

Mixed volumes of hypersimplices

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Abstract

In this paper we consider mixed volumes of combinations of hypersimplices. These numbers, called “mixed Eulerian numbers”, were first considered by A. Postnikov and were shown to satisfy many properties related to Eulerian numbers, Catalan numbers, binomial coefficients, etc. We give a general combinatorial interpretation for mixed Eulerian numbers and prove the above properties combinatorially. In particular, we show that each mixed Eulerian number enumerates a certain set of permutations in S_n . We also prove several new properties of mixed Eulerian numbers using our methods. Finally, we consider a type B analogue of mixed Eulerian numbers and give an analogous combinatorial interpretation for these numbers.

1 Introduction

For integers $1 \leq k \leq n$, the *hypersimplex* $\Delta_{k,n} \subset \mathbb{R}^{n+1}$ is the convex hull of all points of the form

$$e_{i_1} + e_{i_2} + \cdots + e_{i_k}$$

where $1 \leq i_1 < i_2 < \cdots < i_k \leq n+1$ and e_i is the i -th standard basis vector. Thus, $\Delta_{k,n}$ is an n -dimensional polytope which lies in the hyperplane $x_1 + \cdots + x_{n+1} = k$. Given a polytope $P \subset \mathbb{R}^{n+1}$ which lies in a hyperplane $x_1 + \cdots + x_{n+1} = \alpha$ for some $\alpha \in \mathbb{R}$, we define its (normalized) volume $\text{Vol } P$ to be the usual n -dimensional volume of the projection of P onto the first n coordinates. It is a classical result (usually attributed to Laplace [4]) that

$$n! \text{Vol } \Delta_{k,n} = A(n, k),$$

where the *Eulerian number* $A(n, k)$ is the number of permutations on n letters with exactly $k - 1$ descents.

We now define the mixed volume of a set of polytopes. Given a polytope P and a real number $\lambda \geq 0$, let $\lambda P = \{\lambda x \mid x \in P\}$. Given polytopes $P_1, \dots, P_m \subset \mathbb{R}^n$, let their *Minkowski sum* be

$$P_1 + \dots + P_m = \{x_1 + \dots + x_m \mid x_i \in P_i \text{ for all } i\}.$$

For nonnegative real numbers $\lambda_1, \dots, \lambda_m$, the function

$$f(\lambda_1, \dots, \lambda_m) = \text{Vol}(\lambda_1 P_1 + \dots + \lambda_m P_m)$$

is known to be a homogeneous polynomial of degree n in the variables $\lambda_1, \dots, \lambda_m$. Hence there is a unique symmetric function Vol defined on n -tuples of polytopes in \mathbb{R}^n such that

$$f(\lambda_1, \dots, \lambda_m) = \sum_{i_1, \dots, i_n=1}^m \text{Vol}(P_{i_1}, \dots, P_{i_n}) \lambda_{i_1} \dots \lambda_{i_n}.$$

The number $\text{Vol}(P_1, \dots, P_n)$ is called the *mixed volume* of P_1, \dots, P_n . Mixed volumes of lattice polytopes have important connections to algebraic geometry, where they count the number of solutions to generic systems of polynomial equations; see [1]. If $P_1 = \dots = P_n = P$, then $\text{Vol}(P_1, \dots, P_n)$ equals the ordinary volume $\text{Vol}(P)$. If $P_1, \dots, P_m \subset \mathbb{R}^{n+1}$ and each P_i lies in a hyperplane $x_1 + \dots + x_{n+1} = \alpha_i$ for some $\alpha_i \in \mathbb{R}$, then we define the mixed volume $\text{Vol}(P_1, \dots, P_n)$ in terms of the normalized volume defined previously.

Let c_1, c_2, \dots, c_n be nonnegative integers such that $c_1 + \dots + c_n = n$. We define

$$A_{c_1, \dots, c_n} = n! \text{Vol}(\Delta_{1,n}^{c_1}, \Delta_{2,n}^{c_2}, \dots, \Delta_{n,n}^{c_n})$$

where $(\Delta_{1,n}^{c_1}, \Delta_{2,n}^{c_2}, \dots, \Delta_{n,n}^{c_n})$ denotes the n -tuple with c_1 entries $\Delta_{1,n}$, c_2 entries $\Delta_{2,n}$, and so on. The numbers A_{c_1, \dots, c_n} are called *mixed Eulerian numbers*, and were introduced by Postnikov in [6].

As with ordinary volumes of hypersimplices, mixed volumes of hypersimplices appear to satisfy certain combinatorial identities. It is immediate that $A_{0^{k-1}, n, 0^{n-k}} = A(n, k)$, where 0^l denotes l entries 0. Furthermore, the result of Ehrenborg, Readdy, and Steingrímsson [3] states that

$$A_{0^{k-2}, r, n-r, 0^{n-k}}$$

equals the number of permutations $w \in S_{n+1}$ with $k-1$ descents and $w_1 = r+1$. Other properties are listed in Theorem 4.1 and include

$$A_{1, \dots, 1} = n! \quad A_{k, 0, \dots, 0, n-k} = \binom{n}{k} \quad A_{c_1, \dots, c_n} = 1^{c_1} 2^{c_2} \dots n^{c_n} \text{ if } c_1 + \dots + c_i \geq i \text{ for all } i.$$

These results were proven in [6] using algebraic and geometric methods. Additional formulas involving mixed Eulerian numbers and their generalizations to other root systems were derived by Croitoru in [2].

In this paper, the main result is a general combinatorial interpretation for the mixed Eulerian numbers which encompasses the previous results. In particular, we show that

each mixed Eulerian number enumerates a certain well-defined set of permutations in S_n . (When $c_k = n$ and $c_i = 0$ for all $i \neq k$, this set is precisely the set of permutations with $k - 1$ descents.) We show how the above results arise from this result. We also derive some new identities which follow from this interpretation. For example, we show that $A_{c_1, \dots, c_n} \leq 1^{c_1} 2^{c_2} \dots n^{c_n}$ for every mixed Eulerian number. We also show that

$$A_{n-m, 0^{k-3}, r, m-r, 0^{n-k}} = \sum_{i=0}^{n-m} \binom{m+i}{m} A(m, k-i; r) \quad (1)$$

where $A(n, k; r)$ equals the number of permutations $w \in S_{n+1}$ with $k - 1$ descents and $w_1 = r + 1$. This generalizes the result of Ehrenborg, Readdy, and Steingrímsson. The left hand side of (1) with $r = 0$ also appeared in the work of Michałek et. al. [5] during their study of exponential families arising from elementary symmetric polynomials. The authors used the recursions of [2] to obtain the formula

$$A_{n-m, 0^{k-2}, m, 0^{n-k}} = \begin{cases} \sum_{i=0}^{n-k} (n-k+1-i) \binom{n-i}{n-m} k^i A(m-i-1, m-n+k-1) & \text{if } n-m < k-1 \\ k^m & \text{otherwise.} \end{cases}$$

As a secondary result, we define the polytope $\Gamma_{k,n} \subset \mathbb{R}^n$ to be the convex hull of all points of the form

$$\pm e_{i_1} \pm e_{i_2} \pm \dots \pm e_{i_k}$$

where $1 \leq i_1 < \dots < i_k \leq n$. For nonnegative integers c_1, \dots, c_n such that $c_1 + \dots + c_n = n$, define

$$B_{c_1, \dots, c_n} = n! \text{Vol}(\Gamma_{1,n}^{c_1}, \Gamma_{2,n}^{c_2}, \dots, \Gamma_{n,n}^{c_n}).$$

We call the B_{c_1, \dots, c_n} the *type B mixed Eulerian numbers*, whereas the A_{c_1, \dots, c_n} are *type A mixed Eulerian numbers*. We give a combinatorial interpretation for the B_{c_1, \dots, c_n} analogous to that of the A_{c_1, \dots, c_n} and list several identities that follow from this interpretation.

2 Permutohedra and signed permutohedra

We first introduce two polytopes which will be used later in our proofs. Let y_1, \dots, y_{n+1} be real numbers. The *permutohedron* $P(y_1, \dots, y_{n+1})$ is the convex hull of the $(n+1)!$ points of the form $(y_{w(1)}, \dots, y_{w(n+1)})$, where $w \in S_{n+1}$ is a permutation. For example, $\Delta_{k,n} = P(1^k, 0^{n+1-k})$. The permutohedron is an n -dimensional polytope lying in the hyperplane $x_1 + \dots + x_{n+1} = y_1 + \dots + y_{n+1}$.

We have the following characterizations of $P(y_1, \dots, y_{n+1})$; see, for example, [6].

Proposition 2.1. *Let $y_1 \geq \dots \geq y_{n+1}$ be real numbers. Then $P(y_1, \dots, y_{n+1})$ is the set of points $(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ such that for all $1 \leq k \leq n$ and all k -element subsets $\{i_1, \dots, i_k\} \subset \{1, \dots, n+1\}$, we have*

$$x_{i_1} + \dots + x_{i_k} \leq y_1 + \dots + y_k,$$

and

$$x_1 + \cdots + x_{n+1} = y_1 + \cdots + y_{n+1}.$$

Proposition 2.2. *For nonnegative real numbers $\lambda_1, \dots, \lambda_n$, we have*

$$\lambda_1 \Delta_{1,n} + \lambda_2 \Delta_{2,n} + \cdots + \lambda_n \Delta_{n,n} = P(\lambda_1 + \cdots + \lambda_n, \lambda_2 + \cdots + \lambda_n, \dots, \lambda_n, 0).$$

Alternatively, if $y_1 \geq \cdots \geq y_{n+1}$ are real numbers, then $P(y_1, \dots, y_{n+1})$ is a translation by $(y_{n+1}, \dots, y_{n+1})$ of

$$(y_1 - y_2) \Delta_{1,n} + (y_2 - y_3) \Delta_{2,n} + \cdots + (y_n - y_{n+1}) \Delta_{n,n}.$$

Now let y_1, \dots, y_n be real numbers, and define the *signed permutohedron* $SP(y_1, \dots, y_n)$ to be the convex hull of the $2^n n!$ points of the form $(\pm y_{w(1)}, \dots, \pm y_{w(n)})$, where $w \in S_n$ is a permutation. For example, $\Gamma_{k,n} = SP(1^k, 0^{n-k})$. The signed permutohedron is an n -dimensional polytope lying in \mathbb{R}^n .

We have the following characterizations of $SP(y_1, \dots, y_n)$.

Proposition 2.3. *Let $y_1 \geq \cdots \geq y_n \geq 0$ be real numbers. Then $SP(y_1, \dots, y_n)$ is the set of points $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that for all $1 \leq k \leq n$ and all k -element subsets $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, we have*

$$|x_{i_1}| + \cdots + |x_{i_k}| \leq y_1 + \cdots + y_k.$$

Proposition 2.4. *For nonnegative real numbers $\lambda_1, \dots, \lambda_n$, we have*

$$\lambda_1 \Gamma_{1,n} + \lambda_2 \Gamma_{2,n} + \cdots + \lambda_n \Gamma_{n,n} = SP(\lambda_1 + \cdots + \lambda_n, \lambda_2 + \cdots + \lambda_n, \dots, \lambda_n).$$

Alternatively, for real numbers $y_1 \geq \cdots \geq y_n \geq 0$, we have

$$SP(y_1, \dots, y_n) = (y_1 - y_2) \Gamma_{1,n} + (y_2 - y_3) \Gamma_{2,n} + \cdots + (y_{n-1} - y_n) \Gamma_{n-1,n} + y_n \Gamma_{n,n}.$$

3 The main theorem

3.1 C-permutations

Let n be a positive integer, and let S be a totally ordered set with $|S| = n$. Let $C = (C_1, \dots, C_n)$ be a sequence of n pairwise disjoint sets such that

- $C_1 \cup \cdots \cup C_n = S$, and
- $s < t$ whenever $s \in C_i$, $t \in C_j$, and $i < j$.

We will call such a C a *division* of S . Let $|C|$ denote the sequence $(|C_1|, \dots, |C_n|)$.

We say that an element $s \in S$ is *admissible* with respect to C if either s is the smallest element of C_1 , s is the largest element of C_n , or $s \in C_i$ for $i \neq 1, n$. Given an admissible element s , we define the *deletion* of s from C as follows. Let i be such that $s \in C_i$, and let $C_i^- = \{t \in C_i \mid t < s\}$ and $C_i^+ = \{t \in C_i \mid t > s\}$. The deletion of admissible s from C results in a sequence of $n - 1$ sets, denoted by $C^s = (C_1^s, \dots, C_{n-1}^s)$, given as follows:

- If $i = 1$, then $C^s = (C_1^+ \cup C_2, C_3, \dots, C_n)$.
- If $i \neq 1, n$, then $C^s = (C_1, \dots, C_{i-2}, C_{i-1} \cup C_i^-, C_i^+ \cup C_{i+1}, C_{i+2}, \dots, C_n)$.
- If $i = n$, then $C^s = (C_1, \dots, C_{n-2}, C_{n-1} \cup C_n^-)$.

In any case, C^s is a division of $S \setminus \{s\}$.

Suppose $s_1 \in S$ is admissible with respect to C , $s_2 \in S \setminus \{s_1\}$ is admissible with respect to C^{s_1} , $s_3 \in S \setminus \{s_1, s_2\}$ is admissible with respect to $(C^{s_1})^{s_2}$, and so on until s_i . Then we say that the sequence $s_1 s_2 \dots s_i$ is *admissible* with respect to C and write $((C^{s_1})^{s_2} \dots)^{s_i} = C^{s_1 \dots s_i}$. If a permutation $s_1 \dots s_n$ of S is admissible with respect to C , then we call it a C -permutation. Note that the number of C -permutations depends only on $|C|$.

Example 3.1. Suppose $n = 5$ and $C = (\{1\}, \emptyset, \{2, 3\}, \{4\}, \{5\})$. The element 2 is admissible with respect to C , and $C^2 = (\{1\}, \emptyset, \{3, 4\}, \{5\})$. The element 3 is admissible with respect to C^2 , and $C^{23} = (\{1\}, \emptyset, \{4, 5\})$. The element 1 is admissible with respect to C^{23} , and $C^{231} = (\emptyset, \{4, 5\})$. The element 5 is admissible with respect to C^{231} , and $C^{2315} = (\{4\})$. The element 4 is admissible with respect to C^{2315} . Hence 23154 is a C -permutation. The construction of this permutation is visualized below.

$$\begin{array}{cccccc}
 1 & \emptyset & 23 & 4 & 5 \\
 & 1 & \emptyset & 34 & 5 \\
 & & 1 & \emptyset & 45 \\
 & & & \emptyset & 45 \\
 & & & & 4
 \end{array}$$

On the other hand, 23145 is not a C -permutation because 4 is not admissible with respect to $C^{231} = (\emptyset, \{4, 5\})$.

Example 3.2. Suppose $C = (\{1, \dots, n\}, \emptyset, \dots, \emptyset)$. The only element admissible with respect to C is 1, and $C^1 = (\{2, \dots, n\}, \emptyset, \dots, \emptyset)$. The only element admissible with respect to C^1 is 2, and so on. Thus the only C -permutation is $12 \dots n$.

Similarly, if $C = (\emptyset, \dots, \emptyset, \{1, \dots, n\})$, then the only C -permutation is $n(n-1) \dots 1$.

Example 3.3. Suppose C is a division of S and $|C| = (1, \dots, 1)$. Then every element of S is admissible with respect to C . Moreover, for any element $s \in S$, C^s satisfies $|C^s| = (1, \dots, 1)$. So by induction, every permutation of S is a C -permutation.

Example 3.4. Let C be a division of the form $C = (C_1, \emptyset, \dots, \emptyset, C_n)$. Then the only admissible elements with respect to C are the first element of C_1 and the last element of C_n . Furthermore, when we delete either of these elements, the resulting sequence of sets is again of the form $(C'_1, \emptyset, \dots, \emptyset, C'_{n-1})$. So when we construct a C -permutation by successively deleting admissible elements, at each step we delete either the first element of the first set or the last element of the last set. Thus the C -permutations are the permutations where the elements of C_1 appear in ascending order and the elements of C_n appear in descending order.

Example 3.5. We will see from Corollary 4.7 that if $C = (\emptyset^{k-1}, \{1, \dots, n\}, \emptyset^{n-k})$, then a permutation $w \in S_n$ is a C -permutation if and only if it has $k - 1$ descents.

We now state our main result.

Theorem 3.6. *The number of C -permutations is $A_{|C|}$.*

Proof. Let

$$\begin{aligned} f_n(\lambda_1, \dots, \lambda_n) &= \text{Vol}(\lambda_1 \Delta_{1,n} + \lambda_2 \Delta_{2,n} + \dots + \lambda_n \Delta_{n,n}) \\ &= \sum_{c_1 + \dots + c_n = n} \frac{1}{c_1! \dots c_n!} A_{c_1, \dots, c_n} \lambda_1^{c_1} \dots \lambda_n^{c_n} \end{aligned}$$

so that

$$A_{c_1, \dots, c_n} = \partial_1^{c_1} \dots \partial_n^{c_n} f_n.$$

The idea of the proof is to write a recursive formula for f_n . To do this, we make the following observation:

Proposition 3.7. *Let $y_1 \geq \dots \geq y_{n+1}$ be real numbers, and let $P = P(y_1, \dots, y_{n+1})$. Fix a real number $y_{n+1} \leq x \leq y_1$, and let P_x denote the cross section of P with the first coordinate equal to x . Let $1 \leq i \leq n$ be such that $y_{i+1} \leq x \leq y_i$. Then P_x is equal to*

$$\{x\} \times P(y_1, \dots, y_{i-1}, y_i + y_{i+1} - x, y_{i+2}, \dots, y_{n+1}).$$

Proof. By Proposition 2.1, P_x is the set of points $(x, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ such that for all $1 \leq k \leq n - 1$ and k -element subsets $\{i_1, \dots, i_k\} \subset \{2, \dots, n + 1\}$, we have

$$x_{i_1} + \dots + x_{i_k} \leq \min(y_1 + \dots + y_k, y_1 + \dots + y_{k+1} - x)$$

and

$$x_2 + \dots + x_{n+1} = y_1 + \dots + y_{n+1} - x.$$

We have $y_1 + \dots + y_k \leq y_1 + \dots + y_{k+1} - x$ if and only if $x \leq y_{k+1}$. Hence, P_x is the set of points $(x, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ such that for all $1 \leq k \leq n - 1$ and k -element subsets $\{i_1, \dots, i_k\} \subset \{2, \dots, n + 1\}$, we have

$$\begin{aligned} x_{i_1} + \dots + x_{i_k} &\leq y_1 + \dots + y_k && \text{if } x \leq y_{k+1} \\ x_{i_1} + \dots + x_{i_k} &\leq y_1 + \dots + y_{k+1} - x && \text{if } x \geq y_{k+1} \end{aligned}$$

and

$$x_2 + \dots + x_{n+1} = y_1 + \dots + y_{n+1} - x.$$

By Proposition 2.1, this is precisely the description of

$$\{x\} \times P(y_1, \dots, y_{i-1}, y_i + y_{i+1} - x, y_{i+2}, \dots, y_{n+1}),$$

as desired. □

Corollary 3.8. Let $\lambda_1, \dots, \lambda_n$ be nonnegative real numbers. Fix a real number $0 \leq x \leq \lambda_1 + \dots + \lambda_n$, and let $1 \leq i \leq n$ be such that $\lambda_{i+1} + \dots + \lambda_n \leq x \leq \lambda_i + \dots + \lambda_n$ (where $0 \leq x \leq \lambda_n$ if $i = n$). Set $t = \lambda_i + \dots + \lambda_n - x$. Then the cross section of

$$\lambda_1 \Delta_{1,n} + \lambda_2 \Delta_{2,n} + \dots + \lambda_n \Delta_{n,n}$$

with first coordinate equal to x is equal to $\{x\} \times Q$, where Q is the following polytope in the following cases:

- If $i = 1$,

$$(t + \lambda_2) \Delta_{1,n-1} + \lambda_3 \Delta_{2,n-1} + \dots + \lambda_n \Delta_{n-1,n-1}.$$

- If $2 \leq i \leq n-1$,

$$\begin{aligned} & \lambda_1 \Delta_{1,n-1} + \dots + \lambda_{i-2} \Delta_{i-2,n-1} + (\lambda_{i-1} + \lambda_i - t) \Delta_{i-1,n-1} \\ & + (t + \lambda_{i+1}) \Delta_{i,n-1} + \lambda_{i+2} \Delta_{i+1,n-1} + \dots + \lambda_n \Delta_{n-1,n-1}. \end{aligned}$$

- If $i = n$,

$$\lambda_1 \Delta_{1,n-1} + \dots + \lambda_{n-2} \Delta_{n-2,n-1} + (\lambda_{n-1} + \lambda_n - t) \Delta_{n-1,n-1}.$$

Proof. This follows by translating Proposition 3.7 through Proposition 2.2. \square

Corollary 3.8 now gives the following formula for f_n :

Proposition 3.9. We have

$$\begin{aligned} f_n(\lambda_1, \dots, \lambda_n) &= \int_0^{\lambda_1} f_{n-1}(t + \lambda_2, \lambda_3, \dots, \lambda_n) dt \\ &+ \sum_{i=2}^{n-1} \int_0^{\lambda_i} f_{n-1}(\lambda_1, \dots, \lambda_{i-2}, \lambda_{i-1} + \lambda_i - t, t + \lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_n) dt \\ &+ \int_0^{\lambda_n} f_{n-1}(\lambda_1, \dots, \lambda_{n-2}, \lambda_{n-1} + \lambda_n - t) dt. \end{aligned}$$

Now, we wish to use this formula to calculate $\partial_1^{c_1} \dots \partial_n^{c_n} f_n$. We use the “differentiation under the integral” rule: For smooth functions $u(x)$ and $v(x, t)$, we have

$$\frac{d}{dx} \int_0^{u(x)} v(x, t) dt = u'(x) v(x, u(x)) + \int_0^{u(x)} \frac{\partial}{\partial x} v(x, t) dt.$$

It follows that for $2 \leq i \leq n-1$, we have

$$\begin{aligned} & \left(\frac{\partial}{\partial \lambda_i} \right)^{c_i} \int_0^{\lambda_i} f_{n-1}(\lambda_1, \dots, \lambda_{i-1} + \lambda_i - t, t + \lambda_{i+1}, \dots, \lambda_n) dt \\ &= \sum_{r=0}^{c_i-1} \partial_{i-1}^r \partial_i^{c_i-r-1} f_{n-1}(\lambda_1, \dots, \lambda_i + \lambda_{i+1}, \dots, \lambda_n) \\ &+ \int_0^{\lambda_i} \partial_{i-1}^{c_i} f_{n-1}(\lambda_1, \dots, \lambda_{i-1} + \lambda_i - t, t + \lambda_{i+1}, \dots, \lambda_n) dt \quad (2) \end{aligned}$$

and hence

$$\begin{aligned}
\left(\frac{\partial}{\partial \lambda_1}\right)^{c_1} \cdots \left(\frac{\partial}{\partial \lambda_n}\right)^{c_n} \int_0^{\lambda_i} f_{n-1}(\lambda_1, \dots, \lambda_{i-1} + \lambda_i - t, t + \lambda_{i+1}, \dots, \lambda_n) dt \\
= \sum_{r=0}^{c_i-1} \partial_1^{c_1} \cdots \partial_{i-1}^{c_{i-1}} \partial_{i-1}^r \partial_i^{c_i-r-1} \partial_i^{c_{i+1}} \cdots \partial_{n-1}^{c_n} f_{n-1} \\
= \sum_{r=0}^{c_i-1} A_{c_1, \dots, c_{i-2}, c_{i-1}+r, c_i-r-1, c_{i+1}, c_{i+2}, \dots, c_n} \\
= \sum_{s \in C_i} A_{|C^s|}
\end{aligned}$$

where C is a division with $|C| = (c_1, \dots, c_n)$. Note that the final term of (2) vanishes after differentiation because f_{n-1} is a polynomial of degree $n-1$.

By similar (and simpler) calculations, we have

$$\begin{aligned}
\left(\frac{\partial}{\partial \lambda_1}\right)^{c_1} \cdots \left(\frac{\partial}{\partial \lambda_n}\right)^{c_n} \int_0^{\lambda_1} f_{n-1}(t + \lambda_2, \lambda_3, \dots, \lambda_n) dt &= A_{c_1+c_2-1, c_3, \dots, c_n} \\
&= A_{|C^1|}
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{\partial}{\partial \lambda_1}\right)^{c_1} \cdots \left(\frac{\partial}{\partial \lambda_n}\right)^{c_n} \int_0^{\lambda_n} f_{n-1}(\lambda_1, \dots, \lambda_{n-2}, \lambda_{n-1} + \lambda_n - t) dt &= A_{c_1, \dots, c_{n-1}+c_n-1} \\
&= A_{|C^n|}.
\end{aligned}$$

Combining these calculations with Proposition 3.9, we obtain

$$A_{c_1, \dots, c_n} = A_{|C^1|} + \sum_{i=2}^{n-1} \sum_{s \in C_i} A_{|C^s|} + A_{|C^n|}.$$

The desired result now follows by induction with the base case $A_1 = 1$. \square

While C -permutations are defined recursively in general, there are certain cases where more explicit descriptions can be given. This allows us to derive various formulas for mixed Eulerian numbers, which we do in Section 4.

3.2 Index functions and superdiagonality

We will also associate each C -permutation with a function which we call an “index function”. For some applications, this function will be more useful to work with than the permutation itself. This section will only be used in Sections 4.2 and 4.4 and can be skipped until then.

Let $C = (C_1, \dots, C_n)$ be a division of S and let $w = w_1 \dots w_n$ be a C -permutation. For each $1 \leq i \leq n$, the *index* of w_i in w with respect to C is the j such that $w_i \in C_j^{w_1 w_2 \dots w_{i-1}}$.

In other words, j is the index of the set containing w_i immediately before we delete w_i . Let $I_w^C : S \rightarrow \mathbb{N}$ be the function which takes each $s \in S$ to its index in w with respect to C . Note that if $s \in C_i$, then $I_w^C(s) \in \{1, \dots, i\}$. We will call *any* function $I : S \rightarrow \mathbb{N}$ which maps C_i into $\{1, \dots, i\}$ and *index function* of C .

Example 3.10. Let $C = (\{1\}, \emptyset, \{2, 3\}, \{4\}, \{5\})$ and $w = 23154$ as in Example 3.1. Then $I_w^C(2) = 3$, $I_w^C(3) = 3$, $I_w^C(1) = 1$, $I_w^C(5) = 2$, and $I_w^C(4) = 1$.

Example 3.11. Let $C = (\emptyset^{k-1}, \{1, \dots, n\}, \emptyset^{n-k})$ and let w be a C -permutation. By Corollary 4.7, we can uniquely write $w = \overline{w_1} \overline{w_2} \dots \overline{w_k}$ where each $\overline{w_i}$ is an increasing sequence and w is the concatenation of these sequences. Then by Proposition 4.8, if s is a term in $\overline{w_i}$, then $I_w^C(s) = k - i + 1$.

We introduce some final terminology. Call a division C *superdiagonal* if $|C_1| + \dots + |C_i| \geq i$ for all i . Call a division *subdiagonal* if $|C_n| + |C_{n-1}| + \dots + |C_{n-i+1}| \geq i$ for all i . We make the following observation, which is easy to check.

Proposition 3.12. *If C is a superdiagonal (resp., subdiagonal) division of S , then for any admissible $s \in S$, C^s is also superdiagonal (resp., subdiagonal).*

The following is the main result on index functions, which we prove in the next section.

Proposition 3.13. *Let $C = (C_1, \dots, C_n)$ be a division of S . Then the map $w \mapsto I_w^C$ is an injection from the set of C -permutations to the set of index functions of C . This map is a bijection if and only if C is superdiagonal.*

4 Properties of mixed Eulerian numbers

Our main application of C -permutations is to give simple combinatorial proofs of known properties of mixed Eulerian numbers, as well as prove some properties which were unknown before.

Using algebraic and geometric techniques, Postnikov proved the following facts about mixed Eulerian numbers.

Theorem 4.1 (Postnikov [6]). *The mixed Eulerian numbers have the following properties:*

- (a) *The numbers A_{c_1, \dots, c_n} are positive integers defined for $c_1, \dots, c_n \geq 0$, $c_1 + \dots + c_n = n$.*
- (b) *We have $A_{c_1, \dots, c_n} = A_{c_n, \dots, c_1}$.*
- (c) *For $1 \leq k \leq n$, the number $A_{0^{k-1}, n, 0^{n-k}}$ equals the usual Eulerian number $A(n, k)$. Here, 0^l denotes a sequence of l zeroes.*
- (d) *We have $\sum \frac{1}{c_1! \dots c_n!} A_{c_1, \dots, c_n} = (n+1)^{n-1}$, where the sum is over nonnegative integer sequences c_1, \dots, c_n with $c_1 + \dots + c_n = n$.*

- (e) We have $\sum A_{c_1, \dots, c_n} = n!C_n$, where the sum is over nonnegative integer sequences c_1, \dots, c_n with $c_1 + \dots + c_n = n$, and $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number.
- (f) For $2 \leq k \leq n$ and $0 \leq r \leq n$, the number $A_{0^{k-2}, r, n-r, 0^{n-k}}$ is equal to the number of permutations $w \in S_{n+1}$ with $k-1$ descents and $w_1 = r+1$.
- (g) We have $A_{1, \dots, 1} = n!$.
- (h) We have $A_{k, 0, \dots, 0, n-k} = \binom{n}{k}$.
- (i) We have $A_{c_1, \dots, c_n} = 1^{c_1} 2^{c_2} \dots n^{c_n}$ is $c_1 + \dots + c_i \geq i$ for all i .

Theorem 4.2 (Postnikov [6]). Let \sim denote the equivalence relation on the set of nonnegative integer sequences (c_1, \dots, c_n) with $c_1 + \dots + c_n = n$ given by $(c_1, \dots, c_n) \sim (c'_1, \dots, c'_n)$ whenever $(c_1, \dots, c_n, 0)$ is a cyclic shift of $(c'_1, \dots, c'_n, 0)$. Then for a fixed (c_1, \dots, c_n) , we have

$$\sum_{(c'_1, \dots, c'_n) \sim (c_1, \dots, c_n)} A_{c'_1, \dots, c'_n} = n!.$$

Note: There are exactly $C_n = \frac{1}{n+1} \binom{2n}{n}$ equivalence classes.

We now show how these properties arise from the combinatorial interpretation of mixed Eulerian numbers given by Theorem 3.6. We also give the following three additional properties.

Theorem 4.3. We have $A_{c_1, \dots, c_n} \leq 1^{c_1} 2^{c_2} \dots n^{c_n}$, with equality if and only if $c_1 + \dots + c_i \geq i$ for all i .

Theorem 4.4. Let c_1, \dots, c_n be nonnegative integers such that $c_1 + \dots + c_n = n$, and suppose there exists some $0 \leq r \leq n$ such that $c_1 + \dots + c_i \geq i$ for all $1 \leq i \leq r$ and $c_n + c_{n-1} + \dots + c_{n-i+1} \geq i$ for all $1 \leq i \leq n-r$. Then

$$A_{c_1, \dots, c_n} = \binom{n}{c_1 + \dots + c_r} 1^{c_1} 2^{c_2} \dots r^{c_r} 1^{c_n} 2^{c_{n-1}} \dots (n-r)^{c_{r+1}}.$$

Theorem 4.5. We have

$$A_{n-m, 0^{k-3}, r, m-r, 0^{n-k}} = \sum_{i=0}^{n-m} \binom{m+i}{m} A(m, k-i; r)$$

where $A(n, k; r)$ equals the number of permutations $w \in S_{n+1}$ with $k-1$ descents and $w_1 = r+1$. In particular,

$$A_{n-m, 0^{k-2}, m, 0^{n-k}} = \sum_{i=0}^{n-m} \binom{m+i}{m} A(m, k-i)$$

where $A(n, k)$ is defined to be 0 if $k \leq 0$ or $k > n$.

We do not have a combinatorial proof of Theorem 3.1(d), which was proven using the volume of the permutohedron.

The following subsections are mostly independent from each other and can be read in any order.

4.1 Proofs of Theorem 4.1

Property (a) is clear.

Property (b) follows from the fact that if w is a (C_1, \dots, C_n) -permutation, then w is also a (C_n, \dots, C_1) -permutation with the reverse ordering on $C_1 \cup \dots \cup C_n$.

Property (f), which is a generalization of property (c), follows from the following proposition.

Proposition 4.6. *Let $2 \leq k \leq n$ and $0 \leq r \leq n$. Let C be a division of S with $|C| = (0^{k-2}, r, n-r, 0^{n-k})$. Let λ be an element not in S such that $\lambda > s$ for all $s \in C_{k-1}$ and $\lambda < s$ for all $s \in C_k$. Then a permutation $w = w_1 \dots w_n$ of S is a C -permutation if and only if the sequence λ, w_1, \dots, w_n has $k-1$ descents.*

Proof. We induct on n . The argument below will work for $n = 2$ without assuming the inductive hypothesis, so we will have a base case. Assume without loss of generality that $S = \{1, \dots, n\}$. Assume $w = w_1 \dots w_n$ is a C -permutation. First suppose $w_1 \leq r$. If $k > 2$, then $|C^{w_1}| = (0^{k-3}, w_1 - 1, n - w_1, 0^{n-k})$. Since $w_2 \dots w_n$ is a C^{w_1} -permutation, the inductive hypothesis then implies that the sequence w_1, w_2, \dots, w_n has $k-2$ descents. If $k = 2$, then since $w_1 \leq r$ and w_1 is admissible with respect to C , we must have $w_1 = 1$ and $|C^{w_1}| = (n-1, 0^{n-2})$. Thus $w_2 \dots w_n = 2 \dots n$ (see Example 3.2), so $w_1 \dots w_n = 1 \dots n$. In either case, w_1, \dots, w_n has $k-2$ descents. Since $w_1 \leq r$, it follows that λ, w_1, \dots, w_n has $k-1$ descents, as desired. The argument for $w_1 > r$ follows analogously, with $k = n$ being the special case instead of $k = 2$.

Conversely, suppose $w = w_1 \dots w_n$ is a permutation of S such that λ, w_1, \dots, w_n has $k-1$ descents. First suppose $w_1 \leq r$. Hence w_1, w_2, \dots, w_n has $k-2$ descents. If $k > 2$, then w_1 is admissible with respect to C and $|C^{w_1}| = (0^{k-3}, w_1 - 1, n - w_1, 0^{n-k})$. The inductive hypothesis then implies that $w_2 \dots w_n$ is a C^{w_1} -permutation. If $k = 2$, then w_1, \dots, w_n has no descents, so $w = 1 \dots n$. It is easy to see that this is a C -permutation. In either case, we have that w is a C -permutation. The argument for $w_1 > r$ follows analogously. \square

Corollary 4.7. *Let $1 \leq k \leq n$ and let $C = (\emptyset^{k-1}, \{1, \dots, n\}, \emptyset^{n-k})$. Then a permutation $w \in S_n$ is a C -permutation if and only if it has $k-1$ descents.*

Proof. Take $r = 0$ or n in the previous Proposition. \square

We can also consider descents of “unfinished” permutations which are admissible with respect to C . The proof is similarly by induction; we omit it here.

Proposition 4.8. *Let C be a division with $|C| = (0^{k-1}, n, 0^{n-k})$. Suppose that the sequence $s_1 s_2 \dots s_i$ is admissible with respect to C . Let j be the index such that $s_i \in C_j^{s_1 \dots s_{i-1}}$. Then $s_1 s_2 \dots s_i$ has $k-j$ descents.*

Property (e) follows from Theorem 4.2, which is proven in section 4.3.

Property (g) follows from Example 3.3.

Property (h) follows from Example 3.4.

Property (i) is implied by Theorem 4.3, which we prove in the next section.

4.2 Proof of Theorem 4.3

It suffices to prove Proposition 3.13.

We first prove injectivity. Let $w = w_1 \dots w_n$ be a C -permutation, and set $I = I_w^C$. We wish to show that w is determined by I . It suffices to show that w_1 is determined by I . Indeed, if we prove this, then since $w_2 \dots w_n$ is a C^{w_1} -permutation, the same argument would imply that w_2 is determined by $I_{w_2 \dots w_n}^{C^{w_1}}$, and this function is determined as the restriction of I to $S \setminus \{w_1\}$. The terms w_3, w_4, \dots are determined similarly.

Let i_1 be such that $w_1 \in C_{i_1}$. Then $I(w_1) = i_1$. Let i be the largest number such that there exists some $s \in C_i$ with $I(s) = i$, and consider the smallest such s . By definition, $i_1 \leq i$. If $i_1 < i$, then after we delete w_1 from C we have $s \in C_{i-1}^{w_1}$. Hence $I(s) \leq i - 1$, contradicting the definition of s . So $i_1 = i$. Now if $w_1 > s$, then after we delete w_1 from C we obtain $s \in C_{i-1}^{w_1}$, again a contradiction. Hence $w_1 = s$. Thus w_1 is determined by I , as desired.

We now prove surjectivity in the case where C is superdiagonal. We induct on n . The case $n = 1$ is trivial. Suppose C is superdiagonal. Let I be an index function for C . We wish to construct a C -permutation w such that $I_w = I$. First note that $|C_1| \geq 1$, and any element $s \in C_1$ satisfies $I(s) = 1$. Thus we can let i be the largest number such that there exists some $s \in C_i$ with $I(s) = i$, and we consider the smallest such s . Since $|C_n| \leq 1$, it follows that s is admissible with respect to C . By Proposition 3.12, C^s is superdiagonal.

Let $I' : S \setminus \{s\} \rightarrow \mathbb{N}$ be the restriction of I to $S \setminus \{s\}$. We claim that I' is an index function of C^s . Indeed, let $s' \in S \setminus \{s\}$ and let i' be such that $s' \in C_{i'}^s$. We wish to prove $I'(s') \in \{1, \dots, i'\}$. We have either $s' \in C_{i'}$ or $s' \in C_{i'+1}$. In the first case, we are done since $I'(s') = I(s') \in \{1, \dots, i'\}$. In the second case, we must have either $i' + 1 > i$ or $i' + 1 = i$ and $s' < s$. By the definition of i and s , we must therefore have $I(s') \neq i' + 1$, and hence $I'(s') = I(s') \in \{1, \dots, i'\}$, as desired. Thus I' is an index function for C^s .

Since C^s is superdiagonal and I' is an index function for C^s , by the inductive hypothesis there exists a C^s -permutation w' such that $I_{w'}^{C^s} = I'$. Letting $w = sw'$, we have that $I_w^C = I$, as desired. This proves surjectivity.

Conversely, suppose C is not superdiagonal. The function $I : S \rightarrow \mathbb{N}$ with $I(s) = 1$ for all $s \in S$ is clearly an index function of C . Suppose there exists a C -permutation $w = s_1 \dots s_n$ with $I_w = I$. Thus when we successively delete s_1, \dots, s_n from C , we only ever delete from the first set in the current sequence. Hence we only ever delete the smallest remaining element.

Let i be the largest number such that $|C_1| + \dots + |C_i| < i$. Hence $i < n$ and C_{i+1} is nonempty. Let s be the smallest element of C_{i+1} . After deleting $s_1, \dots, s_{|C_1| + \dots + |C_i|}$ from C , the smallest remaining element is s . But $|C_1| + \dots + |C_i| < i$, so after the above deletions, s is not in the first set of the sequence. This contradicts $I(s) = 1$. So there is no w such that $I_w = I$, as desired. This proves Proposition 3.13.

4.3 Proof of Theorem 4.2

Let n be a positive integer and let $C = (C_1, \dots, C_n)$ be a division of $\{1, \dots, n\}$ with $|C| = (c_1, \dots, c_n)$. Set $C_{n+1} = \emptyset$.

We describe a process which is a cyclic version of the construction of C -permutations. Arrange the numbers $1, \dots, n$ around a circle \mathcal{C} clockwise in that order. We will define $n+1$ “blocks” as follows: for each $1 \leq i \leq n+1$, block B_i initially contains the elements of C_i . We view B_1, \dots, B_{n+1} as being arranged around \mathcal{C} in that order, including the empty blocks; i.e. B_i is viewed as being between B_{i-1} and B_{i+1} even if B_i is empty. For any element $s \in \{1, \dots, n\}$, we define the deletion of s from \mathcal{C} as follows. Suppose $s \in B_i$. Let B_i^- be the set of elements in B_i which are to the left of (counterclockwise from) s , and let B_i^+ be the set of elements in B_i which are to the right of (clockwise from) s . To delete s , we remove s and the block B_i from \mathcal{C} , put all the elements of B_i^- into the block to the left of B_i , and put all the elements of B_i^+ into the block to the right of B_i . The order of the undeleted elements remains unchanged. We can then delete another element, and so on. After we delete all n elements, we are left with only one block, which is empty. Since a nonempty block remains nonempty until it is deleted, this final empty block was originally empty and remained so throughout the process.

Let $w = w_1 \dots w_n \in S_n$ be a permutation. Let $r(w)$ be the r such that B_r is the final block that remains when we successively delete w_1, \dots, w_n from \mathcal{C} . It is not hard to see that for each r with $C_r = \emptyset$, the set of w such that $r(w) = r$ is precisely the set of $(C_{r+1}, C_{r+2}, \dots, C_{r-1})$ -permutations, where the indices of the C_i are taken modulo $n+1$ and the elements $\{1, \dots, n\}$ are ordered starting from the first element of C_{r+1} and going cyclically to the last element of C_{r-1} . There are $A_{c_{r+1}, \dots, c_{r-1}}$ such permutations. Hence we have

$$n! = \sum_{c_r=0} A_{c_{r+1}, c_{r+2}, \dots, c_{r-1}}$$

which is exactly what we wanted to prove.

4.4 Proof of Theorem 4.4

Note that the hypotheses on c_1, \dots, c_n imply that $c_1 + \dots + c_r = r$ and $c_{r+1} + \dots + c_n = n-r$. Let $C = (C_1, \dots, C_n)$ be a division with $|C| = (c_1, \dots, c_n)$. Let $S^- = C_1 \cup \dots \cup C_r$ and $S^+ = C_{r+1} \cup \dots \cup C_n$. Let $C^- = (C_1, \dots, C_r)$ and $C^+ = (C_{r+1}, \dots, C_n)$. Hence C^- and C^+ are divisions of S^- and S^+ , respectively, and C^- is superdiagonal and C^+ is subdiagonal. We write $C = (C^-, C^+)$ to indicate that C is the concatenation of the sequences C^-, C^+ .

Suppose $s \in S^-$ is admissible with respect to C . We claim that s is admissible respect to C^- and $C^s = ((C^-)^s, C^+)$. Indeed, this is clearly true if $s \in C_i$ for $i < r$, and it is true if $s \in C_r$ because $|C_r| \leq 1$. Similarly, if $s \in S^+$ is admissible with respect to C , then s is admissible with respect to C^+ and $C^s = (C^-, (C^+)^s)$. Moreover, by Proposition 3.12, C^- and C^+ remain superdiagonal and subdiagonal, respectively, after deleting elements. Hence, successively deleting elements from C is equivalent to successively deleting elements from C^- and C^+ . We can thus bijectively construct any C -permutation $s_1 \dots s_n$ by specifying a C^- -permutation, specifying a C^+ -permutation, and specifying the values of i for which s_i is an element of S^- . There are $\binom{n}{c_1 + \dots + c_r}$ ways to specify the values of i for which s_i is an element of S^- , and by Theorems 4.1(i) and (b), there are $1^{c_1} 2^{c_2} \dots r^{c_r}$ C^- -permutations and $1^{c_{r+1}} 2^{c_{r+2}} \dots (n-r)^{c_{r+1}}$ C^+ -permutations. This gives the desired result.

4.5 Proof of Theorem 4.5

We will in fact prove a more general identity. Fix a division C of S such that

$$|C| = (n - m, 0^{k-3}, r, m - r, 0^{n-k})$$

for some $0 \leq r \leq m \leq n$ and $3 \leq k \leq n$. Suppose s_1, s_2, \dots is a sequence of elements, not necessarily all in S . We call a term s_i of this sequence a C_1 -descent if either

- $s_i \in C_1$, or
- there exists $j > i$ such that $s_j \notin C_1$, $s_i > s_j$, and $s_k \in C_1$ for every $i < k < j$.

Note that if C_1 is empty, then a C_1 -descent is just an ordinary descent.

We can now state the result.

Proposition 4.9. *Let C be as above, and let $w_0 = \lambda$ be an element not in S such that $\lambda > s$ for all $s \in C_{k-1}$ and $\lambda < s$ for all $s \in C_k$. Then a permutation $w = w_1 \dots w_n$ of S is a C -permutation if and only if the sequence $w_0, w_1, w_2, \dots, w_n$ satisfies the following properties:*

- (a) *If $i < j$ and $w_i, w_j \in C_1$, then $w_i < w_j$.*
- (b) *The sequence has at least $k - 1$ C_1 -descents.*
- (c) *If w_i is the $(k - 1)$ -th C_1 -descent, then $w_{i+1}, w_{i+2}, \dots, w_n$ is an increasing sequence.*

Note that if $C_1 = \emptyset$, this proposition becomes Proposition 4.6.

Proof of Proposition 4.9. The proof is similar to that of Proposition 4.6. We induct on n . The below argument will work for $n = 3$ without the inductive hypothesis, so we will have a base case. Call a sequence (t, T) -good if it satisfies properties (a)–(c) with k replaced with t and C_1 replaced with T . Without loss of generality, assume $C_1 = \{1', 2', \dots, (n - m)'\}$ and $C_{k-1} \cup C_k = \{1, 2, \dots, m\}$, with the obvious ordering on these two sets.

Suppose w is a C -permutation. First suppose $w_1 \in C_1$. Then $w_1 = 1'$. If $k > 3$, then $|C^{w_1}| = (n - m - 1, 0^{k-4}, r, m - r, 0^{n-k})$. The inductive hypothesis then implies that the sequence λ, w_2, \dots, w_n is $(k - 1, C_1 \setminus \{1'\})$ -good. It then follows that $\lambda, 1', w_2, \dots, w_n$ is (k, C_1) -good, as desired. If $k = 3$, then $|C^{w_1}| = (n - m + r - 1, m - r, 0^{n-3})$. By Proposition 4.6, it follows that λ, w_2, \dots, w_n has 1 descent in the ordinary sense. It is easy to check that this implies $\lambda, 1', w_2, \dots, w_n$ is $(3, C_1)$ -good.

Now suppose $w_1 \in C_{k-1}$. If $k > 3$, then $|C^{w_1}| = (n - m, 0^{k-4}, w_1 - 1, m - w_1, 0^{n-k})$. By the inductive hypothesis, the sequence w_1, w_2, \dots, w_n is $(k - 1, C_1)$ -good. Since $\lambda > w_1$, it follows that λ, w_1, \dots, w_n is (k, C_1) -good, as desired. If $k = 3$, then $|C^{w_1}| = (n - m + w_1 - 1, m - w_1, 0^{n-k})$. Proposition 4.6 then implies that w_1, w_2, \dots, w_n has 1 descent in the ordinary sense. It is easy to check that this implies λ, w_1, \dots, w_n is $(3, C_1)$ -good.

Finally, suppose $w_1 \in C_k$. If $k < n$, the argument works similarly as in the previous paragraph. Suppose $k = n$. Then $w_1 = m$ and $|C^{w_1}| = (n - m, 0^{n-3}, m - 1)$. By Example 3.4, this implies that in the sequence w_2, \dots, w_n , the elements of $\{1', \dots, (n - m)'\}$ appear in ascending order and the elements $\{1, \dots, m - 1\}$ appear in descending order. Since $w_1 = m$, the same can be said of the sequence w_1, \dots, w_n . This implies that every term except the last term of this sequence is a C_1 -descent. Thus, the sequence is (n, C_1) -good. Since $\lambda < w_1$, the sequence λ, w_1, \dots, w_n is also (n, C_1) -good, as desired.

Conversely, suppose w is a permutation of S such that λ, w_1, \dots, w_n is (k, C_1) -good. First suppose $w_1 \in C_1$. By (a), we must have $w_1 = 1'$. Hence λ, w_2, \dots, w_n is $(k - 1, C_1 \setminus \{1'\})$ -good and w_1 is C -admissible. If $k > 3$, then $|C^{w_1}| = (n - m - 1, 0^{k-4}, r, m - r, 0^{n-k})$, so by the inductive hypothesis, $w_2 \dots w_n$ is a C^{w_1} -permutation. If $k = 3$, then $|C^{w_1}| = (n - m + r - 1, m - r, 0^{n-3})$. Since λ, w_2, \dots, w_n is $(2, C_1 \setminus \{1'\})$ -good, it has exactly 1 descent in the ordinary sense. So by Proposition 4.6, $w_2 \dots w_n$ is a C^{w_1} -permutation. Either way, w is a C -permutation, as desired.

Now suppose $w_1 \in C_{k-1}$. Then the sequence w_1, \dots, w_n is $(k - 1, C_1)$ -good, and w_1 is C -admissible. If $k > 3$, then $|C^{w_1}| = (n - m, 0^{k-4}, w_1 - 1, m - w_1, 0^{n-k})$. The inductive hypothesis then implies $w_2 \dots w_n$ is a C^{w_1} -permutation. If $k = 3$, then w_1, \dots, w_n has exactly one descent in the ordinary sense, and $|C^{w_1}| = (n - m + w_1 - 1, m - w_1, 0^{n-k})$. Proposition 4.6 then implies that $w_2 \dots w_n$ is a C^{w_1} -permutation. Either way, w is a C -permutation.

Finally, suppose $w_1 \in C_k$. If $k < n$, the argument works similarly as in the previous paragraph. Suppose $k = n$. Then the sequence λ, w_1, \dots, w_n has $n - 1$ C_1 -descents. But $\lambda < w_1$, so in this sequence the terms w_1, w_2, \dots, w_{n-1} must all be C_1 -descents. This implies that the elements of $\{1, \dots, m\}$ appear in this sequence in descending order. By (a), the elements of $\{1', \dots, (n - m)'\}$ appear in ascending order. It is easy to check that this implies w is a C -permutation, as desired. \square

We now want a way to enumerate the permutations from Proposition 4.9. Given a set S , define a \star -permutation of S to be a finite sequence $s_1 s_2 \dots$ consisting of elements of S and “ \star ” symbols such that every element of S appears exactly once. A \star -descent of a \star -permutation $s_1 s_2 \dots$ is an index i such that either $s_i = \star$ or there exists some $j > i$ with $s_i, s_j \in S$, $s_i > s_j$, and $s_k = \star$ for every $i < k < j$.

Proposition 4.10. *Let C be a division of S with $|C| = (n - m, 0^{k-3}, r, m - r, 0^{n-k})$, and let λ be a number such that $\lambda > s$ for all $s \in C_{k-1}$ and $\lambda < s$ for all $s \in C_k$. Then the C -permutations are in bijection with \star -permutations $s_1 s_2 \dots$ of $C_{k-1} \cup \{\lambda\} \cup C_k$ for which*

- $s_1 = \lambda$
- The number of \star 's is at most $n - m$.
- The number of \star -descents is equal to $k - 1$.

Proof. Suppose $s = s_1 s_2 \dots$ is a \star -permutation of $C_{k-1} \cup \{\lambda\} \cup C_k$ satisfying the above conditions. Let i be the $(k - 1)$ -th \star -descent of s . We obtain a C -permutation from

s as follows: Begin with the subsequence $s_2 \dots s_i$, and replace the first \star with the first element of C_1 , the second \star with the second element of C_1 , and so on, until all \star 's are replaced. Call the new sequence $w' = w_1 \dots w_{i-1}$. Append to the end of w' the elements of $S \setminus \{w_1, \dots, w_{i-1}\}$ in ascending order. The result is a C -permutation by Proposition 4.9.

Now suppose $w = w_1 \dots w_n$ is a C -permutation. Append λ to the beginning of this permutation, and replace all w_i for which $w_i \in C_1$ with \star 's. Call the resulting \star -permutation s' . Now, delete any \star 's in s' which occur after the $(k-1)$ -th \star -descent of s' . The result is a \star -permutation of $C_{k-1} \cup \{\lambda\} \cup C_k$ satisfying the desired conditions. \square

Corollary 4.11. *We have*

$$A_{n-m, 0^{k-3}, r, m-r, 0^{n-k}} = \sum_{i=0}^{n-m} \binom{m+i}{m} A(m, k-i; r)$$

where $A(n, k; r)$ equals the number of permutations $w \in S_{n+1}$ with $k-1$ descents and $w_1 = r+1$. In particular,

$$A_{n-m, 0^{k-2}, m, 0^{n-k}} = \sum_{i=0}^{n-m} \binom{m+i}{m} A(m, k-i)$$

where $A(n, k)$ is defined to be 0 if $k \leq 0$ or $k > n$.

5 Type B mixed Eulerian numbers

We now give an analogous combinatorial interpretation for the numbers B_{c_1, \dots, c_n} . Let $C = (C_1, \dots, C_n)$ be a division of a set S . We say that an element $s \in S$ is *type B admissible* with respect to C if either s is the smallest element of C_1 or $s \in C_i$ for $i \neq 1$. Given a type B admissible element s , we now define the *type B deletion* of s from C , which by abuse of notation we denote by C^s . Let i be such that $s \in C_i$. If $i \neq n$, then we define C^s to be the same as in the type A case. If $i = n$, then we define

$$C^s = (C_1, \dots, C_{n-2}, C_{n-1} \cup (C_n \setminus \{s\})).$$

Given these definitions of admissibility and deletion, we define a *type B C -permutation* analogously as in the type A case.

Recall that we defined

$$B_{c_1, \dots, c_n} = n! \text{Vol}(\Gamma_{1,n}^{c_1}, \dots, \Gamma_{n,n}^{c_n}).$$

Theorem 5.1. *Let C be a division. Then $B_{|C|}$ equals 2^n times the number of type B C -permutations.*

Proof. Since the proof is analogous to the type A case, we will give an outline and leave details to the reader. Define

$$\begin{aligned} f_n(\lambda_1, \dots, \lambda_n) &= \text{Vol}(\lambda_1 \Gamma_{1,n} + \lambda_2 \Gamma_{2,n} + \dots + \lambda_n \Gamma_{n,n}) \\ &= \sum_{c_1 + \dots + c_n = n} \frac{1}{c_1! \dots c_n!} B_{c_1, \dots, c_n} \lambda_1^{c_1} \dots \lambda_n^{c_n} \end{aligned}$$

so that

$$B_{c_1, \dots, c_n} = \partial_1^{c_1} \cdots \partial_n^{c_n} f_n.$$

We make the following observations, which are proven similarly to Proposition 3.7, Corollary 3.8, and Proposition 3.9.

Proposition 5.2. *Let $y_1 \geq \cdots \geq y_n \geq 0$ be real numbers, and let $SP = SP(y_1, \dots, y_n)$. Fix a real number $-y_1 \leq x \leq y_1$, and let SP_x denote the cross section of SP with first coordinate equal to x . Let $1 \leq i \leq n$ be such that $y_{i+1} \leq |x| \leq y_i$, where we set $y_{n+1} = 0$. Then SP_x is equal to*

$$\{x\} \times SP(y_1, \dots, y_{i-1}, y_i + y_{i+1} - |x|, y_{i+1}, \dots, y_n)$$

if $i \leq n-1$, and

$$\{x\} \times SP(y_1, \dots, y_{n-1})$$

if $i = n$.

Corollary 5.3. *Let $\lambda_1, \dots, \lambda_n$ be nonnegative real numbers. Fix a real number $-(\lambda_1 + \cdots + \lambda_n) \leq x \leq \lambda_1 + \cdots + \lambda_n$, and let $1 \leq i \leq n$ be such that $\lambda_{i+1} + \cdots + \lambda_n \leq |x| \leq \lambda_i + \cdots + \lambda_n$ (where $0 \leq |x| \leq \lambda_n$ if $i = n$). Set $t = \lambda_i + \cdots + \lambda_n - |x|$. Then the cross section of*

$$\lambda_1 \Gamma_{1,n} + \lambda_1 \Gamma_{2,n} + \cdots + \lambda_n \Gamma_{n,n}$$

with first coordinate equal to x is equal to $\{x\} \times Q$, where Q is the following polytope in the following cases:

- If $i = 1$,

$$(t + \lambda_2) \Gamma_{1,n-1} + \lambda_3 \Gamma_{2,n-1} + \cdots + \lambda_n \Gamma_{n-1,n-1}$$

- If $2 \leq i \leq n-1$,

$$\begin{aligned} \lambda_1 \Gamma_{1,n-1} + \cdots + \lambda_{i-2} \Gamma_{i-2,n-1} + (\lambda_{i-1} + \lambda_i - t) \Gamma_{i-1,n-1} \\ + (t + \lambda_{i+1}) \Gamma_{i,n-1} + \lambda_{i+2} \Gamma_{i+1,n-1} + \cdots + \lambda_n \Gamma_{n-1,n-1}, \end{aligned}$$

- If $i = n$,

$$\lambda_1 \Gamma_{i,n-1} + \cdots + \lambda_{n-2} \Gamma_{n-2,n-1} + (\lambda_{n-1} + \lambda_n) \Gamma_{n-1,n-1}.$$

Proposition 5.4. *We have*

$$\begin{aligned} f_n(\lambda_1, \dots, \lambda_n) &= 2 \int_0^{\lambda_1} f_{n-1}(t + \lambda_2, \lambda_3, \dots, \lambda_n) dt \\ &\quad + 2 \sum_{i=2}^{n-1} \int_0^{\lambda_i} f_{n-1}(\lambda_1, \dots, \lambda_{i-2}, \lambda_{i-1} + \lambda_i - t, t + \lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_n) dt \\ &\quad + 2 \int_0^{\lambda_n} f_{n-1}(\lambda_1, \dots, \lambda_{n-2}, \lambda_{n-1} + \lambda_n) dt. \end{aligned}$$

Differentiating this last equation, we obtain

$$B_{c_1, \dots, c_n} = 2 \left(B_{|C^1|} + \sum_{i=2}^{n-1} \sum_{s \in C_i} B_{|C^s|} + \sum_{s \in C_n} B_{|C^s|} \right)$$

where C is a division with $|C| = (c_1, \dots, c_n)$ and all deletions are type B deletions. The desired result follows by induction with the base case $B_1 = 2$. \square

Using Theorem 5.1, we obtain the following properties of type B mixed Eulerian numbers. The proofs are similar to the type A case and we omit them here.

Theorem 5.5. *The type B mixed Eulerian numbers have the following properties.*

- (a) *We have $2^n A_{c_1, \dots, c_n} \leq B_{c_1, \dots, c_n} \leq 2^n 1^{c_1} 2^{c_2} \dots n^{c_n}$. Each inequality is equality if and only if $c_1 + \dots + c_i \geq i$ for all i .*
- (b) *For $1 \leq k \leq n$, the number $B_{0^{k-1}, n, 0^{n-k}}$ is equal to 2^n times the number of permutations in S_n with at most $k-1$ descents.*
- (c) *For $1 \leq k \leq n-1$ and $0 \leq r \leq n$, the number $B_{0^{k-1}, r, n-r, 0^{n-k-1}}$ is equal to 2^n times the number of permutations $w \in S_{n+1}$ with at most k descents and $w_1 = r+1$.*
- (d) *We have $B_{1, \dots, 1} = 2^n n!$.*
- (e) *We have $B_{k, 0, \dots, 0, n-k} = \binom{n}{k} (n-k)!$.*
- (f) *We have $B_{c_1, \dots, c_n} = 2^n 1^{c_1} 2^{c_2} \dots n^{c_n}$ if $c_1 + \dots + c_i \geq i$ for all i .*
- (g) *We have $B_{c_1, \dots, c_n} = 2^n n!$ if $c_n + c_{n-1} + \dots + c_{n-i+1} \geq i$ for all i .*
- (h) *We have*

$$B_{c_1, \dots, c_n} = 2^n \binom{n}{c_1 + \dots + c_r} 1^{c_1} 2^{c_2} \dots r^{c_r} (c_{r+1} + \dots + c_n)!$$

if there exists some $0 \leq r \leq n$ such that $c_1 + \dots + c_i \geq i$ for all $1 \leq i \leq r$ and $c_n + c_{n-1} + \dots + c_{n-i+1} \geq i$ for all $1 \leq i \leq n-r$.

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