Asymptotic enumeration of sparse uniform linear hypergraphs with given degrees

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Abstract

A hypergraph is *simple* if it has no loops and no repeated edges, and a hypergraph is *linear* if it is simple and each pair of edges intersects in at most one vertex. For $n \geqslant 3$, let $r = r(n) \geqslant 3$ be an integer and let $\boldsymbol{k} = (k_1, \ldots, k_n)$ be a vector of nonnegative integers, where each $k_j = k_j(n)$ may depend on n. Let $M = M(n) = \sum_{j=1}^{n} k_j$ for all $n \geqslant 3$, and define the set $\mathcal{I} = \{n \geqslant 3 \mid r(n) \text{ divides } M(n)\}$. We assume that $\mathcal{I}$ is infinite, and perform asymptotics as n tends to infinity along $\mathcal{I}$. Our main result is an asymptotic enumeration formula for linear r-uniform hypergraphs with degree sequence $\boldsymbol{k}$. This formula holds whenever the maximum degree $k_{\max}$ satisfies $r^4 k_{\max}^4 (k_{\max} + r) = o(M)$. Our approach is to work with the incidence matrix of a hypergraph, interpreted as the biadjacency matrix of a bipartite graph, enabling us to apply known enumeration results for bipartite graphs. This approach also leads to a new asymptotic enumeration formula for simple uniform hypergraphs with specified degrees, and a result regarding the girth of random bipartite graphs with specified degrees.

Keywords: hypergraphs; linear hypergraphs; asymptotic enumeration; switchings

1 Introduction

Hypergraphs are combinatorial structures which can model very general relational systems, including some real-world networks [6, 8, 11]. Formally, a *hypergraph* or set system is defined as a pair (V, E), where V is a finite set and E is a multiset of multisubsets of V. (We refer to elements of E as *edges*.) Note that under this definition, a hypergraph may contain repeated edges and an edge may contain repeated vertices.

Any 2-element multisubset of an edge $e \in E$ is called a *link* in e. If a vertex v has multiplicity at least 2 in the edge e, we say that v is a *loop* in e. (So every loop in e is also a link in e.) The *multiplicity* of a link $\{x, y\}$ is the number of edges in E which contain $\{x, y\}$ (counting multiplicities). A *double link* is a link with multiplicity 2.

A hypergraph is *simple* if it has no loops and no repeated edges: that is, E is a set of edges, and each edge is a subset of V. Here it is possible that distinct edges may have more than one vertex in common. (This definition of simple hypergraph appears to be standard, and matches the definition of simple hypergraphs given by Berge [1] in the case of uniform hypergraphs.) A hypergraph is called *linear* if it has no loops and each pair of distinct edges intersect in at most one vertex. (Note that linear hypergraphs are also simple, when $r \geqslant 2$.) Linear hypergraphs have been well-studied in many contexts (sometimes they have been referred to as "simple hypergraphs"). See for example [3, 7, 12, 19].

For a positive integer r, the hypergraph (V, E) is *r-uniform* if each edge $e \in E$ contains exactly r vertices (counting multiplicities). Uniform hypergraphs are a particular focus of study, not least because a 2-uniform hypergraph is precisely a graph. We seek an asymptotic enumeration formula for the number of linear r-uniform hypergraphs with a given degree sequence, when the maximum degree is not too large (the sparse range), and allowing r to grow slowly with n.

To state our result precisely, we need some definitions. Write $[a] = \{1, 2, \ldots, a\}$ for all positive integers a. Given nonnegative integers a, b, let $(a)_b$ denote the falling factorial $a(a-1)\cdots(a-b+1)$. We are given a degree sequence $\boldsymbol{k} = \boldsymbol{k}(n) = (k_1, \ldots, k_n)$ with sum $M = M(n) = \sum_{i=1}^{n} k_i$, and we are also given an integer $r = r(n) \geqslant 3$, for each $n \geqslant 3$. Let $k_{\max} = k_{\max}(n) = \max_{j=1}^{n} k_j$ for all $n \geqslant 3$. For each positive integer t, define

$$M_t = M_t(n) = \sum_{i=1}^{n} (k_i)_t.$$

Then $M_1 = M$ and $M_t \leqslant k_{\max} M_{t-1}$ for $t \geqslant 2$.

Let $\mathcal{H}_r(\boldsymbol{k})$ denote the set of simple r-uniform hypergraphs on the vertex set $[n]$ with degree sequence given by $\boldsymbol{k} = (k_1, \ldots, k_n)$, and let $\mathcal{L}_r(\boldsymbol{k})$ be the set of all linear hypergraphs in $\mathcal{H}_r(\boldsymbol{k})$. Note that $\mathcal{H}_r(\boldsymbol{k})$ and $\mathcal{L}_r(\boldsymbol{k})$ are both empty unless r divides M. Our main theorem is the following.

Theorem 1.1. *For $n \geqslant 3$, let $r = r(n) \geqslant 3$ be an integer and let $\boldsymbol{k} = (k_1, \ldots, k_n)$ be a vector of nonnegative integers, where each $k_j = k_j(n)$ may depend on n. Let $k_{\max} = k_{\max}(n) = \max_{j=1}^{n} k_j$ for all $n \geqslant 3$. Define $M = M(n) = \sum_{j=1}^{n} k_j$ for all $n \geqslant 3$, and suppose that the set*

$$\mathcal{I} = \{n \geqslant 3 \mid r(n) \text{ divides } M(n)\}$$

is infinite. Suppose that $M \to \infty$ and $r^4 k_{\max}^4 (k_{\max} + r) = o(M)$ as n tends to infinity along elements of $\mathcal{I}$. Then

$$|\mathcal{L}_r(\boldsymbol{k})|$$
$$= \frac{M!}{(M/r)!(r!)^{M/r} \prod_{i=1}^n k_i!} \exp\left(-\frac{(r-1)M_2}{2M} - \frac{(r-1)^2 M_2^2}{4M^2} + O\left(\frac{r^4 k_{\max}^4 (k_{\max} + r)}{M}\right)\right).$$

We believe that Theorem 1.1 is the first asymptotic enumeration by degree sequence for r-uniform linear hypergraphs with $r \geqslant 3$, and the first asymptotic enumeration result for sparse hypergraphs which allows r to grow with n. The two (non-error) terms within the exponential arise naturally: the first corresponds to the expected number of loops and the second corresponds to the expected number of double links.

A brief survey of the relevant literature is given in the next subsection. Note that when $r = 2$ (graphs), our result is weaker than the formula given by McKay and Wormald [17], as their expression has smaller error term and more significant terms. In order to improve the accuracy of Theorem 1.1 to a similar level, a more detailed analysis of double links is required. We will present such an analysis in a future paper.

To obtain Theorem 1.1, we treat the incidence matrix of a hypergraph as the biadjacency matrix of a bipartite graph, thereby enabling us to make use of prior enumeration results for bipartite graphs in order to enumerate linear hypergraphs. In Section 2 we show that some undesirable substructures are rare in random bipartite graphs with the appropriate degrees. As a corollary of this, we obtain a new enumeration result for sparse simple uniform hypergraphs. Theorem 1.1 then follows from a switching argument for bipartite graphs which is used to remove 4-cycles, as these correspond to double links in the hypergraph. This switching argument is presented in Section 3, leading to the proof of Theorem 1.1. Finally in Corollary 3.5 we state a consequence of our calculations relating to the girth of bipartite graphs with specified degrees.

Using this approach of translating the problem to one involving bipartite graphs, it should be possible to relax the uniformity condition, perhaps by allowing the number of edges with a given size to be specified up to a maximum edge size (which may grow modestly with n). Such a generalisation has not been performed here.

1.1 History

In the case of graphs, the best asymptotic formula in the sparse range is given by McKay and Wormald [17]. See that paper for further history of the problem. The dense range was treated in [15, 16], but there is a gap between these two ranges in which nothing is known.

An early result in the asymptotic enumeration of hypergraphs was given by Cooper et al. [4], who considered simple k-regular hypergraphs when $k = O(1)$. More recently, Dudek et al. [5] proved an asymptotic formula for simple k-regular hypergraphs with $k = o(n^{1/2})$. In [2] this was extended to irregular sequences, with an improved error bound. We restate this result below.

Theorem 1.2. [2, Theorem 1.1] *Let $r \geqslant 3$ be a fixed integer. Let $\boldsymbol{k}$, M and $k_{\max}$ be defined as in Theorem 1.1. Assume that r divides M for infinitely many values of n. Suppose that $M \to \infty$, $k_{\max} \geqslant 2$ and $k_{\max}^3 = o(M)$ as n tends to infinity along these values. Then*

$$|\mathcal{H}_r(\boldsymbol{k})| = \frac{M!}{(M/r)!\,(r!)^{M/r}\,\prod_{i=1}^n k_i!} \exp\left(-\frac{(r-1)M_2}{2M} + O(k_{\max}^3/M)\right).$$

Kuperberg, Lovett and Peled [13] gave an asymptotic formula for the number of dense simple r-uniform hypergraphs with a given degree sequence.

2 Hypergraphs, incidence matrices and bipartite graphs

Suppose that G is an r-uniform hypergraph with degree sequence $\boldsymbol{k}$ which has no loops (but may have repeated edges). Let A be the $n \times (M/r)$ incidence matrix of G, where the rows of the incidence matrix correspond to vertices $1, 2, \ldots n$ in that order, and the columns correspond to the edges of the hypergraph, in some order. Then A is a 0-1 matrix (as G has no loops), the row sums of A are given by $\boldsymbol{k}$ and each column sum of A equals r.

If G is simple (that is, if $G \in \mathcal{H}_r(\boldsymbol{k})$) then all columns of A are distinct, and hence there are precisely $(M/r)!$ possible (distinct) incidence matrices corresponding to G. Conversely, every 0-1 matrix with rows sums given by $\boldsymbol{k}$, column sums all equal to r and with no repeated columns can be interpreted as the incidence matrix of a hypergraph in $\mathcal{H}_r(\boldsymbol{k})$.

It will be convenient to work with the bipartite graphs whose biadjacency matrices are the incidence matrices of hypergraphs. Let $\mathcal{B}_r(\boldsymbol{k})$ be the set of bipartite graphs with vertex bipartition $\{v_1, \ldots, v_n\} \cup \{e_1, e_2, \ldots, e_{M/r}\}$, such that degree sequence of $(v_1, \ldots, v_n)$ is $\boldsymbol{k}$ and every vertex e_j has degree r. We sometimes say that a vertex v_j is "on the left" and that a vertex e_i is "on the right". An example of a 3-uniform hypergraph, its incidence matrix (with edges ordered in lexicographical order) and corresponding bipartite graph is shown in Figure 1.

Double links will be of particular interest: there are two double links in the hypergraph in Figure 1, and each corresponds to a subgraph of the bipartite graph which is isomorphic to $K_{2,2}$. (One is induced by $\{v_1, v_2, e_1, e_2\}$ and the other by $\{v_5, v_6, e_3, e_4\}$.)

It follows from [10, Theorem 1.3] that

$$|\mathcal{B}_r(\boldsymbol{k})| = \frac{M!}{(r!)^{M/r}\,\prod_{j=1}^n k_j!} \exp\left(-\frac{(r-1)M_2}{2M} + O(r^2 k_{\max}^2/M)\right) \qquad (2.1)$$

whenever $1 \leqslant r k_{\max} = o(M^{1/2})$. (In fact, the result of [10] is more accurate but we are unable to exploit the extra accuracy here, so we state a simplified version.)

Next, let $\mathcal{B}_r^{(0)}(\boldsymbol{k})$ denote the set of all bipartite graphs in $\mathcal{B}_r(\boldsymbol{k})$ such that no two vertices e_{i_1}, e_{i_2} (on the right) have the same neighbourhood. These bipartite graphs correspond to 0-1 matrices with no repeated columns, which in turn can be viewed as incidence matrices of (simple) hypergraphs in $\mathcal{H}_r(\boldsymbol{k})$. Hence

$$(M/r)!\,|\mathcal{H}_r(\boldsymbol{k})| = |\mathcal{B}_r^{(0)}(\boldsymbol{k})|. \qquad (2.2)$$
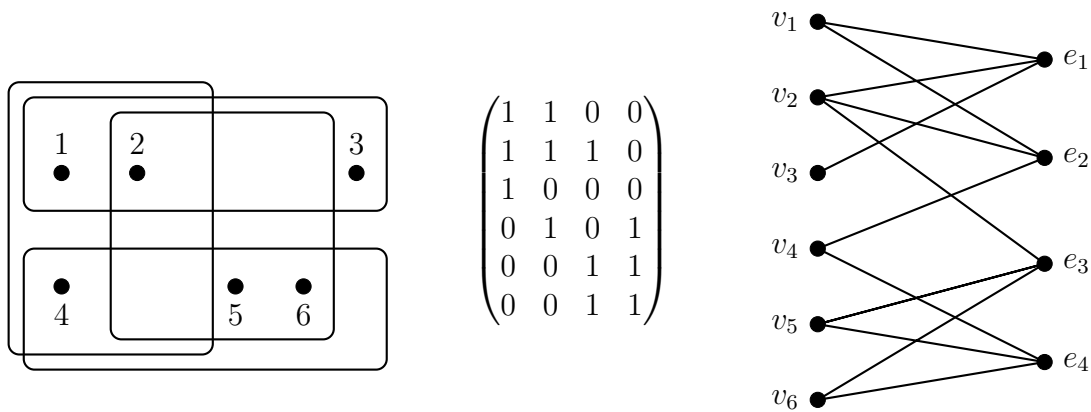
Figure 1: A hypergraph, its incidence matrix and corresponding bipartite graph.

To work towards linear hypergraphs, we identify some desirable properties of the corresponding bipartite graphs. Given an ordered pair of positive integers (a, b), say that the bipartite graph B has a *copy of $K_{a,b}$* if B contains a set of a vertices on the left and b vertices on the right which induce a subgraph isomorphic to $K_{a,b}$. This definition is slightly non-standard, since it is not symmetric with respect to a and b. However, we will mostly be interested in copies of $K_{2,2}$, in which case there is no asymmetry. We will write "B contains a 4-cycle" rather than "B contains a copy of $K_{2,2}$".

Define

$$N_2 = 3 \max \left\{ \lceil \log M \rceil, \ \lceil 2(r-1)^2 M_2^2/M^2 \rceil \right\}$$

and let $\mathcal{B}_r^+(\boldsymbol{k})$ denote the set of all bipartite graphs $B \in \mathcal{B}_r(\boldsymbol{k})$ which satisfy the following properties:

(i) B has no copy of $K_{3,2}$ (with three vertices on the left and two on the right).

(ii) B has no copy of $K_{2,3}$ (with two vertices on the left and three on the right).

(iii) No two 4-cycles in B have a vertex e_j (on the right) in common. (This implies that any 4-cycles in B are edge-disjoint.)

(iv) Any three distinct 4-cycles in B involve at least five vertices on the left. (Together with (iii), this implies that if two distinct 4-cycles share a vertex on the left then any other 4-cycle in B must be vertex-disjoint from the first two.)

(v) The number of 4-cycles in B is at most N_2.

To motivate this definition, note that $B \in \mathcal{B}_r^+(\boldsymbol{k})$ if and only if the corresponding hypergraph $G = G(B)$ satisfies the following properties:

(i)$'$ The intersection of any two edges of G contains at most two vertices.

(ii)′ Any link has multiplicity at most two in G. (That is, the intersection of any three edges of G contains at most one vertex.)

(iii)′ No edge of G contains more than one double link. (That is, if e_1 and e_2 are edges of G which share a double link then e_1 is not involved in any other double link in G, and similarly for e_2.)

(iv)′ No vertex can belong to three double links, and if a vertex v belongs to two double links (say $\{v, x\}$ and $\{v, y\}$ are both double links) then both x and y belong to precisely one double link.

(v)′ There are at most N_2 double links in G.

In particular, as $r \geqslant 3$, any hypergraph $G = G(B)$ with $B \in \mathcal{B}_r^+(\boldsymbol{k})$ has (no loops and) no repeated edges, so is simple.

McKay [14] proved asymptotic formulae for the probability that a randomly chosen bipartite graph with specified degrees contains a fixed subgraph, under certain conditions. We state one of these results below, which will be use repeatedly. (In fact the statement below is a special case of [14, Theorem 3.5(a)], obtained by taking $J = L$ and $H = \varnothing$ in the notation of [14], and with slightly simplified notation.)

Lemma 2.1. ([14, Theorem 3.5(a)]) *Let $\mathcal{B}(\boldsymbol{g})$ denote the set of bipartite graphs with vertex bipartition given by $\{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_m\}$ and degree sequence*

$$\boldsymbol{g} = (g_1, \ldots, g_n; g_1', \ldots, g_m').$$

(Here vertex a_i has degree g_i for $i = 1, \ldots, n$, and vertex b_j has degree g_j' for $j = 1, \ldots, m$.) Let L be a subgraph of the complete bipartite graph on this vertex bipartition, and let $\mathcal{B}(\boldsymbol{g}, L)$ be the set of bipartite graphs in $\mathcal{B}(\boldsymbol{g})$ which contain L as a subgraph. Write $E_{\boldsymbol{g}} = \sum_{i=1}^n g_i$ and $E_{\boldsymbol{\ell}} = \sum_{i=1}^n \ell_i$, where $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n; \ell_1', \ldots, \ell_m')$ is the degree sequence of L. Finally, let $g_{\max}$ and $\ell_{\max}$ denote the maximum degree in $\boldsymbol{g}$ and $\boldsymbol{\ell}$, respectively, and define

$$\Gamma = 2g_{\max}(g_{\max} + \ell_{\max} - 1) + 2.$$

If $E_{\boldsymbol{g}} - \Gamma \geqslant E_{\boldsymbol{\ell}}$ then

$$\frac{|\mathcal{B}(\boldsymbol{g}, L)|}{|\mathcal{B}(\boldsymbol{g})|} \leqslant \frac{\prod_{i=1}^n (g_i)_{\ell_i} \prod_{j=1}^m (g_j')_{\ell_j'}}{(E_{\boldsymbol{g}} - \Gamma)_{E_{\boldsymbol{\ell}}}}.$$

Using this lemma, we now analyse the probability that a uniformly random element of $\mathcal{B}_r(\boldsymbol{k})$ satisfies properties (i)–(v).

Theorem 2.2. *Under the conditions of Theorem 1.1,*

$$\frac{|\mathcal{B}_r^+(\boldsymbol{k})|}{|\mathcal{B}_r(\boldsymbol{k})|} = 1 + O\left(r^5 k_{\max}^4 / M\right).$$

Proof. Throughout this proof, consider a uniformly random element $B \in \mathcal{B}_r(\mathbf{k})$. We will apply Lemma 2.1 several times with $\mathbf{g} = (k_1, \ldots, k_n; r, \ldots, r)$. In each application, L is a subgraph with constant maximum degree. Hence $g_{\max} = \max\{k_{\max}, r\}$ and

$$\Gamma = 2g_{\max}(g_{\max} + \ell_{\max} - 1) + 2 = O(r^2 + k_{\max}^2).$$

For (i), let $v_{j_1}, v_{j_2}, v_{j_3} \in [n]$ be distinct vertices on the left, and let e_{i_1}, e_{i_2} be distinct vertices on the right. Applying Lemma 2.1 with $L = K_{3,2}$, we find that the probability that B has a copy of $K_{3,2}$ on the vertices $\{v_{j_1}, v_{j_2}, v_{j_3}\} \cup \{e_{i_1}, e_{i_1}\}$ is at most

$$\frac{r^2(r-1)^2(r-2)^2}{(M + O(r^2 + k_{\max}^2))_6} (k_{j_1})_2 (k_{j_2})_2 (k_{j_3})_2.$$

By assumption, $k_{\max}^2 + r^2 = o(M)$. Multiplying this by the number of choices for $\{e_{i_1}, e_{i_2}\}$ and summing over all choices of (j_1, j_2, j_3) with $1 \leqslant j_1 < j_2 < j_3 \leqslant n$ shows that the expected number of copies of $K_{3,2}$ in B is at most

$$\binom{M/r}{2} \sum_{j_1 < j_2 < j_3} (k_{j_1})_2 (k_{j_2})_2 (k_{j_3})_2 \, O\left(\frac{r^6}{M^6}\right) = O\left(\frac{r^4 M_2^3}{M^4}\right)$$
$$= O(r^4 k_{\max}^3 / M). \tag{2.3}$$

Hence property (i) fails with probability $O(r^4 k_{\max}^3 / M)$. For future reference, we note that the argument leading to (2.3) still holds under the weaker condition $r^4 k_{\max}^3 = o(M)$ (as this condition still implies that $k_{\max}^2 + r^2 = o(M)$, and all other calculations are unchanged).

Repeating this argument with $L = K_{2,3}$ shows that property (ii) fails with probability $O(r^3 k_{\max}^4 / M)$. Using the subgraphs L shown in Figure 2 (a) and (b) we can establish that property (iii) fails with probability $O(r^5 k_{\max}^4 / M)$. Assuming that properties (i)–(iii)
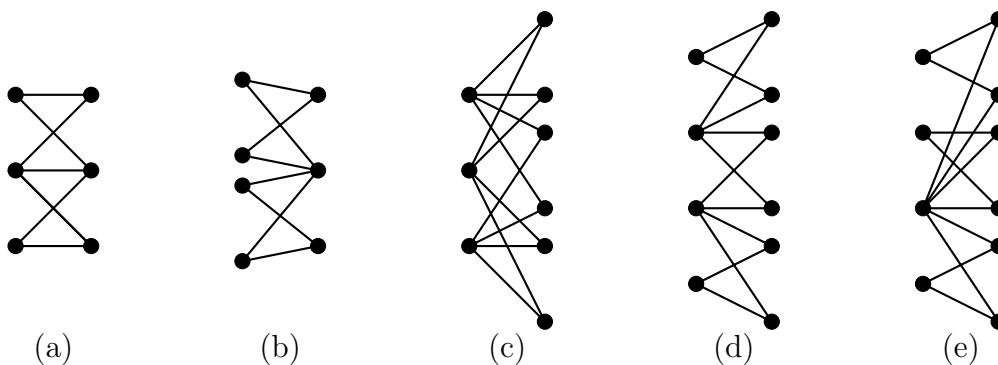

Figure 2: Rare subgraphs

hold, we can prove that property (iv) holds with probability $O(r^2 k_{\max}^3 / M)$ by considering the subgraphs L shown in Figure 2 (c), (d) and (e).

Now we turn to (v). Let $Q_1 = \max\{\lceil \log M \rceil, \lceil 2(r-1)^2 M_2^2/M^2 \rceil\}$ and define $d = Q_1 + 1$. We first show that the expected number of sets of d vertex-disjoint 4-cycles in B is $O(1/M)$. Fix $(j_1, \ldots, j_{2d}) \in [n]^{2d}$ such that $k_{j_\ell} \geqslant 2$ for $\ell = 1, 2, \ldots, 2d$ and $j_{2\ell-1} \neq j_{2\ell}$ for $\ell = 1, 2, \ldots, d$. Let $(i_1, \ldots, i_{2d}) \in \{1, \ldots, M/r\}^{2d}$ be a $(2d)$-tuple of (distinct) edge labels. The probability that there is a 4-cycle on $\{v_{j_{2\ell-1}}, v_{j_{2\ell}}\} \cup \{e_{i_{2\ell-1}}, e_{i_{2\ell}}\}$ for $\ell = 1, \ldots, d$ is

$$\prod_{\ell=1}^{2d} (k_{j_\ell})_2 \; O\left(\frac{(r(r-1))^{2d}}{M^{4d}}\right),$$

by Lemma 2.1. There are at most $(M/r)^{2d}$ choices for $(i_1, \ldots, i_{2d})$, and for an upper bound we can sum over all possible values of $(j_1, \ldots, j_{2d})$. This counts each set of d vertex-disjoint 4-cycles precisely $4^d\, d!$ times. It follows that the expected number of sets of d vertex-disjoint 4-cycles in B is

$$\sum_{(j_1,\ldots,j_{2d})\in[n]^{2d}} \prod_{\ell=1}^{2d} (k_{j_\ell})_2 \; O\left(\frac{(r-1)^{2d}}{4^d\, d!\, M^{2d}}\right) = O\left(\frac{1}{d!}\left(\frac{(r-1)^2\, M_2^2}{4\, M^2}\right)^d\right)$$

$$= O\left(\left(\frac{e\,(r-1)^2\, M_2^2}{4d\, M^2}\right)^d\right)$$

$$= O\left((e/8)^d\right)$$

$$= O(1/M)$$

by choice of d. Next, let $Q_2 = \max\{\lceil \log M \rceil, \lceil (r-1)^4 M_2^2 M_4/M^4 \rceil\}$ and define $b = Q_2 + 1$. Assuming that properties (iii) and (iv) hold, any 4-cycle in B is either vertex-disjoint from all other 4-cycles in B, or shares one vertex on the left with precisely one other 4-cycle in B. In the latter case, call such a pair of 4-cycles a *fused pair*. Arguing as above, the expected number of sets of b fused pairs is at most $O(1/M)$, by choice of b. It follows that with probability $1 + O(r^5 k_{\max}^4/M)$, the number of 4-cycles in B is at most $Q_1 + 2Q_2 \leqslant 3Q_1 = N_2$, completing the proof. $\square$

As a by-product of Theorem 2.2, we obtain a new asymptotic enumeration formula for sparse simple uniform hypergraphs with given degrees, generalising [2, Theorem 1.1] (restated earlier as Theorem 1.2, for ease of comparison): the new formula allows r to grow slowly with n, whereas Theorem 1.2 is only valid for fixed $r \geqslant 3$. (The two results match when r is constant.)

Corollary 2.3. *For $n \geqslant 3$, let $r = r(n) \geqslant 3$ be an integer and let $\boldsymbol{k} = \boldsymbol{k}(n) = (k_1, \ldots, k_n)$ be a vector of nonnegative integers, where each $k_j = k_j(n)$ may depend on n. Let $M = M(n) = \sum_{j=1}^n k_j$ for all $n \geqslant 3$, and suppose that the set*

$$\mathcal{I} = \{n \geqslant 3 \mid r(n) \text{ divides } M(n)\}$$

is infinite. Suppose that $M \to \infty$ and $r^4 k_{\max}^3 = o(M)$ as n tends to infinity along elements of $\mathcal{I}$. Then

$$|\mathcal{H}_r(\boldsymbol{k})| = \frac{M!}{(M/r)!\,(r!)^{M/r}\,\prod_{i=1}^n k_i!}\; \exp\left(-\frac{(r-1)M_2}{2M} + O(r^4\, k_{\max}^3/M)\right).$$

Proof. As noted earlier, the argument leading to (2.3) is still valid when $r^4 k_{\max}^3 = o(M)$. Since $r \geqslant 3$, it follows from (2.3) that $|\mathcal{B}_r^{(0)}(\boldsymbol{k})|/|\mathcal{B}_r(\boldsymbol{k})| = 1 + O(r^4 k_{\max}^3/M)$. Combining this with (2.1) and (2.2) completes the proof. $\qquad\square$

3 Double links

For nonnegative integers d, let $\mathcal{C}_d$ be the set of bipartite graphs in $\mathcal{B}_r^+(\boldsymbol{k})$ which contain precisely d 4-cycles. (The corresponding hypergraph has exactly d double links.) The sets $\mathcal{C}_d$ partition $\mathcal{B}_r^+(\boldsymbol{k})$, and so

$$|\mathcal{B}_r^+(\boldsymbol{k})| = \sum_{d=0}^{N_2} |\mathcal{C}_d|. \tag{3.1}$$

We estimate this sum using a switching operation which we now define.

An 8-tuple of distinct vertices $T = (u_1, u_2, w_1, w_2, f_1, f_2, g_1, g_2)$ is called *suitable* if

$$u_1, u_2, w_1, w_2 \in \{v_1, \ldots, v_n\} \quad \text{and} \quad f_1, f_2, g_1, g_2 \in \{e_1, \ldots, e_{M/r}\}.$$

A *d-switching* from $B \in \mathcal{C}_d$ is described by a suitable 8-tuple T of vertices of B such that

- B has a 4-cycle on $\{u_1, u_2\} \cup \{f_1, f_2\}$,

- $w_1 g_1$ and $w_2 g_2$ are edges in B.

The corresponding d-switching produces a new bipartite graph B' with the same vertex set as B and with edge set

$$E(B') = (E(B) \setminus \{u_1 f_1, \, u_2 f_2, \, w_1 g_1, \, w_2 g_2\}) \cup \{u_1 g_1, \, u_2 g_2, \, w_1 f_1, \, w_2 f_2\}. \tag{3.2}$$

The d-switching operation is illustrated in Figure 3 below. (Note that in the hypergraph setting, the d-switching replaces the four edges f_1, f_2, g_1, g_2 of the original hypergraph with the edges f_1', f_2', g_1', g_2' defined by

$$f_j' = (f_j \setminus \{u_j\}) \cup \{w_j\}, \quad g_j' = (g_j \setminus \{w_j\}) \cup \{u_j\}$$

for $j = 1, 2$.)

We say that a d-switching from $B \in \mathcal{C}_d$ specified by the (suitable) 8-tuple T is *legal* if the resulting bipartite graph B' belongs to $\mathcal{C}_{d-1}$, and otherwise we say that the switching is *illegal*. Let $\mathrm{dist}_{\widehat{B}}(x, y)$ denote the length of the shortest path from x to y in a bipartite graph $\widehat{B}$.

Lemma 3.1. *Suppose that $d \leqslant N_2$ is a positive integer and $B \in \mathcal{C}_d$. With notation as above, if the d-switching from B specified by T is illegal then at least one of the following conditions must hold:*
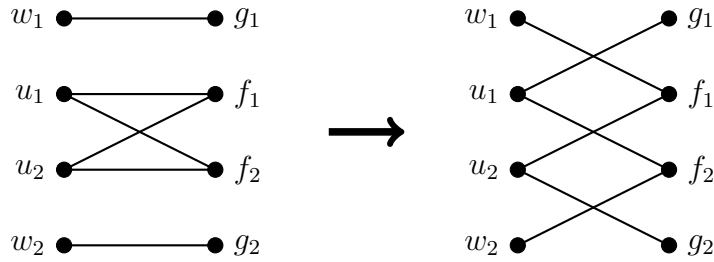
Figure 3: A d-switching

(I) *At least one of g_1 or g_2 belongs to a 4-cycle in B.*

(II) *For some $j \in \{1,2\}$, either $\operatorname{dist}_B(u_j, g_j) \leqslant 3$ or $\operatorname{dist}_B(w_j, f_j) \leqslant 3$.*

(III) $\operatorname{dist}_B(g_1, g_2) = 2$.

Proof. Fix $B \in \mathcal{C}_d$ and let T describe a d-switching from B such that the resulting bipartite graph B' does not belong to $\mathcal{C}_{d-1}$. First, suppose that $B' \in \mathcal{B}_r^+(\boldsymbol{k})$ but that B' contains at most $d - 2$ 4-cycles. Then the d-switching has destroyed more than one 4-cycle, which implies that (I) holds.

Next, suppose that there exists a 4-cycle in B' which contains an edge of $E(B') \setminus E(B)$. Call these *new edges*. Such a 4-cycle has been (accidently) created by the d-switching.

First suppose that B' contains a 4-cycle which involves precisely one new edge. If B' contains a 4-cycle which involves the edge $u_j g_j$ (for some $j \in \{1,2\}$) but does not involve any other new edge then $\operatorname{dist}_B(u_j, g_j) \leqslant 3$, which implies that (II) holds. Similarly, if B' contains a 4-cycle which contains the edge $w_j f_j$ for some $j \in \{1,2\}$, but contains no other new edge, then $\operatorname{dist}_B(w_j, f_j) \leqslant 3$, which again implies that (II) holds. Now suppose that there are 4-cycles in B' which contain at least two new edges. If a 4-cycle in B' contains both $w_1 f_1$ and $w_2 f_2$ then $w_1 f_2 \in E(B)$, which implies that (II) holds. If a 4-cycle in B' contains both $u_1 g_1$ and $u_2 g_2$ then $u_2 g_1 \in E(B)$, so (II) holds. No 4-cycle in B' can contain both $u_j g_j$ and $w_j f_j$ for some $j \in \{1,2\}$, since the edge $u_j f_j$ is not present in B'. Next, suppose that there is a 4-cycle in B' which contains both $u_j g_j$ and $w_\ell f_\ell$, where $\{j, \ell\} = \{1,2\}$. Then $w_\ell g_j \in E(B)$, so (III) holds.

The arguments given above cover the case that $B' \in \mathcal{B}_r^+(\boldsymbol{k})$ but that B' has strictly more than $d - 1$ 4-cycles, since the d-switching must have introduced at least one new 4-cycle. Furthermore, it follows from properties (i)–(v) that if $B' \notin \mathcal{B}_r^+(\boldsymbol{k})$ then there must be a 4-cycle in B' which contains a new edge. (For example, if (i) fails for B' then there is a copy of $K_{3,2}$ involving at least one new edge, but then that new edge is contained in at least one 4-cycle in B'.) Hence this case is also covered by the above arguments, completing the proof. $\square$

A *reverse d-switching* is the reverse of a d-switching. A reverse d-switching from a

bipartite graph $B' \in \mathcal{C}_{d-1}$ is described by a suitable 8-tuple T of vertices such that

$$u_1 g_1, \ u_2 g_2, \ u_1 f_2, \ u_2 f_1, \ w_1 f_1, \ w_2 f_2$$

are all edges of B'. The reverse d-switching produces the bipartite graph B defined by (3.2). This operation is depicted in Figure 3 by following the arrow in reverse.

Given $B' \in \mathcal{C}_{d-1}$, we say that a reverse d-switching from B' specified by the (suitable) 8-tuple T is *legal* if the resulting bipartite graph B belongs to $\mathcal{C}_d$, and otherwise we say that the switching is *illegal*.

The proof of the following is very similar to the proof of Lemma 3.1, but for completeness we give it in full.

Lemma 3.2. *Suppose that $d \leqslant N_2$ is a positive integer and $B' \in \mathcal{C}_{d-1}$. With notation as above, if the reverse switching from B' specified by T is illegal then at least one of the following conditions must hold:*

(I′) *At least one of $u_1, u_2, f_1, f_2, g_1, g_2$ belongs to a 4-cycle in B'.*

(II′) *For some $j \in \{1, 2\}$, either $\mathrm{dist}_{B'}(u_j, f_j) \leqslant 3$ or $\mathrm{dist}_{B'}(w_j, g_j) \leqslant 3$.*

Proof. Fix $B' \in \mathcal{C}_{d-1}$ and let T describe a reverse d-switching from B' such that the resulting bipartite graph B does not belong to $\mathcal{C}_d$. First, suppose that $B \in \mathcal{B}_r^+(\boldsymbol{k})$ but that B contains at most $d - 1$ 4-cycles. Then the reverse d-switching has destroyed at least one 4-cycle, so (I′) holds.

Clearly any new 4-cycle in B created by the reverse d-switching must contain at least one edge of $E(B) \setminus E(B')$. (Again, we call these *new edges*.) Of course, the reverse d-switching is designed to create a new 4-cycle involving the edges $u_1 f_1$, $u_1 f_2$, but here we are only interested in other 4-cycles which may "accidently" be created by the reverse d-switching.

If a 4-cycle in B contains precisely one new edge then (II′) holds in B'. Next suppose that a new 4-cycle in B contains at least two new edges. If any new 4-cycle in B contains both $w_1 g_1$ and $w_2 g_2$ then $w_1 g_2 \in E(B')$. This gives a 4-cycle in B' involving u_2, and so (II′) holds. No new 4-cycle in B can contain both $u_j f_j$ and $w_j g_j$, since $u_j g_j \notin E(B)$, for any $j \in \{1, 2\}$. Next, if $u_j f_j$ and $w_\ell g_\ell$ belong to a 4-cycle in B, where $\{j, \ell\} = \{1, 2\}$, then $u_j g_\ell \in E(B')$ and (II′) holds.

The above argument covers the possibility that $B' \in \mathcal{B}_r^+(\boldsymbol{k})$ but that B contains more than d 4-cycles. Now suppose that B contains precisely d 4-cycles but $B \notin \mathcal{B}_r^+(\boldsymbol{k})$. Note that property (v) holds, by our assumption on d. If property (i) or (ii) fails for B then at least one additional 4-cycle has been created by the reverse d-switching, which was covered by the above argument. If property (iii) fails for B then either f_1 or f_2 must belong to a 4-cycle in B', while if property (iv) fails for B then either u_1 or u_2 must belong to a 4-cycle in B'. Thus (I′) holds in both cases, completing the proof. $\qquad\square$

We will analyse d-switchings to obtain an asymptotic expression for $|\mathcal{C}_d|/|\mathcal{C}_{d-1}|$, and then combine these to find an expression for $|\mathcal{L}_r(\boldsymbol{k})| = |\mathcal{C}_0|/(M/r)!$, which is the quantity of interest. First we analyse one d-switching.

Lemma 3.3. *Assume that the conditions of Theorem 1.1 hold. Let d' be the first value of $d \leqslant N_2$ such that $\mathcal{C}_d = \varnothing$, or $d' = N_2 + 1$ if no such value exists. If $d \in \{1, \ldots, d'-1\}$ then*

$$|\mathcal{C}_d| = |\mathcal{C}_{d-1}| \frac{(r-1)^2 M_2^2}{4dM^2} \left(1 + O\left(\frac{dk_{\max}(k_{\max} + r) + r^2 k_{\max}^3}{M_2}\right)\right).$$

Proof. Fix $d \in \{1, \ldots, d'-1\}$ and let $B \in \mathcal{C}_d$ be given. Let $\mathcal{S}$ be the set of all suitable 8-tuples T such that

- B contains a 4-cycle on $\{u_1, u_2, f_1, f_2\}$,

- the edges $w_1 g_1$, $w_2 g_2$ belong to B, and

- no 4-cycle in B contains g_1 or g_2.

Then $\mathcal{S}$ contains every 8-tuple which defines a legal d-switching from B, so $|\mathcal{S}|$ is an upper bound for the number of legal d-switchings from B. There are precisely d 4-cycles, and 4 ways to order the vertices (u_1, u_2, f_1, f_2). For an upper bound, there are at most M^2 ways to choose the edges $w_1 g_1$, $w_2 g_2$ in order, giving $|\mathcal{S}| \leqslant 4dM^2$. To give a lower bound, we must ensure that all vertices are distinct and that g_1 and g_2 are not contained in any 4-cycle. Given (u_1, u_2, f_1, f_2), there are at least

$$(M - (2rd + 2k_{\max}))(M - ((2d+1)r + 3k_{\max}))$$

good choices for (w_1, w_2, g_1, g_2). Hence

$$|\mathcal{S}| = 4dM^2 \left(1 + O\left(\frac{rd + k_{\max}}{M}\right)\right).$$

We now find obtain an upper bound for the number of 8-tuples in $\mathcal{S}$ which give rise to illegal d-switchings from B, and subtract this value from $|\mathcal{S}|$. By Lemma 3.1 it suffices to find an upper bound for the number of 8-tuples in $\mathcal{S}$ which satisfy Condition (II) or Condition (III). Observe that no 8-tuple in $\mathcal{S}$ satisfies Condition (I), by definition of $\mathcal{S}$. For Condition (II), there are $O(drk_{\max}M)$ 8-tuples in $\mathcal{S}$ such that an edge exists in B from u_1 to g_j or from w_j to f_j, for some $j \in \{1, 2\}$. Similarly, there are $O(dr^2 k_{\max}^2 M)$ 8-tuples for which $\text{dist}_B(u_1, g_j) = 3$ or $\text{dist}_B(w_j, f_j) = 3$, for some $j \in \{1, 2\}$. Hence Condition (II) fails for $O(dr^2 k_{\max}^2 M)$ 8-tuples in $\mathcal{S}$.

Similarly, there are $O(dr^2 k_{\max}^2 M)$ 8-tuples in $\mathcal{S}$ which satisfy Condition (III). Combining these contributions, it follows that there are

$$4dM^2 \left(1 + O\left(\frac{rd + r^2 k_{\max}^2}{M}\right)\right) \tag{3.3}$$

suitable 8-tuples which give a legal d-switching from B.

Next, suppose that $B' \in \mathcal{C}_{d-1}$ (and note that $\mathcal{C}_{d-1}$ is nonempty, by definition of d'). Let $\mathcal{S}'$ be the set of all suitable 8-tuples such that

- $u_1 g_1$, $u_2 g_2$, $u_1 f_2$, $u_2 f_1$, $w_1 f_1$, $w_2 f_2$ are all edges of B', and

- no 4-cycle in B' contains a vertex from $\{u_1, u_2, f_1, f_2, g_1, g_2\}$.

Again, $\mathcal{S}'$ contains every 8-tuple which describes a legal reverse d-switching from B'. Hence there are at most $|\mathcal{S}'|$ legal reverse d-switchings from B'. There are at most M_2 ways to choose (u_1, f_2, g_1) and at most M_2 ways to choose (u_2, f_1, g_2), and then at most $(r-1)^2$ ways to choose (w_1, w_2). Therefore $|\mathcal{S}'| \leqslant (r-1)^2 M_2^2$.

For a lower bound, we must ensure that all vertices are distinct and that we avoid choosing $u_1, u_2, f_1, f_2, g_1, g_2$ from a 4-cycle. We can choose (u_1, f_2, g_1), avoiding vertices contained in 4-cycles, in at least $M_2 - 2(d-1)k_{\max}(k_{\max} + 2r)$ ways. There are still precisely $(r-1)$ choices for w_2 from among all neighbours of f_2 other than u_1. Next, there are at least

$$M_2 - 2(d-1)k_{\max}(k_{\max} + 2r) - 3k_{\max}^2 - 4rk_{\max} - 2rk_{\max}^2$$

ways to choose (u_2, f_1, g_2) avoiding vertices contained in 4-cycles and avoiding those vertices already chosen, such that f_1 is not a neighbour of u_1 or w_2 in B'. This choice of f_1 ensures that all $r-1$ neighbours of f_1 other than u_2 are also distinct from $\{u_1, w_2\}$, so there are still $r-1$ choices for w_2. It follows that

$$|\mathcal{S}'| = (r-1)^2 M_2^2 \left(1 + O\left(\frac{dk_{\max}(k_{\max} + r) + rk_{\max}^2}{M_2} \right) \right).$$

Now we calculate an upper bound for the number of 8-tuples in $\mathcal{S}'$ which give an illegal reverse d-switching from B'. By Lemma 3.2, it suffices to find an upper bound for the number of 8-tuples in $\mathcal{S}'$ which satisfy Condition (II'). (Note that no element of $\mathcal{S}'$ can satisfy Condition (I'), by definition of $\mathcal{S}'$.) There are $O(r^3 k_{\max}^2 M_2)$ elements of $\mathcal{S}'$ such that there is an edge from w_j to g_j or an edge from u_j to f_j, for some $j \in \{1, 2\}$. Similarly, the number of 8-tuples in $\mathcal{S}'$ with $\operatorname{dist}_{B'}(u_j, f_j) = 3$ or with $\operatorname{dist}_{B'}(w_j, g_j) = 3$ for some $j \in \{1, 2\}$ is $O(r^4 k_{\max}^3 M_2)$. Hence Condition (II') fails for $O(r^4 k_{\max}^3 M_2)$ 8-tuples, which (together with the upper bound on $|\mathcal{S}'|$) implies that the number of legal reverse d-switchings from B' is

$$(r-1)^2 M_2^2 \left(1 + O\left(\frac{dk_{\max}(k_{\max} + r) + r^2 k_{\max}^3}{M_2} \right) \right). \tag{3.4}$$

Comparing the error terms from (3.3) and (3.4), we see that the error term from the reverse d-switchings is largest, since $1/M \leqslant k_{\max}/M_2$. Taking the ratio of (3.3) and (3.4) completes the proof. $\qquad \square$

We can now prove our main result. The proof is similar to those in related enumeration results such as [9]. We present the proof in full in order to demonstrate how the factors of r arise in the error bounds (since previous results only dealt with $r = 2$, or assumed that r was constant). The following summation lemma from [10] will be needed. (The statement has been adapted slightly from that given in [10], without affecting the proof given there.)

Lemma 3.4 ([10, Corollary 4.5]). *Let $N \geqslant 2$ be an integer and, for $1 \leqslant i \leqslant N$, let real numbers $A(i)$, $C(i)$ be given such that $A(i) \geqslant 0$ and $A(i) - (i-1)C(i) \geqslant 0$. Define $A_1 = \min_{i=1,\ldots,N} A(i)$, $A_2 = \max_{i=1,\ldots,N} A(i)$, $C_1 = \min_{i=1,\ldots,N} C(i)$ and $C_2 = \max_{i=1,\ldots,N} C(i)$. Suppose that there exists a real number $\hat{c}$ with $0 < \hat{c} < \frac{1}{3}$ such that $\max\{A_2/N, |C_1|, |C_2|\} \leqslant \hat{c}$. Define $n_0, \ldots, n_N$ by $n_0 = 1$ and*

$$n_i = \frac{1}{i}\left(A(i) - (i-1)C(i)\right) n_{i-1}$$

for $1 \leqslant i \leqslant N$. Then

$$\Sigma_1 \leqslant \sum_{i=0}^{N} n_i \leqslant \Sigma_2,$$

where

$$\Sigma_1 = \exp\left(A_1 - \tfrac{1}{2}A_1 C_2\right) - (2e\hat{c})^N,$$
$$\Sigma_2 = \exp\left(A_2 - \tfrac{1}{2}A_2 C_1 + \tfrac{1}{2}A_2 C_1^2\right) + (2e\hat{c})^N.$$

Proof of Theorem 1.1. First we prove that

$$\sum_{d=0}^{N_2} |\mathcal{C}_d| = |\mathcal{C}_0| \exp\left(\frac{(r-1)^2 M_2^2}{4M^2} + O\left(\frac{r^4 k_{\max}^4 (k_{\max} + r)}{M}\right)\right). \tag{3.5}$$

Let d' be the first value of $d \leqslant N_2$ for which $\mathcal{C}_d = \varnothing$, or $d = N_2 + 1$ if no such value of d exists. We saw in Lemma 3.3 that any $B \in \mathcal{C}_d$ can be converted to some $B' \in \mathcal{C}_{d-1}$ using a d-switching. Hence $\mathcal{C}_d = \varnothing$ for $d' \leqslant d \leqslant N_2$. In particular, (3.5) holds if $\mathcal{C}_0 = \varnothing$, so we assume that $d' \geqslant 1$.

By Lemma 3.3, there is some uniformly bounded function α_d such that

$$\frac{|\mathcal{C}_d|}{|\mathcal{C}_0|} = \frac{1}{d} \frac{|\mathcal{C}_{d-1}|}{|\mathcal{C}_0|} \left(A(d) - (d-1)C(d)\right) \tag{3.6}$$

for $1 \leqslant d \leqslant N_2$, where

$$A(d) = \frac{(r-1)^2 M_2^2 - \alpha_d \, r^4 k_{\max}^3 \, M_2}{4M^2}, \quad C(d) = \frac{\alpha_d \, r^2 k_{\max}(k_{\max} + r) \, M_2}{4M^2}$$

for $1 \leqslant d < d'$, and $A(d) = C(d) = 0$ for $d' \leqslant d \leqslant N_2$.

We wish to apply Lemma 3.4. It is clear that $A(d) - (d-1)C(d) \geqslant 0$, from (3.6) if $1 \leqslant d < d'$ or by definition, if $d' \leqslant d \leqslant N_2$. If $\alpha_d \geqslant 0$ then $A(d) \geqslant A(d) - (d-1)C(d) \geqslant 0$ by (3.6), while if $\alpha_d < 0$ then $A(d)$ is nonnegative by definition. Now define A_1, A_2, C_1, C_2 by taking the minimum and maximum of $A(d)$ and $C(d)$ over $1 \leqslant d \leqslant N_2$. Let $A \in [A_1, A_2]$ and $C \in [C_1, C_2]$ and set $\hat{c} = \frac{1}{20}$. Since $A = (r-1)^2 M_2^2/4M^2 + o(1)$ and $C = o(1)$, we

have that $\max\{A/N_2, |C|\} \leqslant \hat{c}$ for M sufficiently large, by the definition of N_2. Hence Lemma 3.4 applies and gives an upper bound

$$\sum_{d=0}^{N_2} \frac{|\mathcal{C}_d|}{|\mathcal{C}_0|} \leqslant \exp\left(\frac{(r-1)^2 M_2^2}{4M^2} + O\left(\frac{r^4 k_{\max}^4 (k_{\max} + r)}{M}\right)\right) + O\left((e/10)^{N_2}\right).$$

Since $(e/10)^{N_2} \leqslant (e/10)^{3\log M} \leqslant M^{-1}$, this gives

$$\sum_{d=0}^{N_2} \frac{|\mathcal{C}_d|}{|\mathcal{C}_0|} \leqslant \exp\left(\frac{(r-1)^2 M_2^2}{4M^2} + O\left(\frac{r^4 k_{\max}^4 (k_{\max} + r)}{M}\right)\right).$$

In the case that $d' = N_2 + 1$, the lower bound given by Lemma 3.4 is the same within the stated error term, which establishes (3.5) in this case.

This leaves the case that $1 \leqslant d' \leqslant N_2$. Considering the analysis of the reverse switchings from Lemma 3.3, this case can only arise if

$$M_2 = O(d' k_{\max}(k_{\max} + r) + r^2 k_{\max}^3) = O\left(k_{\max}(k_{\max} + r)(r^2 k_{\max}^2 + \log M)\right).$$

But then

$$\frac{(r-1)^2 M_2^2}{4M^2} = O\left(\frac{r^2 k_{\max}^2 (k_{\max} + r)^2 (r^2 k_{\max}^2 + \log M)^2}{M^2}\right) = O\left(\frac{r^4 k_{\max}^4}{M}\right),$$

so the trivial lower bound of 1 matches the upper bound within the error term. Hence (3.5) also holds when $1 \leqslant d' \leqslant N_2$.

Therefore (3.5) holds in both cases. Combining (2.1), Theorem 2.2 and (3.1) gives

$$|\mathcal{L}_r(\boldsymbol{k})|$$
$$= \frac{|\mathcal{C}_0|}{(M/r)!}$$
$$= \frac{|\mathcal{B}_r^+(\boldsymbol{k})|}{(M/r)!} \exp\left(-\frac{(r-1)^2 M_2^2}{4M^2} + O\left(\frac{r^4 k_{\max}^4 (k_{\max} + r)}{M}\right)\right)$$
$$= \frac{M!}{(M/r)!\,(r!)^{M/r} \prod_{j=1}^{n} k_j!} \exp\left(-\frac{(r-1)M_2}{2M} - \frac{(r-1)^2 M_2^2}{4M^2} + O\left(\frac{r^4 k_{\max}^4 (k_{\max} + r)}{M}\right)\right),$$

completing the proof. $\qquad\square$

As a corollary of Theorem 2.2 and (3.5) we obtain the following result regarding the girth of bipartite graphs.

Corollary 3.5. *Under the conditions of Theorem 1.1, the probability that a randomly chosen element of $\mathcal{B}_r(\boldsymbol{k})$ has no 4-cycle, and hence has girth at least 6, is*

$$\frac{|\mathcal{C}_0|}{|\mathcal{B}_r(\boldsymbol{k})|} = \exp\left(-\frac{(r-1)^2 M_2^2}{4M^2} + O\left(\frac{r^4 k_{\max}^4 (k_{\max} + r)}{M}\right)\right).$$

McKay, Wormald and Wysocka [18, Corollary 3] proved the following: if $(d-1)^7 = o(n)$ as $n \to \infty$ along the positive even integers then that the probability that a random d-regular bipartite graph on n vertices has girth greater than g is

$$\exp\left(-\frac{(d-1)^4}{4} + o(1)\right).$$

(The conclusion was known much earlier for constant d; see [20].) Corollary 3.5 can be seen as a generalisation of the $g = 4$ case of [18, Corollary 3] to bipartite graphs which are irregular on one side of the vertex bipartition, and are sufficiently sparse. When the bipartite graph is d-regular (with $k_{\max} = r = d$), the condition of Corollary 3.5 becomes $d^8 = o(n)$, which is slightly more restrictive than that of [18].

Acknowledgements

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