,k') \,|\, k > k',\ k \text{ in an earlier row of } R(V) \text{ than } k'\} \\ &= \sum_{j=1}^{r} \text{INV}(R(V_j)) + \ell(u).\end{aligned}$$

Now fix a row V_j of V and consider right inversions in V_j. Let π_i, $\pi_{i'}$ be two paths in this row, with π_i appearing first. Since u belongs to $\mathfrak{S}_\lambda^-$, we have $i < i'$. Let k and k' be the corresponding sink indices. If we have $k > k'$, then the paths cross and form a right inversion in V_j. On the other hand, if we have $k < k'$, then the paths do not form a right inversion in V_j, even if they intersect. Thus we have $\text{RINV}(V_j) = \text{INV}(R(V_j))$ for all j and

$$\text{RINV}(V_1) + \cdots + \text{RINV}(V_r) = \ell(uy) - \ell(u), \tag{6.2}$$

as desired. □

While the final sum in (6.1) has signs, we will use a sign-reversing involution to cancel some of the the terms there, and to obtain a signless sum more amenable to combinatorial interpretation. Fix a partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$, an ordered set partition I of $[n]$ of type λ, and a zig-zag network F, and let $\mathcal{T}_I = \mathcal{T}_I(F)$ be the set of all row-closed, left row-strict F-tableaux U of shape λ such that $L(U)_j = I_j$ (as sets) for $j = 1, \dots, r$. Observe that all row-strict F-tableaux of type e satisfying $L(U)_j = I_j$ (as sets) for $j = 1, \dots, r$ belong to $\mathcal{T}_I$. Let us define an involution

$$\zeta : \mathcal{T}_I \to \mathcal{T}_I$$

as follows.

1. If U is a row-strict tableau of type e, then define $\zeta(U) = U$.

2. Otherwise,

   (a) Let i be the least index such that U_i is not row-strict.

   (b) Let (j, j') be the lexicographically least pair of indices in $L(U)_i$ such that π_j and $\pi_{j'}$ intersect.

   (c) Let (k, k') be the sink indices of paths π_j and $\pi_{j'}$, respectively.

   (d) Define $\zeta(U)$ to be the tableau obtained from U by replacing π_j and $\pi_{j'}$ by the unique paths in F from source j to sink k' and source j' to sink k.

Proposition 6.2. *The involution ζ satisfies* $\text{RINV}(\zeta(U)^\top) = \text{RINV}(U^\top)$.

Proof. For each F-tableau $U \in \mathcal{T}_I$ satisfying $\zeta(U) = U$, the claimed equality is obvious. Now let U be an F-tableau not fixed by ζ. Define the indices i, j, j', k, k' and paths π_j, $\pi_{j'}$ as in the definition of ζ, and let π'_j, $\pi'_{j'}$ be the two new paths created in the final step of the definition of ζ. Since the tableaux $U^\top$ and $\zeta(U)^\top$ agree everywhere except in the two cells in column i containing the paths

$$\pi_j, \quad \pi_{j'}, \quad \pi'_j, \quad \pi'_{j'}, \tag{6.3}$$

it is clear that the right inversions of these tableaux are equal except possibly for inversions involving a path π_h in a column other than i and one of the paths (6.3). We claim that these remaining right inversions in $U^\top$ and $\zeta(U)^\top$ correspond bijectively. In particular, we have

(a) π_h forms a right inversion with π_j in $U^\top$ if and only if it forms a right inversion with $\pi'_{j'}$ in $\zeta(U)^\top$.

(b) π_h forms a right inversion with $\pi_{j'}$ in $U^\top$ if and only if it forms a right inversion with π'_j in $\zeta(U)^\top$.

To see this, observe that $j < j'$, and consider the intersection of π_h with π_j and $\pi_{j'}$. By Lemma 3.5 we have four cases:

1. π_h intersects both π_j and $\pi_{j'}$.
2. π_h intersects only $\pi_{j'}$.
3. π_h intersects only the path whose sink is $\min(k, k')$.
4. π_h intersects neither π_j nor $\pi_{j'}$.

Now considering the intersections of π_h with π'_j and $\pi'_{j'}$, we see that the above cases imply respectively that π_h intersects both π'_j and $\pi'_{j'}$, only $\pi'_{j'}$ (not π'_j), only the path in $\{\pi'_j, \pi'_{j'}\}$ whose sink index is $\min(k, k')$, and neither π'_j nor $\pi'_{j'}$. In all cases, the equivalences (a) and (b) are true. □

Since the tableaux U and $\zeta(U)$ agree except in one row, we have the following.

Proposition 6.3. *Let $U \in \mathcal{T}_I$ be a tableau not fixed by ζ and let i be the index satisfying $U_i \neq \zeta(U)_i$. Then we have*

$$|\mathrm{RINV}(\zeta(U)_j) - \mathrm{RINV}(U_j)| = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Obvious. □

Now we have the following q-analog of Theorem 4.7 (i).

Theorem 6.4. *Let $w \in \mathfrak{S}_n$ avoid the patterns* 3412 *and* 4231*. Then for $\lambda \vdash n$ we have*

$$\epsilon_q^\lambda(q_{e,w}C'_w(q)) = \sum_U q^{\text{INV}(U)},$$

where the sum is over all column-strict F_w-tableaux of type e and shape $\lambda^\top$.

Proof. Let B be the path matrix of F_w and let $(I_1, \dots, I_r)$ be a set partition of $[n]$ of type λ. By (5.2) – (5.3), there is a permutation $u \in \mathfrak{S}_\lambda^-$ corresponding to $(I_1, \dots, I_r)$ such that we have

$$\sigma_B(\det_q(x_{I_1,I_1}) \cdots \det_q(x_{I_r,I_r})) = \sum_{y \in u^{-1}\mathfrak{S}_\lambda u} (-1)^{\ell(uy)-\ell(u)} \sigma_B(q_{u,uy}^{-1} x^{u,uy}).$$

By Proposition 6.1 this is equal to

$$\sum_{(y,\pi)} (-1)^{\ell(uy)-\ell(u)} q^{\text{RINV}(U(u,\pi,\lambda)^\top)}, \tag{6.4}$$

where the sum is over pairs (y, π) such that $y \in u^{-1}\mathfrak{S}_\lambda u$ and π is a path family of type y which covers F_w. If such a path family exists, it is necessarily unique. Thus as y varies over $u^{-1}\mathfrak{S}_\lambda u$ we have that $U(u, \pi, \lambda)$ varies over all tableaux in $\mathcal{T}_I$. Thus by (6.2) this sum is equal to

$$\sum_{V \in \mathcal{T}_I} (-1)^{\text{RINV}(V_1)+\cdots+\text{RINV}(V_r)} q^{\text{RINV}(V^\top)}. \tag{6.5}$$

Now consider a tableau $V \in \mathcal{T}_I$ which satisfies $\zeta(V) \neq V$. By Propositions 6.2 – 6.3, the term of the above sum corresponding to the tableau $\zeta(V)$ is

$$(-1)^{\text{RINV}(\zeta(V)_1)+\cdots+\text{RINV}(\zeta(V)_r)} q^{\text{RINV}(\zeta(V)^\top)} = -(-1)^{\text{RINV}(V_1)+\cdots+\text{RINV}(V_r)} q^{\text{RINV}(V^\top)}.$$

Thus all terms corresponding to tableaux V and $\zeta(V) \neq V$ cancel one another in the sum (6.5), leaving terms only for the tableaux $V \in \mathcal{T}_I$ which are fixed by ζ, i.e., the row-strict tableaux. For these tableaux we have

$$\text{RINV}(V_1) = \cdots = \text{RINV}(V_r) = 0, \qquad \text{RINV}(V^\top) = \text{INV}(V^\top).$$

Furthermore, V is row-strict of shape λ if and only if $V^\top$ is column-strict of shape $\lambda^\top$. Thus we may again rewrite (6.4) as

$$\sum_U q^{\text{INV}(U)},$$

where the sum is over all column-strict F_w-tableaux of shape $\lambda^\top$ satisfying $U_j = I_j$ (as sets). Summing over ordered set partitions and using (6.1), we have the desired result. □

For example, consider the descending star network F_{3421} in (5.10). It is easy to verify that there are exactly two column-strict F_{3421}-tableaux of type e and shape 31:

$$\begin{array}{|c|cc}\hline \pi_4 & & \\ \hline \pi_1 & \pi_2 & \pi_3 \\ \hline \end{array}\ , \qquad \begin{array}{|c|cc}\hline \pi_4 & & \\ \hline \pi_1 & \pi_3 & \pi_2 \\ \hline \end{array}\ , \tag{6.6}$$

where π_i is the unique path from source i to sink i. These tableaux have 2 and 3 inversions, respectively. Since $31^\top = 211$, the tableaux together give $\epsilon_q^{211}(q_{e,3421}C'_{3421}(q)) = q^2+q^3$. It is clear that all twenty-four F_{3421}-tableaux of type e and shape 4 are column-strict. Counting inversions in these tableaux gives $\epsilon_q^{1111}(q_{e,3421}C'_{3421}(q)) = 1 + 3q + 8q^2 + 8q^3 + 3q^4 + q^5$. It is also clear that there are no F_{3421}-tableaux of type e and shapes 22, 211, or 1111. Thus we have $\epsilon_q^{22}(q_{e,3421}C'_{3421}(q)) = \epsilon_q^{31}(q_{e,3421}C'_{3421}(q)) = \epsilon_q^{4}(q_{e,3421}C'_{3421}(q)) = 0$.

Expanding ϵ_q^λ in terms of irreducible characters and Kostka numbers, $\epsilon_q^\lambda = \sum K_{\mu^\top,\lambda}\chi_q^\mu$, and using Haiman's result [11, Lem 1.1], we have that $\epsilon_q^\lambda(q_{e,w}C'_w(q))$ is symmetric and unimodal about $q_{e,w}$ for all $w \in \mathfrak{S}_n$. In the case that w avoids the patterns 3412 and 4231, it would be interesting to explain this phenomenon combinatorially in terms of Theorem 6.4.

Corollary 6.5. *Fix $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ and $w \in \mathfrak{S}_n$ avoiding the patterns* 3412 *and* 4231. *If $w_1 \cdots w_n$ has a decreasing subsequence of length greater than r, then $\epsilon_q^\lambda(q_{e,w}C'_w(q)) = 0$.*

Proof. Let $(\pi_1, \dots, \pi_n)$ be the unique path family of type e which covers F_w. By Theorem 3.7 there exist $r+1$ paths $\pi_i, \dots, \pi_{i+r}$ which share a vertex. No two of these can appear together in a column of a column-strict F_w-tableau. Thus no such tableau has shape $\lambda^\top$. □

More generally, it is known that we have $\epsilon_q^\lambda(q_{e,w}C'_w(q)) = 0$ unless $\lambda \leqslant \mathrm{sh}(w)$ in the majorization order, where $\mathrm{sh}(w)$ is the partition associated to w by the Robinson-Schensted row insertion algorithm. (See, e.g., [11, Prop. 4.1 (3)].) This implies Corollary 6.5, since a decreasing subsequence of length greater than r in $w_1 \dots w_n$ implies that we have $\lambda \not\leqslant \mathrm{sh}(w)$.

7 Chromatic symmetric and quasisymmetric functions

The evaluations of $\mathfrak{S}_n$-class functions and $H_n(q)$ traces at Kazhdan-Lusztig basis elements are closely related to certain symmetric and quasisymmetric functions defined by Stanley [27] and Shareshian and Wachs [23].

Let Λ_n be the $\mathbb{Z}$-module of homogeneous degree n symmetric functions. In [27] Stanley defined certain *chromatic symmetric functions* $\{X_G \,|\, G$ a simple graph on n vertices$\}$ in Λ_n and studied expansions of these in various bases of Λ_n. Given $G = (V, E)$ and defining $\mathbb{P}$ to be the set of positive integers, we call a function $\kappa : V \to \mathbb{P}$ a *proper coloring* of G if $\kappa(u) \neq \kappa(v)$ whenever $(u, v) \in E$. Then we have the definition

$$X_G = \sum_\kappa x_{\kappa(1)} \cdots x_{\kappa(n)},$$

where the sum is over all proper colorings of G. When G is the incomparability graph of a poset P, we will write $X_P = X_{\mathrm{inc}(P)}$. Stanley showed [27, Prop. 2.4] that in this case, we have the equivalent definition

$$X_P = \sum_{\lambda \vdash n} c_{P,\lambda} m_\lambda, \tag{7.1}$$

where $c_{P,\lambda}$ is the number of ordered set partitions of P whose blocks are chains of cardinalities $\lambda_1, \lambda_2, \ldots$. These symmetric functions are related to $\mathfrak{S}_n$-class function evaluations as follows.

Theorem 7.1. *Let P be an n-element unit interval order, let $v \in \mathfrak{S}_n$ be the* 312*-avoiding permutation satisfying $\zeta(F_v) = P$ as in the proof of Theorem 4.1, and let $w \in \mathfrak{S}_n$ be any* 3412*-avoiding,* 4231*-avoiding permutation satisfying $w \sim v$ as in (4.2). Then we have*

$$X_P = \sum_{\lambda \vdash n} \epsilon^\lambda(C'_w(1)) m_\lambda. \tag{7.2}$$

Proof. It is easy to see that $c_{P,\lambda}$ is equal to the number of column-strict P-tableaux of shape $\lambda^\top$. By Theorems 4.1, 4.6, this is the number of column-strict F_w-tableaux of shape $\lambda^\top$. Thus by Theorem 4.7 (i) we have $c_{P,\lambda} = \epsilon^\lambda(C'_w(1))$. □

Expanding X_P in other bases of $\mathbb{Q} \otimes \Lambda_n$, including the forgotten basis $\{f_\lambda \,|\, \lambda \vdash n\}$, we see that other class function evaluations appear as coefficients.

Corollary 7.2. *Let P, v, w be as in Theorem 7.1. Then we have*

$$\begin{aligned} X_P &= \sum_{\lambda \vdash n} \eta^\lambda(C'_w(1)) f_\lambda = \sum_{\lambda \vdash n} \chi^{\lambda^\top}(C'_w(1)) s_\lambda \\ &= \sum_{\lambda \vdash n} \frac{(-1)^{n-\ell(\lambda)}}{z_\lambda} \psi^\lambda(C'_w(1)) p_\lambda = \sum_{\lambda \vdash n} \phi^\lambda(C'_w(1)) e_\lambda. \end{aligned}$$

Proof. The transition matrices relating the class function bases $\{\epsilon^\lambda \,|\, \lambda \vdash n\}$, $\{\eta^\lambda \,|\, \lambda \vdash n\}$, $\{\chi^{\lambda^\top} \,|\, \lambda \vdash n\}$, $\{(-1)^{n-\ell(\lambda)} z_\lambda^{-1} \psi^\lambda \,|\, \lambda \vdash n\}$, $\{\phi^\lambda \,|\, \lambda \vdash n\}$, respectively, are inverse to those relating the symmetric function bases $\{m_\lambda \,|\, \lambda \vdash n\}$, $\{f_\lambda \,|\, \lambda \vdash n\}$, $\{s_\lambda \,|\, \lambda \vdash n\}$, $\{p_\lambda \,|\, \lambda \vdash n\}$, $\{e_\lambda \,|\, \lambda \vdash n\}$, respectively. □

We remark that Theorem 7.1 and Corollary 7.2 do not hold for arbitrary w and P. Not all chromatic symmetric functions X_P can be expressed as $\sum_{\lambda \vdash n} \epsilon^\lambda(C'_w(1)) m_\lambda$ for appropriate $w \in \mathfrak{S}_n$, nor can all symmetric functions of this form be expressed as X_P for an appropriate labeled poset P.

Stanley and Stembridge [27, Conj. 5.1], [30, Conj. 5.5] conjectured that X_P is elementary nonnegative when P is $(\mathbf{3}+\mathbf{1})$-free, and Gasharov [7, Thm. 2] proved the weaker statement that X_P is Schur nonnegative in this case. Guay-Paquet [10, Thm. 5.1] showed that the above conjecture and result are equivalent to the analogous statements in which P is assumed to be a unit interval order. Thus by Theorem 4.6 and Corollary 7.2 the nonnegativity statements are consequences of Haiman's conjecture and result that for all $w \in \mathfrak{S}_n$, $\lambda \vdash n$ we have $\phi_q^\lambda(q_{e,w} C'_w(q)) \in \mathbb{N}[q]$ and $\chi_q^\lambda(q_{e,w} C'_w(q)) \in \mathbb{N}[q]$, respectively [11, Conj. 2.1, Lem. 1.1]. Specifically, the nonnegativity statements are obtained from Haiman's by restricting to the case that w avoids the pattern 312 and by specializing at $q = 1$.

Let QSym_n be the $\mathbb{Z}$-module of homogeneous degree n *quasisymmetric functions* in the commuting indeterminates $x_1, x_2, \ldots$, i.e., the generalization of homogeneous degree

n symmetric functions Λ_n in which monomials of the forms $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ and $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ are required to have the same coefficient only when $i_1 < \cdots < i_k$. In [23, Sec. 4], Shareshian and Wachs defined a q-analog $X_{G,q}$ of the chromatic symmetric function $X_G = X_{G,1}$, with $\{X_{G,q} \,|\, G \text{ a simple labeled graph on } n \text{ vertices}\}$ belonging to $\mathbb{Z}[q] \otimes \mathrm{QSym}_n$, and studied expansions of these functions in various bases. Assume that V is labeled by $[n]$ and let $\mathrm{asc}(\kappa) = \#\{(u,v) \in E \,|\, u < v \text{ and } \kappa(u) < \kappa(v)\}$. Then we have the definition

$$X_{G,q} = \sum_{\kappa} q^{\mathrm{asc}(\kappa)} x_{\kappa(1)} \cdots x_{\kappa(n)},$$

where the sum is over all proper colorings of G. In [24, Thm. 4.5] Shareshian and Wachs showed that this function is in fact symmetric with coefficients in $\mathbb{Z}[q]$ when G is the incomparability graph of an appropriately labeled n-element unit interval order P. Specifically, we require for each pair $x, y \in P$ satisfying

$$\#\{z \in P \,|\, z < x\} - \#\{z \in P \,|\, z > x\} < \#\{z \in P \,|\, z < y\} - \#\{z \in P \,|\, z > y\}, \qquad (7.3)$$

that the label of x be less than that of y. (The equivalence of this requirement to that stated in [24, Thm. 4.5] follows from comparison of [24, Props. 4.1 – 4.2] and the definition of *natural unit interval order* [24, Sec. 4] to results in [6, p. 33] and [26, Obs. 2.1 – 2.3, Prop. 2.4].) When G is the incomparability graph of a labeled poset P, we will write $X_{P,q} = X_{\mathrm{inc}(P),q}$, and we may give an alternate definition of $\{X_{P,q} \,|\, P \text{ an } n\text{-element poset}\}$ which is analogous to (7.1). (See also [24, Eq. (6.2)].)

Proposition 7.3. *Let P be a unit interval order, labeled as in (7.3). Then we have*

$$X_{P,q} = \sum_{\lambda \vdash n} c_{P,\lambda}(q) m_\lambda,$$

where

$$c_{P,\lambda}(q) = \sum_{U} q^{\mathrm{INV}(U)}$$

and the sum is over column-strict P-tableaux of shape $\lambda^\top$.

Proof. Each proper coloring κ of $\mathrm{inc}(P)$ may be viewed as an assignment of colors to elements of P so that each subset of elements having a given color forms a chain. By [24, Thm. 4.5], $X_{P,q}$ is symmetric. Thus for $\lambda \vdash n$, the coefficient in $X_{P,q}$ of m_λ is well-defined: it is the coefficient of $x_1^{\lambda_r} x_2^{\lambda_{r-1}} \cdots x_r^{\lambda_1}$, i.e., the sum of $q^{\mathrm{asc}(\kappa)}$, over all colorings κ that assign color 1 to a λ_r-element chain, color 2 to a λ_{r-1}-element chain, $\dots$, color r to a λ_1-element chain. Each such coloring corresponds to a column-strict P-tableau U of shape λ. Specifically, each λ_{r+1-i}-element chain of color i corresponds to column $r+1-i$ of U, for $i = 1, \dots, r$. Now observe that a pair (u, v) in P, with $u < v$ as integers, forms an ascent of κ if and only if it forms an inversion in U. Specifically, (u, v) is an edge in $\mathrm{inc}(P)$ if and only if u, v are incomparable in P, and we have $\kappa(u) < \kappa(v)$ if and only if u appears in a column of U to the right of the column containing v. $\square$

Just as Stanley's chromatic symmetric functions X_P are related to $\mathfrak{S}_n$-class function evaluations in Theorem 7.1, the Shareshian-Wachs chromatic quasisymmeric functions $X_{P,q}$ are related to $H_n(q)$-trace evaluations.

Theorem 7.4. *Let P be an n-element unit interval order labeled as in (7.3), let $v \in \mathfrak{S}_n$ be the corresponding 312-avoiding permutation as in Theorem 4.1, and let $w \in \mathfrak{S}_n$ be any 3412-avoiding, 4231-avoiding permutation satisfying $w \sim v$ as in (4.2). Then we have*

$$X_{P,q} = \sum_{\lambda \vdash n} \epsilon_q^\lambda (q_{e,w} C'_w(q)) m_\lambda.$$

Proof. By Proposition 7.3, $c_{P,\lambda}(q)$ is equal to the sum of $q^{\mathrm{INV}(U)}$ over column-strict P-tableaux of shape $\lambda^\top$. By Theorems 4.1, 5.6, we may sum over column-strict F_w-tableaux. Now Theorem 6.4 gives $c_{P,\lambda}(q) = \epsilon^\lambda(q_{e,w}C'_w(q))$. □

Expanding X_P in other bases of $\mathbb{Q}[q] \otimes \Lambda_n$ and following the proof of Corollary 7.2, we see that other trace evaluations appear as coefficients.

Corollary 7.5. *Let P, v, w be as in Theorem 7.4. Then we have*

$$\begin{aligned} X_{P,q} &= \sum_{\lambda \vdash n} \eta_q^\lambda (q_{e,w} C'_w(q)) f_\lambda = \sum_{\lambda \vdash n} \chi_q^{\lambda^\top} (q_{e,w} C'_w(q)) s_\lambda \\ &= \sum_{\lambda \vdash n} \frac{(-1)^{n-\ell(\lambda)}}{z_\lambda} \psi_q^\lambda (q_{e,w} C'_w(q)) p_\lambda = \sum_{\lambda \vdash n} \phi_q^\lambda (q_{e,w} C'_w(q)) e_\lambda. \end{aligned}$$

As before, Theorem 7.4 and Corollary 7.5 do not hold for arbitrary w and P. Not all chromatic symmetric functions $X_{P,q}$ can be expressed as $\sum_{\lambda \vdash n} \epsilon_q^\lambda (q_{e,w} C'_w(q)) m_\lambda$ for appropriate $w \in \mathfrak{S}_n$, nor can all symmetric functions of this form be expressed as $X_{P,q}$ for an appropriate labeled poset P.

Shareshian and Wachs conjectured that $X_{P,q}$ belongs to $\mathrm{span}_{\mathbb{N}[q]}\{e_\lambda \,|\, \lambda \vdash n\}$ when P is a unit interval order labeled as in (7.3) [23, Conj. 4.9], and proved that $X_{P,q}$ belongs to $\mathrm{span}_{\mathbb{N}[q]}\{s_\lambda \,|\, \lambda \vdash n\}$ for such posets [24, Thm. 6.3]. These statements do not hold for the more general $(\mathbf{3}+\mathbf{1})$-free posets, since the functions $X_{P,q}$ are not always symmetric in this case. Nevertheless, the result of Guay-Paquet [10, Thm. 5.1] shows that the statements generalize those of Stanley, Stembridge, and Gasharov mentioned after Corrolary 7.2. By Theorem 4.6 and Corollary 7.5, the statements are special cases (corresponding to w avoiding the pattern 312) of Haiman's conjecture and result [11, Conj. 2.1, Lem 1.1] that for all $w \in \mathfrak{S}_n$, $\lambda \vdash n$ we have $\phi_q^\lambda(q_{e,w}C'_w(q)) \in \mathbb{N}[q]$ and $\chi_q^\lambda(q_{e,w}C'_w(q)) \in \mathbb{N}[q]$. Shareshian and Wachs also conjectured [24, Sec. 7], and Athanasiadis proved [1, Thm. 4] that $X_{P,q}$ belongs to $\mathrm{span}_{\mathbb{N}[q]}\{(-1)^{n-\ell(\lambda)} z_\lambda^{-1} p_\lambda \,|\, \lambda \vdash n\}$ when P is a unit interval order labeled as in (7.3). By Theorem 4.6 and Corollary 7.5 this is equivalent to the assertion that we have $\psi_q^\lambda(q_{e,w}C'_w(q)) \in \mathbb{N}[q]$ for w avoiding the pattern 312. Thus this result is a special case of the (unpublished) conjecture that $\psi_q^\lambda(q_{e,w}C'_w(q)) \in \mathbb{N}[q]$ for all $w \in \mathfrak{S}_n$, $\lambda \vdash n$, which is a weakening of Haiman's conjecture [11, Conj. 2.1] since ψ_q^λ is a nonnegative linear combination of monomial traces (1.2).

8 Interpretation of $\chi_q^\lambda(q_{e,w}C'_w(q))$

Combining results in Sections 4, 5, 7 with those of Shareshian and Wachs now leads to the following q-analog of Theorem 4.7 (iii).

Theorem 8.1. *Let $w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231. For $\lambda \vdash n$ we have*

$$\chi_q^\lambda(q_{e,w}C'_w(q)) = \sum_U q^{\mathrm{INV}(U)}, \tag{8.1}$$

where the sum is over all standard F_w-tableaux of type e and shape λ.

Proof. Let $P = P(F_w)$. By Corollary 7.5, $\chi_q^\lambda(q_{e,w}C'_w(q))$ is equal to the coefficient of $s_{\lambda^\top}$ in $X_{P,q}$. By [24, Thm. 6.3] and Theorems 4.1, 5.6, this is precisely the claimed sum. □

For example, consider again the descending star network F_{3421} in (5.10). It is easy to verify that there are exactly two standard F_{3421}-tableaux of type e and shape 31: the two column-strict F_{3421}-tableaux of type e and shape 31 in (6.6) are also row-semistrict. Thus we have $\chi_q^{31}(q_{e,3421}C'_{3421}(q)) = \epsilon_q^{211}(q_{e,3421}C'_{3421}(q)) = q^2 + q^3$. On the other hand, not all twenty-four F_{3421}-tableaux of type e and shape 4 are row-semistrict: the six tableaux with π_4 immediately preceding π_1 are not. It is easy to verify that the eighteen remaining tableaux give $\chi_q^4(q_{e,3421}C'_{3421}(q)) = 1 + 3q + 5q^2 + 5q^3 + 3q^4 + q^5$. Since there are no column-strict F_{3421}-tableaux of shapes 22, 211, or 1111, there are no standard F_{3421}-tableaux of these shapes either, and we have $\chi_q^{22}(q_{e,3421}C'_{3421}(q)) = \chi_q^{211}(q_{e,3421}C'_{3421}(q)) = \chi_q^{1111}(q_{e,3421}C'_{3421}(q)) = 0$.

Combining Theorems 3.7 and 8.1, we have the following analog of Corollary 6.5.

Corollary 8.2. *Fix $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ and $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231. If $w_1 \cdots w_n$ has a decreasing subsequence of length greater than λ_1, then we have $\chi_q^\lambda(q_{e,w}C'_w(q)) = 0$.*

More generally, it is known that we have $\chi_q^\lambda(q_{e,w}C'_w(q)) = 0$ unless $\lambda \geqslant \mathrm{sh}(w)^\top$ in the majorization order. (See the comment following Corollary 6.5.) This implies Corollary 8.2, since a decreasing subsequence of length greater than λ_1 in $w_1 \dots w_n$ implies that we have $\lambda \ngeqslant \mathrm{sh}(w)^\top$.

9 Interpretation of $\psi_q^\lambda(q_{e,w}C'_w(q))$

Combining results in Sections 4, 5, 7 with those of Shareshian, Wachs, and Athanasiadis now leads to q-analogs of Theorem 4.7 (iv-c)–(iv-d). We will sometimes find it useful to reflect path tableaux in a vertical line, and will write U^R for the *reverse* of tableau U. For instance, we have

$$U = \begin{array}{|c|ccc}\hline \pi_1 & \multicolumn{3}{c}{} \\ \hline \pi_2 & \multicolumn{1}{|c|}{\pi_3} & \multicolumn{1}{c|}{\pi_4} \\ \hline \end{array}\,, \qquad U^R = \begin{array}{ccc|c|}\cline{4-4} \multicolumn{3}{c|}{} & \pi_1 \\ \hline \multicolumn{1}{|c|}{\pi_4} & \multicolumn{1}{c|}{\pi_3} & \pi_2 \\ \hline \end{array}\,,$$

where each path π_i retains its original source, sink, and orientation. Thus U_i^R will denote the reverse of the ith row of U. Note that while U^R may not be a tableau, because its cells are right-justified rather than left justified, the functions INV and RINV may still be applied to U^R as at the end of Section 4. We will also use standard notation for the q-analogs of the nonnegative integers and factorial function. For $a \in \mathbb{N}$ we define $[a]_q = 1 + q + \cdots + q^{a-1}$ for $a \geqslant 1$, and $[0]_q = 0$. We also define $[a]_q! = [a]_q[a-1]_q \cdots [1]_q$ for $a \geqslant 1$, and $[0]_q! = 1$.

Theorem 9.1. *Let $w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231. For $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, we have*

$$\psi_q^\lambda(q_{e,w}C'_w(q)) = \sum_U q^{\mathrm{INV}(U_1 \circ \cdots \circ U_r)}, \tag{9.1}$$

where the sum is over all record-free, row-semistrict F_w-tableaux of type e and shape λ, and

$$\psi_q^\lambda(q_{e,w}C'_w(q)) = [\lambda_1]_q \cdots [\lambda_r]_q \sum_U q^{\mathrm{INV}(U_1^R \circ \cdots \circ U_r^R)}, \tag{9.2}$$

where the sum is over all right-anchored, row-semistrict F_w-tableaux of type e and shape λ,

Proof. Let $P = P(F_w)$. By Corollary 7.5, $\psi_q^\lambda(q_{e,w}C'_w(q))$ is equal to $(-1)^{n-r}z_\lambda$ times the coefficient of p_λ in $X_{P,q}$. By [1, Thm. 3.1] and Theorems 4.1, 5.6, this is equal to the right-hand side of (9.1). By [24, Lem. 7.7] and Theorems 4.1, 5.6, it is also equal to the right-hand side of (9.2). $\square$

For example, consider the descending star network F_{3421} in (5.10) and the sum in (9.1). It is easy to verify that there are eighteen record-free, row-semistrict F_{3421}-tableaux of type e and shape 31. Four of these are

$$\begin{array}{|c|cc}\hline \pi_1 & \multicolumn{2}{c}{} \\ \hline \pi_4 & \multicolumn{1}{|c|}{\pi_3} & \multicolumn{1}{c|}{\pi_2} \\ \hline \end{array}\,, \quad \begin{array}{|c|cc}\hline \pi_2 & \multicolumn{2}{c}{} \\ \hline \pi_1 & \multicolumn{1}{|c|}{\pi_3} & \multicolumn{1}{c|}{\pi_4} \\ \hline \end{array}\,, \quad \begin{array}{|c|cc}\hline \pi_3 & \multicolumn{2}{c}{} \\ \hline \pi_2 & \multicolumn{1}{|c|}{\pi_1} & \multicolumn{1}{c|}{\pi_4} \\ \hline \end{array}\,, \quad \begin{array}{|c|cc}\hline \pi_4 & \multicolumn{2}{c}{} \\ \hline \pi_3 & \multicolumn{1}{|c|}{\pi_2} & \multicolumn{1}{c|}{\pi_1} \\ \hline \end{array}\,, \tag{9.3}$$

where π_i represents the unique path from source i to sink i. These tableaux U of shape 31 yield tableaux $U_1 \circ U_2$ of shape 4,

$$\begin{array}{|c|c|c|c|}\hline \pi_4 & \pi_3 & \pi_2 & \pi_1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline \pi_1 & \pi_3 & \pi_4 & \pi_2 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline \pi_2 & \pi_1 & \pi_4 & \pi_3 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline \pi_3 & \pi_2 & \pi_1 & \pi_4 \\ \hline \end{array}\,,$$

which have 5, 2, 2, and 3 inversions, respectively. Together, they contribute $2q^2 + q^3 + q^5$ to $\psi_q^{31}(q_{e,3421}C'_{3421}(q)) = 1 + 3q + 5q^2 + 5q^3 + 3q^4 + q^5$. Now consider the sum in (9.2). It is easy to verify that there are six right-anchored row-semistrict F_{3421}-tableaux of type e and shape 31: the first and fourth tableaux in (9.3) and the four tableaux

$$\begin{array}{|c|cc}\hline \pi_1 & \multicolumn{2}{c}{} \\ \hline \pi_3 & \multicolumn{1}{|c|}{\pi_4} & \multicolumn{1}{c|}{\pi_2} \\ \hline \end{array}\,, \quad \begin{array}{|c|cc}\hline \pi_2 & \multicolumn{2}{c}{} \\ \hline \pi_4 & \multicolumn{1}{|c|}{\pi_3} & \multicolumn{1}{c|}{\pi_1} \\ \hline \end{array}\,, \quad \begin{array}{|c|cc}\hline \pi_3 & \multicolumn{2}{c}{} \\ \hline \pi_4 & \multicolumn{1}{|c|}{\pi_2} & \multicolumn{1}{c|}{\pi_1} \\ \hline \end{array}\,, \quad \begin{array}{|c|cc}\hline \pi_4 & \multicolumn{2}{c}{} \\ \hline \pi_2 & \multicolumn{1}{|c|}{\pi_3} & \multicolumn{1}{c|}{\pi_1} \\ \hline \end{array}\,.$$

These tableaux U of shape 31 yield six tableaux $U_1^R \circ U_2^R$ of shape 4,

$$\begin{array}{|c|c|c|c|}\hline \pi_2 & \pi_3 & \pi_4 & \pi_1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline \pi_1 & \pi_2 & \pi_3 & \pi_4 \\ \hline \end{array}\,,$$

$$\begin{array}{|c|c|c|c|}\hline \pi_2 & \pi_4 & \pi_3 & \pi_1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline \pi_1 & \pi_3 & \pi_4 & \pi_2 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline \pi_1 & \pi_2 & \pi_4 & \pi_3 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline \pi_1 & \pi_3 & \pi_2 & \pi_4 \\ \hline \end{array}\,,$$

which have 2, 0, 3, 2, 1, and 1 inversions, respectively. Together, the six tableaux contribute $1 + 2q + 2q^2 + q^3$ to $[3]_q[1]_q(1 + 2q + 2q^2 + q^3) = \psi_q^{31}(q_{e,3421}C'_{3421}(q))$.

We will state three more combinatorial formulas for $\psi_q^\lambda(q_{e,w}C'_w(q))$ in Theorems 9.4 and 9.13. To justify these, we associate a polynomial $O(F) \in \mathbb{N}[q]$ to a descending star network F, and a path tableau $V(F, I)$ to the pair (F, I), where I is an ordered set partition of $[n]$ of type λ.

Definition 9.1. Let F be a descending star network, and let

$$G_{[c_1,d_1]} \circ \cdots \circ G_{[c_t,d_t]} \tag{9.4}$$

be the concatenation of star networks which corresponds to F as in Section 3. Define the polynomial $O(F) \in \mathbb{N}[q]$ by

$$O(F) = \begin{cases} \dfrac{\displaystyle\prod_{i=1}^{t}[d_i - c_i]_q!}{\displaystyle\prod_{\substack{[c_i,d_i] \prec\!\!\cdot\, [c_j,d_j] \\ c_i < c_j}}[d_i - c_j]_q! \displaystyle\prod_{\substack{[c_i,d_i] \prec\!\!\cdot\, [c_j,d_j] \\ c_j < c_i}}[d_j - c_i]_q!} & \text{if } F \text{ is connected,} \\ 0 & \text{if } F \text{ is disconnected.} \end{cases} \tag{9.5}$$

For example, the connected descending star network F_{3421} in (5.10) corresponds to the concatenation $G = G_{[2,4]} \circ G_{[1,3]}$ and two-element poset of intervals $[2, 4] \prec\!\!\cdot\, [1, 3]$. Thus we have

$$O(F_{3421}) = \frac{[4-2]_q![3-1]_q!}{[3-2]_q!} = \frac{[2]_q![2]_q!}{[1]_q!} = (1+q)^2.$$

Note that for the identity element $e \in \mathfrak{S}_n$ we have $O(F_e) = 1$ if $n = 1$ and $O(F_e) = 0$ otherwise. Letting $\pi = (\pi_1, \ldots, \pi_n)$ be the unique path family of type e covering F, define $V(F, I)$ to be the unique (row-semistrict) π-tableau of shape λ for which $L(V(F, I))$ is a row-strict Young tableau containing indices I_j in row j. For S a k-element subset of $[n]$, let $F|_S$ denote the zigzag network of order k isomorphic to the subnetwork of F covered by paths $\{\pi_i \,|\, i \in S\}$.

Also essential to our proofs of Theorems 9.4 and 9.13 is a map

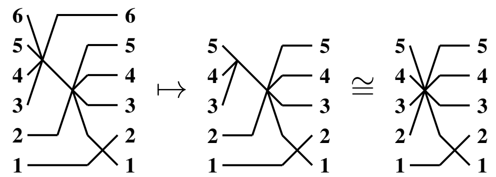
$$\begin{aligned} \iota : \{w \in \mathfrak{S}_n \,|\, w \text{ avoids the patterns 3412 and 4231, } F_w \text{ connected}\} \\ \to \{w \in \mathfrak{S}_{n-1} \,|\, w \text{ avoids the patterns 3412 and 4231, } F_w \text{ connected}\}. \end{aligned}$$

Let F_w be a connected zig-zag network of order $n \geqslant 2$ corresponding to the concatenation (9.4). By Observation 3.2 we may assume that $t = 1$, or $d_1 = n$ and $[c_1, d_1] \prec\!\!\cdot\, [c_2, d_2]$, or

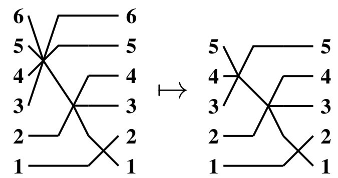
$d_t = n$ and $[c_{t-1}, d_{t-1}] \prec\!\!\cdot\, [c_t, d_t]$. We declare $\iota(w)$ to be the permutation whose descending star network $F_{\iota(w)}$ of order $n-1$ is obtained from F_w by deleting the path from source n to sink n, and, in the case that $d_1 = n$ and $d_2 = n-1$ ($d_t = n$ and $d_{t-2} = n-1$), by contracting one edge whose vertices correspond to the central vertices of $G_{[c_1,n]}$ and $G_{[c_2,n-1]}$ ($G_{[c_{t-1},n-1]}$ and $G_{[c_t,n]}$). Equivalently, $F_{\iota(w)}$ is the zig-zag network corresponding to the concatenation

$$\begin{cases} G_{[c_1,n-1]} & \text{if } t = 1, \\ G_{[c_2,d_2]} \circ \cdots \circ G_{[c_t,d_t]} & \text{if } d_1 = n \text{ and } d_2 = n-1, \\ G_{[c_1,d_1]} \circ \cdots \circ G_{[c_{t-1},d_{t-1}]} & \text{if } d_t = n \text{ and } d_{t-1} = n-1, \\ G_{[c_1,n-1]} \circ G_{[c_2,d_2]} \circ \cdots \circ G_{[c_t,d_t]} & \text{if } d_1 = n \text{ and } d_2 < n-1, \\ G_{[c_1,d_1]} \circ \cdots \circ G_{[c_{t-1},d_{t-1}]} \circ G_{[c_t,n-1]} & \text{if } d_t = n \text{ and } d_{t-1} < n-1. \end{cases} \tag{9.6}$$

For example, let $n = 6$. F_{256431} corresponds to the concatenation $G_{[3,6]} \circ G_{[2,5]} \circ G_{[1,2]}$, with $d_2 = 5 = n-1$. Removing the path from source 6 to sink 6, and contracting the edge whose endpoints correspond to the central vertices of $G_{[3,6]}$ and $G_{[2,5]}$, we obtain $F_{\iota(256431)}$, which can be shown (as in the example preceding Theorem 3.3) to be F_{25431}.

$F_{256431} =$  $= F_{25431}$. (9.7)

Similarly, F_{246531} corresponds to the concatenation $G_{[3,6]} \circ G_{[2,4]} \circ G_{[1,2]}$, with $d_2 = 4 < n-1$. Removing the path from source 6 to sink 6, we obtain $F_{\iota(246531)}$, which can be shown to be F_{24531}.

$F_{246531} =$  $= F_{24531}$. (9.8)

9.1 Right-anchored, row-semistrict F-tableaux and $O(F)$

Inversions in right-anchored, row-semistrict path tableaux are closely related to the polynomials $\{O(F) \mid F \text{ a zig-zag network of order } 1, \ldots, n\}$ in Definition 9.1. In order to state this relationship precisely (Lemma 9.3) and state a third combinatorial formula for $\psi_q^\lambda(q_{e,w} C'_w(q))$ (Theorem 9.4), we define a family of sets $\{Z(F) \mid F \text{ a zig-zag network}\}$ of tableaux and maps between these. For F a zig-zag network of order m, let $Z(F)$ be the set of right-anchored, row-semistrict F-tableaux of type e and shape (m). Note that if F is not connected, then $Z(F) = \emptyset$, since a right-anchored F-tableau cannot be row-semistrict when F is disconnected.

Now let F_w be a connected zig-zag network of order m which corresponds to the concatenation (9.4), and let $[b, m]$ be the unique interval in (9.4) to contain m. Define a map

$$\begin{aligned}\gamma : Z(F_w) &\to Z(F_{\iota(w)}) \times \{0, 1, \ldots, m-b-1\}\\ U &\mapsto (U', k),\end{aligned}$$

by declaring U' to be the tableau obtained from U by deleting π_n, and by declaring k to be the number of indices in the interval $[b, m-1]$ appearing to the left of m in $L(U)$.

For example, consider the network F_{256431} in (9.7) and let U be the tableau

$$\boxed{\pi_4}\boxed{\pi_5}\boxed{\pi_6}\boxed{\pi_3}\boxed{\pi_2}\boxed{\pi_1}\ .$$

Then the unique interval in G containing $m = 6$ is $[3, 6]$, and there are two indices in this interval appearing to the left of 6 in $L(U)$. Thus $\gamma(U) = (U', 2)$ where U' is the tableau

$$\boxed{\pi_4}\boxed{\pi_5}\boxed{\pi_3}\boxed{\pi_2}\boxed{\pi_1}\ .$$

Lemma 9.2. *For each connected zig-zag network F_w of order m, the map γ is a bijection. Furthermore, if $\gamma(U) = (U', k)$ then $\mathrm{INV}(U^R) = \mathrm{INV}(U'^R) + k$.*

Proof. To see that γ is well-defined, fix $U \in Z(F_w)$ and let $L(U) = (i_1, \ldots, i_m = 1)$, where $i_j = m$ and $j < m$. Clearly U' is right-anchored. If $j = 1$ then $U' = (\pi_{i_2}, \ldots, \pi_{i_m})$ is row-semistrict. If $j \geqslant 1$, then $U' = (\pi_{i_1}, \ldots, \pi_{i_{j-1}}, \pi_{i_{j+1}}, \ldots, \pi_{i_m})$ is also row-semistrict, since $\pi_m \not>_{P(F_w)} \pi_{i_{j+1}}$ implies that $\pi_{i_{j-1}} \not>_{P(F_{\iota(w)})} \pi_{i_{j+1}}$.

To invert γ, find an entry i_d of $L(U')$ which belongs to $[b, m-1]$ and has exactly k indices from the interval $[b, m-1]$ to its left. (This is possible, since $0 \leqslant k \leqslant m-b-1$.) Now create a new F_w-tableau by inserting π_m into U' immediately before π_{i_d}. This map is well-defined, because π_{i_d} intersects π_m. It is clear that the map inverts γ. Since U has type e, it is clear that the number of inversions in U^R involving π_m is equal to the number of indices in the interval $[b, m-1]$ appearing to the left of m in $L(U)$. It follows that $\mathrm{INV}(U^R) = \mathrm{INV}(U'^R) + k$. ☐

By the above lemma, we can interpret $O(F)$ as a generating function for inversions in tableaux belonging to $Z(F)$.

Lemma 9.3. *For each $w \in \mathfrak{S}_m$ avoiding the patterns 3412 and 4231, we have*

$$\sum_{U \in Z(F_w)} q^{\mathrm{INV}(U^R)} = O(F_w). \tag{9.9}$$

Proof. When $m = 1$, both sides of (9.9) are 1. Now assume that (9.9) holds for all zig-zag networks corresponding to 3412-avoiding, 4231-avoiding permutations in $\mathfrak{S}_1, \ldots, \mathfrak{S}_{m-1}$, and consider $w \in \mathfrak{S}_m$ avoiding the patterns 3412 and 4231. Let F_w correspond to the concatenation (9.4). If F_w is disconnected, then $Z(F_w) = \emptyset$ and both sides of (9.9) are 0. If F_w is connected, let $[b, m]$ be the unique interval in (9.4) to contain m. By induction and Lemma 9.2 we have

$$\sum_{U \in Z(F_w)} q^{\mathrm{INV}(U^R)} = \sum_{k=0}^{m-b-1} q^k \sum_{U' \in Z(F_{\iota(w)})} q^{\mathrm{INV}(U'^R)} = [m-b]_q O(F_{\iota(w)}). \tag{9.10}$$

By Observation 3.2 we may assume that we have $t = 1$, or $[b, m] = [c_1, d_1] \prec\!\!\cdot\ [c_2, d_2]$, or $[c_{t-1}, d_{t-1}] \prec\!\!\cdot\ [c_t, d_t] = [b, m]$. In the first case, the expression (9.10) is $[m - b]_q!$. In the second case, by Definition 9.1 and (9.6), it is

$$\begin{cases} [m-c_1]_q \dfrac{[m-1-c_1]_q![d_2-c_2]_q!\cdots[d_t-c_t]_q!}{\displaystyle\prod_{\substack{[c_i,d_i]\prec\!\cdot[c_j,d_j]\\ c_i<c_j}}[d_i-c_j]_q! \prod_{\substack{[c_i,d_i]\prec\!\cdot[c_j,d_j]\\ c_j<c_i}}[d_j-c_i]_q!} & \text{if } d_1=m,\ d_2<m-1, \\ \dfrac{[m-c_1]_q!}{[d_2-c_1]_q!}\dfrac{[d_2-c_2]_q!\cdots[d_t-c_t]_q!}{\displaystyle\prod_{\substack{[c_i,d_i]\prec\!\cdot[c_j,d_j]\\ c_i<c_j}}[d_i-c_j]_q! \prod_{\substack{[c_i,d_i]\prec\!\cdot[c_j,d_j]\\ c_j<c_i\\ i\geqslant 2}}[d_j-c_i]_q!} & \text{if } d_1=m,\ d_2=m-1. \end{cases} \tag{9.11}$$

In the third case, we obtain an expression similar to (9.11). In all cases, the expression is equal to $O(F_w)$. ☐

Now we can state the precise relationship between the polynomials $O(F)$ and inversions in right-anchored, row-semistrict path tableaux.

Theorem 9.4. *Let $w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231. For $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, we have*

$$\psi_q^\lambda(q_{e,w}C'_w(q)) = [\lambda_1]_q \cdots [\lambda_r]_q \sum_{\substack{I\vdash[n]\\ \mathrm{type}(I)=\lambda}} q^{\mathrm{INV}(V(F_w,I)_1\circ\cdots\circ V(F_w,I)_r)} O(F_w|_{I_1}) \cdots O(F_w|_{I_r}). \tag{9.12}$$

Proof. By Theorem 9.1, $\psi_q^\lambda(q_{e,w}C'_w(q))$ is equal to the right-hand side of (9.2). Grouping terms in the sum and using (4.3), we may rewrite this expression as

$$[\lambda_1]_q \cdots [\lambda_r]_q \sum_{\substack{I\vdash[n]\\ \mathrm{type}(I)=\lambda}} \sum_U q^{\mathrm{INV}(U_1^R\circ\cdots\circ U_r^R)} = [\lambda_1]_q \cdots [\lambda_r]_q \sum_{\substack{I\vdash[n]\\ \mathrm{type}(I)=\lambda}} \sum_U q^{\mathrm{INV}(U_1^R)+\cdots+\mathrm{INV}(U_r^R)+\mathrm{INV}(U^\top)}, \tag{9.13}$$

where U now varies over the subset of right-anchored, row-semistrict tableaux of type e and shape λ satisfying $U_j = I_j$ for each component of the appropriate ordered set partition $I = (I_1, \ldots, I_r)$. For fixed I, this inner sum can be rewritten as

$$\sum_{W^{(1)}} q^{\mathrm{INV}((W^{(1)})^R)} \cdots \sum_{W^{(r)}} q^{\mathrm{INV}((W^{(r)})^R)} \sum_U q^{\mathrm{INV}(U^\top)},$$

where $W^{(j)}$ varies over right-anchored, row-semistrict $F_w|_{I_j}$-tableaux of shape λ_j and type $e \in \mathfrak{S}_{\lambda_j}$, and U again varies as in (9.13). By Lemma 9.3, the first r sums are equal to $O(F_w|_{I_1}), \ldots, O(F_w|_{I_r})$, and it is easy to see that for any tableau U in the last sum, we have $\mathrm{INV}(U^\top) = \mathrm{INV}(V(F_w, I)_1 \circ \cdots \circ V(F_w, I)_r)$. Thus we obtain the right-hand side of (9.12). ☐

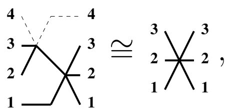
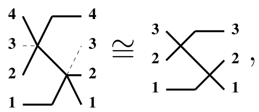
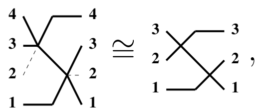
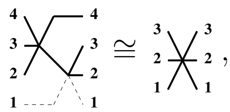
For example, consider again the descending star network F_{3421} in (5.10) and the expression in (9.12). The ordered set partitions of [4] of type 31 are 123|4, 124|3, 134|2, and 234|1. Corresponding to the set partitions I are the tableaux $V(F_{3421}, I)$ of shape 31

$$\begin{array}{|c|cc}\hline \pi_4 & \multicolumn{2}{c}{} \\ \hline \pi_1 & \multicolumn{1}{|c|}{\pi_2} & \multicolumn{1}{c|}{\pi_3} \\ \hline \end{array}\,, \quad \begin{array}{|c|cc}\hline \pi_3 & \multicolumn{2}{c}{} \\ \hline \pi_1 & \multicolumn{1}{|c|}{\pi_2} & \multicolumn{1}{c|}{\pi_4} \\ \hline \end{array}\,, \quad \begin{array}{|c|cc}\hline \pi_2 & \multicolumn{2}{c}{} \\ \hline \pi_1 & \multicolumn{1}{|c|}{\pi_3} & \multicolumn{1}{c|}{\pi_4} \\ \hline \end{array}\,, \quad \begin{array}{|c|cc}\hline \pi_1 & \multicolumn{2}{c}{} \\ \hline \pi_2 & \multicolumn{1}{|c|}{\pi_3} & \multicolumn{1}{c|}{\pi_4} \\ \hline \end{array}\,,$$

respectively, where $(\pi_1, \pi_2, \pi_3, \pi_4)$ is the unique path family of type e covering F_{3421}. These in turn yield tableaux $V(F_{3421}, I)_1 \circ V(F_{3421}, I)_2$ of shape 4

$$\begin{array}{|c|c|c|c|}\hline \pi_1 & \pi_2 & \pi_3 & \pi_4 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline \pi_1 & \pi_2 & \pi_4 & \pi_3 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline \pi_1 & \pi_3 & \pi_4 & \pi_2 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|}\hline \pi_2 & \pi_3 & \pi_4 & \pi_1 \\ \hline \end{array}\,,$$

having 0, 1, 2, and 2 inversions, respectively. The subnetworks $F_{3421}|_{123}$, $F_{3421}|_{124}$, $F_{3421}|_{134}$, $F_{3421}|_{234}$ and polynomials $O(F_{3421}|_{123})$, $O(F_{3421}|_{124})$, $O(F_{3421}|_{134})$, $O(F_{3421}|_{234})$ are

$$[3-1]_q! = 1+q, \qquad \frac{[3-2]_q![2-1]_q!}{[2-2]_q!} = 1, \qquad \frac{[3-2]_q![2-1]_q!}{[2-2]_q!} = 1, \qquad [3-1]_q! = 1+q.$$

On the other hand, each subnetwork $F_{3421}|_i$ is simply a path and satisfies $O(F_{3421}|_i) = 1$. Thus the four set partitions contribute $q^0(1+q)(1)$, $q^1(1)(1)$, $q^2(1)(1)$, $q^2(1+q)(1)$, or a total of $1 + 2q + 2q^2 + q^3$ to $[3]_q[1]_q(1 + 2q + 2q^2 + q^3) = \psi_q^{31}(q_{e,3421}C'_{3421}(q))$.

The special case $\lambda = (n)$ of Theorem 9.4 confirms a conjecture of Haiman [11, Conj. 4.1] concerning evaluations of the form $\phi_q^{(n)}(q_{e,w}C'_w(q)) = \psi_q^{(n)}(q_{e,w}C'_w(q))$.

Proposition 9.5. *Let $w = w_1 \cdots w_n \in \mathfrak{S}_n$ avoid the pattern 312 and define the sequence $(f(1), \ldots, f(n))$ by $f(j) = \max\{w_1, \ldots, w_j\}$. Then we have*

$$\psi_q^{(n)}(q_{e,w}C'_w(q)) = [n]_q[f(1)-1]_q[f(2)-2]_q \cdots [f(n-1)-(n-1)]_q. \tag{9.14}$$

Proof. Setting $\lambda = (n)$ in Theorem 9.4, we have

$$\psi_q^{(n)}(q_{e,w}C'_w(q)) = [n]_q q^0 O(F_w). \tag{9.15}$$

If F_w is not connected, then both sides of (9.15) are 0, and for some index k the prefix $w_1 \cdots w_k$ of w belongs to $\mathfrak{S}_k$. Thus $f(k) = k$ and the right-hand side of (9.14) is 0 as well.

Assume therefore that F_w is connected. Since w avoids the pattern 312, F_w is a descending star network and the intervals in the corresponding concatenation (9.4) form the chain $[c_1, d_1] \prec\!\!\cdot \cdots \prec\!\!\cdot [c_t, d_t]$ with $d_1 = n$, $c_t = 1$. Thus the formula (9.5) for $O(F_w)$ becomes

$$[d_1 - c_1]_q! \frac{[d_2 - c_2]_q!}{[d_2 - c_1]_q!} \cdots \frac{[d_t - c_t]_q!}{[d_t - c_{t-1}]_q!} = \Big([d_1 - c_1]_q[d_1 - (c_1+1)]_q \cdots [d_1 - (n-1)]_q\Big) \cdot$$
$$\Big([d_2 - c_2]_q[d_2 - (c_2+1)]_q \cdots [d_2 - (c_1-1)]_q\Big) \cdots \Big([d_t - 1]_q[d_t - 2]_q \cdots [d_t - (c_{t-1}-1)]_q\Big).$$

Defining $g(j) = \min\{i \,|\, j \in [c_i, d_i]\}$ for $j = 1, \ldots, n$, we may now rewrite (9.15) as

$$\psi_q^{(n)}(q_{e,w} C'_w(q)) = [n]_q [d_{g(1)} - 1]_q [d_{g(2)} - 2]_q \cdots [d_{g(n-1)} - (n-1)]_q. \tag{9.16}$$

Finally we claim that $d_{g(j)} = f(j)$ for $j = 1, \ldots, n-1$. We have $f(j) \leqslant d_{g(j)}$ because there are no paths in F_w from source j to sinks $d_{g(j)} + 1, \ldots, n$ and therefore by Observation 3.4, no paths from sources $1, \ldots, j-1$ to these sinks either. Similarly, we have $f(j) \geqslant d_{g(j)}$ because j belongs to the interval $[c_{g(j)}, d_{g(j)}]$ and $w_{c_{g(j)}} = d_{g(j)}$. ☐

It is straightforward to show that the right-hand side of (9.16) coincides with the expression in [23, Thm. 7.1] for $(-1)^{n-1}n = (-1)^{n-\ell(n)} z_{(n)}$ times the coefficient of p_n in $X_{P,q}$.

9.2 Cylindrical F-tableaux and $O(F)$

In Theorem 9.13, we will prove an analog of Theorem 9.1 in which sums are taken over (left-anchored) cylindrical F-tableaux. To do so, we partition the set of cylindrical F-tableaux into equivalence classes as follows. Fix a permutation $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231, an integer partition $\lambda \vdash n$, and an ordered set partition $I = (I_1, \ldots, I_r)$ of $[n]$ of type λ. Let $\mathcal{C}(I, F_w)$ be the set of cylindrical F_w-tableaux U such that for $j = 1, \ldots, r$, the set of entries of $L(U_j)$ is equal to I_j. Let $\mathcal{C}_L(I, F_w)$ be the subset of these tableaux which are left-anchored. Now the cylindrical analogs of the sums in (9.1) and (9.2) are

$$\begin{aligned}
\sum_U q^{\mathrm{INV}(U_1 \circ \cdots \circ U_r)} &= \sum_{\substack{I \vdash [n] \\ \mathrm{type}(I) = \lambda}} \sum_{U \in \mathcal{C}(I, F_w)} q^{\mathrm{INV}(U_1 \circ \cdots \circ U_r)}, \\
\sum_U q^{\mathrm{INV}(U_1 \circ \cdots \circ U_r)} &= \sum_{\substack{I \vdash [n] \\ \mathrm{type}(I) = \lambda}} \sum_{U \in \mathcal{C}_L(I, F_w)} q^{\mathrm{INV}(U_1 \circ \cdots \circ U_r)},
\end{aligned} \tag{9.17}$$

where the left-hand sums are over cylindrical F_w-tableaux of shape λ and left-anchored cylindrical F_w-tableaux of shape λ, respectively. In both cases, it is easy to show that the inner right-hand sum factors as in Theorem 9.4. To state this factorization explicitly, we relate $\mathrm{INV}(U_1 \circ \cdots \circ U_r)$ to intervals in the concatenation (9.4).

Lemma 9.6. *Let $w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231, and let $[c_1, d_1], \ldots, [c_t, d_t]$ be the intervals appearing in the concatenation (9.4) of star networks that corresponds to F_w. Let U be a cylindrical F_w-tableau having r rows, and fix indices $p_1 < p_2$ in $[r]$. Then we have*

$$\begin{aligned}
\#\{(\pi_a, \pi_b) \in U_{p_2} \times U_{p_1} \,|\, (\pi_b, \pi_a) \text{ an inversion in } U_1 \circ \cdots \circ U_r\} \\
= \#\{(a, b) \in L(U)_{p_2} \times L(U)_{p_1} \,|\, c_j \leqslant a < b \leqslant d_j \text{ for some } j\}.
\end{aligned} \tag{9.18}$$

Proof. Let A and B denote the sets on the left- and right-hand sides of (9.18), respectively. Define a map $\varphi : A \to B$ as follows, assuming $(\pi_a, \pi_b) \in A$. If a and b belong to a common interval $[c_i, d_i]$, then set $\varphi((\pi_a, \pi_b)) = (a, b)$. Otherwise, read U_{p_1} cyclically from left to right starting at π_b, and let π_c be the first path which lies entirely above π_a. Then set $\varphi((\pi_a, \pi_b)) = (a, c)$. We claim that φ is a bijection.

To see that φ is well defined, suppose that a and b belong to no common interval $[c_i, d_i]$, and let π_f be the path in U_{p_1} terminating at sink b. If π_f intersects π_a, then there exists a path in F_w from source a to sink b. Since $a < b$, Observation 3.4 and the comment following it imply that a and b belong to a common interval $[c_i, d_i]$, a contradiction. Thus the set of paths lying strictly above π_a in U_{p_1} is nonempty, and the path π_c is well defined. Suppose a and c belong to no common interval $[c_i, d_i]$, and let π_g be the path in U_{p_1} cyclically preceding π_c. Then π_g terminates at sink $c > a$. By our choice of π_c, the path π_g must intersect π_a. Thus there is a path in F_w from source a to sink c. But this contradicts Observation 3.4.

The inverse ξ of φ may be described as follows, assuming $(a, b) \in B$. If π_a intersects π_b, then set $\xi((a, b)) = (\pi_a, \pi_b)$. Otherwise, read U_{p_1} cyclically from right to left starting at π_b, and let π_c be the first path such that a and c belong to no common interval $[c_i, d_i]$, and $c > a$. Then set $\xi((a, b)) = (\pi_a, \pi_c)$.

To see that ξ is well defined, suppose that π_a does not intersect π_b and that $\xi((a, b)) = (\pi_a, \pi_c)$. Let d be the index of the sink of π_c, so that π_d immediately follows π_c. By our choice of π_c, we have that a and d belong to a common interval $[c_i, d_i]$ or that $d < a$. Thus by Observation 3.4, there is a path in F_w from source a to sink d, and by Lemma 3.5 paths π_a and π_c intersect.

Now we claim that φ and ξ are in fact inverse to one another. This is clear when we restrict to pairs (a, b) belonging to a common interval in (9.4) such that the paths π_a, π_b intersect. Suppose therefore that π_a intersects π_b, and that a, b belong to no common interval in (9.4). Let $\varphi((\pi_a, \pi_b)) = (a, c)$ and let $\xi((a, c)) = (\pi_a, \pi_{b'})$. Since π_c lies entirely above π_a, we have $b' \neq c$. Suppose $b' \neq b$ and let π_f be the path in U_{p_1} terminating at sink b'. By the definitions of φ and ξ, and since a, b belong to no common interval in (9.4), the row U_{p_1} (up to cyclic rotation) has the form

$$\cdots \boxed{\pi_b} \cdots \boxed{\pi_f}\boxed{\pi_{b'}} \cdots \boxed{\pi_c}\,.$$

Since a, b' belong to no common interval and $a < b'$ by the definition of ξ, we have by Observation 3.4 that there is no path in F_w from source a to sink b'. Thus π_a and π_f do not intersect, and π_f must lie entirely above π_a. It follows that $\varphi((\pi_a, \pi_b)) = (a, f) \neq (a, c)$, contradiction. Now suppose that π_a does not intersect π_b, and that a, b belong to some common interval $[c_i, d_i]$ in (9.4). Let $\xi((a, b)) = (\pi_a, \pi_c)$ and let $\varphi((\pi_a, \pi_c)) = (a, b')$. Since a and c belong to no common interval in (9.4), we have $c \neq b'$. Suppose $b' \neq b$ and let g be the sink index of $\pi_{b'}$. By the definitions of φ and ξ, and since π_b lies entirely above π_a, the row U_{p_1} (up to cyclic rotation) has the form

$$\cdots \boxed{\pi_c} \cdots \boxed{\pi_{b'}}\boxed{\pi_g} \cdots \boxed{\pi_b}\,.$$

By our choice of $\pi_{b'}$ we have $g > a$, and there is no path in F_w from source a to sink g. Thus by Observation 3.4 we have that a and g belong to no common interval in (9.4). But this implies that $\xi((a,b)) = (\pi_a, \pi_g) \neq (\pi_a, \pi_c)$, a contradiction. $\square$

Now we can factor the expressions in (9.17) as follows.

Proposition 9.7. *Fix $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231, $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$, and a set partition $I = (I_1, \dots, I_r) \vdash [n]$ of type λ. Let $V = V(F_w, I)$. Then we have*

$$\sum_U q^{\mathrm{INV}(U_1 \circ \cdots \circ U_r)} = q^{\mathrm{INV}(V_1 \circ \cdots \circ V_r))} \Big(\sum_{W^{(1)}} q^{\mathrm{INV}(W^{(1)})} \Big) \cdots \Big(\sum_{W^{(r)}} q^{\mathrm{INV}(W^{(r)})} \Big),$$

where the sums are over $U \in \mathcal{C}(I, F_w)$, $W^{(1)} \in \mathcal{C}(I_1, F_w|_{I_1}), \dots, W^{(r)} \in \mathcal{C}(I_r, F_w|_{I_r})$, or over $U \in \mathcal{C}_L(I, F_w)$, $W^{(1)} \in \mathcal{C}_L(I_1, F_w|_{I_1}), \dots, W^{(r)} \in \mathcal{C}_L(I_r, F_w|_{I_r})$.

Proof. First observe that we have a bijection $\mathcal{C}(I_1, F_w|_{I_1}) \times \cdots \times \mathcal{C}(I_r, F_w|_{I_r}) \to \mathcal{C}(I, F_w)$ defined by joining $W^{(1)}, \dots, W^{(r)}$ into a single tableau U with $U_j = W^{(j)}$. Clearly, the bijection restricts to the corresponding subsets of left-anchored tableaux, and satisfies

$$\mathrm{INV}(U_1 \circ \cdots \circ U_r) = \mathrm{INV}(W^{(1)}) + \cdots + \mathrm{INV}(W^{(r)}) + \mathrm{INV}(U^{\top}). \tag{9.19}$$

Now observe that the number $\mathrm{INV}(U^{\top})$ is equal to the left-hand side of (9.18), summed over pairs (p_1, p_2) with $p_1 < p_2$. Furthermore, by Observation 3.4, Lemma 3.5, and the definition of $V(F_w, I)$, we have that $\mathrm{INV}(V_1 \circ \cdots \circ V_r)$ is equal to the right-hand side of (9.18), summed over pairs (p_1, p_2) with $p_1 < p_2$. Thus we may rewrite (9.19) as

$$\mathrm{INV}(U_1 \circ \cdots \circ U_r) = \mathrm{INV}(W^{(1)}) + \cdots + \mathrm{INV}(W^{(r)}) + \mathrm{INV}(V_1 \circ \cdots \circ V_r),$$

as desired. $\square$

It is clear that a cylindrical tableau U is completely determined by the Young tableau $L(U)$. Under some conditions the insertion or deletion of a greatest letter in the left tableau of a cylindrical tableau yields a valid left tableau of another cylindrical tableau. In these cases, intersecting paths in the two cylindrical tableaux are closely related.

Lemma 9.8. *Let F_w be a connected zig-zag network of order n corresponding to the concatenation (9.4) of star networks (9.4), and let $[b, n]$ be the unique interval containing n.*

1. *For any cylindrical F_w-tableau U of shape (n), if T' is the Young tableau obtained from $L(U)$ by deleting the entry n, then there exists a unique cylindrical $F_{\iota(w)}$-tableau U' such that $T' = L(U')$.*

2. *For any cylindrical $F_{\iota(w)}$-tableau V' of shape $(n-1)$, let T be the Young tableau obtained from $L(V')$ by inserting the entry n cyclically before any element in $[b, n-1]$ if the interval $[b, n]$ is maximal in $\preceq$, and cyclically after any element in $[b, n-1]$ otherwise. Then there exists a unique cylindrical F_w-tableau V such that $L(V) = T$.*

3. *Let tableaux U and U' in (1) contain the path families $(\pi_1, \ldots, \pi_n)$ and $(\pi'_1, \ldots, \pi'_{n-1})$, respectively. For all pairs (i, j), if π_i, π'_i have the same sink index, and π_j, π'_j have the same sink index, then π_i and π_j intersect if and only if π'_i and π'_j intersect.*

Proof. (1) Let (i_1, i_2) be a pair of cyclically consecutive entries in T'. If these entries are also cyclically consecutive in $L(U)$, then there exists a (unique) path π_{i_1} in F_w from source i_1 to sink i_2. Since $F_{\iota(w)}$ differs from F_w by the removal of the unique path from source n to sink n and possibly the contraction of an edge to a single vertex, the image of π_{i_1} is again the unique path in $F_{\iota(w)}$ from source i_1 to sink i_2. If (i_1, i_2) are not cyclically consecutive in $L(U)$, then i_1 cyclically precedes n and i_2 cyclically follows n in $L(U)$. Thus either i_1 or i_2 belongs to $[b, n-1]$. Thus in G there is a path from source i_1 to the central vertex of $G_{[b,n]}$ and a path from this vertex to sink i_2. It follows that there is a path in F_w from source i_1 to sink i_2. Uniqueness of U' follows from uniqueness of source-to-sink paths in zig-zag networks. (See comment following Theorem 3.3.)

(2) Let (i_1, i_2) be a pair of cyclically consecutive entries in T. If these entries are also cyclically consecutive in $L(V')$, then there exists a (unique) path from source i_1 to sink i_2 in $F_{\iota(w)}$. Since each interval in the concatenation corresponding to $F_{\iota(w)}$ is equal to or is contained in an interval in the concatenation corresponding to F_w, we use Observation 3.4 and the fact that both F_w and $F_{\iota(w)}$ are connected to infer that there is a path in F_w from source i_1 to sink i_2. Again, uniqueness of V follows from uniqueness of source-to-sink paths in zig-zag networks.

(3) Let π_i, π'_i, π_j, π'_j satisfy the stated conditions. Then the source and sink indices of these paths are not equal to n. Suppose first that in the concatenation (9.4) corresponding to F_w we have $t = 1$, or $d_1 = n$ and $d_2 < n - 1$, or $d_t = n$ and $d_{t-1} < n - 1$. Then $F_{\iota(w)}$ is the subgraph of F_w obtained by deleting the unique path from source n to sink n. By the uniqueness of paths in descending star networks, we have $\pi_i = \pi'_i$ and $\pi_j = \pi'_j$. Now suppose that in (9.4) we have $d_1 = n$ and $d_2 = n-1$, or $d_t = n$ and $d_{t-1} = n-1$, and let x, y be the vertices in F_w corresponding to the central vertices of the star networks $G_{[b,n]}$ and $G_{[c_2,d_2]}$, respectively, or $G_{[c_{t-1},d_{t-1}]}$ and $G_{[b,n]}$, respectively. Then $F_{\iota(w)}$ is obtained from F_w by deleting the unique path from source n to sink n, and by contracting the edge (x, y) to a single vertex z. Thus if $\pi_i \cap \pi_j$ contains the edge (x, y) then $\pi'_i \cap \pi'_j$ contains the vertex z; if $\pi_i \cap \pi_j$ does not contain the edge (x, y), then $\pi'_i \cap \pi'_j = \pi_i \cap \pi_j$. $\square$

Given a zig-zag network F of order n, define $Y_n(F, i)$ to be the set of cylindrical F-tableaux U of shape (n) in which the first entry of $L(U)$ is i. In terms of our earlier notation, we have

$$\mathcal{C}([n], F_w) = \bigcup_{1 \leqslant i \leqslant n} Y_n(F_w, i), \qquad \mathcal{C}_L([n], F_w) = Y_n(F_w, 1), \tag{9.20}$$

where we interpret $[n]$ as the ordered set partition having one block.

Let us examine the map $U \mapsto U'$ and the numbers $\textsc{inv}(U)$, $\textsc{inv}(U')$ defined by Lemma 9.8 (1) in the case that $U \in Y(F_w, i)$ and $i \leqslant n - 1$. To be precise, for each pair (F_w, i) where F_w is a connected zig-zag network of order at least 2 with corresponding concatenation (9.4), poset $\preceq$ of intervals, and $[b, n]$ the unique interval containing n, and $1 \leqslant i \leqslant n-1$, we define a map

$$\begin{aligned} \delta_1 : Y_n(F_w, i) &\to Y_{n-1}(F_{\iota(w)}, i) \times \{0, 1, \dots, n-b-1\} \\ U &\mapsto (U', k) \end{aligned}$$

by declaring U' to be the cylindrical tableau whose left tableau is obtained from $L(U)$ by deleting n, and by declaring k to be the number of paths following π_n in U whose

$$\begin{cases} \text{sink index belongs to } [b, n-1] & \text{if } [b,n] \text{ maximal in } \preceq, \\ \text{source index belongs to } [b, n-1] & \text{otherwise.} \end{cases}$$

For example, let $n = 6$, recall (9.7), and consider the F_{256431}-tableau and its left Young tableau

$$U = \begin{array}{|c|c|c|c|c|c|} \hline 4{,}6 & 6{,}1 & 1{,}2 & 2{,}5 & 5{,}3 & 3{,}4 \\ \hline \end{array}\,, \qquad T = \begin{array}{|c|c|c|c|c|c|} \hline 4 & 6 & 1 & 2 & 5 & 3 \\ \hline \end{array}\,,$$

with $U \in Y_6(256431, 4)$, and $[3,6]$ not maximal in the poset $[3,6] \prec [2,4] \prec [1,2]$. Removing 6 from T we have the tableaux

$$T' = \begin{array}{|c|c|c|c|c|} \hline 4 & 1 & 2 & 5 & 3 \\ \hline \end{array}\,, \qquad U' = \begin{array}{|c|c|c|c|c|} \hline 4{,}1 & 1{,}2 & 2{,}5 & 5{,}3 & 3{,}4 \\ \hline \end{array}\,,$$

with $T' = L(U')$ and $U' \in Y_5(F_{25431}, 4)$. Since the only paths in U which follow π_6 and have source indices in $[3,5]$ are π_5, π_3, we have $\delta_1(U) = (U', 2)$.

Lemma 9.9. *For each connected zig-zag network F_w of order n, the map δ_1 is a bijection. Furthermore, if $\delta_1(U) = (U', k)$ then we have $\textsc{inv}(U) = \textsc{inv}(U') + k$.*

Proof. Assume that F_w corresponds to the concatenation (9.4) in which the unique interval containing n is $[b,n]$, and let (U', k) be a pair in $Y_{n-1}(F_{\iota(w)}, i) \times \{0, 1, \dots, n-b-1\}$. To invert δ_1, find an entry $j \in [b, n-1]$ in the tableau $L(U')$ with exactly $k-1$ entries in $[b, n-1]$ to its right. (This is possible, since $k \leqslant n-1-b$.) Now create a new Young tableau T by inserting the letter n into $L(U')$ immediately to the left of j if $[b,n]$ is maximal in $\preceq$, or immediately to the right of j otherwise. By Lemma 9.8 (2) there is a unique F_w-tableau U with $L(U) = T$.

To compare inversions in U and U', write

$$\begin{aligned} U &= (\pi_{i_1}, \dots, \pi_{i_n}), \\ U' &= (\pi'_{i_1}, \dots, \pi'_{i_{\ell-1}}, \pi'_{i_{\ell+1}}, \dots, \pi'_{i_n}), \end{aligned}$$

where $\pi_{i_\ell} = \pi_n$. Observe that all paths except $\pi_{i_{\ell-1}}$, π_{i_ℓ} in U have the same sources and sinks as the corresponding paths in U'. Thus by Lemma 9.8 (3), two such paths form an inversion in U if and only if the corresponding paths form an inversion in U'. Consider therefore inversions in U which involve one of the paths $\pi_{i_{\ell-1}}$, π_{i_ℓ}, and inversions in U' which involve the path $\pi'_{i_{\ell-1}}$.

Suppose first that $[b,n]$ is maximal in $\preceq$. By definition, there are k inversions in U of the form (π_{i_ℓ}, π_c), where π_c terminates at a sink having index in $[b, n-1]$. These in fact are the only inversions in U involving π_{i_ℓ}: since there are no paths in F_w from source n to sinks $1, \ldots, b-1$, Observation 3.1 implies that π_{i_ℓ} cannot intersect any path terminating at one of these sinks. Now observe that path $\pi_{i_{\ell-1}}$ terminates at sink n, while paths $\pi'_{i_{\ell-1}}$ and π_n terminate at sink $i_{\ell+1}$. By the maximality of $[b,n]$, we have that $i_{\ell+1}$ belongs to the interval $[b,n]$. Thus the paths $\pi_{i_{\ell-1}}$ and $\pi'_{i_{\ell-1}}$ are identical from their sources up to the vertex of F_w ($F_{\iota(w)}$) corresponding to the central vertex of $G_{[b,n]}$. It follows that any path in U intersects $\pi_{i_{\ell-1}}$ if and only if the corresponding path in U intersects $\pi'_{i_{\ell-1}}$. Therefore we have $\textsc{inv}(U) = \textsc{inv}(U') + k$.

Now suppose that $[b,n]$ is not maximal in $\preceq$. Then it must be minimal, and since $\pi_{i_{\ell-1}}$ terminates at sink n, we have $i_{\ell-1} \geqslant b$. Consider paths π_c with $c \geqslant b$. By the minimality of $[b,n]$ in $\preceq$, we have that π_c intersects $\pi_{i_{\ell-1}}$ and π_{i_ℓ} at the vertex of F_w ($F_{\iota(w)}$) corresponding to the central vertex of $G_{[b,n]}$. Thus $(\pi_c, \pi_{i_{\ell-1}})$ or $(\pi_{i_{\ell-1}}, \pi_c)$ is an inversion in U if and only if the corresponding pair is an inversion in U'. By definition, there are k inversions in U of the form (π_{i_ℓ}, π_c) with $c \geqslant b$. Now consider paths π_c with $c < b$. By Observation 3.1, no pair $(\pi_{i_{\ell-1}}, \pi_c)$ is an inversion in U, since there is no path from source c to sink n in F_w On the other hand, the paths π_{i_ℓ} and $\pi'_{i_{\ell-1}}$ are identical from the vertex of F_w (or $F_{\iota(w)}$) corresonding to the central vertex of $G_{[b,n]}$ to sink $i_{\ell+1}$. Thus each pair (π_{i_ℓ}, π_c) is an inversion in U if and only if $(\pi'_{i_{\ell-1}}, \pi'_c)$ is an inversion in U'. It follows again that $\textsc{inv}(U) = \textsc{inv}(U') + k$. ☐

Note that in the example preceding Lemma 9.9 we have $\textsc{inv}(U) = 6 = \textsc{inv}(U') + 2$, and $\delta_1(U) = (U', 2)$.

Now let us examine a map $U \mapsto U'$ and the numbers $\textsc{inv}(U)$, $\textsc{inv}(U')$ closely related to those defined by Lemma 9.8 (1) in the case that $U \in Y(F_w, n)$. To be precise, for each connected zig-zag network F_w of order at least 2 with corresponding concatenation (9.4), poset $\preceq$ of intervals, and $[b,n]$ the unique interval containing n, we define a map

$$\begin{aligned} \delta_2 : Y_n(F_w, n) &\to \bigcup_{j=b}^{n-1} Y_{n-1}(F_{\iota(w)}, j), \\ U &\mapsto U', \end{aligned} \tag{9.21}$$

by declaring U' to be the cylindrical tableau whose left tableau is obtained from $L(U) = (i_1, \ldots, i_n)$ by

$$L(U') = \begin{cases} (i_2, \ldots, i_n) & \text{if } [b,n] \text{ maximal in } \preceq, \\ (i_n, i_2, \ldots, i_{n-1}) & \text{otherwise.} \end{cases}$$

The tableau U' exists and is unique by Lemma 9.8 (1). If $[b,n]$ is maximal in $\preceq$ then any path beginning at source n must terminate at a sink in this interval. Thus we have $i_2 \in [b, n-1]$. If $[b,n]$ is not maximal in $\preceq$ it must be minimal, and any path terminating at sink n must begin at a source in this interval. Thus we have $i_n \in [b, n-1]$. It follows that U' belongs to the union in (9.21).

For example, let $n = 6$, recall (9.8), and consider the F_{246531}-tableau and its left Young tableau

$$U = \boxed{6,1}\boxed{1,2}\boxed{2,3}\boxed{3,5}\boxed{5,4}\boxed{4,6}\,, \qquad T = \boxed{6}\boxed{1}\boxed{2}\boxed{3}\boxed{5}\boxed{4}\,,$$

with $U \in Y_6(246531, 6)$, and T having rightmost entry 4. Removing 6 from T and moving 4 to the leftmost position, we have the tableaux

$$T' = \begin{array}{|c|c|c|c|c|}\hline 4 & 1 & 2 & 3 & 5 \\ \hline \end{array}\,, \qquad U' = \begin{array}{|c|c|c|c|c|}\hline 4,1 & 1,2 & 2,3 & 3,5 & 5,4 \\ \hline \end{array}\,,$$

with $T' = L(U')$ and $U' \in Y_5(F_{24531}, 4)$.

Lemma 9.10. *For each connected zig-zag network F_w of order n, the map δ_2 is a bijection. Furthermore, if $\delta_2(U) = U' \in Y_{n-1}(F_{\iota(w)}, j)$ and b is as in (9.21) then we have*

$$\textsc{inv}(U) = \begin{cases} \textsc{inv}(U') + n - b & \text{if } [b,n] \text{ maximal in } \preceq, \\ \textsc{inv}(U') + 2n - 2j - 1 & \text{otherwise.} \end{cases}$$

Proof. To invert δ_2, let U' be an element of $Y(F_{\iota(w)}, j)$ for some $1 \leqslant j \leqslant n-1$. Create a Young tableau T from $L(U')$ by inserting n into the leftmost position and if $[b,n]$ is not maximal in $\preceq$ by moving j to the rightmost position. By Lemma 9.8 (2) there is a unique F_w-tableau $U = (\pi_{i_1}, \dots, \pi_{i_n})$ satisfying $L(U) = T$.

To compare inversions in U and U', observe first that the paths $\pi_{i_2}, \dots, \pi_{i_{n-1}}$ in U have the same sources and sinks as the corresponding paths in

$$U' = \begin{cases} (\pi'_{i_2}, \dots, \pi'_{i_n}) & \text{if } [b,n] \text{ is maximal in } \preceq, \\ (\pi'_{i_n} = \pi'_j, \pi'_{i_2}, \dots, \pi'_{i_{n-1}}) & \text{otherwise.} \end{cases}$$

Thus by Lemma 9.8 (3), two such paths form an inversion in U if and only if the corresponding paths form an inversion in U'. Consider therefore inversions in U which involve the paths $\pi_{i_1} = \pi_n$ or π_{i_n}, and inversions in U' which involve the path π'_{i_n}.

If $[b,n]$ is maximal in $\preceq$, the path $\pi_{i_1} = \pi_n$ in U terminates at sink $i_2 \in [b, n-1]$. Thus the path, which precedes all others in U, intersects only those $n-b$ other paths in U which terminate at sinks $[b,n] \smallsetminus \{i_2\}$, and thus also pass through the vertex of F_w corresponding to the central vertex of $G_{[b,n]}$. Now observe that π_{i_n} terminates at sink n of F_w, while π'_{i_n} terminates at sink i_2 of $F_{\iota(w)}$. Since $i_2 \geqslant b$, the paths π_{i_n} and π'_{i_n} are identical up to the vertex of F_w ($F_{\iota(w)}$) which corresponds to the central vertex of $G_{[b,n]}$. Thus any path π_{i_k} in U intersects π_{i_n} if and only if the corresponding path π'_{i_k} in U' intersects π'_{i_n}. It follows that $\textsc{inv}(U) = \textsc{inv}(U') + b - n$ in this case.

If $[b,n]$ is not maximal in $\preceq$, then it is minimal. Since π_{i_n} terminates at sink n, we have that $i_n \geqslant b$. Thus π_{i_n} intersects and follows the $n - i_n + 1$ paths $\pi_{i_n+1}, \dots, \pi_n$ which all intersect at least at the vertex of F_w (or $F_{\iota(w)}$) corresponding to the central vertex of $G_{[b,n]}$. Now observe that the paths π_n and π'_{i_n} both terminate at sink i_2. Thus, since $i_n \geqslant b$, the paths are identical from the vertex of F_w ($F_{\iota(w)}$) which corresponds to the central vertex of $G_{[b,n]}$ until sink i_2. Thus any path π_k in U intersects π_n if and only if the corresponding path π'_k in U' intersects π'_{i_n}. It follows that $\textsc{inv}(U) = \textsc{inv}(U') + n - j + 1$ in this case. ☐

Note that in the example preceding Lemma 9.10, the network F_{246531} (9.8) of order $m = 6$ corresponds to the concatenation $G_{[3,6]} \circ G_{[2,4]} \circ G_{[1,2]}$, and the interval $[3,6]$ is not maximal in the poset $[3,6] \prec [2,4] \prec [1,2]$. The tableaux U, U' satisfy $j = 4$, $\mathrm{INV}(U) = 6$, $\mathrm{INV}(U') = 3$, and $\mathrm{INV}(U) = \mathrm{INV}(U') + 2m - 2j - 1$.

Now we return to the problem of factoring the inner sums in (9.17).

Proposition 9.11. *Fix $w \in \mathfrak{S}_m$ avoiding the patterns 3412 and 4231 and an index $j \in [m]$. Then we have*

$$\sum_{U \in Y_m(F_w, j)} q^{\mathrm{INV}(U)} = q^{j-1} O(F_w). \tag{9.22}$$

Proof. We prove (9.22) by induction on m. The only zig-zag network of order $m = 1$ is F_e, $e \in \mathfrak{S}_1$. Thus the set $Y_1(F_e, 1)$ consists of one tableau of shape (1) having no inversions. The left- and right-hand sides of (9.22) are therefore $q^0 = 1$ and $q^0 O(F_e) = [0]_q! = 1$, respectively.

Now assume (9.22) to hold for zig-zag networks corresponding to 3412-avoiding, 4231-avoiding permutations in $\mathfrak{S}_{m-1}$ and consider F_w with $w \in \mathfrak{S}_m$ avoiding the patterns 3412 and 4231. Let F_w correspond to the concatenation (9.4) of star networks with $[b, m]$ the unique interval containing m, let $u = \iota(w)$, and fix an integer $j \in [m]$. If F_w is disconnected, then the set $Y_m(F_w, j)$ is empty and both sides of (9.22) are 0. Assume therefore that F_w is connected. If $j < m$, then by Lemma 9.9 and induction we have

$$\sum_{U \in Y_m(F_w, j)} q^{\mathrm{INV}(U)} = \sum_{k=0}^{m-b-1} q^k \sum_{U' \in Y_{m-1}(F_u, j)} q^{\mathrm{INV}(U')} = [m-b]_q q^{j-1} O(F_u). \tag{9.23}$$

Similarly, if $j = m$, then by Lemma 9.10 and induction we have

$$\begin{aligned} \sum_{U \in Y_m(F_w, m)} q^{\mathrm{INV}(U)} &= \begin{cases} \displaystyle\sum_{k=b}^{m-1} q^{m-b} q^{k-1} O(F_u) & \text{if } [b,m] \text{ maximal in } \preceq, \\ \displaystyle\sum_{k=b}^{m-1} q^{2m-2k-1} q^{k-1} O(F_u) & \text{otherwise,} \end{cases} \\ &= [m-b]_q q^{m-1} O(F_u). \end{aligned} \tag{9.24}$$

Now applying (9.11) to (9.23), (9.24), we have for $j = 1, \ldots, m$ that

$$\sum_{U \in Y_m(F_w, j)} q^{\mathrm{INV}(U)} = q^{j-1} [m-b]_q O(F_u) = q^{j-1} O(F_w).$$

$\square$

Applying (9.20) to the previous result, we have the following.

Corollary 9.12. *Let $w \in \mathfrak{S}_m$ avoid the patterns 3412 and 4231. Then we have*

$$\sum_{U \in \mathcal{C}([m], F_w)} q^{\mathrm{INV}(U)} = [m]_q O(F_w), \qquad \sum_{U \in \mathcal{C}_L([m], F_w)} q^{\mathrm{INV}(U)} = O(F_w).$$

Now we have the following q-analogs of Theorem 4.7 (iv-a).

Theorem 9.13. *Let $w \in \mathfrak{S}_n$ avoid the patterns* 3412 *and* 4231*. For $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$, we have*

$$\psi_q^\lambda(q_{e,w}C'_w(q)) = \sum_U q^{\mathrm{INV}(U_1 \circ \cdots \circ U_r)}, \tag{9.25}$$

where the sum is over all cylindrical F_w-tableaux of shape λ, and

$$\psi_q^\lambda(q_{e,w}C'_w(q)) = [\lambda_1]_q \cdots [\lambda_r]_q \sum_U q^{\mathrm{INV}(U_1 \circ \cdots \circ U_r)}, \tag{9.26}$$

where the sum is over all left-anchored cylindrical F_w-tableaux of shape λ.

Proof. Rewrite the sums above as in (9.17) and factor the resulting inner sums as in Proposition 9.7. By Corollary 9.12, the right-hand sides of (9.25), (9.26) are both equal to

$$[\lambda_1]_q \cdots [\lambda_r]_q \sum_{\substack{I \vdash [n] \\ \mathrm{type}(I) = \lambda}} q^{\mathrm{INV}(V(F_w, I)_1 \circ \cdots \circ V(F_w, I)_r)} O(F_w|_{I_1}) \cdots O(F_w|_{I_r}).$$

By Theorem 9.4, this is $\psi_q^\lambda(q_{e,w}C'_w(q))$. ☐

For example, consider the descending star network F_{3421} in (5.10) and the sum in (9.25). It is easy to verify that there are eighteen cylindrical F_{3421}-tableaux of shape 31. Four of these are

$$\begin{array}{|c|c|c|} \cline{1-1} 1,1 \\ \hline 4,3 & 3,2 & 2,4 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|} \cline{1-1} 2,2 \\ \hline 1,3 & 3,4 & 4,1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|} \cline{1-1} 3,3 \\ \hline 1,2 & 2,4 & 4,1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|} \cline{1-1} 4,4 \\ \hline 2,3 & 3,1 & 1,2 \\ \hline \end{array}\,, \tag{9.27}$$

where i, j represents the unique path from source i to sink j. These tableaux U of shape 31 yield tableaux $U_1 \circ U_2$ of shape 4,

$$\begin{array}{|c|c|c|c|} \hline 4,3 & 3,2 & 2,4 & 1,1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|} \hline 1,3 & 3,4 & 4,1 & 2,2 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|} \hline 1,2 & 2,4 & 4,1 & 3,3 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|} \hline 2,3 & 3,1 & 1,2 & 4,4 \\ \hline \end{array}\,, \tag{9.28}$$

which have 5, 2, 1, and 2 inversions, respectively. Together, they contribute $q + 2q^2 + q^5$ to $\psi_q^{31}(q_{e,3421}C'_{3421}(q)) = 1 + 3q + 5q^2 + 5q^3 + 3q^4 + q^5$. Now consider the sum in (9.26). It is easy to verify that there are six left-anchored cylindrical F_{3421}-tableaux of shape 31: the second and third tableaux in (9.27) and the four tableaux

$$\begin{array}{|c|c|c|} \cline{1-1} 1,1 \\ \hline 2,3 & 3,4 & 4,2 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|} \cline{1-1} 1,1 \\ \hline 2,4 & 4,3 & 3,2 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|} \cline{1-1} 4,4 \\ \hline 1,2 & 2,3 & 3,1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|} \cline{1-1} 4,4 \\ \hline 1,3 & 3,2 & 2,1 \\ \hline \end{array}\,.$$

These tableaux U of shape 31 yield six tableaux $U_1 \circ U_2$ of shape 4: the second and third tableaux in (9.28) and the four tableaux

$$\begin{array}{|c|c|c|c|} \hline 2,3 & 3,4 & 4,2 & 1,1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|} \hline 2,4 & 4,3 & 3,2 & 1,1 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|} \hline 1,2 & 2,3 & 3,1 & 4,4 \\ \hline \end{array}\,, \quad \begin{array}{|c|c|c|c|} \hline 1,3 & 3,2 & 2,1 & 4,4 \\ \hline \end{array}\,,$$

which have 2, 1, 2, 3, 0, and 1 inversions, respectively. Together, the six tableaux contribute $1 + 2q + 2q^2 + q^3$ to $[3]_q[1]_q(1 + 2q + 2q^2 + q^3) = \psi_q^{31}(q_{e,3421}C'_{3421}(q))$.

As a consequence of Theorem 9.13, we now have the following analog of Corollary 6.5. Say that a partition λ of n is a *refinement* of another partition (or composition) μ of n if λ can be obtained from μ by replacing each part μ_i by an integer partition of μ_i and sorting the results into weakly decreasing order.

Corollary 9.14. *Let $w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231and let the component sizes of F_w be $\mu = (\mu_1, \dots, \mu_s)$. Then we have*

$$\psi_q^\lambda(q_{e,w}C'_w(q)) = 0.$$

if and only if λ is not a refinement of μ.

Proof. It is clear that if λ is not a refinement of μ, then there is no cylindrical F_w-tableau of shape λ. Therefore by Theorem 9.13 we have that $\psi_q^\lambda(q_{e,w}C'_w(q)) = 0$. Suppose on the other hand that $\lambda = (\lambda_1, \dots, \lambda_r)$ is a refinement of μ, and let $J_1, \dots, J_s$ be the subintervals of $[n]$ corresponding to the connected components of F_w. Then there exists an ordered set partition $I = (I_1, \dots, I_r) \vdash [n]$, whose type is a rearrangement of λ, such that each block of J is a union of several consecutive blocks of I. Let $\pi = (\pi_1, \dots, \pi_n)$ be the unique path family of type e covering F_w. It is clear now that we can construct at least one record-free, row-semistrict F_w tableau of type e and shape λ, by creating a row containing the paths $\pi_a, \dots, \pi_b$ (in order) for each block $[a, b]$ of I. By Theorem 9.1, we therefore have $\psi_q^\lambda(q_{e,w}C'_w(q)) \neq 0$. $\square$

It would be interesting to extend Theorem 9.13 to include a q-analog of Theorem 4.7 (iv-b). In particular the fourth identity in Equation (1.2) suggests that an answer to Problem 4.8 and its q-analog are related to a set partition of tableaux counted by ψ^λ. It is not clear whether such a partition is more easily expressed in terms of record-free, row-semistrict F_w-tableaux of type e, right-anchored, row-semistrict F_w-tableaux of type e, cylindrical F_w-tableaux, left-anchored cylindrical F_w-tableaux, or cyclically row-semistrict F_w-tableaux of type e.

Problem 9.15. Find a statistic STAT on F-tableaux such that we have

$$\psi_q^\lambda(q_{e,w}C'_w(q)) = \sum_U q^{\mathrm{STAT}(U)},$$

where the sum is over all cyclically row-semistrict F_w-tableaux of type e and shape λ.

It would also be interesting to show that all stated interpretations of $\psi_q^\lambda(q_{e,w}C'_w(q))$ remain valid if we reverse the order of concatenating rows of tableaux: this may turn out to be the easiest way to link the $H_n(q)$-traces ψ_q^λ and ϕ_q^μ.

Problem 9.16. Show that Theorems 9.1, 9.4, 9.13 remain valid if one replaces the numbers $\mathrm{INV}(U_1 \circ \cdots \circ U_r)$, $\mathrm{INV}(U_1^R \circ \cdots \circ U_r^R)$, $\mathrm{INV}(V(F_w, I)_1 \circ \cdots \circ V(F_w, I)_r)$, with $\mathrm{INV}(U_r \circ \cdots \circ U_1)$, $\mathrm{INV}(U_r^R \circ \cdots \circ U_1^R)$, $\mathrm{INV}(V(F_w, I)_r \circ \cdots \circ V(F_w, I)_1)$, respectively.

10 Results concerning $\phi_q^\lambda(q_{e,w}C'_w(q))$

Recall that the component statements of Theorem 4.7 pertaining to monomial traces are weaker than those pertaining to other traces. To state a q-analog of Theorem 4.7 (v-a), we will use several partial orders, including majorization and refinement of integer partitions. We will use the symbol $\trianglelefteq$ to denote majorization and $\leqslant_R$ to denote refinement, as defined before Corollary 9.14. We begin by stating an analog of Corollaries 6.5, 8.2.

Proposition 10.1. *Fix $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ and $w \in \mathfrak{S}_n$ avoiding the patterns 3412 and 4231. If $w_1 \cdots w_n$ has a decreasing subsequence of length greater than λ_1, then we have $\phi_q^\lambda(q_{e,w}C'_w(q)) = 0$.*

Proof. Let $w \in \mathfrak{S}_n$ have a decreasing subsequence of length greater than λ_1 and recall that there exist integers $\{a_{\lambda,\mu} \mid \lambda, \mu \vdash n\}$ such that

$$\phi_q^\lambda = \sum_{\mu \trianglerighteq \lambda^\top} a_{\lambda,\mu} \epsilon_q^\mu.$$

If $\phi_q^\lambda(q_{e,w}C'_w(q)) \neq 0$ then some partition μ in the above sum satisfies $\epsilon_q^\mu(q_{e,w}C'_w(q)) \neq 0$. But the number of parts of μ is necessarily less than or equal to λ_1. This contradicts Corollary 6.5. $\square$

Similarly, we have a partial analog of Corollary 9.14.

Proposition 10.2. *Let $w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231 and let the component sizes of F_w (in weakly decreasing order) be $\mu = (\mu_1, \ldots, \mu_r)$. Then for each partition $\lambda \vdash n$ not refining μ we have*

$$\phi_q^\lambda(q_{e,w}C'_w(q)) = 0.$$

Proof. Observe that we may rewrite the last equation in (1.2) as

$$\psi_q^\lambda = \sum_{\nu \geqslant_R \lambda} L_{\lambda,\nu} \phi_q^\nu,$$

since no row-constant Young tableau of shape λ has content ν unless λ refines ν. Inverting the matrix $(L_{\lambda,\nu})_{\lambda,\nu \vdash n}$ and evaluating traces at $q_{e,w}C'_w(q)$ we have

$$\phi_q^\lambda(q_{e,w}C'_w(q)) = \sum_{\nu \geqslant_R \lambda} L^{-1}_{\lambda,\nu} \psi_q^\lambda(q_{e,w}C'_w(q)). \tag{10.1}$$

Now suppose that we have $\lambda \not\leqslant_R \mu$. Then each partition ν in (10.1) satisfies $\nu \not\leqslant_R \mu$. By Corollary 9.14, each term on the right-hand side of (10.1) is zero. $\square$

We remark that Propositions 10.1, 10.2 are not new: they follow from [11, Prop. 4.1]. For more facts about these evaluations, see [11, Sec 4].

Now we complete the proof of Theorem 4.7 (v-a) and provide a q-analog of this result. Let $\mathcal{T}_C(F_w, \mu)$ denote the set of column-strict F_w-tableaux of shape μ and type e.

Theorem 10.3. *Let $w \in \mathfrak{S}_n$ avoid the patterns 3412 and 4231. For $\lambda_1 \leqslant 2$ we have*

$$\phi_q^{\lambda}(q_{e,w}C'_w(q)) = \begin{cases} \sum_{U \in \mathcal{T}_C(F_w,\lambda)} q^{\mathrm{INV}(U)} & \text{if for all } \mu \lhd \lambda \text{ we have } \mathcal{T}_C(F_w,\mu) = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{10.2}$$

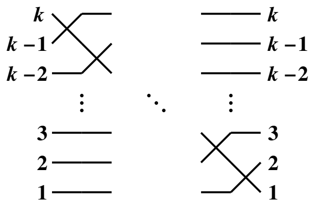
Proof. The claim is true for $\lambda = 1^n$, since $\phi_q^{1^n} = \epsilon_q^{(n)}$ and the claimed formula coincides with that in Theorem 6.4. Suppose that the claim holds for $\lambda = 21^{n-2}, \dots, 2^{k-1}1^{n-2k+2}$ and consider the case $\lambda = 2^k 1^{n-2k}$ ($k \leqslant \lfloor \frac{n}{2} \rfloor$). Then we have

$$\epsilon_q^{(n-k,k)}(q_{e,w}C'_w(q)) = \sum_{i=0}^{k} M_{2^k1^{n-2k},2^i1^{n-2i}} \phi_q^{2^i1^{n-2i}}(q_{e,w}C'_w(q)), \tag{10.3}$$

where $M_{\lambda,\mu}$ is the number of column-strict Young tableaux of shape λ and content μ. It is easy to see that $M_{2^k1^{n-2k},2^i1^{n-2i}}$ is equal to $\binom{n-2i}{k-i}$. By Theorem 6.4, the left-hand side of (10.3) is the sum of $q^{\mathrm{INV}(U)}$ over $U \in \mathcal{T}_C(F_w, 2^k1^{n-2k})$.

If w has a decreasing subsequence of length three, then by Proposition 10.1, the left-hand side of (10.2) is 0. By the proof of Corollary 6.5, we have $\mathcal{T}_C(F_w,\mu) = \emptyset$ for all $\mu \unlhd \lambda$, and the right-hand side of (10.2) is 0 as well.

Assume therefore that w avoids the pattern 321. By Theorem 3.7, every connected component of F_w induces a subposet of $P(F_w)$ which is isomorphic to $P(H_k)$ where

$H_k =$  .

Let $b = b(w)$ be the number of odd components of F_w. Then it is possible to construct an F_w-tableau which has b more paths in column 1 than it has in column 2, but it is not possible to construct an F_w-tableau for which this difference is greater than b. That is, for $j = \frac{n-b}{2}$ we have $\mathcal{T}_C(F_w, 2^j1^{n-2j}) \neq \emptyset$ while

$$\mathcal{T}_C(F_w, 1^n) = \mathcal{T}_C(F_w, 21^{n-2}) = \cdots = \mathcal{T}_C(F_w, 2^{j-1}1^{n-2j+2}) = \emptyset. \tag{10.4}$$

Fix $U \in \mathcal{T}_C(F_w, 2^j1^{n-2j})$, and let the interval $[p_1, p_{2m+1}]$ of sources and sinks define an odd component of F_w. Then paths indexed by $p_1, p_3, \dots, p_{2m+1}$ belong to the first column of U while those indexed by $p_2, \dots, p_{2m}$ belong to the second column. Note that swapping the columns of the two sets of paths creates a valid column-strict F_w-tableau of shape $2^{j+1}1^{n-2j-2}$ which has the same number of inversions as U. Thus for each such tableau U, we may create a column-strict F_w-tableau U' of shape 2^k1^{n-2k} by choosing $k-j$ of the odd components and swapping the columns of the even and odd indexed paths within these components. There are $\binom{n-2j}{k-j}$ ways to do this. Conversely, every tableau $U' \in \mathcal{T}_C(F_w, 2^k1^{n-2k})$ arises in this way. Thus we have

$$\epsilon_q^{(n-k,k)}(q_{e,w}C'_w(q)) = \binom{n-2j}{k-j} \sum_{U \in \mathcal{T}_C(F_w, 2^j1^{n-2j})} q^{\mathrm{INV}(U)}. \tag{10.5}$$

If $j < k$, then we have by induction and (10.4) that $\phi_q^{2^i 1^{n-2i}}(q_{e,w}C'_w(q)) = 0$ for $i < k$, $i \neq j$. Now (10.5) implies that $\phi_q^{2^k 1^{n-2k}}(q_{e,w}C'_w(q)) = 0$, and the claim is true. If $j = k$, then we have by (10.3) – (10.4) that $\epsilon_q^{(n-k,k)}(q_{e,w}C'_w(q)) = \phi_q^{2^k 1^{n-2k}}(q_{e,w}C'_w(q))$ and by (10.5) the claim again is true. □

We remark that the obvious q-analogs of Theorem 4.7 (v-b) are false. Consider the permutation $w = 3142$ and the evaluation $\phi_q^{22}(q_{e,w}C'_w(q)) = q + q^2$. F_w is the penultimate zig-zag network in (3.4), and there are two column-strict cylindrical F_w-tableaux of shape 22,

$$\begin{array}{|c|c|}\hline 4,3 & 3,4 \\ \hline 2,1 & 1,2 \\ \hline \end{array}\ , \qquad \begin{array}{|c|c|}\hline 3,4 & 4,3 \\ \hline 1,2 & 2,1 \\ \hline \end{array}\ .$$

Unfortunately, as U varies over these tableaux we have

$$\sum_U q^{\mathrm{INV}(U)} = 1 + q^3, \qquad \sum_U q^{\mathrm{INV}(U_1 \circ U_2)} = 1 + q^2, \qquad \sum_U q^{\mathrm{INV}(U_2 \circ U_1)} = q + q^3.$$

Perhaps a correct q-analog of Theorem 4.7 (v-b) would help with the formulation of an interpretation of $\phi_q^\lambda(q_{e,w}C'_w(q))$ when w avoids the patterns 3412 and 4231. Given Theorems 5.4, 6.4, 8.1, 9.1, 9.4, and 9.13, it seems reasonable to hope that F_w-tableaux can play an important role in such an interpretation.

Problem 10.4. Find a property X of F_w-tableaux and a statistic $\textsc{stat}$ such that for $\lambda \vdash n$ and w avoiding the patterns 3412 and 4231 we have

$$\phi_q^\lambda(q_{e,w}C'_w(q)) = \sum_U q^{\mathrm{STAT}(U)},$$

where the sum is over all F_w-tableaux U of shape λ having property X.

Acknowledgements

The authors are grateful to Kaitlyn Peterson, Daniel Studenmund, and Michelle Wachs for helpful conversations, to Lehigh University, the University of Miami, and Universidad de los Andes for financial support and hospitality. The authors are also grateful to an anonymous referee for helpful comments.

References

[1] C. Athanasiadis. Power sum expansion of chromatic quasisymmetric functions (2014). arXiv:1409.2595.

[2] D. Beck, J. Remmel, and T. Whitehead. The combinatorics of transition matrices between the bases of the symmetric functions and the b_n analogues. *Discrete Math.*, **153** (1996) pp. 3–27.

[3] S. Billey and V. Lakshmibai. *Singular loci of Schubert varieties*, vol. 182 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA (2000).

[4] S. Clearman, B. Shelton, and M. Skandera. Path tableaux and combinatorial interpretations of immanants for class functions on S_n. In *23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011)*, Discrete Math. Theor. Comput. Sci. Proc., AO. Assoc. Discrete Math. Theor. Comput. Sci., Nancy (2011), pp. 233–244.

[5] R. A. Dean and G. Keller. Natural partial orders. *Canad. J. Math*, **20** (1968) pp. 535–554.

[6] P. C. Fishburn. *Interval Graphs and Interval Orders*. Wiley, New York (1985).

[7] V. Gasharov. Incomparability graphs of $(\mathbf{3}+\mathbf{1})$-free posets are s-positive. *Discrete Math.*, **157** (1996) pp. 211–215.

[8] I. Gessel and G. Viennot. Determinants and plane partitions (1989). Preprint.

[9] I. P. Goulden and D. M. Jackson. Immanants, Schur functions, and the MacMahon master theorem. *Proc. Amer. Math. Soc.*, **115**, 3 (1992) pp. 605–612.

[10] M. Guay-Paquet. A modular relation for the chromatic symmetric functions of $(\mathbf{3}+\mathbf{1})$-free posets (2013). arXiv:1306.2400.

[11] M. Haiman. Hecke algebra characters and immanant conjectures. *J. Amer. Math. Soc.*, **6**, 3 (1993) pp. 569–595.

[12] S. Karlin and G. McGregor. Coincidence probabilities. *Pacific J. Math.*, **9** (1959) pp. 1141–1164.

[13] D. Kazhdan and G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, **53** (1979) pp. 165–184.

[14] S. V. Kerov and A. M. Vershik. Characters, factor representations and K-functor of the infinite symmetric group. In *Operator algebras and group representations, Vol. II (Neptun, 1980)*, vol. 18 of *Monogr. Stud. Math.*. Pitman, Boston, MA (1984), pp. 23–32.

[15] R. C. King and B. G. Wybourne. Representations and traces of the Hecke algebras $H_n(q)$ of type A_{n-1}. *J. Math. Phys.*, **33**, 1 (1992) pp. 4–14.

[16] M. Konvalinka and M. Skandera. Generating functions for Hecke algebra characters. *Canad. J. Math.*, **63**, 2 (2011) pp. 413–435.

[17] V. Lakshmibai and B. Sandhya. Criterion for smoothness of Schubert varieties in $SL(n)/B$. *Proc. Indian Acad. Sci. (Math Sci.)*, **100**, 1 (1990) pp. 45–52.

[18] B. Lindström. On the vector representations of induced matroids. *Bull. London Math. Soc.*, **5** (1973) pp. 85–90.

[19] D. E. Littlewood. *The Theory of Group Characters and Matrix Representations of Groups*. Oxford University Press, New York (1940).

[20] R. Merris and W. Watkins. Inequalities and identities for generalized matrix functions. *Linear Algebra Appl.*, **64** (1985) pp. 223–242.

[21] A. Ram. A Frobenius formula for the characters of the Hecke algebras. *Invent. Math.*, **106**, 3 (1991) pp. 461–488.

[22] B. Sagan. *The Symmetric Group*. Springer, New York (2001).

[23] J. Shareshian and M. Wachs. Chromatic quasisymmetric functions and Hessenberg varieties. In *Configuration Spaces* (A. Bjorner, F. Cohen, C. D. Concini, C. Procesi, and M. Salvetti, eds.). Edizione Della Normale, Pisa (2012), pp. 433–460.

[24] J. Shareshian and M. Wachs. Chromatic quasisymmetric functions (2014). to appear in *Adv. Math.* arXiv:1405.4629v1.

[25] M. Skandera. On the dual canonical and Kazhdan-Lusztig bases and 3412, 4231-avoiding permutations. *J. Pure Appl. Algebra*, **212** (2008).

[26] M. Skandera and B. Reed. Total nonnegativity and $(\mathbf{3}+\mathbf{1})$-free posets. *J. Combin. Theory Ser. A*, **103** (2003) pp. 237–256.

[27] R. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, **111** (1995) pp. 166–194.

[28] R. Stanley. *Enumerative Combinatorics*, vol. 1. Cambridge University Press, Cambridge (1997).

[29] R. Stanley. *Enumerative Combinatorics*, vol. 2. Cambridge University Press, Cambridge (1999).

[30] R. Stanley and J. R. Stembridge. On immanants of Jacobi-Trudi matrices and permutations with restricted positions. *J. Combin. Theory Ser. A*, **62** (1993) pp. 261–279.

[31] J. Stembridge. Immanants of totally positive matrices are nonnegative. *Bull. London Math. Soc.*, **23** (1991) pp. 422–428.

[32] J. Stembridge. Some conjectures for immanants. *Can. J. Math.*, **44**, 5 (1992) pp. 1079–1099.

[33] J. Van der Jeugt. An algorithm for characters of Hecke algebras $H_n(q)$ of type A_{n-1}. *J. Phys. A*, **24**, 15 (1991) pp. 3719–3725.

[34] H. L. Wolfgang, III. *Two interactions between combinatorics and representation theory: Monomial immanants and Hochschild cohomology*. ProQuest LLC, Ann Arbor, MI (1997). Thesis (Ph.D.)–Massachusetts Institute of Technology.