Even More Infinite Ball Packings from Lorentzian Coxeter Systems

Hao Chen *

Institut für Mathematik
Freie Universität Berlin
Arnimallee 2, 14195, Berlin, Germany

hao.chen.math@gmail.com

Submitted: Jan 22, 2015; Accepted: Jul 22, 2016; Published: Aug 5, 2016
Mathematical Subject Classification: 52C17, 20F55

Abstract

Boyd (1974) proposed a class of infinite ball packings that are generated by inversions. Later, Maxwell (1983) interpreted Boyd's construction in terms of root systems in Lorentz spaces. In particular, he showed that the space-like weight vectors correspond to a ball packing if and only if the associated Coxeter graph is of "level 2". In Maxwell's work, the simple roots form a basis of the representations space of the Coxeter group. In several recent studies, the more general based root system is considered, where the simple roots are only required to be positively independent. In this paper, we propose a geometric version of "level" for root systems to replace Maxwell's graph theoretical "level". Then we show that Maxwell's results naturally extend to the more general root systems with positively independent simple roots. In particular, the space-like extreme rays of the Tits cone correspond to a ball packing if and only if the root system is of level 2. We also present a partial classification of level-2 root systems, namely the Coxeter d-polytopes of level-2 with $d + 2$ facets.

Keywords: Ball packing, hyperbolic Coxeter group, Coxeter polytope

1 Introduction

The title refers to a paper of Boyd titled "A new class of infinite sphere packings" [6], in which he described a class of infinite ball packings that are generated by inversions, generalising the famous Apollonian disk packing.

*This research was supported by the Deutsche Forschungsgemeinschaft within the Research Training Group "Methods for Discrete Structures" (GRK 1408) and by the ERC Advanced Grant number 247029 "SDModels" while the author was at Freie Universität Berlin.

Maxwell [31] generalized Boyd's construction by interpreting a ball packing as the space-like weights of an infinite root system in a Lorentz space. In particular, Maxwell defined the "level" of a Coxeter graph as the smallest integer l such that the deletion of any l vertices leaves a Coxeter graph for finite or affine Coxeter system. He then proved that the space-like weights correspond to a ball packing if and only if the associated Coxeter graph is of level 2.

Labbé and the author [11] revisited Maxwell's work, and found connections with recent works on limit roots. *Limit roots* are accumulation points of the roots in projective space. The notion was introduced and studied in [18], where it was proved that limit roots lie on the isotropic cone of the quadratic space. The relations between limit roots and the imaginary cone are investigated in [13] and [14].

For root systems in a Lorentz space, the set of limit roots is equal to the limit set of the Coxeter group seen as a Kleinian group acting on the hyperbolic space [19]. In [11], we proved that the accumulation points of the roots and of the weights coincide on the light cone in the projective space. As a consequence, when the Coxeter graph is of level 2, the set of limit roots is the residue set of the ball packing described by Boyd and Maxwell. Furthermore, we gave a geometric interpretation for Maxwell's notion of level, described the tangency graph of the Boyd–Maxwell ball packing in terms of the Coxeter complex, and completed the enumeration of 326 Coxeter graphs of level 2.

Comparing to [31], the root systems considered in most studies of limit roots (e.g. [14, 18]) are more general in several ways:

First, the root systems considered in [14, 18] are not necessarily in a Lorentz space. For non-Lorentzian root systems, we conjectured in [11] that the accumulation points of roots still coincide with the accumulation points of weights.

Second, even if the root system is Lorentzian, the associated Coxeter graph is not necessarily of level 2. The cases of level $\neq 2$ were also investigated in [11]. It turns out that no ball appears if the Coxeter graph is of level 1, and balls may intersect if the Coxeter graph is of level > 2. In either case, it remains true that the set of limit roots is the residue set of the balls corresponding to the space-like weights.

The current paper deals with a third gap. Maxwell only considered the case where the simple roots form a basis of the representation space, so one can define the fundamental weights as the dual basis. However, in [14] and [18], a more general based root system is considered, which only requires the simple roots to be positively independent, but not necessarily linearly independent.

In order to extend Maxwell's results, we propose the notion of "level" for root systems to replace Maxwell's graph theoretical "level"; see Definition 2.2. The definition is based on the geometric interpretation in [11]. When the simple roots are linearly independent as in [31], our "level" for root systems and Maxwell's "level" for the Coxeter graphs coincide. Moreover, in place of the weight vectors, we will look at the extreme rays of the Tits cone. Then we show in Section 3 that all the results in [31] and [11] extend to the more general setting with positively independent simple roots. In particular, the following theorem generalizes [31, Theorem 3.2].

Theorem 1.1. *For a Lorentzian based root system, the space-like extreme rays of the Tits cone correspond to a ball packing if and only if the based root system is of level* 2*.*

This correspondence will be explained in Section 3.2. We consider the ball packing as associated to the based root system in the theorem, and obtain the following generalization of [11, Theorem 3.4].

Theorem 1.2. *The set of limit roots of an irreducible Lorentzian root system of level* 2 *is equal to the residual set of the associated ball packing.*

Maxwell's proofs rely on the decomposition of vectors into basis vectors (i.e. the simple roots), which is not possible in our setting. Hence many proofs need to be revised. Our proofs will make heavy use of projective geometry. Lorentzian root systems of level $\geqslant 3$ are discussed in the end of Section 3.

For many Lorentzian root systems of level 2, the associated Coxeter graph is not of level 2 in Maxwell's sense. So our results imply many new infinite ball packings generated by inversions. In Section 4, we provide a partial classification of Lorentzian root systems of level 2. More specifically, we try to classify the Coxeter d-polytopes of level 2 with $d+2$ facets. For this, we follow the approach of [23, 15, 40] for enumerating hyperbolic Coxeter d-polytopes with $d+2$ facets, and take advantage of previous enumerations of Coxeter systems, such as [25, 9, 11].

For explicit images of ball packings, the readers are referred to artworks of Leys' [26]. The 3-dimensional ball packings in Leys' paper (and also on his website) are inspired from [4]. Similar idea was also proposed by Bullett and Mantica [7, 30], who also noticed generalizations in higher dimensions. However, the packings considered in these literatures are very limited. In our language, the Coxeter polytopes associated to these packings have the combinatorial type of pyramid over regular polytopes. In [7], the authors were aware of Maxwell's work, but explained that:

> Our approach via limit sets of Kleinian groups is more naive, replacing arguments about weight vectors in Minkowsky N-space by elementary geometric arguments involving polygonal tiles on the Poincare disc: it mirrors the algorithm we use to construct the circle-packings and seems well adapted to computation of the exponent of the packing and other scaling constants.

On the contrary, one easily verifies that the height of a weight vector, defined in the end of Section 2.6, is asymptotically equivalent to the curvature of the corresponding ball. Hence weight vectors only make the computation of the *exponent* much easier.

Remark. The Hausdorff dimension of the residual set of infinite ball packings are usually approximated by computing the exponent. In the literature, Boyd's works (e.g. [6]) are often cited to support this numeric estimate. However, this was not fully justified until recently by Oh and Shah [35].

2 Geometric Coxeter systems and levels

2.1 Lorentz space

For general references on Lorentz spaces, we refer the readers to the books [8] and [17]. In particular, these books discuss the representation of spheres in Lorentz spaces, which would be very helpful for understanding this paper.

A *quadratic space* is a pair $(V, \mathcal{B})$ where V is a real vector space and $\mathcal{B}$ is a symmetric bilinear form on V. Two vectors $\mathbf{x}, \mathbf{y} \in V$ are said to be *orthogonal* if $\mathcal{B}(\mathbf{x}, \mathbf{y}) = 0$. For a subspace $U \subseteq V$, its *orthogonal companion* is the set

$$U^\perp = \{\mathbf{x} \in V \mid \mathcal{B}(\mathbf{x}, \mathbf{y}) = 0 \text{ for all } \mathbf{y} \in U\}.$$

The orthogonal companion $V^\perp$ of the whole space V is called the *radical*. We say that $(V, \mathcal{B})$ is *degenerate* if the radical $V^\perp$ contains non-zero vectors. In this case, the matrix of $B = (\mathcal{B}(\mathbf{e}_i, \mathbf{e}_j))$ is singular.

The signature of $(V, \mathcal{B})$ is the triple (n_+, n_0, n_-) indicating the number of positive, zero and negative eigenvalues of the matrix B. For non-degenerate spaces we have $n_0 = 0$. A non-degenerate space is an *Euclidean space* if $n_- = 0$ (i.e. B is positive definite), or a *Lorentz space* if $n_- = 1$.

The group of linear transformations of V that preserve the bilinear form $\mathcal{B}$ is called an *orthogonal group*, and is denoted by $O_\mathcal{B}(V)$. The orthogonal group of a Lorentz space is called a *Lorentz group*.

The set

$$Q = \{\mathbf{x} \in V \mid \mathcal{B}(\mathbf{x}, \mathbf{x}) = 0\}$$

is called the *isotropic cone*, and vectors in Q are said to be *isotropic*. In a Lorentz space, the isotropic cone is called the *light cone*, and isotropic vectors are said to be *light-like*. Two light-like vectors are orthogonal if and only if one is the scalar multiple of the other. A non-isotropic vector $\mathbf{x} \in V$ is said to be *space-like* (resp. *time-like*) if $\mathcal{B}(\mathbf{x}, \mathbf{x}) > 0$ (resp. < 0). A subspace $U \subseteq V$ is said to be *space-like* if its non-zero vectors are all space-like, *light-like* if it contains some non-zero light-like vector but no time-like vector, or *time-like* if it contains time-like vectors.

For a non-isotropic vector $\alpha \in V$, the *reflection* in α is defined as the map

$$s_\alpha(\mathbf{x}) = \mathbf{x} - 2\frac{\mathcal{B}(\alpha, \mathbf{x})}{\mathcal{B}(\alpha, \alpha)}\alpha, \quad \text{for all } \mathbf{x} \in V.$$

The orthogonal companion of $\mathbb{R}\alpha$,

$$H_\alpha = \{\mathbf{x} \in V \mid \mathcal{B}(\mathbf{x}, \alpha) = 0\},$$

is the hyperplane fixed by the reflection s_α, and is called the *reflecting hyperplane* of α. One verifies that α is space-like (resp. time-like) if and only if H_α is time-like (resp. space-like).

2.2 Based root systems

Based root systems have been the framework of several recent studies of infinite Coxeter systems, including [20, 24, 14, 18] etc., and traces back to Vinberg [44].

Recall that an abstract *Coxeter system* is a pair (W, S), where S is a finite set of generators and the *Coxeter group* W is generated by S with the relations $(st)^{m_{st}} = e$ where $s, t \in S$, $m_{ss} = 1$ and $m_{st} = m_{ts} \geqslant 2$ or $= \infty$ if $s \neq t$. The cardinality $n = |S|$ is the *rank* of the Coxeter system (W, S). For an element $w \in W$, the *length* of w, denoted by $\ell(w)$, is the smallest natural number k such that $w = s_1 s_2 \dots s_k$ for $s_i \in S$. Readers unfamiliar with Coxeter groups are invited to consult [5, 21] for basics.

Let $(V, \mathcal{B})$ be a quadratic space. A *root basis* Δ in $(V, \mathcal{B})$ is a *finite* set of vectors in V such that

1. $\mathcal{B}(\alpha, \alpha) = 1$ for all $\alpha \in \Delta$;
2. $\mathcal{B}(\alpha, \beta) \in]-\infty, -1] \cup \{-\cos(\pi/k),\ k \in \mathbb{Z}_{\geqslant 2}\}$ for all $\alpha \neq \beta \in \Delta$;
3. Δ is *positively independent*. That is, a linear combination of Δ with non-negative coefficient only vanishes when all the coefficients vanish.

We assume that Δ spans V. If this is not the case, we replace V by the subspace $\mathrm{Span}(\Delta)$, and $\mathcal{B}$ by its restriction on $\mathrm{Span}(\Delta)$.

Following [20], we call Δ a *free root basis* if it also forms a basis of V. Following [24], we call a free root basis Δ *classical* if $\mathcal{B}(\alpha, \beta) \geqslant -1$ for all $\alpha, \beta \in \Delta$. The classical root basis is present in textbooks such as [5, 21].

Let $S = \{s_\alpha \mid \alpha \in \Delta\}$ be the set of reflections in vectors of Δ, and W be the reflection subgroup of $O_{\mathcal{B}}(V)$ generated by S. Then (W, S) is a *Coxeter system*, where the order of $s_\alpha s_\beta$ is k if $\mathcal{B}(\alpha, \beta) = -\cos(\pi/k)$, or ∞ if $\mathcal{B}(\alpha, \beta) \leqslant -1$. Let $\Phi := W(\Delta)$ be the orbit of Δ under the action of W, then the pair (Δ, Φ) is called a *based root system in* $(V, \mathcal{B})$ *with associated Coxeter system* (W, S). Vectors in Δ are called *simple roots*, and vectors in Φ are called *roots*. The roots Φ are partitioned into *positive roots* $\Phi^+ = \mathrm{Cone}(\Delta) \cap \Phi$ and *negative roots* $\Phi^- = -\Phi^+$. The *rank* of (Δ, Φ) is the cardinality of Δ. In the following, we write $w(\mathbf{x})$ for the action of $w \in W$ on V, and the word "based" is often omitted.

A based root system in Euclidean space has a classical root basis, in which case the associated Coxeter system is of *finite type*, and we say the root system is *finite*. If $(V, \mathcal{B})$ is degenerate yet $\mathcal{B}(\mathbf{v}, \mathbf{v}) \geqslant 0$ for all $\mathbf{v} \in V$, the based root system (Δ, Φ) in $(V, \mathcal{B})$ also has a classical root basis; in this case, we say that (Δ, Φ) is *affine* since the associated Coxeter system must be of affine type. If $(V, \mathcal{B})$ is a Lorentz space, we say that the root system (Δ, Φ) is *Lorentzian*. Finally, (Δ, Φ) is said to be non-degenerate if $(V, \mathcal{B})$ is.

A based root system is *irreducible* if there is no proper partition $\Delta = \Delta_I \sqcup \Delta_J$ such that $\mathcal{B}(\alpha, \beta) = 0$ for all $\alpha \in \Delta_I$ and $\beta \in \Delta_J$, in which case the associated Coxeter group is also irreducible.

2.3 Geometric representations of Coxeter systems

Given an abstract Coxeter system (W, S) of rank n. We introduce a matrix B such that

$$B_{st} = \begin{cases} -\cos(\pi/m_{st}) & \text{if} \quad m_{st} < \infty, \\ -c_{st} & \text{if} \quad m_{st} = \infty, \end{cases}$$

for $s, t \in S$, where c_{st} are chosen arbitrarily with $c_{st} = c_{ts} \geqslant 1$. Note that if m_{st} is finite for all $s, t \in S$, there is only one choice for the matrix B. This is the case when, for instance, (W, S) is of finite or affine type (with the exception of I_∞). In [11], the Coxeter system (W, S) associated with the matrix B is referred to as a *geometric Coxeter system*, and denoted by $(W, S)_B$.

With the matrix B, there is a canonical way to associate (W, S) to a root system with free root basis. Let V be a real vector space of dimension n with basis $\{\mathbf{e}_s\}_{s\in S}$ indexed by the elements in S. Then the matrix B defines a bilinear form $\mathcal{B}$ on V by $\mathcal{B}(\mathbf{e}_s, \mathbf{e}_t) = \mathbf{e}_s^\intercal B \mathbf{e}_t$ for $s, t \in S$. The basis $\{\mathbf{e}_s\}_{s\in S}$ form a free root basis in $(V, \mathcal{B})$. The homomorphism that maps $s \in S$ to the reflection in $\mathbf{e}_s$ is a faithful (cf. [24, Theorem 1.2.2(b)]) *geometric representation* of the Coxeter group W as a discrete reflection subgroup of the orthogonal group $O_\mathcal{B}(V)$. This is the representation considered in Maxwell [31].

The current paper investigates based root systems in Lorentz spaces, which are non-degenerate. Hence we will focus on non-degenerate root systems. We will be particularly interested in root basis that is not linearly independent. If the free root basis given above is degenerate, there is a canonical way to obtain a non-degenerate root system by "dividing out the radical" [24, Section 6.1].

If the matrix B is singular with rank d, then the dimension of the *radical* $V^\perp$ is $n - d$. The bilinear form $\mathcal{B}$ restricted on the quotient space $U = V/V^\perp$ is non-degenerate. Let α_s be the projection of $\mathbf{e}_s$ onto U for all $s \in S$. If the root system induced by $\{\mathbf{e}_s\}_{s\in S}$ is not affine, the vectors $\Delta = \{\alpha_s \mid s \in S\}$ are positively independent and form a root basis in $(U, \mathcal{B})$ such that $\mathcal{B}(\alpha_s, \alpha_t) = B_{st}$ [24, Proposition 6.1.2]. The homomorphism that maps $s \in S$ to the reflection in α_s is a faithful (cf. [24, Theorem 1.2.2(b)]) *geometric representation* of the Coxeter group W as a discrete reflection subgroup of the orthogonal group $O_\mathcal{B}(U)$.

This process does not work for affine root systems. We call a root system *canonical* if it is non-degenerate or affine. The process described in the previous paragraph is called *canonicalization*. Unless otherwise stated, root systems in this paper are all canonical.

We adopt Vinberg's convention[1] to encode the matrix B into the Coxeter graph. That is, if $c_{st} > 1$ the edge st is dashed and labeled by $-c_{st}$. This convention is also used by Abramenko–Brown [1, Section 10.3.3] and Maxwell [31, Section 1]. A Coxeter graph is connected if and only if the Coxeter system it represents is irreducible.

Maxwell proposed the following definition:

Definition 2.1 (Level of a Coxeter graph)**.** The *level of a Coxeter graph* is the smallest integer l such that deletion of any l vertices leaves a Coxeter graph of a finite or affine root system.

[1]This convention traces back to [44] according to [31], [45] according to [12]

2.4 Facials subsets and the level of root systems

The readers are assumed to be familiar with convex cones. Otherwise we recommend [37] or the appendix of [13] for reference.

Let (Δ, Φ) be a canonical root basis in $(V, \mathcal{B})$ with associated Coxeter system (W, S). Because of the positive independence, the cone $\mathcal{C} = \mathrm{Cone}(\Delta)$ is a pointed polyhedral cone. We call $\mathcal{C}$ the *positive cone* since it is also spanned by the positive roots, i.e. $\mathcal{C} = \mathrm{Cone}(\Phi^+)$. The extreme rays of $\mathcal{C}$ are spanned by the simple roots in Δ.

A subset Δ' of Δ is said to be *k-facial*, $1 \leqslant k \leqslant d$, if $\mathrm{Cone}(\Delta')$ is a face of codimension k of $\mathcal{C}$. In this case, we say that $I = \{s_\alpha \mid \alpha \in \Delta'\} \subset S$ is a *k-facial* subset of S, and use the notation Δ_I in place of Δ'. We also write $W_I = \langle I \rangle$, $V_I = \mathrm{Span}(\Delta_I)$ and $\Phi_I = W_I(\Delta_I)$. Then a k-facial subset I of S induce a based root system (Δ_I, Φ_I) in $(V_I, \mathcal{B})$ with associated Coxeter system (W_I, I). We call (Δ_I, Φ_I) a k-facial root subsystem. We agree on the convention that Δ and S themselves are 0-facial.

We now define the central notion of this paper.

Definition 2.2 (Level of a root system)**.** Let (Δ, Φ) be a root system in $(V, \mathcal{B})$ with canonical root basis Δ and associated Coxeter system (W, S). The *level* of (Δ, Φ) is the smallest integer l such that every l-facial subsystem is (after canonicalization if necessary) finite or affine.

So root systems of finite or affine type are of level 0. If a root system is of level l and every l-facial subsystem is of finite type, we say that it is *strictly* of level l.

Remark. We only defined the notion of “facial” and “level” for canonical root basis. However, it is possible that a facial root subsystem is not canonical. This does not happen for Lorentzian root systems, whose facial root subsystems are either finite, affine or Lorentzian. Hence in this paper, the canonicalization in the definition of “level” is never needed. For non-Lorentzian root systems, we propose for the moment to first canonicalize a facial root subsystem if necessary. This is however open for future discussions.

For free root basis, the level of the root system equals the level of the Coxeter graph. Otherwise, the two levels are different in general.

Given a root system of level l, the bilinear form $\mathcal{B}$ is positive semidefinite on $\mathrm{Span}(\Delta_I)$ for every l-facial $I \subset S$, and indefinite for at least one $(l-1)$-facial $I \subset S$. If the root system (Δ, Φ) is Lorentzian, we can reformulate Definition 2.2 in terms of its positive cone $\mathcal{C}$: The level of (Δ, Φ) is 1+ the maximum codimension of the time-like faces of $\mathcal{C}$. Here, we use the conventions that a face of codimension 0 is the polytope itself.

2.5 Fundamental cone, Tits cone and imaginary cone

In this part, we assume that the root system is non-degenerate, so that we can identify the dual space V^* with V. For definitions for degenerate root systems, see the remark in [13, §1.9].

The dual cone $\mathcal{C}^*$ of the positive cone is called the *fundamental chamber*. Recall that

$$\mathcal{C}^* = \{\mathbf{x} \in V \mid \mathcal{B}(\mathbf{x}, \mathbf{y}) \geqslant 0 \text{ for all } \mathbf{y} \in \mathcal{C}\}.$$

The fundamental chamber is the fundamental domain for the action of W on V.

Since $(V, \mathcal{B})$ is non-degenerate, $\mathcal{C}^*$ is also a pointed polyhedral cone. The supporting hyperplanes at the facets of $\mathcal{C}^*$ are the reflecting hyperplanes of the simple roots. So the Coxeter group W is isomorphic to the group generated by reflections in the facets of $\mathrm{Cone}(\Delta^*)$, and the stabilizer of a face of $\mathrm{Cone}(\Delta^*)$ is generated by reflections in the facets that contains this face.

The extreme rays of $\mathcal{C}^*$ correspond to the facets of $\mathcal{C}$. For each 1-facial subset $I \subset S$, we define the vector ω_I by

$$\mathcal{B}(\alpha, \omega_I) = \begin{cases} = 0, & \alpha \in \Delta_I \\ > 0, & \alpha \notin \Delta_I \end{cases}$$

and

$$\min_{\alpha \notin \Delta_I} \mathcal{B}(\alpha, \omega_I) = 1.$$

So ω_I is a representative on the extremal ray of $\mathcal{C}^*$ associated to the facet $\mathrm{Cone}(\Delta_I)$ of $\mathcal{C}$. We write $\Delta^* := \{\omega_I \mid I \text{ is a 1-facial subset of } S\}$. If Δ is a free root basis, then Δ^* is the dual basis, i.e. the set of *fundamental weights*. We use Ω to denote the orbit $W(\Delta^*)$ under the action of W.

We define the Tits cone as the union of the orbits of $\mathcal{C}^*$ under the action of W, i.e.

$$\mathcal{T} = \bigcup_{w \in W} w(\mathcal{C}^*).$$

It is equal to $\mathrm{Cone}(\Omega)$. For finite root systems, the Tits cone is the entire representation space V [1, § 2.6.3]. Otherwise, for infinite non-affine root systems, $\mathcal{T}$ is a pointed cone. See [13, Lemma 1.10] for more properties of Tits cone.

For Lorentzian root systems, the dual of the closure $\overline{\mathcal{T}}$ is the closure of the *imaginary cone* $\mathcal{I}$, which is equal to the intersection of the orbits of $\mathcal{C}$ under the action of W [13, § 3.1 and Theorem 5.1(a,b)], i.e.

$$\overline{\mathcal{T}}^* = \overline{\mathcal{I}} = \bigcap_{w \in W} w(\mathcal{C}).$$

2.6 Projective picture and limit roots

As observed in [18, Section 2.1], the roots Φ have no accumulation point. To study the asymptotic behavior of roots, we pass to the projective representation space $\mathbb{P}V$, i.e. the space of 1-subspaces of V; see [18, Remark 2.2]. For a non-zero vector $\mathbf{x} \in V$, we denote by $\widehat{\mathbf{x}} \in \mathbb{P}V$ the 1-subspace spanned by $\mathbf{x}$. The geometric representation then induces a *projective representation*

$$w \cdot \widehat{\mathbf{x}} = \widehat{w(\mathbf{x})}, \quad w \in W, \quad \mathbf{x} \in V.$$

For a set $X \subset V$, we have the corresponding projective set

$$\widehat{X} := \{\widehat{\mathbf{x}} \in \mathbb{P}V \mid \mathbf{x} \in X\}$$

In this sense, we have the projective simple roots $\widehat{\Delta}$, projective roots $\widehat{\Phi}$ and the projective isotropic cone $\widehat{Q}$. We use $\operatorname{Conv}(\widehat{X})$ and $\operatorname{Aff}(\widehat{X})$ to denote $\widehat{\operatorname{Cone}(X)}$ and $\widehat{\operatorname{Span}(X)}$ respectively.

The *limit roots* are the accumulation points of $\widehat{\Phi} \subset \mathbb{P}V$. In [18], it was proved that the limit roots lies on $\widehat{Q}$. For free root basis, it was proved in [11] that limit roots are also the accumulation points of $\widehat{\Omega}$. Limit roots are the projectivization of the extreme rays of the imaginary cone [13, 14].

The projective space $\mathbb{P}V$ can be identified with an affine subspace plus a *hyperplane at infinity*. We usually fix a vector $\mathbf{t}$ and take the affine subspace

$$H_{\mathbf{t}}^1 = \{\mathbf{x} \in V \mid \mathcal{B}(\mathbf{t}, \mathbf{x}) = 1\}.$$

Then for a vector $\mathbf{x} \in V$, we represent $\widehat{\mathbf{x}} \in \mathbb{P}V$ by the vector $\mathbf{x}/\mathcal{B}(\mathbf{t}, \mathbf{x}) \in H_{\mathbf{t}}^1$ if $\mathcal{B}(\mathbf{t}, \mathbf{x}) \neq 0$, or some point at infinity if $\mathcal{B}(\mathbf{t}, \mathbf{x}) = 0$.

For Lorentzian root systems, it is convenient to choose a time-like $\mathbf{t} \in -\mathcal{C} \cap \mathcal{C}^*$. Such a vector always exists; see [14, § 2.2] and [13, § 3.2].

Since $\mathbf{t}$ is time-like, the subspace $H_{\mathbf{t}} = \mathbf{t}^\perp$ is space-like and divides the space into two parts, each containing half of the light cone. Vectors on the same side as $\mathbf{t}$ are said to be *past-directed*; they have negative inner products with $\mathbf{t}$. Those on the other side are said to be *future directed*. It then makes sense to call $\mathbf{t}$ the *direction of past*.

Since $\mathbf{t} \in -\mathcal{C}$, the fundamental chamber $\mathcal{C}^*$ is past-directed. Hence on the affine hyperplane $H_{\mathbf{t}}^1$, the light cone $\widehat{Q}$ appears as a closed surface projectively equivalent to a sphere, and the fundamental chamber appears as a bounded polytope. We call $\mathcal{P} = \widehat{\mathcal{C}}^*$ the Coxeter polytope. We can view the interior of $\widehat{Q}$ as the Klein model for the hyperbolic space. Then the study on Lorentzian root systems can be seen as a study of hyperbolic Coxeter polytopes.

Since $\mathbf{t} \in \mathcal{C}^*$, one verifies that $\mathcal{B}(\mathbf{t}, \alpha) > 0$ for all the simple roots $\alpha \in \Phi^+$, so the affine hyperplane $H_{\mathbf{t}}^1$ is transverse for the positive cone $\mathcal{C}$. This fact is useful for the visualization of the root system, see [18, § 5.2], and also for a technical reason. An important technical notion in [11] is the height $h(\mathbf{x})$ for a vector $\mathbf{x} \in V$; see [11, § 3.1] and [18, Theorem 2.7]. For non-degenerate root systems, we define $h(\mathbf{x}) = \mathcal{B}(\mathbf{t}, \mathbf{x})$. One verifies that $h(\mathbf{x})$ is a L_1-norm on the positive roots Φ^+. If the root basis is free, our definition coincides with the definition in [11], where $h(\mathbf{x})$ is the sum of the coordinates with simple roots as basis.

3 Extending Maxwell's results

In this part, we extend one by one the major results of [31] to canonical root systems. Detailed proofs are given only if there is a significant difference from Maxwell's proof. For the statement of the results, we mimic intentionally the formulations in [31]. Remember that the level of a root system is in general different from the level of its Coxeter graph.

First of all, the following two results are proved in [31] for free root basis, and are extended to canonical root basis in [20]; see also [13, §9.4]. They generalize, respectively, Proposition 1.2 and Corollary 1.3 of [31].

Proposition 3.1 ([20, Proposition 3.4]). *For every vector* $\mathbf{x}$ *in the dual of the Tits cone,* $\mathcal{B}(\mathbf{x}, \mathbf{x}) \leqslant 0$.

Corollary 3.2 ([20, Proposition 3.7]). *The Tits cone of a Lorentzian root system contains one component of the light cone* $Q \setminus \{0\}$.

We will need the following lemma:

Lemma 3.3. *Let* $\mathcal{P}$ *be a polytope of dimension* $d \geqslant 3$, *and* x *be a point in the exterior of* $\mathcal{P}$. *Then there is a line* L *that passes through* x *and two points* $u \in F$ *and* $v \in G$, *where* F *and* G *are two disjoint faces of* $\mathcal{P}$. *Moreover, the following are equivalent:*

(i) *For any* L *with the forementioned property, either* u *or* v *is in the interior of a facet.*

(ii) x *is not on the affine hull of any facet, and is either beyond all but one facet, or beneath all but one facet.*

Here, we say that x is beyond (resp. beneath) a facet F of $\mathcal{P}$ if x is on the opposite (resp. same) side of $\mathrm{Aff}(F)$ as the interior of $\mathcal{P}$; see [16].

Proof. Let H be any hyperplane that separates x and $\mathcal{P}$. For any point w on the boundary of $\mathcal{P}$, let $\pi(w)$ be the projection of w on H from x, i.e. $\pi(w)$ is the intersection point of H with the segment $[x, w]$. So $\pi(\mathcal{P})$ models the polytope $\mathcal{P}$ seen from x; see Figure 1 for an example. If the segment $[x, w]$ is disjoint from the interior of $\mathcal{P}$, we say that w is *visible* (from x). In Figure 1, visible edges are solid while invisible edges are dashed.

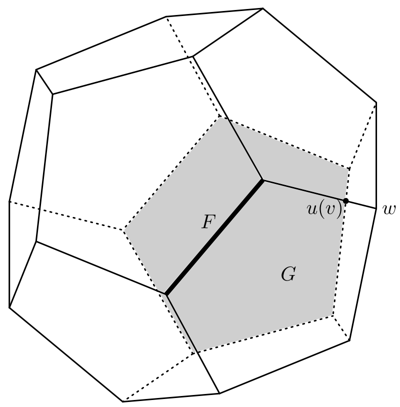


Figure 1:

The existence of L with the given property follows from a generalization of Radon's theorem [3], which guarantees two disjoint faces, F and G, of $\mathcal{P}$ such that $\pi(F) \cap \pi(G) \neq \emptyset$. Then L is given by any point in the intersection. For the equivalence, we only need to prove that (i) implies (ii) by contraposition. The other direction is obvious.

If $x \in \mathrm{Aff}(F)$ for some facet F of $\mathcal{P}$, we apply the generalized Radon's theorem to F, and conclude that there are two disjoint faces of F whose images under π intersect. Any point in the intersection gives a line violating (i).

Assume that x is beneath at least two facets and beyond at least two other facets, i.e. there are at least two visible facets and two invisible facets. Let F and G be two disjoint faces predicted by the generalized Radon's theorem, where F is visible while G is not. By going to the boundary if necessary, we may assume that one of them, say F, is of dimension $< d-1$. If G is also of dimension $< d-1$, any point in the intersection gives a line violating (i), so G must be a facet. We further assume $\pi(F) \subset \operatorname{Int} \pi(G)$, otherwise $\pi(F)$ intersects the boundary of $\pi(G)$ and any point in the intersection gives a line violating (i). In Figure 1, F is the thick edge, and G is the gray face.

Consider the visible faces of dimension $< d-1$ that are disjoint from G. In Figure 1, they happen to be the solid edges and their vertices. The projection of their union is a connected set (because $d \geqslant 3$) that contains $\pi(F) \subset \operatorname{Int} \pi(G)$. Since G is not the only invisible facet, there is a visible vertex w on the boundary of $\pi(\mathcal{P})$ such that $\pi(w) \notin \pi(G)$. So the projection of the union intersects the boundary of $\pi(G)$. Any point in the intersection gives a line violating (i).

To conclude, whenever x violates (ii), we are able to find a line violating (i), which finishes the proof. $\square$

3.1 Lorentzian root systems of level 1 or 2

Two vectors $\mathbf{x}, \mathbf{y}$ in $(V, \mathcal{B})$ are said to be *disjoint* if $\mathcal{B}(\mathbf{x}, \mathbf{y}) \leqslant 0$ and $\mathcal{B}$ is *not* positive definite on the subspace $\operatorname{Span}\{\mathbf{x}, \mathbf{y}\}$.

The proofs of the following results are apparently very different from Maxwell's argument [31]. Indeed, while Maxwell's proofs make heavy use of basis, our proofs rely primarily on projective geometry. However, the basic idea is the same. In the case of free root basis, our proofs are just geometric interpretations of Maxwell's proofs.

The following result generalizes [31, Proposition 1.4].

Proposition 3.4. *If* (Δ, Φ) *is level* 1*, then it is Lorentzian, and vectors in* Δ^* *are pairwise disjoint while none are space-like.*

Proof. Let (Δ, Φ) be a canonical root system of level 1 in $(V, \mathcal{B})$ with associated Coxeter system (W, S). For a vector $\mathbf{x}$ such that $\mathcal{B}(\mathbf{x}, \mathbf{x}) \leqslant 0$, we claim that $\widehat{\mathbf{x}} \in \widehat{\mathcal{C}}$. Assume without loss of generality that $h(\mathbf{x}) \geqslant 0$. If the claim is not true, we can find two vectors $\mathbf{x}_+ \in \operatorname{Cone}(\Delta_I)$ and $\mathbf{x}_- \in \operatorname{Cone}(\Delta_J)$ such that $\mathbf{x} = \mathbf{x}_+ - \mathbf{x}_-$ for two disjoint facial subset $I, J \subset S$; see Figure 2 (left) for the projective picture.

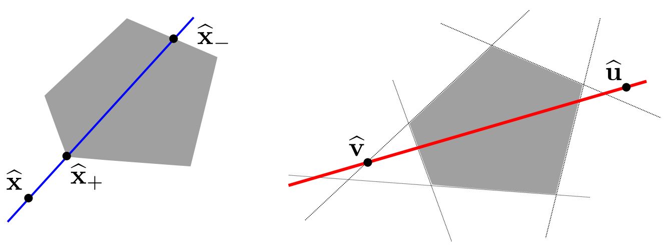


Figure 2:

Since $\mathcal{B}(\mathbf{x},\mathbf{x}) = \mathcal{B}(\mathbf{x}_+,\mathbf{x}_+) + \mathcal{B}(\mathbf{x}_-,\mathbf{x}_-) - 2\mathcal{B}(\mathbf{x}_+,\mathbf{x}_-) \leqslant 0$, and $\mathcal{B}(\mathbf{x}_+,\mathbf{x}_-) < 0$ because I and J are disjoint, we have either $\mathcal{B}(\mathbf{x}_+,\mathbf{x}_+) < 0$ or $\mathcal{B}(\mathbf{x}_-,\mathbf{x}_-) < 0$, both contradict the fact that (Δ,Φ) is of level 1. Our claim is then proved. If $\mathcal{B}(\mathbf{x},\mathbf{x}) < 0$, since $\mathcal{B}$ is positive semidefinite on the facets, $\mathbf{x}$ must be in the interior of $\mathcal{C}$. If $\mathcal{B}(\mathbf{x},\mathbf{x}) = 0$, it is possible that $\mathbf{x}$ is in the interior of a facet of $\mathcal{C}$.

Now assume that the representation space V is not a Lorentz space. Then, as noticed by Maxwell [31], there is a pair of *orthogonal* vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathcal{B}(\mathbf{u},\mathbf{u}) < 0$ and $\mathcal{B}(\mathbf{v},\mathbf{v}) = 0$. For any linear combination $\mathbf{x} = \lambda\mathbf{u} + \mu\mathbf{v}$ we have $\mathcal{B}(\mathbf{x},\mathbf{x}) < 0$ as long as $\lambda \neq 0$. The subspace $\operatorname{Span}\{\mathbf{u},\mathbf{v}\}$ intersect the hyperplanes $\operatorname{Span}(\Delta_I)$ at a ray $\mathbb{R}_+\mathbf{x}_I$ for every 1-facial subset $I \subset S$; see Figure 2 (right) for the projective picture (modeled on $H^1_{\mathbf{t}}$). This means that each of these supporting hyperplanes contains a ray $\mathbb{R}_+\mathbf{x}_I$ where $\mathcal{B}(\mathbf{x}_I,\mathbf{x}_I) < 0$, with at most one exception (namely $\mathbb{R}_+\mathbf{v}$). This contradicts the fact that (Δ,Φ) is of level 1. We then proved that (Δ,Φ) is Lorentzian.

Let I be any 1-facial subset of S. Since (Δ,Φ) is of level 1, the subspace $\operatorname{Span}(\Delta_I)$ is not time-like, so its orthogonal companion $\mathbb{R}\omega_I$ is not space-like. This proves that no vector in Δ^* is space-like.

It is clear that any two vectors in Δ^* span a Lorentz space. For the disjointness, we only needs to prove that $\mathcal{B}(\omega_I,\omega_J) \leqslant 0$ for any two 1-facial subsets $I \neq J \subset S$. Since ω_I is not space-like, we have seen that $\omega_I \in \mathcal{C}$, so $\mathcal{B}(\omega_I,\omega_J)$ has the same sign (possibly 0) for all 1-facial $J \subset S$, which is $\leqslant 0$ when ω_I is time-like (take the sign of $\mathcal{B}(\omega_I,\omega_I)$). If ω_I is light-like, notice that $\omega_I \in \operatorname{Cone}\Delta_I$, i.e. ω_I can be written as a linear combination of Δ_I with coefficients of the same sign (possibly 0). For any $s \notin I$, we have $\mathcal{B}(\omega_I,\alpha_s) > 0$ and $\mathcal{B}(\alpha_t,\alpha_s) \leqslant 0$ for any $t \in I$, so ω_I must be a negative combination of Δ_I. We then conclude that $\mathcal{B}(\omega_I,\omega_J) \leqslant 0$ since $\mathcal{B}(\alpha_t,\omega_J)$ are all $\geqslant 0$. □

As a consequence, the Tits cone of a level-1 root system equals the set of non-space-like vectors; see also [13, Proposition 9.4].

The following result generalizes [31, Proposition 1.6].

Proposition 3.5. *If (Δ,Φ) is of level 2, then it is Lorentzian, and vectors in Δ^* are pairwise disjoint. A vector $\omega_I \in \Delta^*$ is space-like if and only if the 1-facial root system (Δ_I,Φ_I) is of level 1, in which case we have $\mathcal{B}(\omega_I,\omega_I) \leqslant 1$.*

Proof. Let (Δ,Φ) be a canonical root system of level 2 in $(V,\mathcal{B})$ with associated Coxeter system (W,S). If V is of dimension 3, it is immediate that (Δ,Φ) is Lorentzian. So we assume that the dimension of V is at least 4.

We argue as in the proof of Proposition 3.4. If the representation space V is not a Lorentz space, it contains a pair of orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathcal{B}(\mathbf{u},\mathbf{u}) < 0$ and $\mathcal{B}(\mathbf{v},\mathbf{v}) = 0$. Then, for any linear combination $\mathbf{x} = \lambda\mathbf{u} + \mu\mathbf{v}$, we have $\mathcal{B}(\mathbf{x},\mathbf{x}) < 0$ as long as $\lambda \neq 0$. The subspace $\operatorname{Span}(\mathbf{u},\mathbf{v})$ intersect the hyperplanes $\operatorname{Span}(\Delta_I)$ at a ray $\mathbb{R}_+\mathbf{x}_I$ for every 1-facial subset $I \subset S$. Since the 1-facial root subsystems are all of level $\leqslant 1$, we see from the proof of Proposition 3.4 that, whenever $\operatorname{Span}(\Delta_I)$ contains a time-like ray $\mathbb{R}_+\mathbf{x}_I$, it must be in the interior of the facet $\operatorname{Cone}(\Delta_I)$. In the projective picture (modeled on $H^1_{\mathbf{t}}$), $\operatorname{Span}\{\mathbf{u},\mathbf{v}\}$ appears as a line L intersecting every facet of $\widehat{\mathcal{C}}$ such that every intersection point are in the interior of a facet, with at most one exception (namely $\widehat{\mathbf{v}}$) on the boundary of a facet. But L intersects the boundary of $\widehat{\mathcal{C}}$ only at two points. Now that one is in the interior of a facet, the other must be the intersection of all other facets, hence a vertex. The only possibility is that $\widehat{\mathcal{C}}$ is a pyramid, in which case L pass through the apex and an interior point of the base facet. But the apex is a projective simple root $\widehat{\alpha}$, and $\mathcal{B}(\alpha, \alpha) > 0$, so $\widehat{\alpha} \notin L$. This contradiction proves that V is a Lorentz space.

Let $\mathbf{x}$ be a vector with $\mathcal{B}(\mathbf{x}, \mathbf{x}) \leqslant 0$ and assume $h(\mathbf{x}) \geqslant 0$. Contrary to the case of level 1, it is possible that $\widehat{\mathbf{x}} \notin \widehat{\mathcal{C}}$. In this case, we can again write $\mathbf{x} = \mathbf{x}_+ - \mathbf{x}_-$ where $\mathbf{x}_+ \in \operatorname{Cone}(\Delta_I)$ and $\mathbf{x}_- \in \operatorname{Cone}(\Delta_J)$ and I, J are two disjoint facial subset of S. And again, since $\mathcal{B}(\mathbf{x}_+, \mathbf{x}_-) < 0$, we have either $\mathcal{B}(\mathbf{x}_+, \mathbf{x}_+)$ or $\mathcal{B}(\mathbf{x}_-, \mathbf{x}_-) < 0$. Since (Δ, Φ) is of level 2, the 1-facial root subsystems are all of level $\leqslant 1$, so we must have either $\mathbf{x}_+$ or $\mathbf{x}_-$ in the interior of a facet of $\mathcal{C}$ by Proposition 3.4. In the projective picture, it means that for *any* line through $\widehat{\mathbf{x}}$ that intersects two disjoint faces of $\widehat{\mathcal{C}}$, one intersection point must be in the interior of a facet. We then conclude from Lemma 3.3 that $\mathcal{B}(\mathbf{x}, \omega)$ are non zero, and have the same sign for all but one $\omega \in \Delta^*$. On the other hand, if $\widehat{\mathbf{x}} \in \widehat{\mathcal{C}}$, only when $\mathcal{B}(\mathbf{x}, \mathbf{x}) = 0$ is it possible that $\widehat{\mathbf{x}}$ lies on a codimension-2 face of $\widehat{\mathcal{C}}$.

Consequently, the intersection $\operatorname{Span}(\Delta_I) \cap \operatorname{Span}(\Delta_J)$ is not time-like for any two 1-facial subsets $I \neq J \subset S$. The orthogonal companion of the intersection is the subspace $\operatorname{Span}\{\omega_I, \omega_J\}$, which is not space-like. For proving the disjointness, one still needs to prove that $\mathcal{B}(\omega_I, \omega_J) \leqslant 0$.

Assume that ω_I is not space-like. A similar argument as in the proof of Proposition 3.4 shows that $\mathcal{B}(\omega_I, \omega_J) \leqslant 0$ for all $J \neq I$ with at most one exception. Let $K \neq I$ be this exception. Pick a generator $s \in I \setminus K$, we can write $\omega_I = \lambda\alpha_s - \omega'_I$, where ω'_I is a linear combination of Δ_K with coefficients of same sign, which is also the sign of λ. We have $\mathcal{B}(\alpha_s, \omega_I) = 0$ by definition, but this is not the case since $\mathcal{B}(\alpha_s, \alpha_t) \leqslant 0$ for $t \in K$ while $\mathcal{B}(\alpha_s, \alpha_s) = 1$. Therefore, the exception K does not exist.

If ω_I is space-like, then the subspace $\operatorname{Span}(\Delta_I)$ is time-like, so $(\operatorname{Span}(\Delta_I), \mathcal{B})$ is a (non-degenerate) Lorentz space. This proves that (Δ_I, Φ_I) is of level 1. Then, for a simple root $\alpha \notin \Delta_I$, let

$$\alpha' = \alpha - \frac{\mathcal{B}(\alpha, \omega_I)}{\mathcal{B}(\omega_I, \omega_I)}\omega_I$$

be the projection of α on $\operatorname{Span}(\Delta_I)$. Since $\mathcal{B}(\alpha', \beta) \leqslant 0$ for all $\beta \in \Delta_I$, $\widehat{\alpha}'$ is in the Coxeter polytope $\widehat{\mathcal{C}}^*_I$ for (Δ_I, Φ_I). Since (Δ_I, Φ_I) is of level 1, α' is not space-like by Proposition 3.4, i.e.

$$\mathcal{B}(\alpha', \alpha') = \mathcal{B}(\alpha, \alpha) - \frac{\mathcal{B}(\alpha, \omega_I)^2}{\mathcal{B}(\omega_I, \omega_I)} = 1 - \frac{\mathcal{B}(\alpha, \omega_I)^2}{\mathcal{B}(\omega_I, \omega_I)} \leqslant 0.$$

Since this is true for all $\alpha \notin \Delta_I$, it proves that

$$\mathcal{B}(\omega_I, \omega_I) \leqslant \min_{\alpha \notin \Delta_I} \mathcal{B}(\alpha, \omega_I)^2 = 1. \tag{1}$$

Let J be a 1-facial subset such that $\alpha \in \Delta_J$, and

$$\omega'_J = \omega_J - \frac{\mathcal{B}(\omega_J, \omega_I)}{\mathcal{B}(\omega_I, \omega_I)}\omega_I$$

be the projection of ω_J on $\mathrm{Span}(\Delta_I)$. Then, since $\alpha' \in \widehat{\mathcal{C}}^*$,

$$\mathcal{B}(\alpha', \omega'_J) = \mathcal{B}(\alpha, \omega_J) - \frac{\mathcal{B}(\alpha, \omega_I)\mathcal{B}(\omega_J, \omega_I)}{\mathcal{B}(\omega_I, \omega_I)} = -\frac{\mathcal{B}(\alpha, \omega_I)\mathcal{B}(\omega_J, \omega_I)}{\mathcal{B}(\omega_I, \omega_I)} = \mathcal{B}(\alpha', \omega_J) \geqslant 0,$$

which proves that $\mathcal{B}(\omega_I, \omega_J) \leqslant 0$. Since J can be chosen as any 1-facial subset $J \neq I \subset S$, this finish the proof of disjointness. $\square$

The proposition has an interesting consequence.

Corollary 3.6. *Let Δ be a canonical root basis of level 2 in $(V, \mathcal{B})$, then the set*

$$\Delta \cup \{-\omega/\sqrt{\mathcal{B}(\omega, \omega)} \mid \omega \in \Delta^*, \mathcal{B}(\omega, \omega) > 0\}$$

is a canonical root basis of level 1.

The following proposition generalizes [31, Corollary 1.8]. It was proved for free root basis in, for instance, [5][Ch. V, § 4.4] and [1, Lemma 2.58]. An extension for canonical root basis can be found in [24, Theorem 1.2.2(b)], who refers to [44] for proof. Note that $\mathcal{C}^*$ is closed in the present paper, so the inequalities are not strict.

Proposition 3.7. *For $\mathbf{x} \in \mathcal{C}^*$, $w \in W$ and $\alpha_s \in \Delta$, $\mathcal{B}(w(\mathbf{x}), \alpha_s) \geqslant 0$ if $\ell(sw) > \ell(w)$ and $\mathcal{B}(w(\mathbf{x}), \alpha_s) \leqslant 0$ if $\ell(sw) < \ell(w)$.*

We now prove the generalization of [31, Theorem 1.9].

Theorem 3.8. *The followings are equivalent:*

(a) (Δ, Φ) is of level 1 or 2;

(b) (Δ, Φ) is Lorentzian and any two vectors in Ω are disjoint.

Maxwell's proof applies with only slight modification. Yet we present here a complete proof.

Proof. (a)$\Rightarrow$(b): It is already proven in Proposition 3.4 and 3.5 that (Δ, Φ) is Lorentzian. So we only need to prove the disjointness.

We first prove that, for any $\omega_I, \omega_J \in \Delta^*$ and $w \in W$ such that $\omega_I \neq w(\omega_J)$,

$$\mathcal{B}(\omega_I, w(\omega_J)) \leqslant 0. \tag{2}$$

The proof is by induction on the length of $w \in W$. The case of $w = e$ is already known. One may assume that $\ell(tw) > \ell(w)$ for all $t \in I$, otherwise one may replace w by tw in (2). So $w = sw'$ for some $s \notin I$ and $\ell(w) > \ell(w')$. We then have

$$\mathcal{B}(\omega_I, w(\omega_J)) = \mathcal{B}(s(\omega_I), w'(\omega_J)) = \mathcal{B}(\omega_I, w'(\omega_J)) - 2\mathcal{B}(\alpha_s, \omega_I)\mathcal{B}(\alpha_s, w'(\omega_J)).$$

If $\omega_I \neq w'(\omega_J)$, (2) is proved since $\mathcal{B}(\omega_I, w'(\omega_J)) \leqslant 0$ by inductive hypothesis, $\mathcal{B}(\alpha_s, \omega_I) > 0$ by definition, and $\mathcal{B}(\alpha_s, w'(\omega_J)) \geqslant 0$ by Proposition 3.7. Otherwise, if $\omega_I = w'(\omega_J)$, we have

$$\mathcal{B}(\omega_I, w(\omega_J)) = \mathcal{B}(\omega_I, \omega_I) - 2\mathcal{B}(\alpha_s, \omega_I)^2 \leqslant -1 < 0 \tag{3}$$

by (1) and definition of ω_I (recall that $s \notin I$).

It remains to prove that $\mathcal{B}$ is not positive definite on the subspace $\operatorname{Span}(\omega_I, w(\omega_J))$. Assume the opposite. Then ω_I is space-like hence (Δ_I, Φ_I) is of level 1. Let $\mathbf{v}$ be the projection of $w(\omega_J)$ on $\omega_I^\perp = V_I$. The subspace $\mathbf{v}^\perp$ in V_I is the time-like intersection $\omega_I^\perp \cap w(\omega_J)^\perp$ in V, so $\mathbf{v}$ must be space-like. On the other hand, for all $t \in I$, $\mathcal{B}(\alpha_t, w(\omega_J)) \geqslant 0$ because $\ell(tw) \geqslant \ell(w)$. We then conclude that $\mathbf{v}$ is in the Coxeter polytope of (Δ_I, Φ_I), so $\mathbf{v}$ must be time-like. This contradiction finishes the proof of disjointness.

(b)⇒(a): Since $\mathcal{B}$ is not positive definite on $\operatorname{Span}\{\omega_I, \omega_J\}$ for any $\omega_I \neq \omega_J \in \Delta^*$, the orthogonal companion of these subspaces, which contains the codimension-2 faces of $\mathcal{C}^*$, are not time-like. So $\mathcal{B}$ is positive semidefinite on all codimension-2 faces, which proves that (Δ, Φ) is of level 1 or 2. □

3.2 Infinite ball packings

For a *space-like* vector $\mathbf{x}$ in a Lorentz space $(V, \mathcal{B})$, the *normalized vector* $\overline{\mathbf{x}}$ of $\mathbf{x}$ is given by

$$\overline{\mathbf{x}} = \mathbf{x}/\sqrt{\mathcal{B}(\mathbf{x}, \mathbf{x})}.$$

It lies on the one-sheet hyperboloid $\mathcal{H} = \{\mathbf{x} \in V \mid \mathcal{B}(\mathbf{x}, \mathbf{x}) = 1\}$. Note that $\widehat{\mathbf{x}} = \widehat{-\mathbf{x}}$ is the same point in $\mathbb{P}V$, but $\overline{\mathbf{x}}$ and $\overline{-\mathbf{x}}$ are two different vectors in opposite directions in V. One verifies that two space-like vectors $\mathbf{x}, \mathbf{y}$ are disjoint if and only if $\mathcal{B}(\overline{\mathbf{x}}, \overline{\mathbf{y}}) \leqslant -1$.

A correspondence between space-like directions in $(d+2)$-dimensional Lorentz space $(V, \mathcal{B})$ and d-dimensional balls is introduced in [31, §2], see also [17, § 1.1] and [8, § 2.2]. Fix a time-like direction of past $\mathbf{t}$ so that the projective light cone $\widehat{Q}$ appears as a closed sphere on $H^1_{\mathbf{t}}$. Then in the affine picture, given a space-like vector $\mathbf{x} \in V$, the intersection of $\widehat{Q}$ with the half-space $\widehat{H}^-_{\mathbf{x}} = \{\mathbf{x}' \in H^1_{\mathbf{t}} \mid \mathcal{B}(\mathbf{x}, \mathbf{x}') \leqslant 0\}$ is a closed ball (spherical cap) on $\widehat{Q}$. We denote this ball by $\mathsf{Ball}(\mathbf{x})$; see Figure 3.

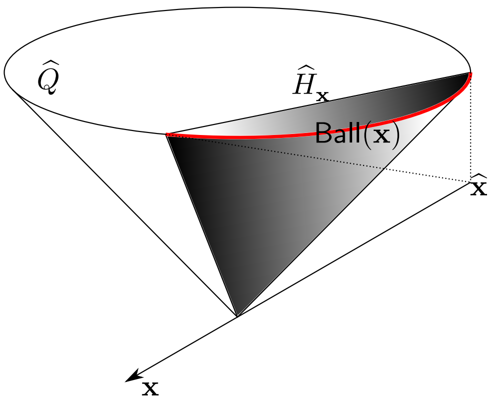


Figure 3:

After a stereographic projection, $\mathsf{Ball}(\mathbf{x})$ becomes an d-dimensional ball in Euclidean space. Here, we also regard closed half-spaces as closed balls of curvature 0, and complement of open balls as closed balls of negative curvature. For two past-directed space-like vectors $\mathbf{x}$ and $\mathbf{y}$, we have

- $\mathsf{Ball}(\mathbf{x})$ and $\mathsf{Ball}(\mathbf{y})$ are disjoint if $\mathcal{B}(\overline{\mathbf{x}}, \overline{\mathbf{y}}) < -1$;
- $\mathsf{Ball}(\mathbf{x})$ is tangent to $\mathsf{Ball}(\mathbf{y})$ if $\mathcal{B}(\overline{\mathbf{x}}, \overline{\mathbf{y}}) = -1$;
- $\mathsf{Ball}(\mathbf{x})$ and $\mathsf{Ball}(\mathbf{y})$ *overlap* (i.e. their interiors intersect) if $\mathcal{B}(\overline{\mathbf{x}}, \overline{\mathbf{y}}) > -1$;
- One of $\mathsf{Ball}(\mathbf{x})$ and $\mathsf{Ball}(\mathbf{y})$ is contained in the other if $\mathcal{B}(\overline{\mathbf{x}}, \overline{\mathbf{y}}) \geqslant 1$.

A *ball packing* is a collection of balls with disjoint interiors. It is then clear that a ball packing correspond to a set of space-like vectors $X \in V$, with at most one future-directed vector, such that any two vectors are disjoint. Conversely, Maxwell proved that every such set of space-like vectors correspond to a ball packing [31, Proposition 3.1]. So Theorem 1.1, which we restate below, follows directly from Theorem 3.8.

Theorem 3.9. *Let Ω_r be the set of space-like vectors in Ω, then $\{\mathsf{Ball}(\omega) \mid \omega \in \Omega_r\}$ is a ball packing if and only if the associated Lorentzian root system is of level* 2.

Remark. It is not obvious that the statement is equivalent to Theorem 1.1. A vector $\omega \in \Omega_r$ is in the form of $w(\omega_I)$ for some $w \in W$ and $\omega_I \in \Delta^*$. The stabilizer of ω is then the parabolic facial subgroup in the form of wW_Iw^{-1}. Since ω_I is space-like, W_I is of level 1, and its Coxeter graph must be connected. Facial subgroups with no finite irreducible components are said to be *special* in [13]; see also [27]. Hence by [13, §10.3], vectors in Ω_r span the space-like extreme rays of the Tits cone $\mathcal{T}$, which proves the equivalence. As we will see later, this does not hold for root systems of level $\geqslant 3$.

In the following, the ball packing in the theorem is referred to as the *Boyd–Maxwell packing* associated to the Lorentzian root system. A ball packing is maximal if one can not add any additional ball into the packing without overlapping other balls.

Theorem 3.10. *The Boyd–Maxwell packing associated to a level-*2 *Lorentzian root system is maximal.*

In the case of free root basis, Theorem 3.10 was proved in [31, Theorem 3.3] under a hypothesis asserting that $\overline{\operatorname{Cone}\Omega_r} = \overline{\operatorname{Cone}\Omega}$ when the root system is of level 2. The proof of this part applies word for word to canonical root basis, so we will not repeat it here. To complete the proof of Theorem 3.10, it suffices to verify this hypothesis. For free root basis of level $\geqslant 2$, the hypothesis was confirmed in [32, Theorem 6.1]. We now prove the hypothesis for canonical root basis.

Proposition 3.11. *For Lorentzian root systems of level $\geqslant 2$, we have $\overline{\operatorname{Cone}\Omega_r} = \overline{\operatorname{Cone}\Omega}$.*

Maxwell's proof applies here with slight modification, so we only give a sketch.

Sketch of proof. What we need to prove is that if $\omega_I \in \Omega$ is not space-like, then $\omega_I \in \overline{\operatorname{Cone}\Omega_r}$.

If $\mathcal{B}(\omega_I, \omega_I) < 0$, W_I is finite. Choose any space-like $\omega \in \Delta^*$. Then $\mathbf{v} = \sum_{w \in W_I} w(\omega)$ is fixed by W_I, hence $\mathbf{v}$ is a scalar multiple of ω_I.

If $\mathcal{B}(\omega_I, \omega_I) = 0$, then (Δ_I, Φ_I) is affine and ω_I spans the radical of $(V_I, \mathcal{B})$. Again, choose any space-like $\omega \in \Delta^*$. Then, by a similar argument as in the proof of [32, Proposition 5.15], we have $\omega_I \in \overline{\operatorname{Cone}(W_I\omega)}$. $\square$

We have extended most of the major results of [31]. We now continue to extend results from [11].

The *residual set* of a collection of balls is the complement of the interiors of the balls. With the modified height function $h(\mathbf{x})$ in Section 2.6, the proofs in [11] applies directly. We then obtain Theorem 1.2 for level 2.

The projective polytopes in the orbit $W \cdot \widehat{\mathcal{C}}^*$ are called *chambers*. Analogous to the situation of free root basis ($\widehat{\mathcal{C}}^*$ is a simplex) [1], the chambers form a cell decomposition $\mathscr{C}$ of the projective Tits cone $\widehat{\mathcal{T}}$ called *Coxeter complex*. It is a pure polyhedral cell complex of dimension $d-1$ (dimension of $\mathbb{P}V$). The set of vertices of $\mathscr{C}$ is $\widehat{\Omega}$. The 1-cells of $\mathscr{C}$ are called *edges*, and $(d-2)$-cells are called *panels*.

Two chambers are *adjacent* if they share a panel. A *gallery* is a sequence of chambers $(\widehat{C}_0, \dots, \widehat{C}_k)$ such that consecutive chambers are adjacent, and k is the *length* of the gallery. We say that a gallery $(\widehat{C}_0, \dots, \widehat{C}_k)$ *connects* two simplices $\widehat{A}$ and $\widehat{A}'$ of $\mathscr{C}$ if $\widehat{A} \subseteq \widehat{C}_0$ and $\widehat{A}' \subseteq \widehat{C}_k$. The *gallery distance* $d(\widehat{A}, \widehat{A}')$ between two simplices $\widehat{A}$ and $\widehat{A}'$ is the minimum length of a gallery connecting $\widehat{A}$ and $\widehat{A}'$. A gallery connecting $\widehat{A}$ and $\widehat{A}'$ with length $d(\widehat{A}, \widehat{A}')$ is called a *minimal gallery*. For an element $w \in W$, its length $\ell(w) = d(\widehat{C}, w \cdot \widehat{C})$. We refer the readers to [1, Section 1.4.9] for more details.

Since $\widehat{\mathcal{C}}^*$ is the fundamental domain for the action of W on the projective Tits cone $\widehat{T}$, the orbit of two different fundamental weights are disjoint. So the vertices of the Coxeter complex admits a coloring by Δ^*, i.e. a vertex u is colored by $\omega \in \Delta^*$ if $u \in W \cdot \widehat{\omega}$. Panels are orbits of the facets of Δ^*, therefore they can be colored by the simple roots, i.e. a panel is colored by $\alpha \in \Delta$ if it is the orbit of the facet of Δ^* corresponding to α.

The *tangency graph* of a ball packing takes the balls as vertices and the tangent pairs as edges. We now try to describe the tangency graph in term of the Coxeter complex. Vertices with time- or light-like colors are called *imaginary vertices*; vertices with space-like colors are called *real vertices* because they correspond to balls in the packing, and are therefore vertices in the tangency graph. An edge of the Coxeter complex connecting two real vertices of color ω and ω' is said to be *real* if $\mathcal{B}(\overline{\omega}, \overline{\omega}') = -1$. Real edges correspond to tangent pairs in the packing, and are therefore edges in the tangency graph. For a Lorentzian root system of level 2, real vertices colored by $\omega \in \Delta^*$ such that $\mathcal{B}(\omega, \omega) = 1$ are said to be *surreal*. Two distinct surreal vertices of the same color ω are said to be *adjacent* if they are vertices of two chambers sharing a panel of color α such that $\mathcal{B}(\omega, \alpha) = 1$. One verifies that pairs of adjacent surreal vertices are also edges in the tangency graph.

The following theorem was proved in [11, Theorem 3.5] for free root basis.

Theorem 3.12. *The tangency graph of the ball packing associated to a Lorentzian root system of level 2 takes the real vertices of the Coxeter complex as vertices; two vertices u and v are connected in the tangency graph if and only if either uv is a real edge of the Coxeter complex, in which case u and v are of different colors; or u and v are adjacent surreal vertices, in which case u and v are of the same color.*

Proof. We have seen that real vertices represent balls in the Boyd–Maxwell packing. Recall from Theorem 3.8 that

$$\mathcal{B}(\omega_I, w(\omega_J)) \leqslant -\sqrt{\mathcal{B}(\omega_I,\omega_I)\mathcal{B}(\omega_J,\omega_J)} \tag{4}$$

whenever $\omega_I \neq w(\omega_J)$. We need to prove that the equality only holds if the two vertices $u = \widehat{\omega}_I$ and $v = w \cdot \widehat{\omega}_J$ are either vertices of a real edge, or a pair of adjacent surreal vertices. In any other case, the inequality is strict, meaning that the balls are not tangent.

If u and v are in the same chamber ($d(u,v) = 0$), they are necessarily of different colors, i.e. $I \neq J$. If they correspond to a pair of tangent balls, they must form an edge of the chamber [10, Theorem 5.2] and this edge must be real.

If u and v are in adjacent chambers, we may assume that $w = s_\alpha \notin I \cup J$. If $I \neq J$, and the strict inequality follows from

$$\mathcal{B}(\omega_I, s_\alpha\omega_J) = \mathcal{B}(\omega_I,\omega_J) - 2\mathcal{B}(\omega_I,\alpha)\mathcal{B}(\omega_J,\alpha) < \mathcal{B}(\omega_I,\omega_J) \leqslant -\sqrt{\mathcal{B}(\omega_I,\omega_I)\mathcal{B}(\omega_J,\omega_J)}.$$

Otherwise if $I = J$, the calculation was done in Eq. (3), where one sees immediately that $\mathcal{B}(\omega_I, s_\alpha\omega_I) = -\mathcal{B}(\omega_I,\omega_I)$ if and only if $\mathcal{B}(\omega_I,\omega_I) = 1$ and $\mathcal{B}(\omega_I,\alpha_s) = 1$. Hence two vertices in adjacent chambers correspond to tangent balls if and only if they are a pair of adjacent surreal vertices.

It remains to prove that the inequality in Eq. (4) is strict if $d(u,v) \geqslant 2$. Let $(C_0, \dots, C_{d(u,v)})$ be a minimal gallery connecting u and v, and α_i be the color of the panel shared by C_{i-1} and C_i. We may take $w = s_1 \dots s_{d(u,v)}$ where s_i is the reflection in α_i. Note that $s_1 \notin I$, $s_{d(u,v)} \notin J$, and $\ell(w) = d(u,v)$. Write $w = s_1 s_2 w'$ with $w' = s_3 \dots s_{d(u,v)}$. Then

$$\begin{aligned}\mathcal{B}(\omega_I, w\omega_J) = \mathcal{B}(s_1\omega_I, s_2 w'\omega_J) = \mathcal{B}(s_1\omega_I, w'\omega_J)&\\ - 2\mathcal{B}(w'\omega_J,\alpha_2)[\mathcal{B}(\omega_I,\alpha_2) - 2\mathcal{B}(\omega_I,\alpha_1)\mathcal{B}(\alpha_1,\alpha_2)]&\end{aligned}$$

If $I \neq J$ or $w' \neq s_1$, $\mathcal{B}(w'\omega_J,\alpha_2) \geqslant 0$ by Proposition 3.7, but the equality is ruled out by the minimality of the gallery. Moreover, either $\mathcal{B}(\omega_I,\alpha_2) > 0$ or $\mathcal{B}(\alpha_1,\alpha_2) < 0$ (s_1 and s_2 do not commute); if both are 0, one may find a smaller gallery, contradicting the minimality. The strict inequality in Eq. (4) is then proved since the second line is strictly negative.

If $I = J$ and $w' = s_1$, we continue the calculation and obtain

$$\begin{aligned}\mathcal{B}(\omega_I, w\omega_I) = \mathcal{B}(\omega_I, s_1 s_2 s_1\omega_I) = \mathcal{B}(\omega_I,\omega_I)&\\ - 2\mathcal{B}(\omega_I,\alpha_2)^2 + 8\mathcal{B}(\omega_I,\alpha_1)\mathcal{B}(\omega_I,\alpha_2)\mathcal{B}(\alpha_1,\alpha_2) - 8\mathcal{B}(\omega_I,\alpha_1)^2\mathcal{B}(\alpha_1,\alpha_2)^2&\end{aligned}$$

Note that either $\mathcal{B}(\omega_I, \alpha_2) > 0$ or $\mathcal{B}(\alpha_1, \alpha_2) < -1/2$ (order of $s_1 s_2$ is bigger than 3); otherwise there is a smaller gallery, contradicting the minimality. Hence $\mathcal{B}(\omega_I, w\omega_I) < -1 \leqslant -\mathcal{B}(\omega_I, \omega_I)$, which proves the strict inequality in Eq. (4). □

Finally, the following generalizes [11, Corollary 3.1], whose proof applies here word for word.

Corollary 3.13. *The projective Tits cone $\widehat{\mathcal{T}}$ of a Lorentzian root system of level 2 is an edge-tangent infinite polytope, i.e. its edges are all tangent to the projective light cone. Furthermore, the 1-skeleton of $\widehat{\mathcal{T}}$ is the tangency graph of the ball packing associated to the root system.*

3.3 Lorentzian root systems of higher levels

In this part we investigate Lorentzian Coxeter systems of level $\geqslant 3$. The space-like vectors in Ω still correspond to balls, and we call $\{\mathsf{Ball}(\omega) \mid \omega \in \Omega_r\}$ a *Boyd–Maxwell ball cluster*. It turns out that, in a Boyd–Maxwell cluster, either the balls may overlap hence do not form a packing, or one may extract a subset that form a packing but lose the maximality.

Remark. If the level $\geqslant 3$, the balls do not correspond to the extreme rays as in Theorem 1.1. It is possible that the stabilizer of some space-like $\omega_I \in \Delta^*$ has a finite component, hence not a special subgroup. In this case, the rays spanned by $W(\omega_I)$ might not be extreme for $\mathcal{T}$.

In [11, §3.4], it was falsely claimed for level $\geqslant 3$ that the balls in $\{\mathsf{Ball}(\omega) \mid \omega \in \Omega_r\}$ can only intersect at acute angles. The correct statement is the following, which is also valid for canonical root basis:

Lemma 3.14. *In the Boyd–Maxwell cluster associated to a level $\geqslant 3$ Lorentzian root system, no ball contains another.*

Though the mistake plays a minor role in [11], we include a short proof here for completeness.

Proof. If a ball is contained in another, the corresponding space-like weight ω would be in the interior of the Tits cone $\overline{\operatorname{Cone}\Omega_r} = \overline{\operatorname{Cone}\Omega}$. Then the stabilizer of ω in W must be finite, so ω can not be space-like. This contradiction proves the lemma. □

A set of overlapping balls is *maximal* if any extra ball, once introduced into the packing, would contain, be contained in, or overlap at an obtuse angle with an existing ball. The following results from [11], together with their proofs, extend word for word to canonical root systems.

Theorem 3.15. *The Boyd–Maxwell cluster associated to a Lorentzian root system of level $\geqslant 2$ is maximal. And the set of limit roots is equal to the residual set of the cluster.*

We have seen that the balls corresponding to Ω_r do not form a packing. However, one could extract a packing from the Boyd–Maxwell cluster, in the price of losing maximality. Let $\Delta_1^* \subseteq \Delta^*$ be the set of space-like vectors ω_I in Δ^* such that the 1-facial root systems (Δ_I, Φ_I) are of level 1.

Theorem 3.16. *For a Lorentzian root system of level* $\geqslant 3$, $\mathcal{B}(\omega_I, \omega_I) \leqslant 1$ *for all* $\omega_I \in \Delta_1^*$, *and the balls* $\{\mathsf{Ball}(\omega) \mid \omega \in W(\Delta_1^*) \subset \Omega_r\}$ *form a packing.*

The proofs for Eq. (1) and Theorem 3.8 applies here word for word, so we will not repeat it. Examples of such packings appear in a study of Bianchi groups by Stange [38], to whom the author is grateful for helpful discussions.

4 Partial classification of level-2 Coxeter polytopes

To provide examples of new infinite ball packings, we devote the last section to a partial enumeration of based root systems of level 2. More precisely, we will enumerate the Coxeter d-polytopes with $d+2$ facets.

The enumeration is implemented in the computer algebra system Sage [39]. Commented source code is made public at http://github.com/Dr-How/L2Graph. Our algorithm is an enhanced combination of the techniques used in previous enumerations such as [23, 15, 40]. Since a thorough description of the algorithm would cost too much space yet provide little mathematical insight, we decide to present only main ideas of the procedure. Interested readers are encouraged to consult the cited literatures and the source code for mathematical and technical details.

For convenience, we will talk about the level for Coxeter polytopes and, by abuse of language, for Coxeter graphs, but we actually mean the level of the associated root system. For Coxeter graphs, this leads to a confusion with Maxwell's definition, but should not cause any problem within this section.

4.1 Background

Recall that a simple system Δ in $(V, \mathcal{B})$ can be represented by the Coxeter graph G with Vinberg's convention. Simple roots correspond to vertices of G. If two simple roots $\alpha, \beta \in \Delta$ are not orthogonal, they are connected by an edge, which is solid with label $3 \leqslant m < \infty$, if $\mathcal{B}(\alpha, \beta) = -\cos(\pi/m)$; with label ∞ if $\mathcal{B}(\alpha, \beta) = -1$; or dashed with label $-c$ if $\mathcal{B}(\alpha, \beta) = -c < -1$. The label 3 on solid edges are often omitted.

If we consider the Coxeter polytope $\mathcal{P}$, then vertices of the Coxeter graph G correspond to facets of $\mathcal{P}$. A solid edge of G with integer label means that the intersection of two facets is time-like; a solid edge with label ∞ means that the intersection is light-like; and a dashed edge means that the intersection is space-like.

For a based root system (Δ, Φ), the *corank* of it's Coxeter polytope $\mathcal{P}$ is defined as the nullity of the Gram matrix B of Δ. A Coxeter polytope of dimension d and corank k has $d+k+1$ facets. In particular, a Coxeter polytope of corank 0 is a simplex. In this case, the level of the Coxeter graph coincide with Maxwell's definition. For convenience, a Coxeter polytope of level l and corank k is abbreviated as (l,k)-polytope. Such abbreviation is also used for Coxeter graphs.

In the affine picture of the projective space $\mathbb{P}V$, the projective light cone $\widehat{Q}$ appears as a closed surface that is projectively equivalent to a sphere. We can consider the interior of the sphere (time-like part) as the Klein model of the hyperbolic space. With this point of view, a Coxeter polytope $\mathcal{P} \in \mathbb{P}V$ is a hyperbolic polytope, and is the fundamental domain of the hyperbolic reflection group generated by the reflections in its facets [47]. By Proposition 3.4, Coxeter polytopes of level 1 correspond to finite-volume hyperbolic polytopes, or even compact if the level is strict; Coxeter polytopes of level $\geqslant 2$, on the other hand, are hyperbolic polytopes of infinite volume.

Vinberg [46] proved that there is no strict level-1 Coxeter polytopes of dimension 30 or higher, and Prokhorov [36] proved that there is no level-1 Coxeter polytopes in hyperbolic spaces of dimension 996 or higher. On the other hand, Allcock [2] proved that there are infinitely many level-1 (resp. strictly level-1) Coxeter polytopes in every hyperbolic space of dimension 19 (resp. 6) or lower, which suggests that a complete enumeration of level-1 Coxeter polytopes is hopeless.

Nevertheless, there are many interesting partial enumerations. The $(1,0)$-polytopes have been completely enumerated by Chein [9]. They are hyperbolic simplices of finite volume. The list of Chein also comprises strict $(1,0)$-polytopes, which was first enumerated by Lannér [25]. The $(1,1)$-polytopes have been enumerated by Kaplinskaja [23] for simplicial prisms, Esselmann [15] for compact polytopes and Tumarkin [40] for finite-volume polytopes. Tumarkin also studied strict $(1,2)$- and $(1,3)$-polytopes [41, 42]. Mcleod [34] finished the classification of all pyramids of level 1. Our algorithm of enumeration is based on the techniques and results in these works.

Recall that a Coxeter polytope is of level 2 if all its edges are time-like or light-like, but some vertices are space-like. In view of Corollary 3.6, we deduce immediately from the result of [36] that there is no level-2 Coxeter polytopes in hyperbolic spaces of dimension 996 or higher. However, in the shadow of [2], there might be infinitely many level-2 Coxeter polytopes in lower dimensions, so a complete classification may be hopeless. A $(2,0)$-graph is either a connected graph, or a disjoint union of an isolated vertex and a $(1,0)$-graph. The enumeration of connected $(2,0)$-graphs was initiated in [31] and completed in [11]. In this section, we would like to enumerate $(2,1)$-graphs.

Remark. We notice that the "doubling trick" used in Allcock's construction produces Coxeter subgroups of finite index, so the infinitely many hyperbolic Coxeter groups constructed in [2] are all commensurable. It has been noticed in [31, § 4] that commensurable Coxeter groups of level-2 correspond to the same ball packing. Indeed, if two Coxeter groups are commensurable, their Coxeter complex is the subdivision of the same coarser Coxeter complex. Therefore, it makes more sense to investigate commensurable classes of Coxeter groups, as Maxwell did in [31, Table II].

For root systems of corank 0, the commensurable classes and subgroup relations have been studied for level 1 and 2 in [33], and are completely determined for level 1 by Johnson et al. [22]. Despite of Allcock's result, we may still ask: Are there infinitely many commensurable classes for level-l Coxeter groups acting on lower dimensional hyperbolic spaces? For level 1 Coxeter groups, the answer is “yes” in dimension 2 (triangle groups), 3 [28, § 4.7.3], 4 and 5 [29] [47, § 5.4]. The constructions in dimension 3–5 made use of level-1 polytopes of low corank.

4.2 Strategy of enumeration

Let F be a k-face of a d-polytope P. The *face figure* of F, denoted by P/F, is the projection of P onto the quotient space $\operatorname{Span}(P)/\operatorname{Span}(F)$. F is said to be *simple* if P/F is a simplex, or *almost simple* if P/F is the direct product of simplices. A polytope is said to be k-simple (resp. almost k-simple) if all its k-faces are simple (resp. almost simple). For a hyperbolic Coxeter polytope $\mathcal{P}$, time-like faces are simple, and light-like faces are almost simple; this is an easy generalization of [43, Lemma 3]. We then conclude the following proposition from the definition of level.

Proposition 4.1. *A Coxeter polytope of level l is almost l-simple, or l-simple if the level is strict.*

By the Gale diagram analysis [16, §6.3] [40, §2] and Proposition 4.1, we conclude that

Proposition 4.2. *The $(2,1)$-polytope falls into one of the four cases:*

- *the direct product of two simplices.*
- *the pyramid over the direct product of two simplices, with light-like apex.*
- *the pyramid over the direct product of two simplices, with space-like apex.*
- *the two-fold pyramid over the direct product of two simplices.*

Polytopes in each case will be enumerated separately. Our enumeration, like previous enumerations, follows a “nomination–recognition” strategy: We first generate a reasonably short list of candidates, then pass the list to a recognition program to screen out the $(2,1)$-graphs. Hence for each case, we will come up with a lemma characterizing the Coxeter graphs. This lemma serves as the guide for nominating candidates. We will present the lemmata without proof. Interested readers are referred to Lemmata 2 and 4 of [40] for the idea of proofs. Basically, the lemmata follow from the definition of level, and make use of the following theorem:

Theorem 4.3 ([47, Theorem 3.1])**.** *A subgraph of Coxeter graph G corresponds to the face figure of a time-like face of $\mathcal{P}$ if and only if it is of finite type.*

For some cases of low rank, due to the large number of graphs, we do not give explicit lists in this paper. Interested readers are welcome to check the source code at http://github.com/Dr-How/L2Graph.

Remark (Notation). Let G be a Coxeter graph, G_1 and G_2 be two subgraphs of G. We use $G_1 + G_2$ to denote the subgraph induced by the vertices of G_1 and G_2, use $G_1 - G_2$ to denote the subgraph induced by the vertices of G_1 that are not in G_2. A subgraph with only one vertex is denoted by the vertex.

4.3 Products of two simplices

In this case, the following lemma guides the nomination of candidates:

Lemma 4.4. *If a* $(2,1)$*-polytope is combinatorial equivalent to the product of two simplices, then its Coxeter graph* G *is connected. It consists of two* $(1,0)$*-subgraphs, say* G_1 *and* G_2*, corresponding to the two simplices. For any* $v_1 \in G_1$*, the graphs* $G_2 + v_1$ *is a* $(2,0)$*-graph. Moreover,* G_2 *is strictly* $(1,0)$*, and* $G_2 + v_1$ *is strictly* $(2,0)$*, unless* G_1 *contains only two vertices.*

If one simplex is a segment, the product is a simplicial prism. As argued in [23], a prism can be cut into two orthogonally based prisms (one of whose base facets is orthogonal to all the lateral facets). The Coxeter graph of an orthogonally based $(2,1)$-prism is obtained from a $(2,0)$-graph by attaching a vertex with a dashed edge. For the polytope, this corresponds to truncating a space-like vertex. A candidate nominated in this way is of level 1 or 2 depending on whether the truncated vertex is the only space-like vertex. Moreover, we shall verify the label for the dashed edge, which should be < -1; otherwise the truncating face contains an unexpected vertex, and the combinatorial type of the polytope is the pyramid over a simplicial prism.

The list of $(2,0)$-graphs with at least five vertices can be found in [11], where some vertices are colored in white or grey. By attaching a vertex to each of these white or gray vertices, we obtain 655 candidate Coxeter graphs. Among them, 17 graphs correspond to orthogonally based simplicial prisms of level 1, as also enumerated in [23]; for 129 graphs, the label of the dashed edge is -1, hence the polytope is a pyramid. The remaining 509 graphs are confirmed by the recognition program as orthogonally based $(2,1)$-prisms.

If neither simplex is segment, we enumerate 28 $(2,1)$-graphs in Table 1. The table is used in conjunction with Figure 4 in the following manner: Figure 4 lists some strict $(1,0)$-graphs, with some vertices colored in white and marked with numbers. For each entry in Table 1, the first two columns give the positions of two $(1,0)$-graphs, say G_1 and G_2, in Figure 4. A Coxeter graph G is obtained by connecting G_1 with G_2 by adding the edges indicated in the last column in the format (vertex in G_1, vertex in G_2, label). We claim that

Proposition 4.5. *The* 28 *Coxeter graphs in Table 1 are the only* $(2,1)$*-graphs whose Coxeter polytope is combinatorially equivalent to the product of two simplices, both of dimension* > 1.

Remark. Figure 4 excludes hyperbolic triangle groups with label 7, 9 or $\geqslant 11$. By the same technique as in [15, § 4.1, Step 3) 4)], we verified by computer that these triangle groups can not be used to form any Coxeter graph of positive corank.

G_1	G_2	Edges between G_1 and G_2	G_1	G_2	Edges between G_1 and G_2
2	13	(1,0,3), (2,2,3)	4	19	(0,0,3), (3,1,3)
6	11	(0,1,3)	6	13	(0,0,3), (0,2,3)
10	17	(0,1,3), (3,1,3)	10	22	(0,0,3), (3,1,3)
11	14	(1,0,3)	11	22	(1,2,3)
12	12	(0,1,3)	12	12	(2,2,3)
12	15	(0,0,3), (0,2,3)	12	15	(2,1,3)
12	19	(0,0,3), (0,1,3)	12	19	(1,2,3)
12	24	(2,0,3), (2,1,3)	12	27	(0,0,3), (2,1,3)
13	14	(0,0,3), (2,0,3)	13	22	(0,0,3), (2,1,3)
13	26	(0,0,3), (2,2,3)	15	15	(0,0,3), (2,2,3)
15	15	(1,1,3)	15	19	(0,0,3), (2,1,3)
16	30	(0,0,3), (2,1,3)	17	22	(0,0,3), (2,1,3)
17	26	(0,0,3), (2,2,3)	18	18	(0,1,3), (1,0,3)
24	24	(0,0,3), (1,1,3), (2,2,3)	25	25	(0,0,3), (0,0,4), (1,1,3), (1,1,4)

Table 1: The first two columns are the positions of G_1 and G_2 in Figure 4, and the third columns are the edges connecting G_1 and G_2. The white vertices in Figure 4 are numbered, so the edges are represented in the format of (vertex in G_1, vertex in G_2, label). By connecting G_1 and G_2 by the indicated edges, we obtain the $(2,1)$-graphs for the products of two simplices (both of dimension > 1).

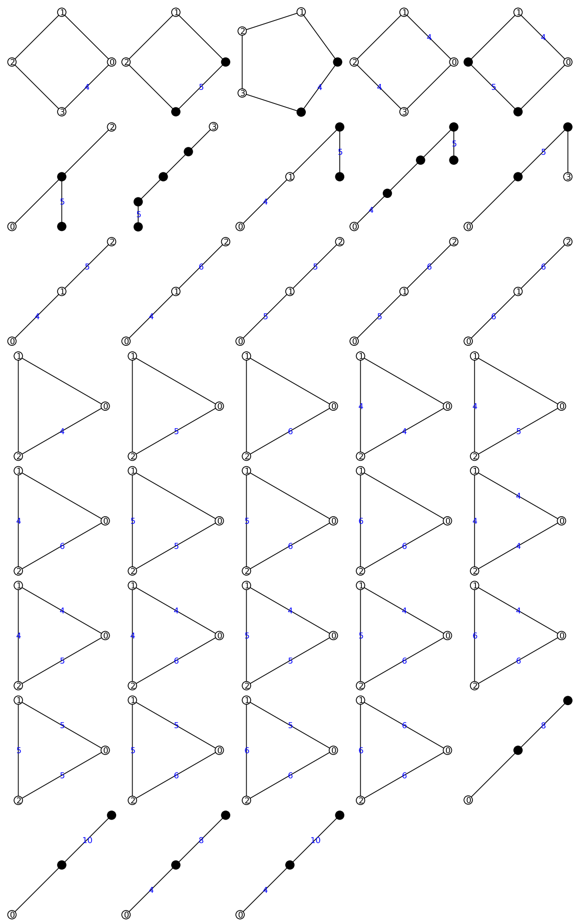


Figure 4: Strict $(1,0)$-graphs of $\geqslant 3$ vertices used in Table 1.

4.4 Pyramids over products of two simplices, with light-like apex

In this case, the following lemma guides the nomination of candidates:

Lemma 4.6. *If a* $(2,1)$*-polytope is combinatorial equivalent to the pyramid over the product of two simplices, and the apex is light-like, then its Coxeter graph* G *consists of two* $(1,0)$*-subgraphs, say* G_1 *and* G_2*, sharing a common vertex* v*. The subgraphs* $G_1 - v$ *and* $G_2 - v$ *are affine Coxeter graphs, and are disjoint; they correspond to the two simplices. If* $v_2 \in G_2 - v$ *is a neighbor of* v*, then* $G_1 + v_2$ *is a* $(2,0)$*-graph. Moreover,* $G_1 + v_2$ *is strictly* $(2,0)$*, and* v *is the unique vertex of* G_2 *whose removal leaves an affine graph, unless* G_1 *contains only two vertices.*

If one of the simplices is a segment, we construct a candidate $(2,1)$-graph as follows: we take a non-strict $(1,0)$-graph G, find a vertex v whose removal leaves an affine graph, then attach two vertices u and u' to v, and finally connect u and u' by a solid edge with label ∞. The edge uu' corresponds to the segment.

The list of non-strict $(1,0)$-graph with $\geqslant 4$ vertices can be found in [9]. The procedure above then gives, up to graph isomorphism, 358 graphs of level 1 or 2. Among them, 89 are of level 1 as also enumerated in [40]. The remaining 269 graphs correspond to pyramids of level 2, and 129 of them were previously discovered when enumerating orthogonally based simplicial prisms.

If neither simplex is segment, we enumerate 65 $(2,1)$-graphs in Table 2. The table is used in conjunction with Figure 5 in the following manner: Figure 5 lists some non-strict $(1,0)$-graphs with at least 4 vertices. Each of them has a white vertex. For each entry in Table 2, the two numbers are the positions of two $(1,0)$-graphs, say G_1 and G_2, in Figure 5. A Coxeter graph G is obtained by combining G_1 and G_2 by identifying the white vertex. We claim that:

Proposition 4.7. *The* 65 *Coxeter graphs in Table 2 are the only* $(2,1)$*-graphs whose Coxeter polytope is combinatorially equivalent to the pyramid over the product of two simplices, both of dimension* > 1.

1–2	1–5	1–9	1–12	1–15	1–16	1–19	1–22	1–24	1–27
2–3	2–7	2–8	2–23	3–5	3–13	3–15	3–16	3–20	3–28
4–9	4–12	4–19	4–22	4–24	4–27	5–7	5–8	5–23	7–9
7–12	7–15	7–16	7–19	7–22	7–24	7–27	8–9	8–12	8–15
8–16	8–19	8–22	8–24	8–27	9–26	10–12	10–19	10–27	11–12
11–19	11–27	12–18	12–26	13–23	15–23	16–23	18–19	18–27	19–26
20–23	22–26	23–28	24–26	26–27					

Table 2: For each pair i–j in the list, by identifying the white vertices of the i-th and the j-th graph in Figure 5, we obtain the $(2,1)$-graph of a pyramid over the product of two simplices (both of dimension > 1).

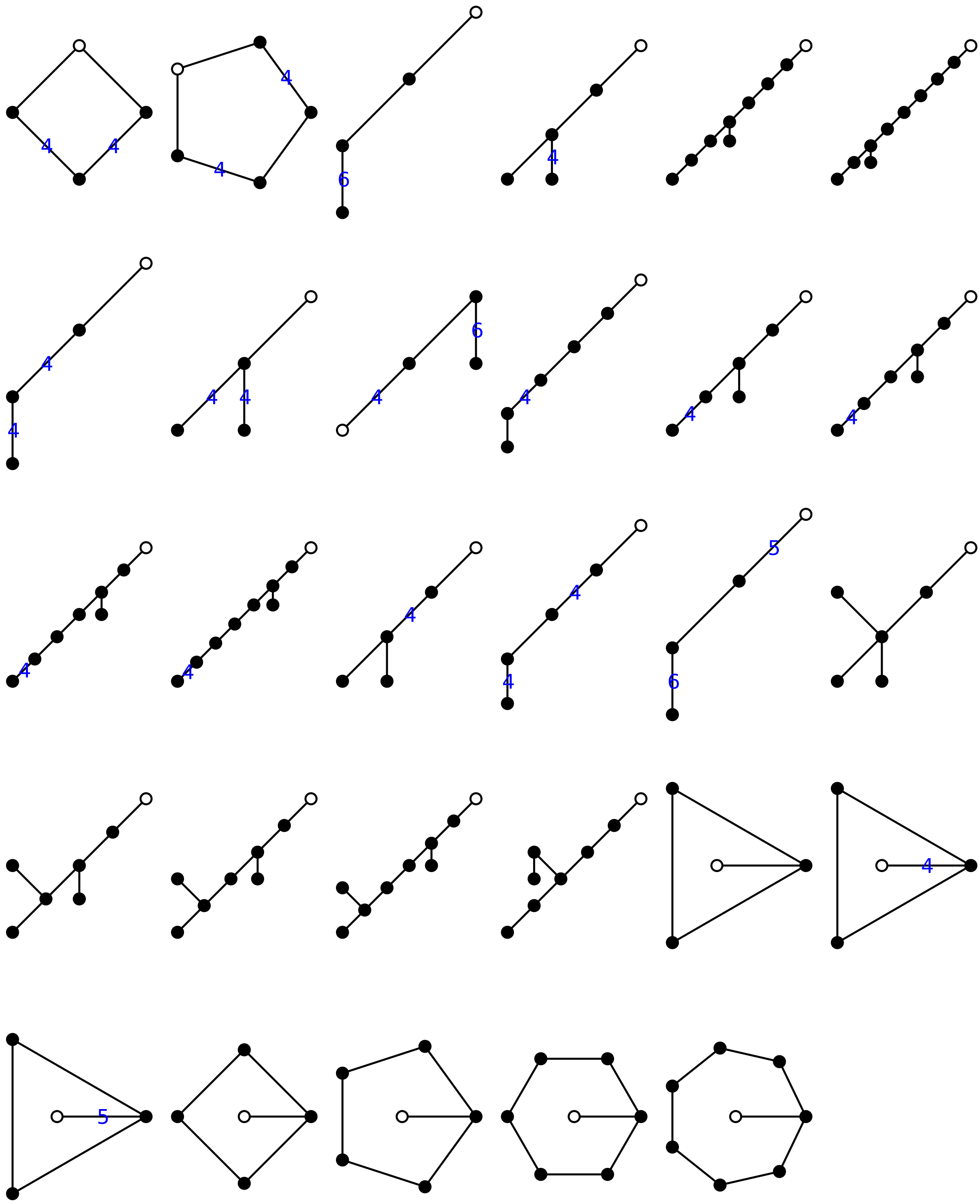


Figure 5: Non-strict (1, 0)-graphs of $\geqslant 4$ vertices used in Table 2.

4.5 Pyramids over products of two simplices, with space-like apex

In this case, the following lemma guides the nomination of candidates:

Lemma 4.8. *If a $(2,1)$-polytope is combinatorially equivalent to the pyramid over the product of two simplices, and the apex is space-like, then its Coxeter graph G consists of two strict $(2,0)$-graphs, say G_1 and G_2, sharing a common vertex v. The subgraph $G - v$ is a $(1,1)$-graph, hence essentially connected. In both G_1 and G_2, v is the unique vertex whose removal leaves a $(1,0)$-graph. For any $v_2 \in G_2 - v$, $G_1 - v + v_2$ is a strict $(2,0)$-graph in which v_2 is the unique vertex whose removal leaves a strict $(1,0)$-graph, and $G_1 + v_2$ is a strict $(3,0)$-graph in which v and v_2 are the only two vertices whose removal leaves a $(2,0)$-graph.*

If one of the simplices is a segment, the base facet is a $(1,1)$-prism. Coxeter graphs for $(1,1)$-prisms are classified in [23], where a list of orthogonally based $(1,1)$-prisms is given. The disjoint union of an isolated vertex with the graph of a $(1,1)$-prism is indeed a $(2,1)$-pyramid. Otherwise, we claim that

Proposition 4.9. *The 18 Coxeter graphs in Figure 6 are the only connected $(2,1)$-graphs with $\geqslant 7$ vertices whose Coxeter polytope is combinatorially equivalent to the pyramid over a prism and whose apex is space-like.*

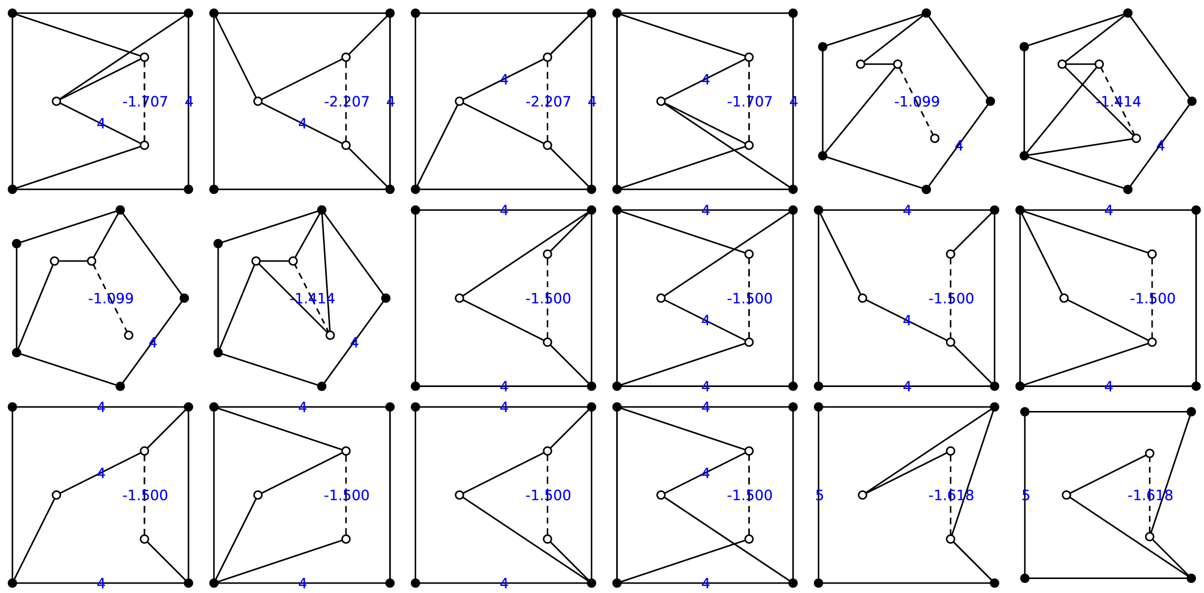
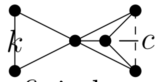


Figure 6: All the 18 connected $(2,1)$-graphs of rank $\geqslant 7$ whose Coxeter polytope has the type of a pyramid over a prism and whose apex is space-like.

For 4-dimensional pyramids over triangular prisms, we obtain 266 connected $(2,1)$-graphs from triangle graphs with labels at most 6. For triangle graphs with a label $k \geqslant 7$, the Coxeter graph is necessarily in the form of 

. The unlabeled edges can not have label $\geqslant 7$, so for a given k, there are only finitely many possibilities for the labels. For each of them, the value of $-c$ is determined by k. We then use Sage to find the expressions of the determinant in terms of k, and find no integer root that is $\geqslant 7$ for these expressions. So we believe that the labels on solid edges are at most 6 for this type of $(2,1)$-graphs. However, the author thinks that this is the point to question the reliability of computer enumeration, and an analytic explanation is welcomed.

If neither simplex is segment, the Coxeter graph for the base facet falls in the list in [15] and [40]. The two lists contain eight graphs in total. Disjoint union of an isolated vertex with any of the eight graphs clearly yields a $(2,1)$-graph. Otherwise, we claim that

Proposition 4.10. *The three Coxeter graphs in Figure 7 are the only connected $(2,1)$-graphs whose Coxeter polytope is combinatorially equivalent to the pyramid over the product of two simplices, both of dimension >1, and whose apex is space-like.*

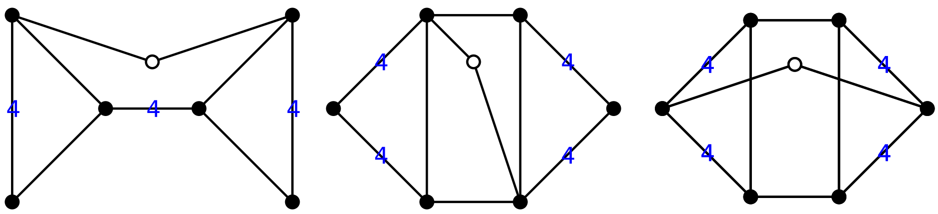


Figure 7: All the three connected $(2,1)$-graphs whose Coxeter polytope has the type of $\mathrm{Pyr}(\triangle \times \triangle)$ with space-like apex. The white vertex corresponds to the base facet.

4.6 Two-fold pyramids over products of two simplices

In this case, the following lemma guides the nomination of candidates:

Lemma 4.11. *If a $(2,1)$-polytope is combinatorially equivalent to the 2-fold pyramid over a product of two simplices, then its Coxeter graph G consists of two $(2,0)$-graphs, say G_1 and G_2, sharing two common vertices u and v. The subgraph $G_1 - u - v$ and $G_2 - u - v$ are affine graphs, and are not connected to each other. In both G_1 and G_2, u and v are the unique pair of vertices whose removal leaves an affine graph. For any $v_2 \in G_2$, the subgraph $G_1 + v_2$ is a $(3,0)$-graph in which u, v and v_2 are the only three vertices whose removal leaves a $(2,0)$-graph.*

If one of the simplices is a segment, we enumerate 49 $(2,1)$-graphs in Table 3. The table is used in conjunction with Figure 8 in the following manner: Figure 8 lists some $(2,0)$-graphs with at least 5 vertices; each of them has a white vertex and a gray vertex. Each entry of table 3 is in the format of $i:(a,b)(c,d)$. A Coxeter graph is obtained as follows: Take the i-th graph G in Figure 8. Let u be the gray vertex and v be the white vertex, introduce to new vertices w and w' connected by a solid edge with label ∞, and connect them to G by solid edges wu with label a, wv with label b, $w'u$ with label c and $w'v$ with label d. We claim that

Proposition 4.12. *The* 49 *Coxeter graphs in Table 3 are the only* $(2,1)$*-graphs with* $\geqslant 7$ *vertices whose Coxeter polytope is combinatorially equivalent to the two-fold pyramid over a simplicial prism.*

4:(2,3)(3,2)	4:(2,4)(4,2)	5:(2,3)(3,2)	6:(2,3)(3,2)	15:(2,2)(3,3)
15:(2,3)(3,2)	15:(2,4)(4,2)	15:(3,3)(3,3)	15:(3,4)(4,3)	24:(2,3)(4,3)
28:(2,2)(3,3)	28:(2,3)(3,2)	28:(3,3)(3,3)	32:(2,2)(3,3)	32:(2,3)(3,2)
32:(2,4)(4,2)	32:(3,3)(3,3)	37:(2,3)(3,2)	38:(2,2)(3,3)	38:(2,3)(3,2)
38:(3,3)(3,3)	39:(2,3)(3,2)	40:(2,2)(3,3)	40:(2,3)(3,2)	40:(3,3)(3,3)
41:(2,2)(3,3)	41:(3,3)(3,3)	42:(2,2)(3,3)	42:(3,3)(3,3)	48:(3,2)(3,3)
49:(2,2)(4,3)	49:(2,3)(4,2)	49:(3,2)(3,4)	49:(4,3)(4,3)	57:(2,3)(4,3)
59:(2,2)(3,3)	59:(3,3)(3,3)	61:(2,2)(3,3)	61:(2,3)(3,2)	61:(2,4)(4,2)
61:(3,3)(3,3)	65:(2,3)(3,2)	66:(2,2)(3,3)	66:(2,3)(3,2)	66:(3,3)(3,3)
67:(2,3)(3,2)	68:(2,2)(3,3)	68:(2,3)(3,2)	68:(3,3)(3,3)	

Table 3: For each entry i:$(a,b)(c,d)$ in the list, take the i-th graph $H+u+v$ in Figure 8, where u is the gray vertex and v is the white vertex. Introduce two new vertices w and w', and connect them to H such that wu has label a, wv has label b, $w'u$ has label c, $w'v$ has label d, and finally label the edge ww' by ∞. The result is the $(2,1)$-graph of a 2-fold pyramid over a prism.

If neither simplex is segment. We enumerate 36 $(2,1)$-graphs in Table 4. The table is used in conjunction with Figure 8 in the following manner: For each entry in the table, the two numbers are the positions of two $(2,0)$-graphs, say G_1 and G_2, in Figure 8. A Coxeter graph is obtained by combining G_1 and G_2 by identifying the non-black vertices. It turns out that, for every entry in the table, there is a unique way combination up to graph isomorphism. We claim that:

Proposition 4.13. *The* 36 *Coxeter graphs in Table 4 are the only* $(2,1)$*-graphs whose Coxeter polytope is combinatorially equivalent to the two-fold pyramid over the product of two simplices, both of dimension* > 1.

4–4	8–15	8–22	8–56	8–62	13–13
13–49	15–15	15–22	15–32	15–35	15–54
15–56	15–61	15–62	22–22	22–32	22–35
22–54	22–56	22–61	22–62	32–56	32–62
35–56	35–62	38–56	49–49	54–56	54–62
56–56	56–61	56–62	56–66	61–62	62–62

Table 4: For each pair i–j in the list, by identifying the white/light-gray vertices of the i-th and the j-th graph in Figure 8, we obtain the $(2,1)$-graph of a 2-fold pyramid over the product of two simplices (both of dimension > 1).

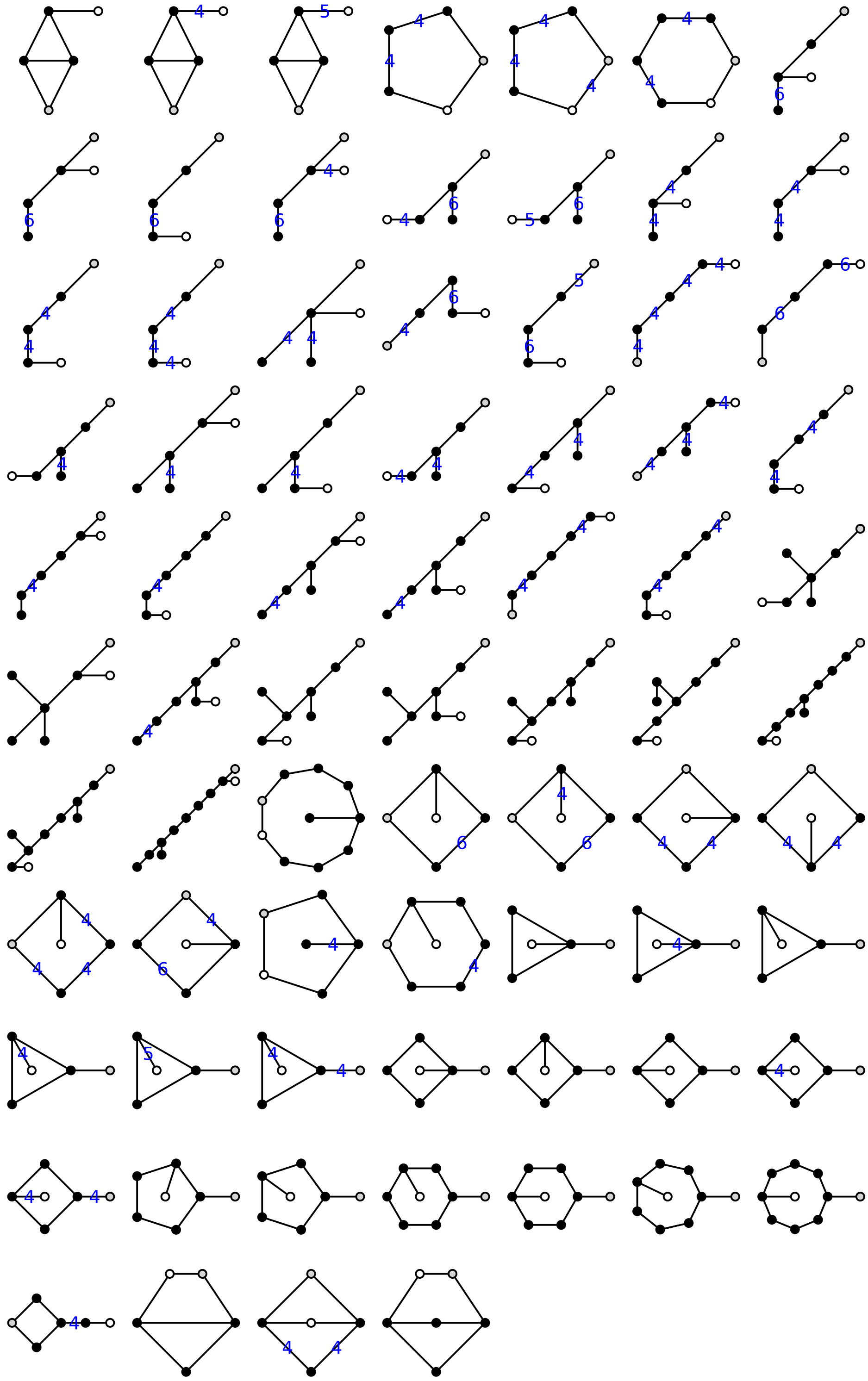


Figure 8: $(2,0)$-graphs of $\geqslant 5$ vertices used in Tables 3 and 4.

Acknowledgement

The author is grateful to Pavel Tumarkin for inspiring discussions and interesting references during my one-day visit at Durham university, which helped improving the paper and the program. I would also like to thank Jean-Philippe Labbé for helpful discussions, as well as Michael Kapovich and the anonymous referee for suggestions on preliminary versions of the paper.

References

[1] P. Abramenko and K. S. Brown. *Buildings*, volume 248 of *GTM*. Springer, New York, 2008.

[2] D. Allcock. Infinitely many hyperbolic Coxeter groups through dimension 19. *Geom. Topol.*, 10:737–758, 2006.

[3] E. G. Bajmóczy and I. Bárány. On a common generalization of Borsuk's and Radon's theorem. *Acta Math. Acad. Sci. Hungar.*, 34(3-4):347–350 (1980), 1979.

[4] R. M. Baram and H. J. Herrmann. Self-similar space-filling packings in three dimensions. *Fractals*, 12(3):293–301, 2004.

[5] N. Bourbaki. *Groupes et algèbres de Lie. Chapitre 4-6.* Paris: Hermann, 1968.

[6] D. W. Boyd. A new class of infinite sphere packings. *Pacific J. Math.*, 50:383–398, 1974.

[7] S. Bullett and G. Mantica. Group theory of hyperbolic circle packings. *Nonlinearity*, 5(5):1085–1109, 1992.

[8] T. E. Cecil. *Lie Sphere Geometry.* Universitext. Springer, New York, 2 edition, 2008.

[9] M. Chein. Recherche des graphes des matrices de Coxeter hyperboliques d'ordre $\leqslant 10$. *Rev. Française Informat. Recherche Opérationnelle*, 3(Ser. R-3):3–16, 1969.

[10] H. Chen. Apollonian Ball Packings and Stacked Polytopes. *Discrete Comput. Geom.*, 55(4):801–826, 2016.

[11] H. Chen and J.-P. Labbé. Lorentzian Coxeter systems and Boyd-Maxwell ball packings. *Geom. Dedicata*, 174:43–73, 2015.

[12] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, third edition, 1999.

[13] M. Dyer. Imaginary cone and reflection subgroups of Coxeter groups. 2013. preprint, arXiv:1210.5206v2.

[14] M. Dyer, C. Hohlweg, and V. Ripoll. Imaginary cones and limit roots of infinite Coxeter groups. *Math. Z.*, pages 1–66, 2016. online first.

[15] F. Esselmann. The classification of compact hyperbolic Coxeter d-polytopes with $d+2$ facets. *Comment. Math. Helv.*, 71(2):229–242, 1996.

[16] B. Grünbaum. *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

[17] U. Hertrich-Jeromin. *Introduction to Möbius Differential Geometry*, volume 300 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.

[18] C. Hohlweg, J.-P. Labbé, and V. Ripoll. Asymptotical behaviour of roots of infinite Coxeter groups. *Canad. J. Math.*, 66(2):323–353, 2014.

[19] C. Hohlweg, J.-P. Préaux, and V. Ripoll. On the limit set of root systems of Coxeter groups and Kleinian groups. 2013. preprint, `arXiv:1305.0052v2`.

[20] R. B. Howlett, P. J. Rowley, and D. E. Taylor. On outer automorphism groups of Coxeter groups. *Manuscripta Math.*, 93(4):499–513, 1997.

[21] J. E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1992.

[22] N. W. Johnson, R. Kellerhals, J. G. Ratcliffe, and S. T. Tschantz. Commensurability classes of hyperbolic Coxeter groups. *Linear Algebra Appl.*, 345:119–147, 2002.

[23] I. M. Kaplinskaya. Discrete groups generated by reflections in the faces of symplicial prisms in lobachevskian spaces. *Mat. Zametki*, 15(1):159–164, 1974. English translation: Math. Notes, 15(1):88–91, (1974).

[24] D. Krammer. The conjugacy problem for Coxeter groups. *Group. Geom. Dynam.*, 3(1):71–171, 2009.

[25] F. Lannér. On complexes with transitive groups of automorphisms. *Comm. Sém., Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.]*, 11:71, 1950.

[26] J. Leys. Sphere inversion fractals. *Computers & Graphics*, 29(3):463–466, 2005.

[27] E. Looijenga. Invariant theory for generalized root systems. *Invent. Math.*, 61(1):1–32, 1980.

[28] C. Maclachlan and A. W. Reid. *The arithmetic of hyperbolic 3-manifolds*, volume 219 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003.

[29] V. S. Makarov. The Fedorov groups of four-dimensional and five-dimensional Lobačevskiĭ space. In *Studies in General Algebra, No. 1 (Russian)*, pages 120–129. Kišinev. Gos. Univ., Kishinev, 1968.

[30] G. Mantica and S. Bullett. Plato, Apollonius, and Klein: playing with spheres. *Phys. D*, 86(1-2):113–121, 1995. Chaos, order and patterns: aspects of nonlinearity—the "gran finale" (Como, 1993).

[31] G. Maxwell. Sphere packings and hyperbolic reflection groups. *J. Algebra*, 79(1):78–97, 1982.

[32] G. Maxwell. Wythoff's construction for Coxeter groups. *J. Algebra*, 123(2):351–377, 1989.

[33] G. Maxwell. Euler characteristics and imbeddings of hyperbolic Coxeter groups. *J. Austral. Math. Soc. Ser. A*, 64(2):149–161, 1998.

[34] J. Mcleod. Hyperbolic coxeter pyramids. *Advances in Pure Mathematics*, 3:78–82, 2013.

[35] H. Oh and N. Shah. The asymptotic distribution of circles in the orbits of Kleinian groups. *Invent. Math.*, 187(1):1–35, 2012.

[36] M. N. Prokhorov. Absence of discrete groups of reflections with a noncompact fundamental polyhedron of finite volume in a Lobachevskiĭ space of high dimension. *Izv. Akad. Nauk SSSR Ser. Mat.*, 50(2):413–424, 1986. English translation: Math. USSR Izv., 28(2):401–411, 1987.

[37] R. T. Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[38] K. E. Stange. The Apollonian structure of Bianchi groups. 2015. preprint, arXiv:1505.03121v2.

[39] W. A. Stein et al. *Sage Mathematics Software (Version 6.2)*. The Sage Development Team, 2014. http://www.sagemath.org.

[40] P. V. Tumarkin. Hyperbolic Coxeter n-polytopes with $n + 2$ facets. *Mat. Zametki*, 75(6):909–916, 2004. English translation: Math. Notes, 75(6):848–854, 2004.

[41] P. V. Tumarkin. Compact hyperbolic Coxeter n-polytopes with $n+3$ facets. *Electron. J. Combin.*, 14(1):Research Paper 69, 36 pp. (electronic), 2007.

[42] P. V. Tumarkin and A. Felikson. On bounded hyperbolic d-dimensional Coxeter polytopes with $d + 4$ hyperfaces. *Tr. Mosk. Mat. Obs.*, 69:126–181, 2008. English translation: Trans. Moscow Math. Soc., 69:105–151, 2008.

[43] È. B. Vinberg. Discrete groups generated by reflections in Lobačevskiĭ spaces. *Mat. Sb. (N.S.)*, 72(114)(3):471–488; correction, ibid. 73(115)(2) (1967), 303, 1967. English translation: Math. USSR Sb. 1(3):429–444, 1967.

[44] È. B. Vinberg. Discrete linear groups generated by reflections. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35(5):1072–1112, 1971. English translation: Math. USSR Izv., 5(5):1083–1119, 1971.

[45] È. B. Vinberg. Some arithmetical discrete groups in Lobačevskiĭ spaces. In *Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973)*, pages 323–348. Oxford Univ. Press, Bombay, 1975.

[46] È. B. Vinberg. The non-exitence of crystallographic groups of reflections in Lobachevskij spaces of large dimension. *Tr. Mosk. Mat. Obs.*, 47:68–102, 1984. English translation: Trans. Moskow Math. Soc., 47:75–112, 1985.

[47] È. B. Vinberg. Hyperbolic groups of reflections. *Uspekhi Mat. Nauk*, 40(1(241)):29–66, 1985. English translation: Russian Math. Surveys, 40(1):31–75, 1985.