Refined dual stable Grothendieck polynomials and generalized Bender-Knuth involutions

Pavel Galashin
Department of Mathematics
Massachusetts Institute of Technology
Cambridge
Massachusetts, U.S.A.
galashin@mit.edu

Darij Grinberg
Department of Mathematics
Massachusetts Institute of Technology
Cambridge
Massachusetts, U.S.A.
darij@mit.edu

Gaku Liu
Department of Mathematics
Massachusetts Institute of Technology
Cambridge
Massachusetts, U.S.A.
gakuliu@math.mit.edu

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Abstract

The dual stable Grothendieck polynomials are a deformation of the Schur functions, originating in the study of the K-theory of the Grassmannian. We generalize these polynomials by introducing a countable family of additional parameters, and we prove that this generalization still defines symmetric functions. For this fact, we give two self-contained proofs, one of which constructs a family of involutions on the set of reverse plane partitions generalizing the Bender-Knuth involutions on semistandard tableaux, whereas the other classifies the structure of reverse plane partitions with entries 1 and 2.

Keywords: Bender-Knuth involutions, reverse plane partitions, dual stable Grothendieck polynomials, symmetric functions.

1 Introduction

Grothendieck polynomials were introduced by Lascoux and Schützenberger [LasSch82] to model classes of structure sheaves of Schubert varieties in the K-theory of flag manifolds. *Stable Grothendieck polynomials* were introduced by Fomin and Kirillov [FomKir96] as stable limits of certain sequences of Grothendieck polynomials. In particular, the stable Grothendieck polynomials G_λ represent classes of structure sheaves of Schubert varieties in the K-theory of the Grassmannian, and can be seen as K-theory analogues of Schur functions. Buch [Buch02] gave a combinatorial formula for G_λ in terms of set-valued tableaux and described a Littlewood-Richardson rule for their structure coefficients.

Thomas Lam and Pavlo Pylyavskyy, in [LamPyl07, §9.1], and earlier Mark Shimozono and Mike Zabrocki in unpublished work of 2003, studied *dual stable Grothendieck polynomials*, the dual basis to the stable Grothendieck polynomials with respect to the Hall inner product. Lam and Pylyavskyy related these functions to the K-homology of the Grassmannian and gave a combinatorial definition of these functions in terms of *reverse plane partitions*. We now briefly recount their definition.

Let λ/μ be a skew partition. The Schur function $s_{\lambda/\mu}$ is a multivariate generating function for the semistandard tableaux of shape λ/μ. In the same vein, the dual stable Grothendieck polynomial $g_{\lambda/\mu}$ is a generating function for the reverse plane partitions of shape λ/μ; these, unlike semistandard tableaux, are only required to have their entries increase *weakly* down columns (and along rows). More precisely, $g_{\lambda/\mu}$ is a formal power series in countably many commuting indeterminates $x_1, x_2, x_3, \ldots$ defined by

$$g_{\lambda/\mu} = \sum_{\substack{T \text{ is a reverse plane} \\ \text{partition of shape } \lambda/\mu}} \mathbf{x}^{\mathrm{ircont}(T)},$$

where $\mathbf{x}^{\mathrm{ircont}(T)}$ is the monomial $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ whose i-th exponent a_i is the number of *columns* of T containing the entry i. Note that we are counting columns rather than cells here; thus, $g_{\lambda/\mu}$ is not homogeneous in general. As proven in [LamPyl07, §9.1], this power series $g_{\lambda/\mu}$ is a symmetric function.

We devise a common generalization of the dual stable Grothendieck polynomial $g_{\lambda/\mu}$ and the classical skew Schur function $s_{\lambda/\mu}$. Namely, if $t_1, t_2, t_3, \ldots$ are countably many indeterminates, then we set

$$\widetilde{g}_{\lambda/\mu} = \sum_{\substack{T \text{ is a reverse plane} \\ \text{partition of shape } \lambda/\mu}} \mathbf{t}^{\mathrm{ceq}(T)} \mathbf{x}^{\mathrm{ircont}(T)},$$

where $\mathbf{t}^{\mathrm{ceq}(T)}$ is the product $t_1^{b_1} t_2^{b_2} t_3^{b_3} \cdots$ whose i-th exponent b_i is the number of cells in the i-th row of T whose entry equals the entry of their neighbor cell directly below them. This $\widetilde{g}_{\lambda/\mu}$ becomes $g_{\lambda/\mu}$ when all the t_i are set to 1, and becomes $s_{\lambda/\mu}$ when all the t_i are set to 0.

Our main result, Theorem 5, states that $\widetilde{g}_{\lambda/\mu}$ is a symmetric function (in the variables $x_1, x_2, x_3, \ldots$). This generalizes [LamPyl07, Theorem 9.1].

We prove this result first using an elaborate generalization of the classical Bender-Knuth involutions to reverse plane partitions, and then for a second time by analyzing the structure of reverse plane partitions whose entries lie in $\{1, 2\}$. The second proof reflects back on the first, in particular providing an alternative definition of the generalized Bender-Knuth involutions constructed in the first proof, and showing that these involutions are "the only reasonable choice" in a sense that we clarify. We notice that both our proofs are explicitly bijective, unlike the proof of [LamPyl07, Theorem 9.1] which relies on computations in an algebra of operators.

The present paper is organized as follows: In Section 2, we recall classical definitions and introduce notations pertaining to combinatorics and symmetric functions. In Section 3, we define the refined dual stable Grothendieck polynomials $\widetilde{g}_{\lambda/\mu}$; we state our main result that they are symmetric functions, and do the first steps of its proof, by reducing it to a purely combinatorial statement about the existence of an involution with certain properties. In Section 4, we describe the idea of constructing this involution in an elementary way without proofs. In Section 5, we prove various properties of this involution advertised in Section 4, thus finishing the proof of our main result. In Section 6, we recapitulate the definition of the classical Bender-Knuth involution, and show that our involution is a generalization of the latter. Finally, in Section 7 we study the structure of reverse plane partitions with entries belonging to $\{1, 2\}$, which gives us an explicit formula for the $\mathbf{t}$-coefficients of $\widetilde{g}_{\lambda/\mu}(x_1, x_2, 0, 0, \ldots; \mathbf{t})$. As a consequence, we observe that the involution constructed in Sections 4 and 5 is the unique involution that shares certain natural properties with the classical Bender-Knuth involutions.

An extended abstract of this paper, omitting the proofs, is to appear as [GaGrLi16].

1.1 Acknowledgments

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2 Notations and definitions

Let us begin by defining our notations.

2.1 Partitions and tableaux

We set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$.

A sequence $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ of nonnegative integers is called a *weak composition* if the sum of its entries is finite. This sum is denoted by $|\alpha|$. We shall always write α_i for the i-th entry of a weak composition α.

A *partition* is a weak composition $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ satisfying $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \cdots$. As usual, we often omit trailing zeroes when writing a weak composition. For instance, the partition $(5, 2, 1, 0, 0, 0, \ldots)$ can be also written as $(5, 2, 1, 0)$ or $(5, 2, 1)$.

We identify each partition λ with the subset $\left\{(i, j) \in \mathbb{N}_+^2 \mid j \leqslant \lambda_i\right\}$ of $\mathbb{N}_+^2$, called *the Young diagram of* λ. We draw this subset as a left-aligned table of empty boxes, where the box $(1, 1)$ is in the top-left corner while the box $(2, 1)$ is directly below it; this is the *English notation*, also known as the *matrix notation*. See [Fulton97] for the detailed definition.

$$\begin{array}{|c|c|c|}\hline & 6 & 3 \\ \hline 2 & 4 & \\ \hline 3 & 4 & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & 3 & 3 \\ \hline 2 & 3 & \\ \hline 3 & 4 & \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline & 3 & 3 \\ \hline 2 & 4 & \\ \hline 3 & 7 & \\ \hline \end{array}$$

(a) (b) (c)

Figure 1: Fillings of $(3,2,2)/(1)$: (a) is not an rpp as it has a 4 below a 6, (b) is an rpp but not a semistandard tableau as it has a 3 below a 3, (c) is a semistandard tableau (and hence also an rpp).

A *skew partition* λ/μ is a pair (λ, μ) of partitions satisfying $\mu \subseteq \lambda$ (as subsets of the plane). In this case, we shall also often use the notation λ/μ for the set-theoretic difference of λ and μ; this difference is called the *Young diagram* of λ/μ.

If λ/μ is a skew partition, then a *filling* of λ/μ means a map $T : \lambda/\mu \to \mathbb{N}_+$. It is visually represented by drawing λ/μ and filling each box c with the entry $T(c)$. Three examples of a filling can be found on Figure 1.

A filling $T : \lambda/\mu \to \mathbb{N}_+$ of λ/μ is called a *reverse plane partition of shape* λ/μ if its values increase weakly in each row of λ/μ from left to right and in each column of λ/μ from top to bottom. If, in addition, the values of T increase strictly down each column, then T is called a *semistandard tableau of shape* λ/μ. See Fulton's [Fulton97] for an exposition of properties and applications of semistandard tableaux. We denote the set of all reverse plane partitions of shape λ/μ by $\mathrm{RPP}(\lambda/\mu)$. We abbreviate reverse plane partitions as *rpps*.

Examples of an rpp, of a non-rpp and of a semistandard tableau can be found on Figure 1.

2.2 Symmetric functions

A *symmetric function* is defined to be a power series in countably many indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbb{Z}$ that is invariant under permutations of $x_1, x_2, x_3, \ldots$ with finite support and that has the degrees of its monomials bounded from above.

The symmetric functions form a ring, which is called the *ring of symmetric functions* and denoted by Λ. In [LamPyl07] this ring is denoted by Sym, while the notation Λ is reserved for the set of all partitions. Much research has been done on symmetric functions and their relations to Young diagrams and tableaux; see [Stan99, Chapter 7], [Macdon95] and [GriRei15, Chapter 2] for expositions.

Given a filling T of a skew partition λ/μ, its *content* is a weak composition $\mathrm{cont}(T) = (r_1, r_2, r_3, \ldots)$, where $r_i = |T^{-1}(i)|$ is the number of entries of T equal to i. For a skew partition λ/μ, we define the *Schur function* $s_{\lambda/\mu}$ to be the formal power series

$$s_{\lambda/\mu}(x_1, x_2, \ldots) = \sum_{\substack{T \text{ is a semistandard} \\ \text{tableau of shape } \lambda/\mu}} \mathbf{x}^{\mathrm{cont}(T)} \in \mathbb{Z}[[x_1, x_2, x_3, \ldots]].$$

Here, for every weak composition $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$, we define a monomial $\mathbf{x}^\alpha$ to be $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$. These Schur functions are symmetric:

Proposition 1. *We have $s_{\lambda/\mu} \in \Lambda$ for every skew partition λ/μ.*

This result appears, e.g., in [Stan99, Theorem 7.10.2] and [GriRei15, Proposition 2.11]; it is commonly proven bijectively using the so-called *Bender-Knuth involutions*. We shall recall the definitions of these involutions in Section 6.

Replacing "semistandard tableau" by "rpp" in the definition of a Schur function in general gives a non-symmetric function. Nevertheless, Lam and Pylyavskyy [LamPyl07, §9] have been able to define symmetric functions from rpps, albeit using a subtler construction instead of the content $\operatorname{cont}(T)$.

Namely, for a filling T of a skew partition λ/μ, we define its *irredundant content* (or, by way of abbreviation, its *ircont statistic*) as the weak composition $\operatorname{ircont}(T) = (r_1, r_2, r_3, \ldots)$ where r_i is the number of *columns* of T that contain an entry equal to i. For instance, if T_a, T_b, and T_c are the fillings from Figure 1, then their irredundant contents are

$$\operatorname{ircont}(T_a) = (0,1,2,1,0,1), \ \operatorname{ircont}(T_b) = (0,1,3,1), \ \operatorname{ircont}(T_c) = (0,1,3,1,0,0,1),$$

because, for example, T_a has one column with a 4 in it (so $(\operatorname{ircont}(T_a))_4 = 1$) and T_b contains three columns with a 3 (so $(\operatorname{ircont}(T_b))_3 = 3$).

Notice that if T is a semistandard tableau, then $\operatorname{cont}(T)$ and $\operatorname{ircont}(T)$ coincide.

For the rest of this section, we fix a skew partition λ/μ. Now, the *dual stable Grothendieck polynomial* $g_{\lambda/\mu}$ is defined to be the formal power series

$$\sum_{\substack{T \text{ is an rpp} \\ \text{of shape } \lambda/\mu}} \mathbf{x}^{\operatorname{ircont}(T)}.$$

Unlike the Schur function $s_{\lambda/\mu}$, it is (in general) not homogeneous, because whenever a column of an rpp T contains an entry several times, the corresponding monomial $\mathbf{x}^{\operatorname{ircont}(T)}$ "counts" this entry only once. It is fairly clear that the highest-degree homogeneous component of $g_{\lambda/\mu}$ is $s_{\lambda/\mu}$. Therefore, $g_{\lambda/\mu}$ can be regarded as an inhomogeneous deformation of the Schur function $s_{\lambda/\mu}$.

Lam and Pylyavskyy, in [LamPyl07, §9.1], have shown the following fact:

Proposition 2. *We have $g_{\lambda/\mu} \in \Lambda$ for every skew partition λ/μ.*

They prove this proposition using generalized plactic algebras [FomGre06, Lemma 3.1], and also give a second, combinatorial proof for the case $\mu = \varnothing$ by explicitly expanding $g_{\lambda/\varnothing}$ as a sum of Schur functions.

In the next section, we shall introduce a refinement of these $g_{\lambda/\mu}$, and later we will reprove Proposition 2 in a bijective and elementary way.

3 Refined dual stable Grothendieck polynomials

3.1 Definition

Let $\mathbf{t} = (t_1, t_2, t_3, \ldots)$ be a sequence of further indeterminates. For any weak composition α, we define $\mathbf{t}^\alpha$ to be the monomial $t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} \cdots$.

If T is a filling of a skew partition λ/μ, then a *redundant cell* of T is a cell of λ/μ whose entry is equal to the entry directly below it. That is, a cell (i, j) of λ/μ is redundant if $(i+1, j)$ is also a cell of λ/μ and $T(i, j) = T(i+1, j)$. Notice that a semistandard tableau is the same thing as an rpp which has no redundant cells.

If T is a filling of λ/μ, then we define the *column equalities vector* (or, by way of abbreviation, the *ceq statistic*) of T to be the weak composition $\operatorname{ceq}(T) = (c_1, c_2, c_3, \ldots)$ where c_i is the number of $j \in \mathbb{N}_+$ such that (i, j) is a redundant cell of T. Visually speaking, $(\operatorname{ceq}(T))_i$ is the number of columns of T whose entry in the i-th row equals their entry in the $(i+1)$-th row. For instance, for fillings T_a, T_b, T_c from Figure 1 we have $\operatorname{ceq}(T_a) = (0, 1)$, $\operatorname{ceq}(T_b) = (1)$, and $\operatorname{ceq}(T_c) = ()$.

Notice that $|\operatorname{ceq}(T)|$ is the number of redundant cells in T, so we have

$$|\operatorname{ceq}(T)| + |\operatorname{ircont}(T)| = |\lambda/\mu| \tag{1}$$

for all rpps T of shape λ/μ.

Let now λ/μ be a skew partition. We set

$$\widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = \sum_{\substack{T \text{ is an rpp} \\ \text{of shape } \lambda/\mu}} \mathbf{t}^{\operatorname{ceq}(T)} \mathbf{x}^{\operatorname{ircont}(T)} \in \mathbb{Z}[t_1, t_2, t_3, \ldots][[x_1, x_2, x_3, \ldots]].$$

Let us give some examples of $\widetilde{g}_{\lambda/\mu}$.

Example 3.

(a) If λ/μ is a single row with n cells, then for each rpp T of shape λ/μ we have $\operatorname{ceq}(T) = (0, 0, \ldots)$ and $\operatorname{ircont}(T) = \operatorname{cont}(T)$. In fact, any rpp of shape λ/μ is a semistandard tableau in this case. Therefore we get

$$\widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = h_n(\mathbf{x}) = \sum_{a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n} x_{a_1} x_{a_2} \cdots x_{a_n}.$$

Here $h_n(\mathbf{x})$ is the n-th complete homogeneous symmetric function.

(b) If λ/μ is a single column with n cells, then, by (1), for all rpps T of shape λ/μ we have $|\operatorname{ceq}(T)| + |\operatorname{ircont}(T)| = n$, so in this case

$$\widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = \sum_{k=0}^{n} e_k(t_1, t_2, \ldots, t_{n-1})\, e_{n-k}(x_1, x_2, \ldots) = e_n(t_1, t_2, \ldots, t_{n-1}, x_1, x_2, \ldots),$$

where $e_i(\xi_1, \xi_2, \xi_3, \ldots)$ denotes the i-th elementary symmetric function in the indeterminates $\xi_1, \xi_2, \xi_3, \ldots$.

The power series $\widetilde{g}_{\lambda/\mu}$ generalize the power series $g_{\lambda/\mu}$ and $s_{\lambda/\mu}$ studied before. The following proposition is clear:

Proposition 4. *Let λ/μ be a skew partition.*

(a) *Specifying* $\mathbf{t} = (1, 1, 1, \ldots)$ *yields* $\widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = g_{\lambda/\mu}(\mathbf{x})$.

(b) *Specifying* $\mathbf{t} = (0, 0, 0, \ldots)$ *yields* $\widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = s_{\lambda/\mu}(\mathbf{x})$.

3.2 The symmetry statement

Our main result is now the following:

Theorem 5. *Let λ/μ be a skew partition. Then $\widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t})$ is symmetric in $\mathbf{x}$.*

Here, "symmetric in $\mathbf{x}$" means "invariant under all permutations of the indeterminates $x_1, x_2, x_3, \ldots$ with finite support", while $t_1, t_2, t_3, \ldots$ remain unchanged. Clearly, Theorem 5 implies the symmetry of $g_{\lambda/\mu}$ and $s_{\lambda/\mu}$ due to Proposition 4. We shall prove Theorem 5 bijectively, and the core of our proof will be the following restatement of Theorem 5:

Theorem 6. *Let λ/μ be a skew partition and let $i \in \mathbb{N}_+$. Then, there exists an involution $\mathbf{B}_i : \operatorname{RPP}(\lambda/\mu) \to \operatorname{RPP}(\lambda/\mu)$ which preserves the ceq statistics and acts on the ircont statistic by the transposition of its i-th and $i+1$-th entries.*

This involution $\mathbf{B}_i$ is a generalization of the i-th Bender-Knuth involution defined for semistandard tableaux (see, e.g., [GriRei15, proof of Proposition 2.11]), but its definition is more complicated than that of the latter. Defining it and proving its properties will take a significant part of this paper. We will compare our involution $\mathbf{B}_i$ with the i-th Bender-Knuth involution in Section 6.

Proof of Theorem 5 using Theorem 6. We need to prove that $\widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t})$ is invariant under all finite permutations of the indeterminates $x_1, x_2, x_3, \ldots$. The group of such permutations is generated by $s_1, s_2, s_3, \ldots$, where for each $i \in \mathbb{N}_+$, we define s_i as the permutation of $\mathbb{N}_+$ which transposes i with $i+1$ and leaves all other positive integers unchanged. Hence, it suffices to show that $\widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t})$ is invariant under each of the permutations $s_1, s_2, s_3, \ldots$. In other words, it suffices to show that $s_i \cdot \widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = \widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t})$ for each $i \in \mathbb{N}_+$.

So fix $i \in \mathbb{N}_+$. In order to prove $s_i \cdot \widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = \widetilde{g}_{\lambda/\mu}(\mathbf{x}; \mathbf{t})$, it suffices to find a bijection $\mathbf{B}_i : \operatorname{RPP}(\lambda/\mu) \to \operatorname{RPP}(\lambda/\mu)$ with the property that every $T \in \operatorname{RPP}(\lambda/\mu)$ satisfies $\operatorname{ceq}(\mathbf{B}_i(T)) = \operatorname{ceq}(T)$ and $\operatorname{ircont}(\mathbf{B}_i(T)) = s_i \cdot \operatorname{ircont}(T)$. Theorem 6 yields precisely such a bijection, which turns out to be an involution. $\square$

3.3 Reduction to 12-rpps

Fix a skew partition λ/μ. We shall make one further simplification before we step to the actual proof of Theorem 6. We define a *12-rpp* to be an rpp whose entries all belong to the set $\{1, 2\}$. We let $\mathrm{RPP}^{12}(\lambda/\mu)$ be the set of all 12-rpps of shape λ/μ.

Lemma 7. *There exists an involution* $\mathbf{B} : \mathrm{RPP}^{12}(\lambda/\mu) \to \mathrm{RPP}^{12}(\lambda/\mu)$ *which preserves the ceq statistic and switches the number of columns containing a* 1 *with the number of columns containing a* 2*. In other words,* $\mathbf{B}$ *switches the first two entries of the ircont statistic.*

This Lemma implies Theorem 6: for any $i \in \mathbb{N}_+$ and for T an rpp of shape λ/μ, we construct $\mathbf{B}_i(T)$ as follows:

- Ignore all entries of T not equal to i or $i + 1$.
- Replace all entries i by 1 and all entries $i + 1$ by 2. We get a 12-rpp T' of some smaller shape, which is still a skew partition. (The skew partition itself is not always uniquely determined, but its Young diagram is, which is all that matters to us here.)
- Replace T' by $\mathbf{B}(T')$.
- In $\mathbf{B}(T')$, replace back all entries 1 by i and all entries 2 by $i + 1$.
- Finally, restore the remaining entries of T that were ignored on the first step.

It is clear that this operation acts on $\mathrm{ircont}(T)$ by a transposition of the i-th and $i+1$-th entries. The fact that it does not change $\mathrm{ceq}(T)$ is also not hard to show: the set of redundant cells remains the same.

4 Construction of B

In this section we are going to sketch the definition of $\mathbf{B}$ and state some of its properties. We postpone the proofs until the next section.

For the whole Sections 4 and 5, we shall be working in the situation of Lemma 7. In particular, we fix a skew partition λ/μ.

A *12-table* means a filling $T : \lambda/\mu \to \{1, 2\}$ of λ/μ such that the entries of T are weakly increasing down columns. We do not require them to be weakly increasing along rows. Every column of a 12-table is a sequence of the form $(1, 1, \ldots, 1, 2, 2, \ldots, 2)$. We say that such a sequence is

- *1-pure* if it is nonempty and consists purely of 1's,
- *2-pure* if it is nonempty and consists purely of 2's,
- *mixed* if it contains both 1's and 2's.

Definition 8. For a 12-table T, we define $\mathrm{flip}(T)$ to be the 12-table obtained from T by changing each column of T as follows:

- **If** this column is 1-pure, we replace all its entries by 2's, so that it becomes 2-pure.

  Otherwise, if this column is 2-pure, we replace all its entries by 1's, so that it becomes 1-pure.

  Otherwise this column is either mixed or empty, in which case we do not change it.

If T is a 12-rpp then $\mathrm{flip}(T)$ need not be a 12-rpp, because it can contain a 2 to the left of a 1 in some row. We say that a positive integer k is a *descent* of a 12-table P if there is a 2 in the column k and there is a 1 to the right of it in the column $k+1$. We will encounter three possible kinds of descents depending on the types of columns k and $k+1$:

(M1) The k-th column of P is mixed and the $(k+1)$-th column of P is 1-pure.

(2M) The k-th column of P is 2-pure and the $(k+1)$-th column of P is mixed.

(21) The k-th column of P is 2-pure and the $(k+1)$-th column of P is 1-pure.

For an arbitrary 12-table it can happen also that two mixed columns form a descent, but such a descent will never arise in our process.

For each of the three types of descents, we will define what it means to *resolve* this descent. This is an operation which transforms the 12-table P by changing the entries in its k-th and $(k+1)$-th columns. These changes can be informally explained by Figure 2:

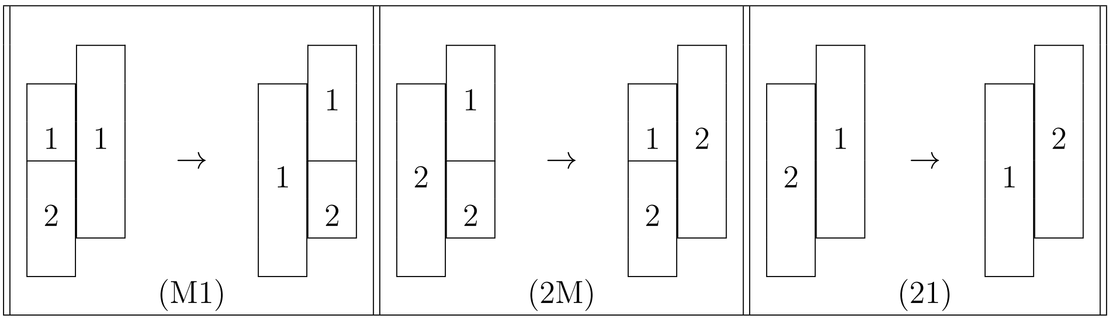


Figure 2: The three descent-resolution steps

For example, if k is a descent of type (M1) in a 12-table P, then we define the 12-table $\mathrm{res}_k P$ as follows: the k-th column of $\mathrm{res}_k P$ is 1-pure; the $(k+1)$-th column of $\mathrm{res}_k P$ is mixed and the highest 2 in it is in the same row as the highest 2 in the k-th column of P; all other columns of $\mathrm{res}_k P$ are copied over from P unchanged. The definitions of $\mathrm{res}_k P$ for the other two types of descents are similar, and will be elaborated upon in Subsection 5.3. We say that $\mathrm{res}_k P$ is obtained from P by *resolving* the descent k, and we say that passing from P to $\operatorname{res}_k P$ constitutes a *descent-resolution step*. Of course, a 12-table P can have several descents and thus offer several ways to proceed by descent-resolution steps.

Now the map $\mathbf{B}$ is defined as follows: take any 12-rpp T and apply flip to it to get a 12-table flip(T). Next, apply descent-resolution steps to flip(T) in arbitrary order until we get a 12-table with no descents left. Put $\mathbf{B}(T) := P$. A rigorous statement of this is Definition 19.

In the next section we will see that $\mathbf{B}(T)$ is well-defined: the process terminates after a finite number of descent-resolution steps, and the result does not depend on the order of steps. We will also see that $\mathbf{B}$ is an involution $\operatorname{RPP}^{12}(\lambda/\mu) \to \operatorname{RPP}^{12}(\lambda/\mu)$ that satisfies the claims of Lemma 7. An alternative proof of all these facts can be found in Section 7.

5 Proof of Lemma 7

We shall now prove Lemma 7 in detail.

Recall that every column of a 12-table is a sequence of the form $(1, 1, \ldots, 1, 2, 2, \ldots, 2)$. If s is a sequence of the form $(1, 1, \ldots, 1, 2, 2, \ldots, 2)$, then we define the *signature* $\operatorname{sig}(s)$ of s to be

$$\operatorname{sig}(s) = \begin{cases} 0, & \text{if } s \text{ is 2-pure or empty;} \\ 1, & \text{if } s \text{ is mixed;} \\ 2, & \text{if } s \text{ is 1-pure} \end{cases} .$$

Definition 9. For any 12-table T, we define a nonnegative integer $\ell(T)$ by

$$\ell(T) = \sum_{h \in \mathbb{N}_+} h \cdot \operatorname{sig}(\text{the } h\text{-th column of } T).$$

For instance, if T is the 12-table

$$\begin{array}{|c|c|c|c|c|c|} \hline & & 1 & 2 & 1 & 2 \\ \hline & 1 & 1 & 2 & & \\ \hline 2 & 1 & 1 & 2 & & \\ \hline 2 & 2 & & & & \\ \hline \end{array} \qquad (2)$$

then $\ell(T) = 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 0 + 5 \cdot 2 + 6 \cdot 0 + 7 \cdot 0 + 8 \cdot 0 + \cdots = 18$.

5.1 Descents, separators, and benign 12-tables

In Subsection 4, we have defined a "descent" of a 12-table. Let us reword this definition in more formal terms: If T is a 12-table, then we define a *descent* of T to be a positive integer i such that there exists an $r \in \mathbb{N}_+$ satisfying $(r, i) \in \lambda/\mu$, $(r, i+1) \in \lambda/\mu$, $T(r, i) = 2$ and $T(r, i+1) = 1$. For instance, the descents of the 12-table shown in (2) are 1 and 4. Clearly, a 12-rpp of shape λ/μ is the same as a 12-table which has no descents.

If T is a 12-table, and if $k \in \mathbb{N}_+$ is such that the k-th column of T is mixed, then we define $\operatorname{sep}_k T$ to be the smallest $r \in \mathbb{N}_+$ such that $(r, k) \in \lambda/\mu$ and $T(r, k) = 2$. Thus, every 12-table T, every $r \in \mathbb{N}_+$ and every $k \in \mathbb{N}_+$ such that the k-th column of T is mixed and such that $(r, k) \in \lambda/\mu$ satisfy

$$T(r, k) = \begin{cases} 1, & \text{if } r < \operatorname{sep}_k T; \\ 2, & \text{if } r \geqslant \operatorname{sep}_k T. \end{cases} \tag{3}$$

If T is a 12-table, then we let $\operatorname{seplist} T$ denote the list of all values $\operatorname{sep}_k T$, in the order of increasing k. Here k ranges over all positive integers for which the k-th column of T is mixed. For instance, if T is

		1	1	1
	2	1	1	2
1	2	1		
2	2	2		

then $\operatorname{sep}_1 T = 4$, $\operatorname{sep}_3 T = 4$, and $\operatorname{sep}_5 T = 2$, and there are no other k for which $\operatorname{sep}_k T$ is defined; thus, $\operatorname{seplist} T = (4, 4, 2)$.

We say that a 12-table T is *benign* if the list $\operatorname{seplist} T$ is weakly decreasing. For example, the 12-table in (2) is benign, but replacing its third column by $(1, 2, 2)$ and its fourth column by $(1, 1, 2)$ would yield a 12-table which is not benign.

All 12-rpps are benign 12-tables, but the converse is not true. If T is a benign 12-table, then

$$\text{there exists no descent } k \text{ of } T \text{ such that both the } k\text{-th column of } T \\ \text{and the } (k+1)\text{-th column of } T \text{ are mixed.} \tag{4}$$

Let $\mathrm{BT}^{12}(\lambda/\mu)$ denote the set of all benign 12-tables; we have $\mathrm{RPP}^{12}(\lambda/\mu) \subseteq \mathrm{BT}^{12}(\lambda/\mu)$.

Recall the map flip defined for 12-tables in Definition 8. If $T \in \mathrm{BT}^{12}(\lambda/\mu)$ then $\operatorname{flip}(T) \in \mathrm{BT}^{12}(\lambda/\mu)$ as well because T and $\operatorname{flip}(T)$ have the same mixed columns. Thus, the map flip restricts to a map $\mathrm{BT}^{12}(\lambda/\mu) \to \mathrm{BT}^{12}(\lambda/\mu)$ which we will also denote flip.

Remark 10. *It is clear that* flip *is an involution on* $\mathrm{BT}^{12}(\lambda/\mu)$ *that preserves* ceq *and* seplist *but switches the first two entries of* ircont. *In other words, if some* $T \in \mathrm{BT}^{12}(\lambda/\mu)$ *has* $\operatorname{ircont}(T) = (a, b, 0, 0, 0, \ldots)$, *then* $\operatorname{ircont}(\operatorname{flip}(T)) = (b, a, 0, 0, 0, \ldots)$.

5.2 Plan of the proof

Let us now briefly sketch the ideas behind the rest of the proof before we go into them in detail. The map $\operatorname{flip} : \mathrm{BT}^{12}(\lambda/\mu) \to \mathrm{BT}^{12}(\lambda/\mu)$ does not generally send 12-rpps to 12-rpps, i.e. it does not restrict to a map $\mathrm{RPP}^{12}(\lambda/\mu) \to \mathrm{RPP}^{12}(\lambda/\mu)$. However, we shall amend this by defining a way to transform any benign 12-table into a 12-rpp by what we call "resolving descents". The process of "resolving descents" will be a stepwise process, and will be formalized in terms of a binary relation $\Rrightarrow$ on the set $\mathrm{BT}^{12}(\lambda/\mu)$ which we will soon introduce. The intuition behind saying "$P \Rrightarrow Q$" is that the benign 12-table P has a descent, resolving which yields the benign 12-table Q. Starting with a benign 12-table P, we can repeatedly resolve descents until this is no longer possible. We have some freedom in performing this process, because at any step there can be a choice of several descents to resolve; but we will see that the final result does not depend on the process. Hence, the final result can be regarded as a function of P. We will denote it by $\operatorname{norm} P$, and we will see that it is a 12-rpp. We will then define a map $\mathbf{B} : \mathrm{RPP}^{12}(\lambda/\mu) \to \mathrm{RPP}^{12}(\lambda/\mu)$ by $\mathbf{B}(T) = \operatorname{norm}(\operatorname{flip} T)$, and show that it is an involution satisfying the properties that we want it to satisfy.

5.3 Resolving descents

Now we come to the details.

Let $k \in \mathbb{N}_+$. Let $P \in \mathrm{BT}^{12}(\lambda/\mu)$. Assume for the whole Subsection 5.3 that k is a descent of P. Thus, the k-th column of P must contain at least one 2. Hence, the k-th column of P is either mixed or 2-pure. Similarly, the $(k+1)$-th column of P is either mixed or 1-pure. But the k-th and the $(k+1)$-th columns of P cannot both be mixed (by (4), because P is benign). Thus, exactly one of the following three statements holds:

(M1) The k-th column of P is mixed and the $(k+1)$-th column of P is 1-pure.

(2M) The k-th column of P is 2-pure and the $(k+1)$-th column of P is mixed.

(21) The k-th column of P is 2-pure and the $(k+1)$-th column of P is 1-pure.

Now, we define a new 12-table $\operatorname{res}_k P$ as follows (see Figure 2 for illustration):

- If we have (M1), then $\operatorname{res}_k P$ is the 12-table defined as follows: The k-th column of $\operatorname{res}_k P$ is 1-pure; the $(k+1)$-th column of $\operatorname{res}_k P$ is mixed and satisfies $\operatorname{sep}_{k+1}(\operatorname{res}_k P) = \operatorname{sep}_k P$; all other columns of $\operatorname{res}_k P$ are copied over from P unchanged. We encourage the reader to check that this 12-table is well-defined.
- If we have (2M), then $\operatorname{res}_k P$ is the 12-table defined as follows: The k-th column of $\operatorname{res}_k P$ is mixed and satisfies $\operatorname{sep}_k(\operatorname{res}_k P) = \operatorname{sep}_{k+1} P$; the $(k+1)$-th column of $\operatorname{res}_k P$ is 2-pure; all other columns of $\operatorname{res}_k P$ are copied over from P unchanged.
- If we have (21), then $\operatorname{res}_k P$ is the 12-table defined as follows: The k-th column of $\operatorname{res}_k P$ is 1-pure; the $(k+1)$-th column of $\operatorname{res}_k P$ is 2-pure; all other columns of $\operatorname{res}_k P$ are copied over from P unchanged.

In either case, $\operatorname{res}_k P$ is a well-defined 12-table. It is furthermore clear that

$$\operatorname{seplist}(\operatorname{res}_k P) = \operatorname{seplist} P.$$

Thus, $\operatorname{res}_k P$ is benign, since P is benign; that is, $\operatorname{res}_k P \in \mathrm{BT}^{12}(\lambda/\mu)$. We say that $\operatorname{res}_k P$ is the 12-table obtained by *resolving* the descent k in P.

Example 11. Let P be the 12-table on the left:

$$
\begin{array}{ccccc} & & 1 & 2 & 1 \\ & 1 & 1 & 2 & \\ 2 & 1 & 1 & & \\ 2 & 2 & 1 & & \\ 2 & & & & \end{array}
\qquad
\begin{array}{ccccc} & & 1 & 2 & 1 \\ & 2 & 1 & 2 & \\ 1 & 2 & 1 & & \\ 2 & 2 & 1 & & \\ 2 & & & & \end{array}
\qquad
\begin{array}{ccccc} & & 1 & 2 & 1 \\ & 1 & 1 & 2 & \\ 2 & 1 & 1 & & \\ 2 & 1 & 2 & & \\ 2 & & & & \end{array}
\qquad
\begin{array}{ccccc} & & 1 & 1 & 2 \\ & 1 & 1 & 1 & \\ 2 & 1 & 1 & & \\ 2 & 2 & 1 & & \\ 2 & & & & \end{array}
$$

$$P \qquad \operatorname{res}_1 P \qquad \operatorname{res}_2 P \qquad \operatorname{res}_4 P$$

Then P is a benign 12-table, and its descents are 1, 2 and 4. We have $\operatorname{sep}_2 P = 4$.

If we set $k = 1$ then we have (2M), if we set $k = 2$ then we have (M1), and if we set $k = 4$ then we have (21). We can resolve each of these three descents; the results are the three 12-tables on the right.

We notice that each of the three 12-tables $\operatorname{res}_1 P$, $\operatorname{res}_2 P$ and $\operatorname{res}_4 P$ still has descents. In order to get a 12-rpp from P, we will have to keep resolving these descents until none remain.

We now observe some further properties of $\operatorname{res}_k P$:

Proposition 12. *Let $P \in \mathrm{BT}^{12}(\lambda/\mu)$ and $k \in \mathbb{N}_+$ be such that k is a descent of P.*

(a) *The 12-table $\operatorname{res}_k P$ differs from P only in columns k and $k+1$.*

(b) *The k-th and the $(k+1)$-th columns of $\operatorname{res}_k P$ depend only on the k-th and the $(k+1)$-th columns of P.*

(c) *We have*
$$\operatorname{ceq}(\operatorname{res}_k P) = \operatorname{ceq}(P).$$

(d) *We have*
$$\operatorname{ircont}(\operatorname{res}_k P) = \operatorname{ircont}(P).$$

(e) *The integer k is a descent of $\operatorname{flip}(\operatorname{res}_k P)$, and we have*
$$\operatorname{res}_k(\operatorname{flip}(\operatorname{res}_k P)) = \operatorname{flip}(P).$$

(f) *Recall that we defined a nonnegative integer $\ell(T)$ for every 12-table T in Definition 9. We have*
$$\ell(P) > \ell(\operatorname{res}_k P).$$

Proof of Proposition 12. All parts of Proposition 12 follow from straightforward arguments using the definitions of res_k and flip and (3). $\square$

5.4 The descent-resolution relation $\Rrightarrow$

Definition 13. Let us now define a binary relation $\Rrightarrow$ on the set $\mathrm{BT}^{12}(\lambda/\mu)$ as follows: Let $P \in \mathrm{BT}^{12}(\lambda/\mu)$ and $Q \in \mathrm{BT}^{12}(\lambda/\mu)$. If $k \in \mathbb{N}_+$, then we write $P \underset{k}{\Rrightarrow} Q$ if k is a descent of P and we have $Q = \mathrm{res}_k P$. We write $P \Rrightarrow Q$ if there exists a $k \in \mathbb{N}_+$ such that $P \underset{k}{\Rrightarrow} Q$.

Proposition 12 translates into the following properties of this relation $\Rrightarrow$:

Lemma 14. *Let $P \in \mathrm{BT}^{12}(\lambda/\mu)$ and $Q \in \mathrm{BT}^{12}(\lambda/\mu)$ be such that $P \Rrightarrow Q$. Then:*

(a) *We have* $\mathrm{ceq}(Q) = \mathrm{ceq}(P)$.

(b) *We have* $\mathrm{ircont}(Q) = \mathrm{ircont}(P)$.

(c) *The benign 12-tables* $\mathrm{flip}(P)$ *and* $\mathrm{flip}(Q)$ *have the property that* $\mathrm{flip}(Q) \Rrightarrow \mathrm{flip}(P)$.

(d) *We have* $\ell(P) > \ell(Q)$.

We now define $\overset{*}{\Rrightarrow}$ to be the reflexive-and-transitive closure of the relation $\Rrightarrow$. This relation $\overset{*}{\Rrightarrow}$ is reflexive and transitive, and extends the relation $\Rrightarrow$. Lemma 14 thus yields:

Lemma 15. *Let $P \in \mathrm{BT}^{12}(\lambda/\mu)$ and $Q \in \mathrm{BT}^{12}(\lambda/\mu)$ be such that $P \overset{*}{\Rrightarrow} Q$. Then:*

(a) *We have* $\mathrm{ceq}(Q) = \mathrm{ceq}(P)$.

(b) *We have* $\mathrm{ircont}(Q) = \mathrm{ircont}(P)$.

(c) *The benign 12-tables* $\mathrm{flip}(P)$ *and* $\mathrm{flip}(Q)$ *have the property that* $\mathrm{flip}(Q) \overset{*}{\Rrightarrow} \mathrm{flip}(P)$.

(d) *We have* $\ell(P) \geqslant \ell(Q)$.

We now state the following crucial lemma:

Lemma 16. *Let A, B and C be three elements of $\mathrm{BT}^{12}(\lambda/\mu)$ satisfying $A \Rrightarrow B$ and $A \Rrightarrow C$. Then, there exists a $D \in \mathrm{BT}^{12}(\lambda/\mu)$ such that $B \overset{*}{\Rrightarrow} D$ and $C \overset{*}{\Rrightarrow} D$.*

Proof of Lemma 16. If $B = C$, then we can simply choose $D = B = C$; thus, we assume that $B \neq C$.

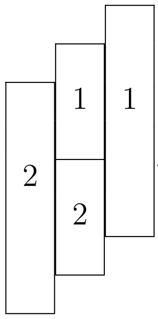
Let $u, v \in \mathbb{N}_+$ be such that $A \underset{u}{\Rrightarrow} B$ and $A \underset{v}{\Rrightarrow} C$. Thus, $B = \mathrm{res}_u A$ and $C = \mathrm{res}_v A$. Since $B \neq C$, we have $u \neq v$. Without loss of generality, assume that $u < v$. We are in one of the following two cases:

Case 1: We have $u = v - 1$.

Case 2: We have $u < v - 1$.

Let us deal with Case 2 first. In this case, $\{u, u+1\} \cap \{v, v+1\} = \varnothing$. It follows that $\operatorname{res}_v(\operatorname{res}_u A)$ and $\operatorname{res}_u(\operatorname{res}_v A)$ are well-defined and $\operatorname{res}_u(\operatorname{res}_v A) = \operatorname{res}_v(\operatorname{res}_u A)$. Setting $D = \operatorname{res}_u(\operatorname{res}_v A) = \operatorname{res}_v(\operatorname{res}_u A)$ completes the proof in this case.

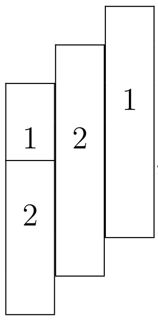
Now, let us consider Case 1. The v-th column of A must contain a 1 (since $v-1 = u$ is a descent of A) and a 2 (since v is a descent of A). Hence, the v-th column of A is mixed. Since A is benign but has $v-1$ and v as descents, it thus follows that the $(v-1)$-th column of A is 2-pure and the $(v+1)$-th column of A is 1-pure. We can represent the relevant portion (that is, the $(v-1)$-th, v-th and $(v+1)$-th columns) of the 12-table A as follows:

$A =$  . (5)

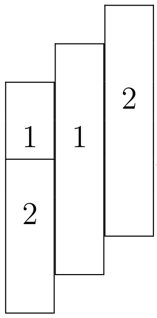
Notice that the separating line which separates the 1's from the 2's in column v is lower than the upper border of the $(v-1)$-th column (since $v-1$ is a descent of A), and higher than the lower border of the $(v+1)$-th column (since v is a descent of A).

Let $s = \operatorname{sep}_v A$. Then, the cells $(s, v-1)$, (s, v), $(s, v+1)$, $(s+1, v-1)$, $(s+1, v)$, $(s+1, v+1)$ all belong to λ/μ, due to what we just said about separating lines. We shall refer to this observation as the "six-cells property".

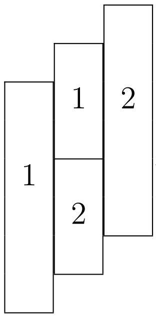
Now, $B = \operatorname{res}_u A = \operatorname{res}_{v-1} A$, so B is represented as follows:

$B =$  ,

where $\operatorname{sep}_{v-1} B = s$. In other words, the separating line in the $(v-1)$-th column of B is between the cells $(s, v-1)$ and $(s+1, v-1)$. Now, v is a descent of B. Resolving this descent yields a 12-table $\operatorname{res}_v B$ which is represented as follows:

$\operatorname{res}_v B =$  .

This, in turn, shows that $v-1$ is a descent of $\operatorname{res}_v B$ by the six-cells property. Resolving this descent yields a 12-table $\operatorname{res}_{v-1}(\operatorname{res}_v B)$ which is represented as follows:

$\operatorname{res}_{v-1}(\operatorname{res}_v B) =$  , (6)

where $\operatorname{sep}_v(\operatorname{res}_{v-1}(\operatorname{res}_v B)) = s$.

On the other hand, $C = \operatorname{res}_v A$. We can apply a similar argument as above to show that the 12-table $\operatorname{res}_v(\operatorname{res}_{v-1} C)$ is well-defined, and is exactly equal to the 12-table in (6). Hence, $\operatorname{res}_{v-1}(\operatorname{res}_v B) = \operatorname{res}_v(\operatorname{res}_{v-1} C)$, and setting D equal to this 12-table completes the proof in Case 1. $\square$

5.5 The normalization map

The following proposition is the most important piece in our puzzle:

Proposition 17. *For every $T \in \mathrm{BT}^{12}(\lambda/\mu)$, there exists a unique $N \in \mathrm{RPP}^{12}(\lambda/\mu)$ such that $T \overset{*}{\Rrightarrow} N$.*

Proof of Proposition 17. For every $T \in \mathrm{BT}^{12}(\lambda/\mu)$, let $\operatorname{Norm}(T)$ denote the set

$$\left\{ N \in \mathrm{RPP}^{12}(\lambda/\mu) \ \mid \ T \overset{*}{\Rrightarrow} N \right\}.$$

Thus, in order to prove Proposition 17, we need to show that for every $T \in \mathrm{BT}^{12}(\lambda/\mu)$ this set $\operatorname{Norm}(T)$ is a one-element set.

We shall prove this by strong induction on $\ell(T)$. Fix some $T \in \mathrm{BT}^{12}(\lambda/\mu)$, and assume that

$$\operatorname{Norm}(S) \text{ is a one-element set for every } S \in \mathrm{BT}^{12}(\lambda/\mu) \text{ satisfying } \ell(S) < \ell(T). \quad (7)$$

We then need to prove that $\operatorname{Norm}(T)$ is a one-element set.

Let $\mathbf{Z} = \left\{S \in \mathrm{BT}^{12}(\lambda/\mu) \ \mid \ T \Rrightarrow S\right\}$. In other words, $\mathbf{Z}$ is the set of all benign 12-tables S which can be obtained from T by resolving one descent. If $\mathbf{Z}$ is empty, then $T \in \mathrm{RPP}^{12}(\lambda/\mu)$, so that $\operatorname{Norm}(T) = \{T\}$ and we are done. Hence, we can assume that $\mathbf{Z}$ is nonempty. Therefore $T \notin \mathrm{RPP}^{12}(\lambda/\mu)$.

Thus, every $N \in \mathrm{RPP}^{12}(\lambda/\mu)$ satisfying $T \overset{*}{\Rrightarrow} N$ must satisfy $Z \overset{*}{\Rrightarrow} N$ for some $Z \in \mathbf{Z}$. In other words, every $N \in \operatorname{Norm}(T)$ must belong to $\operatorname{Norm}(Z)$ for some $Z \in \mathbf{Z}$. The converse of this clearly holds as well. Hence,

$$\operatorname{Norm}(T) = \bigcup_{Z \in \mathbf{Z}} \operatorname{Norm}(Z). \tag{8}$$

Let us now notice that:

- By Lemma 14 **(d)** and (7), for every $Z \in \mathbf{Z}$, the set $\operatorname{Norm}(Z)$ is a one-element set.
- By Lemma 16, for every $B \in \mathbf{Z}$ and $C \in \mathbf{Z}$, we have $\operatorname{Norm}(B) \cap \operatorname{Norm}(C) \neq \varnothing$. (In more detail: Let $B \in \mathbf{Z}$ and $C \in \mathbf{Z}$. By Lemma 16, applied to $A = T$, there exists a $D \in \mathrm{BT}^{12}(\lambda/\mu)$ such that $B \overset{*}{\Rrightarrow} D$ and $C \overset{*}{\Rrightarrow} D$. This D has $\ell(T) > \ell(B) \geqslant \ell(D)$, by Lemma 14 **(d)** and Lemma 15 **(d)**, respectively. Hence, by (7), the set $\operatorname{Norm}(D)$ is a one-element set. Its unique element clearly lies in both $\operatorname{Norm}(B)$ and $\operatorname{Norm}(C)$, so $\operatorname{Norm}(B) \cap \operatorname{Norm}(C) \neq \varnothing$.)

Hence, (8) shows that $\operatorname{Norm}(T)$ is a union of one-element sets, any two of which have a nonempty intersection, and thus are identical. Moreover, this union is nonempty, since $\mathbf{Z}$ is nonempty. Hence, $\operatorname{Norm}(T)$ itself is a one-element set. This completes our induction. □

Definition 18. Let $T \in \mathrm{BT}^{12}(\lambda/\mu)$. Proposition 17 shows that there exists a unique $N \in \mathrm{RPP}^{12}(\lambda/\mu)$ such that $T \overset{*}{\Rrightarrow} N$. We define $\operatorname{norm}(T)$ to be this N.

5.6 Definition of B

Definition 19. Let us define a map $\mathbf{B} : \mathrm{RPP}^{12}(\lambda/\mu) \to \mathrm{RPP}^{12}(\lambda/\mu)$ as follows: For every $T \in \mathrm{RPP}^{12}(\lambda/\mu)$, set $\mathbf{B}(T) = \operatorname{norm}(\operatorname{flip}(T))$.

In order to complete the proof of Lemma 7, we need to show that $\mathbf{B}$ is an involution, preserves the ceq statistic, and switches the number of columns containing a 1 with the number of columns containing a 2. At this point, all of this is easy:

$\mathbf{B}$ *is an involution.* Let $T \in \mathrm{RPP}^{12}(\lambda/\mu)$. We have $\operatorname{flip}(T) \overset{*}{\Rrightarrow} \operatorname{norm}(\operatorname{flip}(T)) = \mathbf{B}(T)$. Lemma 15 **(c)** thus yields $\operatorname{flip}(\mathbf{B}(T)) \overset{*}{\Rrightarrow} \operatorname{flip}(\operatorname{flip} T) = T$.

But $\mathbf{B}(\mathbf{B}(T)) = \operatorname{norm}(\operatorname{flip}(\mathbf{B}(T)))$ is the unique $N \in \mathrm{RPP}^{12}(\lambda/\mu)$ such that

$$\operatorname{flip}(\mathbf{B}(T)) \overset{*}{\Rrightarrow} N.$$

Since $T \in \mathrm{RPP}^{12}(\lambda/\mu)$, we have $\mathbf{B}(\mathbf{B}(T)) = T$, as desired. □

$\mathbf{B}$ *preserves* ceq. Let $T \in \mathrm{RPP}^{12}(\lambda/\mu)$. As above, $\operatorname{flip}(T) \overset{*}{\Rrightarrow} \mathbf{B}(T)$. Lemma 15 **(a)** and Remark 10 thus yield $\operatorname{ceq}(\mathbf{B}(T)) = \operatorname{ceq}(\operatorname{flip}(T)) = \operatorname{ceq}(T)$. $\square$

$\mathbf{B}$ *switches the numbers of columns containing 1 and 2.* Let $T \in \mathrm{RPP}^{12}(\lambda/\mu)$. As above, $\operatorname{flip}(T) \overset{*}{\Rrightarrow} \mathbf{B}(T)$. Lemma 15 **(b)** thus yields $\operatorname{ircont}(\mathbf{B}(T)) = \operatorname{ircont}(\operatorname{flip}(T))$. Due to Remark 10, this is the result of switching the first two entries of $\operatorname{ircont}(T)$. $\square$

Lemma 7 is now proven.

6 The classical Bender-Knuth involutions

Fix a skew partition λ/μ and a positive integer i. We claim that the involution $\mathbf{B}_i : \mathrm{RPP}(\lambda/\mu) \to \mathrm{RPP}(\lambda/\mu)$ we have constructed in the proof of Theorem 6 is a generalization of the i-th Bender-Knuth involution defined for semistandard tableaux. First, we shall define the i-th Bender-Knuth involution, following [GriRei15, proof of Proposition 2.11] and [Stan99, proof of Theorem 7.10.2].

Let $\mathrm{SST}(\lambda/\mu)$ denote the set of all semistandard tableaux of shape λ/μ. We define a map $\mathbf{BK}_i : \mathrm{SST}(\lambda/\mu) \to \mathrm{SST}(\lambda/\mu)$ as follows:

Let $T \in \mathrm{SST}(\lambda/\mu)$. Then every column of T contains at most one i and at most one $i+1$. If a column contains both an i and an $i+1$, we will mark its entries as "ignored". Now, let $k \in \mathbb{N}_+$. The k-th row of T is a weakly increasing sequence of positive integers; thus, it contains a (possibly empty) string of i's followed by a (possibly empty) string of $(i+1)$'s. These two strings together form a substring of the k-th row which looks as follows:

$$(i, i, \ldots, i, i+1, i+1, \ldots, i+1).$$

Some of the entries of this substring are "ignored"; it is easy to see that the "ignored" i's are gathered at the left end of the substring whereas the "ignored" $(i+1)$'s are gathered at the right end of the substring. So the substring looks as follows:

$$\left(\underbrace{i, i, \ldots, i}_{\substack{a \text{ many } i\text{'s which} \\ \text{are "ignored"}}}, \underbrace{i, i, \ldots, i}_{\substack{r \text{ many } i\text{'s which} \\ \text{are not "ignored"}}}, \underbrace{i+1, i+1, \ldots, i+1}_{\substack{s \text{ many } (i+1)\text{'s which} \\ \text{are not "ignored"}}}, \underbrace{i+1, i+1, \ldots, i+1}_{\substack{b \text{ many } (i+1)\text{'s which} \\ \text{are "ignored"}}}\right)$$

for some $a, r, s, b \in \mathbb{N}$. Now, we change this substring into

$$\left(\underbrace{i, i, \ldots, i}_{\substack{a \text{ many } i\text{'s which} \\ \text{are "ignored"}}}, \underbrace{i, i, \ldots, i}_{\substack{s \text{ many } i\text{'s which} \\ \text{are not "ignored"}}}, \underbrace{i+1, i+1, \ldots, i+1}_{\substack{r \text{ many } (i+1)\text{'s which} \\ \text{are not "ignored"}}}, \underbrace{i+1, i+1, \ldots, i+1}_{\substack{b \text{ many } (i+1)\text{'s which} \\ \text{are "ignored"}}}\right).$$

We do this for every $k \in \mathbb{N}_+$. At the end, we have obtained a new semistandard tableau of shape λ/μ. We define $\mathbf{BK}_i(T)$ to be this new tableau.

Proposition 20. *The map* $\mathbf{BK}_i : \operatorname{SST}(\lambda/\mu) \to \operatorname{SST}(\lambda/\mu)$ *thus defined is an involution. It is known as the i-th Bender-Knuth involution.*

Now, every semistandard tableau of shape λ/μ is also an rpp of shape λ/μ. Hence, $\mathbf{B}_i(T)$ is defined for every $T \in \operatorname{SST}(\lambda/\mu)$. Our claim is the following:

Proposition 21. *For every* $T \in \operatorname{SST}(\lambda/\mu)$*, we have* $\mathbf{BK}_i(T) = \mathbf{B}_i(T)$*.*

Proof of Proposition 21. Recall that the map $\mathbf{B}_i$ comes from the map $\mathbf{B}$ we defined on 12-rpps in Section 5. We could have constructed the map $\mathbf{BK}_i$ from the map $\mathbf{BK}_1$ in an analogous way. We define a *12-sst* to be a semistandard tableau whose entries all belong to the set $\{1, 2\}$. Clearly, to prove Proposition 21, it suffices to prove that $\mathbf{BK}_1(T) = \mathbf{B}(T)$ for all 12-ssts T.

Let T be a 12-sst, and let $k \in \mathbb{N}_+$. The k-th row of T has the form

$$\left(\underbrace{1,1,\ldots,1}_{\substack{a\text{ 1's which are in}\\ \text{mixed columns}}}, \underbrace{1,1,\ldots,1}_{\substack{r\text{ 1-pure}\\ \text{columns}}}, \underbrace{2,2,\ldots,2}_{\substack{s\text{ 2-pure}\\ \text{columns}}}, \underbrace{2,2,\ldots,2}_{\substack{b\text{ 2's which are in}\\ \text{mixed columns}}}\right)$$

where we use the observation that each 1-pure and each 2-pure column contains only one entry. Thus, the k-th row of $\operatorname{flip}(T)$ is

$$\left(\underbrace{1,1,\ldots,1}_{\substack{a\text{ 1's which are in}\\ \text{mixed columns}}}, \underbrace{2,2,\ldots,2}_{\substack{r\text{ 2-pure}\\ \text{columns}}}, \underbrace{1,1,\ldots,1}_{\substack{s\text{ 1-pure}\\ \text{columns}}}, \underbrace{2,2,\ldots,2}_{\substack{b\text{ 2's which are in}\\ \text{mixed columns}}}\right).$$

We can now repeatedly apply descent-resolution steps to obtain a tableau whose k-th row is

$$\left(\underbrace{1,1,\ldots,1}_{\substack{a\text{ 1's which are in}\\ \text{mixed columns}}}, \underbrace{1,1,\ldots,1}_{\substack{s\text{ 1-pure}\\ \text{columns}}}, \underbrace{2,2,\ldots,2}_{\substack{r\text{ 2-pure}\\ \text{columns}}}, \underbrace{2,2,\ldots,2}_{\substack{b\text{ 2's which are in}\\ \text{mixed columns}}}\right).$$

Repeating this process for every row of $\operatorname{flip}(T)$, we obtain a 12-rpp. By the definition of $\mathbf{B}$, this rpp must equal $\mathbf{B}(T)$. By the above description, it is also equal to $\mathbf{BK}_1(T)$, because the ignored entries in the construction of $\mathbf{BK}_1(T)$ are precisely the entries lying in mixed columns. This completes the proof. $\square$

7 The structure of 12-rpps

In this section, we restrict ourselves to the two-variable dual stable Grothendieck polynomial $\widetilde{g}_{\lambda/\mu}(x_1, x_2, 0, 0, \ldots; \mathbf{t})$ defined as the result of substituting $0, 0, 0, \ldots$ for $x_3, x_4, x_5, \ldots$ in $\widetilde{g}_{\lambda/\mu}$. We can represent it as a polynomial in $\mathbf{t}$ with coefficients in $\mathbb{Z}[x_1, x_2]$:

$$\widetilde{g}_{\lambda/\mu}(x_1, x_2, 0, 0, \ldots; \mathbf{t}) = \sum_{\alpha \in \mathbb{N}^{\mathbb{N}_+}} \mathbf{t}^\alpha Q_\alpha(x_1, x_2),$$

					1	2
			1	1	**1**	2
		1	**1**	1	2	2
1	2	2	2			
2	2	2	2			

$\mathrm{NR}(T_1) = \{(4,1), (3,3), (3,4), (2,6)\}$
$\mathrm{seplist}(T_1) = (4,3,3,2)$

					1	1
			1	1	1	**1**
		1	**1**	1	1	2
1	**1**	2	2			
2	2	2	2			

$\mathrm{NR}(T_2) = \{(4,2), (3,3), (3,4), (2,7)\}$
$\mathrm{seplist}(T_2) = (4,3,3,2)$

Figure 3: Two 12-rpps of the same shape and with the same seplist-partition.

where the sum ranges over all weak compositions α, and all but finitely many $Q_\alpha(x_1, x_2)$ are 0.

We shall show that each $Q_\alpha(x_1, x_2)$ is either zero or has the form

$$Q_\alpha(x_1, x_2) = (x_1x_2)^M P_{n_0}(x_1, x_2) P_{n_1}(x_1, x_2) \cdots P_{n_r}(x_1, x_2), \tag{9}$$

where M, r and $n_0, n_1, \ldots, n_r$ are nonnegative integers naturally associated to α and λ/μ and

$$P_n(x_1, x_2) = \frac{x_1^{n+1} - x_2^{n+1}}{x_1 - x_2} = x_1^n + x_1^{n-1}x_2 + \cdots + x_1x_2^{n-1} + x_2^n.$$

We fix the skew partition λ/μ throughout the whole section. We will have a running example with $\lambda = (7,7,7,4,4)$ and $\mu = (5,3,2)$.

7.1 Irreducible components

We recall that a *12-rpp* means an rpp whose entries all belong to the set $\{1, 2\}$.

Given a 12-rpp T, consider the set $\mathrm{NR}(T)$ of all cells $(i,j) \in \lambda/\mu$ such that $T(i,j) = 1$ but $(i+1, j) \in \lambda/\mu$ and $T(i+1, j) = 2$. In other words, $\mathrm{NR}(T)$ is the set of all non-redundant cells in T which are filled with a 1 and which are not the lowest cells in their columns. Clearly, $\mathrm{NR}(T)$ contains at most one cell from each column; thus, let us write $\mathrm{NR}(T) = \{(i_1, j_1), (i_2, j_2), \ldots, (i_s, j_s)\}$ with $j_1 < j_2 < \cdots < j_s$. Because T is a 12-rpp, it follows that the numbers $i_1, i_2, \ldots, i_s$ decrease weakly, therefore they form a partition which we denoted

$$\mathrm{seplist}(T) := (i_1, i_2, \ldots, i_s)$$

in Section 5.1. This partition will be called the *seplist-partition of* T. An example of calculation of $\mathrm{seplist}(T)$ and $\mathrm{NR}(T)$ is illustrated on Figure 3.

We would like to answer the following question: for which partitions $\nu = (i_1 \geqslant \cdots \geqslant i_s > 0)$ does there exist a 12-rpp T of shape λ/μ such that $\mathrm{seplist}(T) = \nu$?

A trivial necessary condition for this to happen is that there should exist some numbers $j_1 < j_2 < \cdots < j_s$ such that

$$(i_1, j_1), (i_1+1, j_1), (i_2, j_2), (i_2+1, j_2), \ldots, (i_s, j_s), (i_s+1, j_s) \in \lambda/\mu. \tag{10}$$

Until the end of Section 7, we make an assumption: namely, that the skew partition λ/μ is connected as a subgraph of $\mathbb{Z}^2$ (where two nodes are connected if and only if their cells have an edge in common), and that it has no empty columns. This is a harmless assumption, since every skew partition λ/μ can be written as a disjoint union of such connected skew partitions and the corresponding seplist-partition splits into several independent parts, the polynomials $\widetilde{g}_{\lambda/\mu}$ get multiplied and the right hand side of (9) changes accordingly.

For each integer i, the set of all integers j such that $(i,j),(i+1,j) \in \lambda/\mu$ is just an interval $[\mu_i+1,\lambda_{i+1}]$, which we call *the support of* i and denote $\operatorname{supp}(i) := [\mu_i+1,\lambda_{i+1}]$.

Let $\#\kappa$ denote the length of a partition κ.

We say that a partition ν is *admissible* if every k satisfies $\operatorname{supp}(i_k) \neq \varnothing$. This is clearly satisfied when there exist $j_1 < j_2 < \cdots < j_s$ satisfying (10), but also in other cases. Assume that $\nu = (i_1 \geqslant \cdots \geqslant i_s > 0)$ is an admissible partition. For two integers $a < b$, we let $\nu\big|_{\subseteq[a,b)}$ denote the subpartition $(i_r, i_{r+1}, \dots, i_{r+q})$ of ν, where $[r, r+q]$ is the (possibly empty) set of all k for which $\operatorname{supp}(i_k) \subseteq [a,b)$. Thus, the number of entries in $\nu\big|_{\subseteq[a,b)}$ is $\#\nu\big|_{\subseteq[a,b)} = q+1$. Similarly, we set $\nu\big|_{\cap[a,b)}$ to be the subpartition $(i_r, i_{r+1}, \dots, i_{r+q})$ of ν, where $[r, r+q]$ is the set of all k for which $\operatorname{supp}(i_k) \cap [a,b) \neq \emptyset$. For example, for $\nu = (4,3,3,2)$ and λ/μ as on Figure 3, we have

$$\operatorname{supp}(3) = [3,4],\ \operatorname{supp}(2) = [4,7],\ \operatorname{supp}(4) = [1,4],$$

$$\nu\big|_{\subseteq[2,7)} = (3,3),\ \nu\big|_{\subseteq[2,8)} = (3,3,2),\ \nu\big|_{\subseteq[4,8)} = (2),\ \nu\big|_{\cap[4,5)} = (4,3,3,2),\ \#\nu\big|_{\subseteq[2,7)} = 2.$$

Remark 22. *If ν is not admissible, that is, if* $\operatorname{supp}(i_k) = \emptyset$ *for some k, then i_k belongs to $\nu\big|_{\subseteq[a,b)}$ for any a, b, so $\nu\big|_{\subseteq[a,b)}$ might no longer be a contiguous subpartition of ν. On the other hand, if ν is an admissible partition, then the partitions $\nu\big|_{\subseteq[a,b)}$ and $\nu\big|_{\cap[a,b)}$ are clearly admissible as well. For the rest of this section, we will only work with admissible partitions.*

We introduce several definitions: An admissible partition $\nu = (i_1 \geqslant \cdots \geqslant i_s > 0)$ is called

- *non-representable* if for some $a < b$ we have $\#\nu\big|_{\subseteq[a,b)} > b-a$;
- *representable* if for all $a < b$ we have $\#\nu\big|_{\subseteq[a,b)} \leqslant b-a$;

a representable partition ν is called

- *irreducible* if for all $a < b$ we have $\#\nu\big|_{\subseteq[a,b)} < b-a$;
- *reducible* if for some $a < b$ we have $\#\nu\big|_{\subseteq[a,b)} = b-a$.

For example, $\nu = (4,3,3,2)$ is representable but reducible because we have $\nu\big|_{\subseteq[3,5)} = (3,3)$ so $\#\nu\big|_{\subseteq[3,5)} = 2 = 5-3$.

Note that these notions depend on the skew partition; thus, when we want to use a skew partition $\widetilde{\lambda/\mu}$ rather than λ/μ, we will write that ν is non-representable/irreducible/etc. *with respect to* $\widetilde{\lambda/\mu}$, and we denote the corresponding partitions by $\nu\big|_{\subseteq[a,b)}^{\widetilde{\lambda/\mu}}$.

These definitions can be motivated as follows. Suppose that a partition ν is non-representable, so there exist integers $a < b$ such that $\#\nu\big|_{\subseteq[a,b)} > b-a$. Recall that

$\nu\big|_{\subseteq[a,b)} =: (i_r, i_{r+1}, \dots, i_{r+q})$ contains all entries of ν whose support is a subset of $[a,b)$. Thus in order for condition (10) to be true there must exist some integers $j_r < j_{r+1} < \cdots < j_{r+q}$ such that

$$(i_r, j_r), (i_r+1, j_r), \dots, (i_{r+q}, j_{r+q}), (i_{r+q}+1, j_{r+q}) \in \lambda/\mu.$$

On the other hand, by the definition of the support, we must have $j_k \in \mathrm{supp}(i_k) \subseteq [a,b)$ for all $r \leqslant k \leqslant r+q$. Therefore we get $q+1$ distinct elements of $[a,b)$ which is impossible if $q+1 = \#\nu\big|_{\subseteq[a,b)} > b-a$. It means that a non-representable partition ν is never a seplist-partition of a 12-rpp T.

Suppose now that a partition ν is reducible, so for some $a < b$ we get an equality $\#\nu\big|_{\subseteq[a,b)} = b-a$. Then these integers $j_r < \cdots < j_{r+q}$ should still all belong to $[a,b)$ and there are exactly $b-a$ of them, hence

$$j_r = a,\ j_{r+1} = a+1,\ \cdots,\ j_{r+q} = a+q = b-1. \tag{11}$$

Because $\mathrm{supp}(i_r) \subseteq [a,b)$ but $\mathrm{supp}(i_r) \neq \varnothing$ (since ν is admissible), we have $(i_r, a-1) \notin \lambda/\mu$. Thus, placing a 1 into (i_r, a) and 2's into $(i_r+1, a), (i_r+2, a), \dots$ does not put any restrictions on entries in columns $1, \dots, a-1$. And the same is true for columns $b, b+1, \dots$ when we place a 2 into $(i_{r+q}+1, b-1)$ and 1's into all cells above. Thus, if a partition ν is reducible, then the filling of columns $a, a+1, \dots, b-1$ is uniquely determined by (11), and the filling of the rest can be arbitrary – the problem of existence of a 12-rpp T such that $\mathrm{seplist}(T) = \nu$ reduces to two smaller independent problems of the same kind: one for the columns $1, 2, \dots, a-1$, the other for the columns $b, b+1, \dots, \lambda_1$. Recall that a 12-rpp of shape λ/μ cannot have any nonempty column beyond the λ_1'th one.

One can continue this reduction process and end up with several independent irreducible components separated from each other by mixed columns. An illustration of this phenomenon can be seen on Figure 3: the columns 3 and 4 must be mixed for any 12-rpps T with $\mathrm{seplist}(T) = (4,3,3,2)$.

More explicitly, we have thus shown that every nonempty interval $[a,b) \subseteq [1, \lambda_1+1)$ satisfying $\#\nu\big|_{\subseteq[a,b)} = b-a$ splits our problem into two independent subproblems. But if two such intervals $[a,b)$ and $[c,d)$ satisfy $a \leqslant c \leqslant b \leqslant d$ then their union $[a,d)$ is another such interval, because in this case, inclusion-exclusion gives $\#\nu\big|_{\subseteq[a,d)} \geqslant \#\nu\big|_{\subseteq[a,b)} + \#\nu\big|_{\subseteq[c,d)} - \#\nu\big|_{\subseteq[c,b)}$, but $\#\nu\big|_{\subseteq[c,b)} \leqslant b-c$ by representability of ν. Hence, the maximal (with respect to inclusion) among all such intervals are pairwise disjoint and separated from each other by at least a distance of 1. This yields part **(a)** of the following lemma:

Lemma 23. *Let ν be a representable partition.*

- ***(a)*** *There exist unique integers* $(1 = b_0 \leqslant a_1 < b_1 < a_2 < b_2 < \cdots < a_r < b_r \leqslant a_{r+1} = \lambda_1 + 1)$ *satisfying the following two conditions:*

  - *(a) For all $1 \leqslant k \leqslant r$, we have $\#\nu\big|_{\subseteq[a_k,b_k)} = b_k - a_k$.*

(b) The set $\bigcup_{k=0}^{r}[b_k, a_{k+1})$ is minimal (with respect to inclusion) among all sequences ($1 = b_0 \leqslant a_1 < b_1 < a_2 < b_2 < \cdots < a_r < b_r \leqslant a_{r+1} = \lambda_1 + 1$) satisfying property 1.

Furthermore, for these integers, we have:

***(b)** The partition ν is the concatenation*

$$\left(\nu\big|_{\cap[b_0,a_1)}\right)\left(\nu\big|_{\subseteq[a_1,b_1)}\right)\left(\nu\big|_{\cap[b_1,a_2)}\right)\left(\nu\big|_{\subseteq[a_2,b_2)}\right)\cdots\left(\nu\big|_{\cap[b_r,a_{r+1})}\right)$$

(where we regard a partition as a sequence of positive integers, with no trailing zeroes).

***(c)** The partitions $\nu\big|_{\cap[b_k,a_{k+1})}$ are irreducible with respect to $\lambda/\mu\big|_{[b_k,a_{k+1})}$, which is the skew partition λ/μ with columns $1, 2, \ldots, b_k - 1, a_{k+1}, a_{k+1} + 1, \ldots$ removed.*

Proof. Part **(a)** has already been proven.

(b) Let $\nu = (i_1 \geqslant \cdots \geqslant i_s > 0)$. If $\operatorname{supp}(i_r) \subseteq [a_k, b_k)$ for some k, then i_r appears in exactly one of the concatenated partitions, namely, $\nu\big|_{\subseteq[a_k,b_k)}$. Otherwise there is an integer k such that $\operatorname{supp}(i_r) \cap [b_k, a_{k+1}) \neq \emptyset$. It remains to show that such k is unique, that is, that $\operatorname{supp}(i_r) \cap [b_{k+1}, a_{k+2}) = \emptyset$. Assume the contrary. The interval $[a_{k+1}, b_{k+1})$ is nonempty, therefore there is an entry i of ν with $\operatorname{supp}(i) \subseteq [a_{k+1}, b_{k+1})$. It remains to note that we get a contradiction: we get two numbers i, i_r with $\operatorname{supp}(i_r)$ being both to the left and to the right of $\operatorname{supp}(i)$.

(c) Fix k. Let J denote the restricted skew partition $\lambda/\mu\big|_{[b_k,a_{k+1})}$, and let $\nu' = \nu\big|_{\cap[b_k,a_{k+1})}$. We need to show that if $[c, d)$ is a nonempty interval contained in $[b_k, a_{k+1})$, then $\#\nu'\big|^J_{\subseteq[c,d)} < d - c$. We are in one of the following four cases:

- *Case 1: We have $c > b_k$ (or $k = 0$) and $d < a_{k+1}$ (or $k = r$).* In this case, every i_p with $\operatorname{supp}^J(i_p) \subseteq [c, d)$ must satisfy $\operatorname{supp}(i_p) \subseteq [c, d)$. Hence, $\nu'\big|^J_{\subseteq[c,d)} = \nu\big|_{\subseteq[c,d)}$, so that $\#\nu'\big|^J_{\subseteq[c,d)} = \#\nu\big|_{\subseteq[c,d)} < d - c$, and we are done.

- *Case 2: We have $c = b_k$ and $k > 0$ (but not $d = a_{k+1}$ and $k < r$).* Assume for the sake of contradiction that $\#\nu'\big|^J_{\subseteq[c,d)} \geqslant d - c$. Then, the i_p satisfying $\operatorname{supp}^J(i_p) \subseteq [c, d)$ must satisfy $\operatorname{supp}(i_p) \subseteq [a_k, d)$, since otherwise, $\operatorname{supp}(i_p)$ would intersect both $[b_{k-1}, a_k)$ and $[b_k, a_{k+1})$, something we have ruled out in the proof of **(b)**. Thus, $\#\nu\big|_{\subseteq[a_k,d)} \geqslant (d - c) + (b_k - a_k) = d - a_k$, which contradicts the minimality of $\bigcup_{k=0}^{r}[b_k, a_{k+1})$: we could increase b_k to d.

- *Case 3: We have $d = a_{k+1}$ and $k < r$ (but not $c = b_k$ and $k > 0$).* The argument here is analogous to Case 2.

- *Case 4: Neither of the above.* Exercise.

$\square$

Definition 24. In the context of Lemma 23, for $0 \leqslant k \leqslant r$ the subpartitions $\nu\big|_{\cap[b_k,a_{k+1})}$ are called *the irreducible components of* ν and the nonnegative integers $n_k := a_{k+1} - b_k - \#\nu\big|_{\cap[b_k,a_{k+1})}$ are called their *degrees*. For T with $\mathrm{seplist}(T) = \nu$, the k-th degree n_k is equal to the number of pure columns of T inside the corresponding k-th irreducible component. All n_k are positive, except for n_0 if $a_1 = 1$ and n_r if $b_r = \lambda_1 + 1$.

Example 25. For $\nu = (4, 3, 3, 2)$ we have $r = 1, b_0 = 1, a_1 = 3, b_1 = 5, a_2 = 8$. The irreducible components of ν are (4) and (2) and their degrees are $3 - 1 - 1 = 1$ and $8 - 5 - 1 = 2$ respectively. We have $\nu\big|_{\cap[1,3)} = (4), \nu\big|_{\subseteq[3,5)} = (3, 3), \nu\big|_{\cap[5,8)} = (2)$.

7.2 The structural theorem and its applications

It is easy to see that for a 12-rpp T, the number $\#\mathrm{seplist}(T)$ is equal to the number of mixed columns in T.

Recall that $\mathrm{RPP}^{12}(\lambda/\mu)$ denotes the set of all 12-rpps T of shape λ/μ, and let $\mathrm{RPP}^{12}(\lambda/\mu; \nu)$ denote its subset consisting of all 12-rpps T with $\mathrm{seplist}(T) = \nu$. Now we are ready to state a theorem that completely describes the structure of irreducible components:

Theorem 26. *Let ν be an irreducible partition. Then for all $0 \leqslant m \leqslant \lambda_1 - \#\nu$ there is exactly one 12-rpp $T \in \mathrm{RPP}^{12}(\lambda/\mu; \nu)$ with $\#\nu$ mixed columns, m 1-pure columns and $(\lambda_1 - \#\nu - m)$ 2-pure columns. Moreover, these are the only elements of $\mathrm{RPP}^{12}(\lambda/\mu; \nu)$. In other words, for an irreducible partition ν we have*

$$\sum_{T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)} \mathbf{x}^{\mathrm{ircont}(T)} = (x_1x_2)^{\#\nu} P_{\lambda_1 - \#\nu}(x_1, x_2). \tag{12}$$

Before we proceed to the proof, let us discuss some applications and examples.

Example 27. Each of the two 12-rpps on Figure 3 has two irreducible components. One of them is supported on the first two columns and the other one is supported on the last three columns. Here are all possible 12-rpps for each component:

$$\begin{array}{|c|c|}\hline 1 & \mathbf{1} \\ \hline 1 & 2 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline \mathbf{1} & 2 \\ \hline 2 & 2 \\ \hline\end{array}$$

$\lambda = (2, 2);\ \mu = ();\ \nu = (4)$

$$\begin{array}{c|c|c|}\cline{2-3} & 1 & 1 \\ \hline \multicolumn{1}{|c|}{1} & 1 & \mathbf{1} \\ \hline \multicolumn{1}{|c|}{1} & 1 & 2 \\ \hline\end{array} \quad \begin{array}{c|c|c|}\cline{2-3} & 1 & 2 \\ \hline \multicolumn{1}{|c|}{1} & \mathbf{1} & 2 \\ \hline \multicolumn{1}{|c|}{1} & 2 & 2 \\ \hline\end{array} \quad \begin{array}{c|c|c|}\cline{2-3} & 2 & 2 \\ \hline \multicolumn{1}{|c|}{\mathbf{1}} & 2 & 2 \\ \hline \multicolumn{1}{|c|}{2} & 2 & 2 \\ \hline\end{array}$$

$\lambda = (3, 3, 3);\ \mu = (1);\ \nu = (2)$.

After decomposing into irreducible components, we can obtain a formula for general representable partitions:

Corollary 28. *Let ν be a representable partition. Then*

$$\sum_{T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)} \mathbf{x}^{\mathrm{ircont}(T)} = (x_1x_2)^M P_{n_0}(x_1, x_2) P_{n_1}(x_1, x_2) \cdots P_{n_r}(x_1, x_2), \tag{13}$$

where the numbers $M, r, n_0, \dots, n_r$ are defined above: $M = \#\nu$, $r+1$ is the number of irreducible components of ν and $n_0, n_1, \dots, n_r$ are their degrees.

Proof of Corollary 28. The restriction map

$$\mathrm{RPP}^{12}(\lambda/\mu;\nu) \to \prod_{k=0}^{r} \mathrm{RPP}^{12}\left(\lambda/\mu\big|_{[b_k,a_{k+1})};\nu\big|_{\cap[b_k,a_{k+1})}\right)$$

is injective, since, as we know, the entries of a $T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ in any column outside of the irreducible components are uniquely determined. It is also surjective, as one can "glue" rpps together. Now use Theorem 26. □

For a 12-rpp T, the vectors $\mathrm{seplist}(T)$ and $\mathrm{ceq}(T)$ uniquely determine each other: if $(\mathrm{ceq}(T))_i = h$ then $\mathrm{seplist}(T)$ contains exactly $\lambda_{i+1} - \mu_i - h$ entries equal to i, and this correspondence is one-to-one. Therefore, the polynomials on both sides of (13) are equal to $Q_\alpha(x_1, x_2)$ where the vector α is the one that corresponds to ν.

Note that the polynomials $P_n(x_1, x_2)$ are symmetric for all n. Since the question about the symmetry of $\widetilde{g}_{\lambda/\mu}$ can be reduced to the two-variable case, Corollary 28 gives an alternative proof of the symmetry of $\widetilde{g}_{\lambda/\mu}$:

Corollary 29. *The polynomials $\widetilde{g}_{\lambda/\mu} \in \mathbb{Z}[t_1, t_2, \dots][[x_1, x_2, x_3, \dots]]$ are symmetric.*

Of course, our standing assumption that λ/μ is connected can be lifted here, because in general, $\widetilde{g}_{\lambda/\mu}$ is the product of the analogous power series corresponding to the connected components of λ/μ. So we have obtained a new proof of Theorem 5.

Another application of Theorem 26 is a complete description of Bender-Knuth involutions on rpps.

Corollary 30. *Let ν be an irreducible partition. Then there is a unique map*

$$b : \mathrm{RPP}^{12}(\lambda/\mu;\nu) \to \mathrm{RPP}^{12}(\lambda/\mu;\nu)$$

such that for all $T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)$, the sequence $\mathrm{ircont}(b(T))$ is obtained from $\mathrm{ircont}(T)$ by switching the first two entries. This unique map b is an involution on $\mathrm{RPP}^{12}(\lambda/\mu;\nu)$. So, for irreducible partition ν the corresponding Bender-Knuth involution exists and is unique.

Take any 12-rpp $T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ and recall that a 12-table $\mathrm{flip}(T)$ is obtained from T by simultaneously replacing all entries in 1-pure columns by 2 and all entries in 2-pure columns by 1.

Corollary 31. *If ν is an irreducible partition, then, no matter in which order one resolves descents in $\mathrm{flip}(T)$, the resulting 12-rpp T' will be the same. The map $T \mapsto T'$ is the unique Bender-Knuth involution on $\mathrm{RPP}^{12}(\lambda/\mu;\nu)$.*

Proof of Corollary 31. Descent-resolution steps applied to $\mathrm{flip}(T)$ in any order eventually give an element of $\mathrm{RPP}^{12}(\lambda/\mu;\nu)$ with the desired ircont. There is only one such element. So we get a map $\mathrm{RPP}^{12}(\lambda/\mu;\nu) \to \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ that satisfies the assumptions of Corollary 30. □

Finally, notice that, for a general representable partition ν, descents in a 12-table T with $\mathrm{seplist}(T) = \nu$ may only occur inside each irreducible component independently. Thus, we conclude the chain of corollaries by stating that our constructed involutions are canonical in the following sense:

Corollary 32. *For a representable partition ν, the map*

$$\mathbf{B} : \mathrm{RPP}^{12}(\lambda/\mu;\nu) \to \mathrm{RPP}^{12}(\lambda/\mu;\nu)$$

is the unique involution that interchanges the number of 1*-pure columns with the number of* 2*-pure columns inside each irreducible component.*

7.3 The proof

Let $\nu = (i_1, \ldots, i_s)$ be an irreducible partition. We start with the following simple observation:

Lemma 33. *Let $T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ for an irreducible partition ν. Then any* 1*-pure column of T is to the left of any* 2*-pure column of T.*

Proof of Lemma 33. Suppose it is false and we have a 1-pure column to the right of a 2-pure column. Among all pairs (a, b) such that column a is 2-pure and column b is 1-pure, and $b > a$, consider the one with smallest $b-a$. Then, the columns $a+1, \ldots, b-1$ must all be mixed. Therefore the set $\mathrm{NR}(T)$ contains $\{(i_{p+1}, a+1), (i_{p+2}, a+2), \ldots, (i_{p+b-1-a}, b-1)\}$ for some $p \in \mathbb{N}$. And because a is 2-pure and b is 1-pure, each i_{p+k} for $k = 1, \ldots, b-1-a$ must be $\leqslant$ to the y-coordinate of the highest cell in column a and $>$ than the y-coordinate of the lowest cell in column b. Thus, the support of any i_{p+k} for $k = 1, \ldots, b-1-a$ is a subset of $[a+1, b)$, which contradicts the irreducibility of ν. □

Proof of Theorem 26. We proceed by strong induction on the number of columns in λ/μ. If the number of columns is 1, then the statement of Theorem 26 is obvious. Suppose that we have proven that for all skew partitions $\widetilde{\lambda/\mu}$ with less than λ_1 columns and for all partitions $\widetilde{\nu}$ irreducible with respect to $\widetilde{\lambda/\mu}$ and for all $0 \leqslant \widetilde{m} \leqslant \widetilde{\lambda}_1 - \#\widetilde{\nu}$, there is exactly one 12-rpp $\widetilde{T}$ of shape $\widetilde{\lambda/\mu}$ with exactly $\widetilde{m}$ 1-pure columns, exactly $\#\widetilde{\nu}$ mixed columns and exactly $(\widetilde{\lambda}_1 - \#\widetilde{\nu} - \widetilde{m})$ 2-pure columns. Now we want to prove the same for λ/μ.

Take any 12-rpp $T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ with $\mathrm{seplist}(T) = \nu$ and with m 1-pure columns for $0 \leqslant m \leqslant \lambda_1 - \#\nu$. Suppose first that $m > 0$. Then there is at least one 1-pure column in T. Let $q \geqslant 0$ be such that the leftmost 1-pure column is column $q+1$. Then by Lemma 33 the columns $1, 2, \ldots, q$ are mixed. If $q > 0$ then the supports of $i_1, i_2, \ldots, i_q$ are all contained inside $[1, q+1)$ and we get a contradiction with the irreducibility of ν. The only remaining case is that $q = 0$ and the first column of T is 1-pure. Let $\widetilde{\lambda/\mu}$ denote λ/μ with the first column removed. Then ν is obviously admissible but may not be irreducible with respect to $\widetilde{\lambda/\mu}$, because it may happen that $\#\nu\big|^{\widetilde{\lambda/\mu}}_{\subseteq[2,b+1)} = b-1$ for some $b > 1$. In this case we can remove these $b-1$ nonempty columns from $\widetilde{\lambda/\mu}$ and remove the first $b-1$ entries from ν to get an irreducible partition again[1], to which we can apply the induction hypothesis. We are done with the case $m > 0$. If $m < \lambda_1 - \#\nu$ then we can apply a mirrored argument to the last column, and it remains to note that the cases $m > 0$ and $m < \lambda_1 - \#\nu$ cover everything, since the irreducibility of ν shows that $\lambda_1 - \#\nu > 0$.

This inductive proof shows the uniqueness of the 12-rpp with desired properties. Its existence follows from a parallel argument, using the observation that the first $b-1$ columns of $\widetilde{\lambda/\mu}$ can actually be filled in. This amounts to showing that for a representable ν, the set $\mathrm{RPP}^{12}(\lambda/\mu;\nu)$ is non-empty in the case when $\lambda_1 = \#\nu$, that is, all columns of $T \in \mathrm{RPP}^{12}(\lambda/\mu;\nu)$ must be mixed.

This is clear when there is just one column. To prove it in the general case, we proceed by induction on the number of columns:

If we had $1 \notin \mathrm{supp}(\nu_1)$, then we would have $\mathrm{supp}(\nu_1) \subseteq [2, \lambda_1+1)$, and thus $\mathrm{supp}(\nu_j) \subseteq [2, \lambda_1+1)$ for every j, since ν is weakly decreasing and since $\mathrm{supp}(\nu_1)$ is nonempty. This would lead to $\nu\big|_{\subseteq[2,\lambda_1+1)} = \nu$ and thus $\#\nu\big|_{\subseteq[2,\lambda_1+1)} = \#\nu = \lambda_1 > \lambda_1 + 1 - 2$, contradicting the representability of ν. Hence, we have $1 \in \mathrm{supp}(\nu_1)$, so that we can fill the first column of λ/μ with 1's and 2's in such a way that it becomes mixed and the 1's are displaced by 2's at level ν_1. Now, let $\widetilde{\lambda/\mu}$ be the skew partition λ/μ without its first column, and $\widetilde{\nu}$ be the partition $(\nu_2, \nu_3, \ldots)$. Then, the partition $\widetilde{\nu}$ is representable with respect to $\widetilde{\lambda/\mu}$. Otherwise we would have $\#\nu\big|^{\widetilde{\lambda/\mu}}_{\subseteq[2,b+1)} > b-1$ for some $b \geqslant 1$, but then we would have $\mathrm{supp}(\nu_1) \subseteq [1, b+1)$ as well and therefore $\#\nu\big|_{\subseteq[1,b+1)} > (b-1)+1 = b$, contradicting the representability of λ/μ. Thus we can fill in the entries in the cells of $\widetilde{\lambda/\mu}$ by induction. □

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[1]This follows from Lemma 23 **(c)** (applied to the skew shape $\widetilde{\lambda/\mu}$ and $k = 1$). Here we are using the fact that if we apply Lemma 23 **(a)** to $\widetilde{\lambda/\mu}$ instead of λ/μ, then we get $r = 1$ (because if $r \geqslant 2$, then $\#\nu\big|_{\subseteq[a_2,b_2)} = \#\nu\big|^{\widetilde{\lambda/\mu}}_{\subseteq[a_2,b_2)} = b_2 - a_2$ in contradiction to the irreducibility of λ/μ).

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