On the Cohen-Macaulay property for quadratic tangent cones

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Abstract

Let H be an n-generated numerical semigroup such that its tangent cone $\operatorname{gr}_{\mathfrak{m}} K[H]$ is defined by quadratic relations. We show that if $n < 5$ then $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Cohen-Macaulay, and for $n = 5$ we explicitly describe the semigroups H such that $\operatorname{gr}_{\mathfrak{m}} K[H]$ is not Cohen-Macaulay. As an application we show that if the field K is algebraically closed and of characteristic different from two, and $n \leqslant 5$ then $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Koszul if and only if (possibly after a change of coordinates) its defining ideal has a quadratic Gröbner basis.

Keywords: numerical semigroup ring, tangent cone, Cohen-Macaulay, Koszul, G-quadratic, h-vector

Introduction

A numerical semigroup H is a subset of $\mathbb{N}$ containing 0 and which is closed under addition such that the gcd of all elements in H is 1, or equivalently, such that $|\mathbb{N} \setminus H| < \infty$. We denote $Gen(H)$ its unique minimal generating set. The embedding dimension of H is defined as $\operatorname{emb}\dim(H) = |Gen(H)|$ and the multiplicity of H is $e(H) = \min Gen(H)$.

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Let K be an infinite field. The additive relations among the generators of H are captured by the defining ideal I_H of the semigroup ring $K[H] = K[t^h : h \in H] \subset K[t]$. Namely, if $Gen(H) = \{a_1, \ldots, a_n\}$ and we let $S = K[x_1, \ldots, x_n]$, then $I_H = \operatorname{Ker} \phi$, where $\phi : S \to K[H]$ is the K-algebra map with $\phi(x_i) = t^{a_i}$, for $1 \leqslant i \leqslant n$.

Another algebra that is associated to H is its tangent cone

$$\operatorname{gr}_{\mathfrak{m}} K[H] = \oplus_{i \geqslant 0} \mathfrak{m}^i / \mathfrak{m}^{i+1},$$

which is the associated graded ring of $K[H]$ with respect to the maximal ideal $\mathfrak{m} = (t^h : h \in H)$. The defining ideal of $\operatorname{gr}_{\mathfrak{m}} K[H]$ is I_H^*, the ideal of initial forms in I_H, see [13, §15.10.3] and [15, §3.4].

It is a classical topic to study algebraic properties of $\operatorname{gr}_{\mathfrak{m}} K[H]$ like being Cohen-Macaulay or complete intersection (CI for short) in terms of the arithmetic of H, see [18], [22], [3], [10].

Algebras defined by quadratic relations occur naturally in algebraic geometry from varieties cut out by quadrics and they have been the initial framework for formulating several strong conjectures, e.g. what is currently known as the Eisenbud-Green-Harris conjecture introduced in [14, Section 4].

In recent work ([19]), J. Herzog and the author gave effective bounds for the multiplicity of a numerical semigroup H such that $\operatorname{gr}_{\mathfrak{m}} K[H]$ is defined by quadrics. The motivation for the current paper came from the puzzling observation that all such numerical semigroups that we had obtained by blind computer search have the property that $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Cohen-Macaulay.

Koszul algebras are an important class of quadratic algebras. Recall that a graded K-algebra $R = \oplus_{i \geqslant 0} R_i$ is called Koszul if K has a graded R-linear resolution. R is called G-quadratic if there exists a graded isomorphism $R \cong K[x_1, \ldots, x_n]/I$ where I has a quadratic Gröbner basis with respect to some term order. It is well known that if R is G-quadratic, then it is Koszul. We refer to [7] and [15] for proofs and related results on Koszul algebras.

For brevity, we say that a numerical semigroup H is quadratic, Koszul, or G-quadratic, if $\operatorname{gr}_{\mathfrak{m}} K[H]$ has the respective property. Note that the quadratic property for $\operatorname{gr}_{\mathfrak{m}} K[H]$ depends on H alone (see [21, Theorem 6.8]), however the other two might depend on the field K. It will be clear from the context, mainly in Section 2, which are our extra assumptions on K.

Let H be a quadratic numerical semigroup. Using a criterion obtained independently by J. Herzog ([18]) and A. Garcia ([17]), and also our results from [19], we show in Proposition 5 that if $\operatorname{emb\,dim}(H) < 5$, then $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Cohen-Macaulay. It requires a bit more work to prove in Theorem 8 that if $\operatorname{emb\,dim}(H) = 5$, then $\operatorname{gr}_{\mathfrak{m}} K[H]$ is not Cohen-Macaulay precisely when H is generated as

$$\begin{gathered}\langle 8, 4u', 4u + 2u', 4u'' + 2u + u', 6u + 7u' + 4u'' - 8 \rangle, \text{ or} \\ \langle 8, 4u', 4u + 2u', 4u'' + 2u + 3u', 6u + 9u' + 4u'' - 8 \rangle,\end{gathered}$$

with u, u', u'' positive integers and $u' > 1$ is odd. Extending these exemples, in Proposition 7 for any $n > 5$ we construct infinitely many G-quadratic numerical semigroups H with

$\operatorname{emb\,dim}(H) = n$ and $\operatorname{gr}_{\mathfrak{m}} K[H]$ not Cohen-Macaulay. It would be interesting to figure out if such constructions cover all the cases when $\operatorname{gr}_{\mathfrak{m}} K[H]$ is quadratic and not Cohen-Macaulay.

In the terminology introduced by Rossi and Valla in [23], for H in the above families the ideals I_H provide first examples of 2-isomultiple ideals such that $\operatorname{gr}_{\mathfrak{m}} S/I_H$ is not Cohen-Macaulay. Their existence was also questioned in [23, Remark 2.3].

For a standard graded K-algebra $R = \oplus_{i\geqslant 0} R_i$ its h-polynomial $h(z) = \sum_{i\geqslant 0} h_i z^i$ is the numerator of the Hilbert series $H_R(z) = \sum_{i\geqslant 0} \dim_K R_i z^i$ when we write $H_R(z) = h(z)/(1-z)^d$ with $h(1) \neq 0$. The h-vector of R is the vector of coefficients $(h_0, h_1, \dots)$ of the h-polynomial. Also, the (Hilbert-Samuel) multiplicity of R is defined as $e(R) = h(1)$. It is known that for a numerical semigroup H its multiplicity equals the multiplicity of the tangent cone $\operatorname{gr}_{\mathfrak{m}} K[H]$.

By work of Backelin, Conca and others, small values of h_2 imply the Koszul or the G-quadratic property of R, see [1], [4], [6], [9] and Lemma 10. If R is Cohen-Macaulay and the field K is infinite, we can mod out by a regular sequence of linear forms and the h-vector and the multiplicity are preserved. In case H is a numerical semigroup and $R = \operatorname{gr}_{\mathfrak{m}} K[H]$ is Cohen-Macaulay, we may use $t^{e(H)}$ as a regular element.

As an application, in Section 2 we show that if $\operatorname{emb\,dim}(H) < 5$, then H is quadratic if and only if it is G-quadratic. The first examples of quadratic and non-Koszul semigroups occur in embedding dimension 5 having multiplicity 9, e.g. $H = \langle 9, 17, 20, 23, 25\rangle$.

In a similar way, in [24] Roos and Sturmfels considered the Koszul property for quadratic projective monomial curves. Namely, given the relatively prime integers $0 = a_1 < a_1 < \cdots < a_n$, let $R = K[t_1^{a_i} t_2^{a_n - a_i} : 1 \leqslant i \leqslant n] \subset K[t_1, t_2]$. According to Table 1 in [24] obtained by a computer search, the first time when R is quadratic and not Koszul is for $n = 6$, and for $n = 8$ occurs the first example where R is Koszul and the associated toric ideal has no quadratic Gröbner basis.

Under the assumption that the field K is algebraically closed and of characteristic $\neq 2$, we show in Theorem 12 that if $\operatorname{emb\,dim}(H) = 5$, then H is Koszul if and only if it is G-quadratic. The proof works on the possible h-vectors of $\operatorname{gr}_{\mathfrak{m}} K[H]$ when H is quadratic, employing a result of Eisenbud, Green, and Harris in [14]. The assumptions on the field K are due to Conca's results on the G-quadratic property for quadratic algebras with $h_2 \leqslant 3$, see [4] and [6]. Screening the possible ideals $J = I_H^* \bmod x_1$ we found only two possible situations without a quadratic Gröbner basis, described in Remark 15. However, experimentally we found no quadratic semigroup H producing such ideals.

We summarize our findings in Table 1 in Section 2 where we give a maximal list of 12 possible h-vectors of quadratic 5-generated numerical semigroups. Note that experimentally we could not obtain the h-vectors $(1,4,3,1)$ and $(1,4,5)$. Nevertheless, we can conclude that if H is quadratic and $\operatorname{emb\,dim}(H) \leqslant 5$, the Hilbert function of $\operatorname{gr}_{\mathfrak{m}} K[H]$ is non-decreasing. This topic has been recently considered by D'Anna, Di Marca and Micale in [11] and by Oneto, Strazzanti and Tamone in [20].

1 The Cohen-Macaulay condition

In this section we study the Cohen-Macaulay property for the tangent cone of a quadratic numerical semigroup.

For further reference we first recall from our joint work with J. Herzog [19] some restrictions that we found on the multiplicity of a quadratic numerical semigroup.

Theorem 1. *([19, 1.1, 1.9, 1.12]) Let H be a quadratic numerical semigroup minimally generated by $n > 1$ elements and $K[H]$ its semigroup ring. Then*

(i) $n \leqslant e(H) \leqslant 2^{n-1}$;

(ii) $e(H) = n \iff I_H^*$ *has a linear resolution;*

(iii) $e(H) = 2^{n-1} \iff I_H^*$ *is a CI ideal* $\iff I_H$ *is a CI ideal;*

(iv) if $\operatorname{gr}_{\mathfrak{m}} K[H]$ *is Cohen-Macaulay and* $e(H) < 2^{n-1}$, *then* $e(H) \leqslant 2^{n-1} - 2^{n-3}$.

Moreover, if we are in any of the situations from (ii), (iii) or if $\operatorname{gr}_{\mathfrak{m}} K[H]$ *is Cohen-Macaulay and* $e(H) = 2^{n-1} - 2^{n-3}$ *then H is G-quadratic, hence Koszul.*

Remark 2. With notation as above, if $e(H) = n$, then $\operatorname{gr}_{\mathfrak{m}} K[H]$ has minimal multiplicity and by Sally's [25, Theorem 2] we get that $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Cohen-Macaulay. We refer to the proof of Proposition 1.3 in [19] for related properties.

The following arithmetic result appeared in [19].

Lemma 3. *([19, Lemma 1.6]) Let H be a numerical semigroup minimally generated by $a_1 < a_2 < \cdots < a_n$ with $n > 1$. If H is quadratic, then*

(i) there exist $k, \ell \geqslant 2$ *such that* $a_1 | a_k + a_\ell$.

(ii) $2a_i \in \langle a_1, \ldots a_{i-1}, a_{i+1}, \ldots a_n \rangle$, *for all* $2 \leqslant i \leqslant n$.

For the numerical semigroup H minimally generated by $a_1 < \cdots < a_n$ we denote

$$c_i = \min\{k > 0 : ka_i \in \langle Gen(H) \setminus \{a_i\} \rangle\}, \text{ for } i = 1, \ldots, n.$$

With this notation one has the following characterization proved independently by Herzog ([18]) and Garcia ([17]).

Proposition 4. *(Herzog [18, pp.189-190], Garcia [17, Theorem 24]) The tangent cone* $\operatorname{gr}_{\mathfrak{m}} K[H]$ *is Cohen-Macaulay if and only if for all integers* $0 \leqslant \nu_i < c_i$ *and* $2 \leqslant i \leqslant n$ *such that*

$$\sum_{i=2}^{n} \nu_i a_i \in a_1 + H,$$

there exist integers $\mu_1 > 0, \mu_2 \geqslant 0, \ldots, \mu_n \geqslant 0$ *such that*

$$\sum_{i=2}^{n} \nu_i a_i = \sum_{i=1}^{n} \mu_i a_i \quad \text{and} \quad \sum_{i=2}^{n} \nu_i \leqslant \sum_{i=1}^{n} \mu_i.$$

If H is quadratic, by Lemma 3(ii) we have $c_i = 2$ for $i = 2, \ldots, n$. This observation, together with Proposition 4, gives the next result.

Proposition 5. *If H is a quadratic semigroup and* $\operatorname{emb\,dim}(H) < 5$ *then* $\operatorname{gr}_{\mathfrak{m}} K[H]$ *is Cohen-Macaulay.*

Proof. If $\operatorname{emb\,dim}(H) = 2$ then $H = \langle 2, \ell \rangle$ with $\ell > 1$ odd. Hence $\operatorname{gr}_{\mathfrak{m}} K[H] \cong K[x_1, x_2]/(x_2^2)$ which is Cohen-Macaulay.

If $\operatorname{emb\,dim}(H) = 3$, by Theorem 1(i) we have $3 \leqslant e(H) \leqslant 4$, and by (ii) and (iii) in loc.cit. $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Cohen-Macaulay.

If $\operatorname{emb\,dim}(H) = 4$ let

$$\nu_2 a_2 + \nu_3 a_3 + \nu_4 a_4 = \mu_1 a_1 + \mu_2 a_2 + \mu_3 a_3 + \mu_4 a_4$$

for some $\mu_1 > 0, \mu_2, \mu_3, \mu_4 \geqslant 0$ and $\nu_2, \nu_3, \nu_4 \in \{0, 1\}$. It is enough to consider the case when not both of ν_2 and μ_2, of ν_3 and μ_3, and of ν_4 and μ_4 are positive. Note that since $\operatorname{emb\,dim}(H) = 4$ at least two of the ν_i's need to be positive.

If $\nu_2 = \nu_3 = 1$ and $\nu_4 = 0$ then in the equation $a_2 + a_3 = \mu_1 a_1 + \mu_4 a_4$ we have either $\mu_4 > 0$, hence $\mu_1 + \mu_4 \geqslant 2 = \nu_2 + \nu_3$, or $\mu_4 = 0$, hence $a_2 + a_3 = \mu_1 a_1$ with $\mu_1 > 2$. The cases $\nu_3 = \nu_4 = 1, \nu_2 = 0$ and $\nu_2 = \nu_4 = 1, \nu_3 = 0$ are treated similarly.

If $\nu_2 = \nu_3 = \nu_4 = 1$ then in the equation $a_2 + a_3 + a_4 = \mu_1 a_1$ we have $\mu_1 > 3$.

By Proposition 4 it follows that $\operatorname{gr}_{\mathfrak{m}} K[H]$ is Cohen-Macaulay. $\square$

Example 6. The statement of Proposition 5 is no longer true when $\operatorname{emb\,dim}(H)$ is at least 5. We can check with Singular ([16]) that for $H = \langle 8, 12, 13, 18, 35 \rangle$ the ideal I_H^* has a quadratic Gröbner basis with respect to revlex, however $\operatorname{gr}_{\mathfrak{m}} K[H]$ is not Cohen-Macaulay.

Indeed, the toric ideal I_H is minimally generated by

$$\begin{aligned} I_H = (x_3^2 - x_1x_4,\ x_2x_4^2 - x_3x_5,\ x_2x_3x_4 - x_1x_5,\ x_2^3 - x_4^2, \\ x_1^3 - x_2^2,\ x_1^2x_3x_4 - x_2x_5,\ x_1^2x_4^3 - x_5^2,\ x_1^2x_2^2x_3 - x_4x_5). \end{aligned}$$

A revlex Gröbner basis for I_H^* is given by

$$I_H^* = (x_5^2, x_4x_5, x_3x_5, x_2x_5, x_1x_5, x_4^2, x_3^2 - x_1x_4, x_2^2),$$

and $(I_H^* : x_5) = (x_1, x_2, x_3, x_4, x_5)$, hence $\operatorname{depth}_{\mathfrak{m}} K[H] = 0$.

This is not an isolated example. For any given embedding dimension $n > 4$ we construct infinitely many G-quadratic numerical semigroups whose tangent cone is not Cohen-Macaulay. But first we recall a useful construction.

Let L be a numerical semigroup, ℓ an odd integer in L and $H = \langle 2L, \ell \rangle$. By [19, Definition 2.2], the semigroup H is called a quadratic gluing of L. It is proved in [19, Corollary 2.7] that L and H are quadratic, Koszul, respectively G-quadratic, at the same time. It is also known by Delorme's work [12] that if L is a complete intersection (CI), then so is H. We refer to Section 2 in [19] for more details about the CI property for quadratic numerical semigroups.

Proposition 7. *Given $n \geqslant 3$ and the positive integers u_i, $i = 1, \ldots, n+1$, where $u_1 \geqslant 3$ is odd, let*

$$\begin{aligned}
a_1 &= 2^n, \\
a_2 &= 2^{n-1}u_1, \\
a_3 &= 2^{n-1}u_2 + 2^{n-2}u_1, \\
\ldots & \\
a_{n+1} &= 2^{n-1}u_n + 2^{n-2}u_{n-1} + \cdots + u_1, \\
a_{n+2} &= a_2 + \cdots + a_{n+1} - a_1.
\end{aligned}$$

The semigroup $H = \langle a_1, \ldots, a_{n+2}\rangle$ is a G-quadratic numerical semigroup of embedding dimension $n+2$ and $\mathrm{gr}_{\mathfrak{m}} K[H]$ is not Cohen-Macaulay.

Letting $n = 3$, $u_1 = u_2 = 3$ and $u_3 = 1$ in the construction above, we obtain the quadratic semigroup $H = \langle 8, 12, 18, 13, 35\rangle$ from Example 6. Note that in Proposition 7 the listed generators a_i are not necessarily in increasing order, however we always have $a_1 < a_i < a_{n+2}$ for $2 \leqslant i \leqslant n+1$.

Proof. Denote $H_i = \langle a_1, \ldots, a_i\rangle$ for $1 \leqslant i \leqslant n+2$. It is easy to see that $\gcd(a_1, \ldots, a_i) = 2^{n-i+1}$ for $i = 1, \ldots, n+1$, hence H_{n+1} and H are numerical semigroups. From the defining relations we infer that

$$2a_{i+1} = 2^n u_i + a_i \quad \text{for } 2 \leqslant i \leqslant n, \tag{1}$$

hence the (not necessarily numerical) semigroups $H_1, \ldots, H_{n+1}$ are obtained by quadratic gluings, are CI and G-quadratic. Also, from the equations (1) we see that

$$I_{H_i} = (x_2^2 - x_1^{u_1}) + (x_{j+1}^2 - x_1^{u_j} x_j : 2 \leqslant j < i) \quad \text{for } 2 \leqslant i \leqslant n+1.$$

Next we compute I_H and I_H^*. We note that $a_{n+2} + a_1 = a_2 + \cdots + a_{n+1}$. For $2 \leqslant i \leqslant n+1$ using (1) repeatedly we get

$$\begin{aligned}
a_{n+2} + a_i =& a_2 + \cdots + a_{i-1} + 2a_i + \sum_{j=i+1}^{n+1} a_j - 2^n \\
=& a_2 + \cdots + a_{i-1} + (a_{i-1} + 2^n u_{i-1}) + \sum_{j=i+1}^{n+1} a_j - 2^n \\
=& a_2 + \cdots + a_{i-2} + (a_{i-2} + 2^n u_{i-2}) + 2^n u_{i-1} + \sum_{j=i+1}^{n+1} a_j - 2^n \\
& \ldots \\
=& 2^n (u_{i-1} + \cdots + u_1 - 1) + \sum_{j=i+1}^{n+1} a_j \\
=& (u_{i-1} + \cdots + u_1 - 1) a_1 + \sum_{j=i+1}^{n+1} a_j.
\end{aligned}$$

Arguing similarly we obtain

$$2a_{n+2} = (u_1 + \cdots + u_n - 2)a_1 + \sum_{i=2}^{n} a_i.$$

Each of these relations produces a binomial in I_H:

$$\begin{aligned} f_1 &= x_1 x_{n+2} - \prod_{j=2}^{n+1} x_j, \\ f_i &= x_i x_{n+2} - x_1^{(\sum_{j=1}^{i-1} u_j)-1} \prod_{j=i+1}^{n+1} x_j, \text{ for } 2 \leqslant i \leqslant n+1, \text{ and} \\ f_{n+2} &= x_{n+2}^2 - x_1^{(\sum_{j=1}^{n} u_j)-2} \prod_{j=2}^{n} x_j. \end{aligned}$$

By inspecting these relations we remark that we can always choose a generating set for I_H consisting of binomials such that in each monomial in the support, different from x_{n+2}^2, the variable x_{n+2} has degree at most one. Therefore, $I_H = (I_{H_{n+1}}, f_1, \ldots, f_{n+2})$.

Since $n \geqslant 3$ and $u_1 \geqslant 3$ it is easy to see that $f_i^* = x_i x_{n+2}$ for $i = 1, \ldots, n+2$. Arguing as above we derive

$$I_H^* = I_{H_{n+1}}^* + x_{n+2}(x_1, \ldots, x_{n+2}).$$

This gives $I_H^* : x_{n+2} = (x_1, \ldots, x_{n+2})$ and $\operatorname{depth} \operatorname{gr}_{\mathfrak{m}} K[H] = 0$, hence $R = \operatorname{gr}_{\mathfrak{m}} K[H]$ is not Cohen-Macaulay.

Since $x_{n+2}R_1 = 0$, by Conca's [4, Lemma 4.(1)] we have that R is G-quadratic if and only if

$$R/(t^{a_{n+2}}) \cong K[x_1, \ldots, x_{n+2}]/(I_H^*, x_{n+2}) \cong K[x_1, \ldots, x_{n+1}]/I_{H_{n+1}}^* \cong \operatorname{gr}_{\mathfrak{m}} K[H_{n+1}]$$

is G-quadratic, which is true since H_{n+1} is a quadratic CI, see Theorem 1. □

Our next goal is to identify the quadratic numerical semigroups H of embedding dimension 5 and $\operatorname{gr}_{\mathfrak{m}} K[H]$ not Cohen-Macaulay.

Theorem 8. *Let H be a quadratic numerical semigroup with $\operatorname{emb\,dim}(H) = 5$. Then $\operatorname{gr}_{\mathfrak{m}} K[H]$ is not Cohen-Macaulay if and only if H is of any of the following forms:*

(i) $H = \langle 8, 4u', 4u+2u', 4u''+2u+u', 6u+7u'+4u''-8 \rangle$ with u, u', u'' positive integers and $u' > 1$ is odd, or

(ii) $H = \langle 8, 4u', 4u+2u', 4u''+2u+3u', 6u+9u'+4u''-8 \rangle$ with u, u', u'' positive integers and $u' > 1$ is odd.

Whenever H is in any of these two families, it is also G-quadratic.

Proof. We first assume H is in any of the specified families and we show that $\mathrm{gr}_{\mathfrak{m}} K[H]$ is not Cohen-Macaulay. We label $a_1, \ldots, a_5$ the generators of H in the given ordering.

For (i) we are in the situation described in Proposition 7 for $n = 3$, $u_1 = u'$, $u_2 = u$ and $u_3 = u''$, hence the conclusion follows.

For (ii) we note that the semigroup $L = \langle a_1, a_2, a_3, a_4 \rangle = \langle 2\langle 4, 2u', 2u + u' \rangle, a_4 \rangle$ is obtained by a quadratic gluing since $a_4 = u'' \cdot 4 + (2u + u') + (2u')$ is odd. By [19, Proposition 3.6] the semigroup $\langle 4, 2u', 2u + u' \rangle$ is a quadratic complete intersection, and the same holds for L by Delorme's [12, Proposition 9] and by [19, Corollary 2.7].

It is straightforward to check that

$$\begin{aligned} a_5 + a_1 &= a_2 + a_3 + a_4, \\ a_5 + a_2 &= (u' - 1)a_1 + a_3 + a_4, \\ a_5 + a_3 &= (u + u' - 1)a_1 + a_4, \\ a_5 + a_4 &= (u + u'' - 1)a_1 + 3a_2, \\ 2a_5 &= (u + 2u' + u'' - 2)a_1 + a_3, \end{aligned}$$

hence $I_H^* : x_5 = (x_1, \ldots, x_5)$ and $\mathrm{gr}_{\mathfrak{m}} K[H]$ is not Cohen-Macaulay. Arguing as in the proof of Proposition 7 we get that H is G-quadratic.

The direct implication is proved separately in Section 3. □

2 Koszul and G-quadraticity

As an application of Theorem 8, under some restrictions on the field K, we prove that if H is a numerical semigroup and $\operatorname{emb\,dim}(H) \leqslant 5$, then H is Koszul if and only if it is G-quadratic. We wonder if this statement holds for arbitrary embedding dimension.

Let $R = \oplus_{i \geqslant 0} R_i$ be a standard graded K-algebra. A *Koszul filtration* for R is a family $\mathcal{F}$ of ideals of R generated by linear forms such that 0 and the maximal homogeneous ideal of R belong to $\mathcal{F}$ and for every $I \in \mathcal{F}$ different from 0, there exists $J \in \mathcal{F}$ such that $J \subset I$, I/J is cyclic and $J : I \in \mathcal{F}$. A Koszul filtration that is totally ordered with respect to inclusion is called a *Gröbner flag*. It is known that if R has a Koszul filtration, then it is Koszul. Also, by [5, Theorem 2.4], if R has a Gröbner flag, then R is G-quadratic. We refer to the original papers [8], [5] and to the recent survey [7] of Conca et al. for more properties.

For easier reference we group in the following lemma some known results about lifting Koszul-like properties modulo a linear form.

Lemma 9. *Let R be a standard graded K-algebra and x a linear form that is regular on R. If $R/(x)$ has property $(\mathcal{P})$, then so does R, where $\mathcal{P}$ stands for Koszul, G-quadratic, admits a Koszul filtration, or a Gröbner flag.*

Proof. The statements for Koszul and G–quadraticity are due to Backelin and Fröberg in [2, Lemma 2], respectively to Conca in [4, Lemma 4.(2)].

That any Gröbner flag may be lifted from $R/(x)$ to R is proved in [5, Lemma 2.11.(a)]. Using the same idea one can produce a Koszul filtration for R from a Koszul filtration of $R/(x)$. □

For R as above its h-polyonomial is defined as the numerator $h(t)$ in the writing of the Hilbert series $H_R(t) = \Sigma_{i\geqslant 0} \dim_K R_i t^i = h(t)/(1-t)^d$ with $h(1) \neq 0$. If $h(t) = \Sigma_{i\geqslant 0} h_i t^i$, the sequence of coefficients $(h_0, h_1, \dots)$ is called the h-vector of R.

It is clear that if x is a linear form which is regular on R, the h-polynomial and the h-vector of R and $R/(x)$ are the same. In this context, the Cohen-Macaulay property for R facilitates the computation of the h-vector of R by reduction to the artinian case.

For a quadratic (artinian) K-algebra the Koszul property or the existence of a quadratic Gröbner basis, or of a Gröbner flag may sometimes be deduced by inspecting the h-vector. We collect some results on this topic that we will use later on.

Lemma 10. *Let $R = \oplus_{i\geqslant 0} R_i$ be a quadratic standard graded K-algebra.*

(i) (Conca, Rossi, Valla, [5, Proposition 2.12]) If $H_R(t) = 1 + nt + t^2$ with $n > 1$, then R has a Gröbner flag.

(ii) (Backelin, [1, Theorem 4.8]) If $\dim_K R_2 \leqslant 2$, then R is Koszul.

Assume the field K is algebraically closed and of characteristic $\neq 2$.

(iii) (Conca, [4, Theorem 1]) If $\dim_K R_2 \leqslant 2$, then R is G-quadratic if and only if it is not graded isomorphic to the K-algebra $K[x, y, z]/(x^2, xy, y^2 - xz, yz)$ or its trivial fiber extentions.

(iv) (Conca, [6, Theorem 1.1]) If R is artinian and $\dim_K R_2 = 3$, then R is Koszul. Moreover, R is G-quadratic if and only if it is not a trivial fiber extension of $K[x, y, z]/I$, where I is a complete intersection of three quadrics not containing the square of a linear form.

It is now easier to prove the announced statement for the case when $\operatorname{emb}\dim(H)$ is less than 5.

Theorem 11. *Let H be a numerical semigroup. If $\operatorname{emb}\dim(H) < 5$, then H is quadratic if and only if H is G-quadratic.*

Proof. Denote $n = \operatorname{emb}\dim(H)$ and $R = \operatorname{gr}_{\mathfrak{m}} K[H]$. Assume H is quadratic. If $n = 2$ then R is a hypersurface ring, and we are done. If $n = 3$, then $e(H) \in \{3, 4\}$ and the result follows from Theorem 1.

Assume $n = 4$ and then R is Cohen-Macaulay by Proposition 5. Denote $\bar{R} = R/(x_1)$. Using Theorem 1 we get that $e(H) \in \{4, 5, 6, 8\}$. If $e(H) \neq 5$, then we are in one of the cases covered by Theorem 1 and the conclusion follows. If $e(H) = 5$, then $\ell(\bar{R}) = e(H)$ and $\bar{R}$ has the h-vector $(1, 3, 1)$. By Lemma 10(i) we obtain that $\bar{R}$ has a Gröbner flag which by Lemma 9 produces a Gröbner flag for R. Hence R is G-quadratic. $\square$

For a 5-generated quadratic numerical semigroup there are more possible multiplicities for which the Koszul property does not follow easily from Theorem 1. Our analysis depends on the possible h-polynomial of $R = \operatorname{gr}_{\mathfrak{m}} K[H]$ when H is quadratic. We combine the results listed in Lemma 10 and Theorem 8 with computer testing in Singular ([16])

for the remaining situations. Before giving the main result, we describe our screening strategy performed on the computer.

Working under the assumption that x_1 is regular on R, we analysed the possible quadratic ideals $J = I_H^* \bmod(x_1)$ in $K[x_2, x_3, x_4, x_5]$ generated by monomials and binomials and subject to some restrictions due to our setup. Choose $\mathcal{G}$ any minimal generating set for I_H consisting of binomials. If we denote $\mathcal{G}^*$ the collection of their initial forms, by [19, Lemma 1.5] $\mathcal{G}^*$ generates I_H^* minimally. Since x_1 is regular on R, the set $\mathcal{H}$ obtained by letting $x_1 = 0$ in $\mathcal{G}^*$ is a minimal generating set for J consisting of quadratic monomials and possibly of binomials.

Since the variables correspond to the generators of H taken in increasing order, there is only a short list of possible binomials in $\mathcal{H}$:

$$p_0 = x_2x_5 - x_3x_4, \quad p_1 = x_3^2 - x_2x_4, \quad p_2 = x_3^2 - x_2x_5,$$
$$p_3 = x_4^2 - x_2x_5, \quad p_4 = x_4^2 - x_3x_5.$$

Clearly p_1 and p_2 can not occur at the same time in $\mathcal{H}$, otherwise $p_1 - p_2 = x_2(x_4 - x_5) \in I_H$, hence $x_4 - x_5 \in I_H$, which is false. Similarly, p_3 and p_4 may not both occur $\mathcal{H}$. Hence at most 3 binomials may occur simultaneously in $\mathcal{H}$.

On the other hand, if p_0 and p_1 occur in $\mathcal{H}$, these occur in I_H, too. Hence $a_2 + a_5 = a_3 + a_4$ and $2a_3 = a_2 + a_4$. Adding these equations we get $a_3 + a_5 = 2a_4$, therefore $p_4 \in I_H$. We get that either $p_4 \in \mathcal{H}$, or that the monomials in its support are in $\mathcal{H}$.

Arguing similarly we see that if any two of p_0, p_1 and p_4 occur in $\mathcal{H}$, then the remaining one is in I_H and in J.

By Lemma 3 we see that $x_2^2, x_5^2 \in \mathcal{H}$. Also, if $x_3^2 \notin \mathcal{H}$ then either p_1 or p_2 is in $\mathcal{H}$. Similarly, if $x_4^2 \notin \mathcal{H}$, then either p_3 or p_4 is in $\mathcal{H}$.

Theorem 12. *Let H be a numerical semigroup with* $\operatorname{emb}\dim(H) = 5$.

If the field K is algebraically closed and of characteristic $\neq 2$, then H is Koszul if and only if it is G-quadratic.

Proof. By Theorem 8, if $R = S/I_H^* \cong \operatorname{gr}_{\mathfrak{m}} K[H]$ is not Cohen-Macaulay then it is G-quadratic. So it is enough to consider the case when x_1 is a nonzero divisor on R. For $i = 2, \dots, 5$, there exist distinct polynomials in I_H^* of the form $f_i = x_i^2 - g_i$, where g_i is either zero or a quadratic monomial which is not a pure power. Denoting by an overbar $^-$ the image in $\bar{R} = R/(x_1)$ and $<$ the revlex term order induced by $x_5 > x_4 > \dots$, we have $\operatorname{in}_<(\bar{f}_i) = x_i^2$ for $2 \leqslant i \leqslant 5$. Therefore the Hilbert series of the artinian graded algebras $\bar{R}$ and $K[x_2, \dots, x_5]/\operatorname{in}_<(J)$ coincide and moreover $\operatorname{in}_<(J)$ contains the squares of all the variables.

In this situation, as noted by Eisenbud, Green and Harris in [14, Section 4], for any m, if $h_m = \dim_K \bar{R}_m$ has the binomial decomposition

$$h_m = \binom{b_m}{m} + \binom{b_{m-1}}{m-1} + \dots + \binom{b_1}{1}$$

with $b_m > b_{m-1} > \dots > b_1 \geqslant 0$, then

$$h_{m+1} \leqslant \binom{b_m}{m+1} + \binom{b_{m-1}}{m} + \dots + \binom{b_1}{2}. \tag{2}$$

The h-vector of $\bar{R}$ is $(1,4,h_2,h_3,h_4)$. By (2) we have $0 \leqslant h_2 \leqslant \binom{4}{2} = 6$. Our analysis depends on the possible values for h_2.

- If $h_2 = 0$, then $\ell(\bar{R}) = e(H) = 5$, and by Theorem 1, H is G-quadratic.

- If $h_2 = 1$, then by (2) we get $h_3 = 0$, hence the h-vector of $\bar{R}$ is $(1,4,1)$. By Lemma 10(i) and Lemma 9 we get that $\bar{R}$ and R have a Gröbner flag, hence they are G-quadratic.

- If $h_2 = 2 = \binom{2}{2} + \binom{1}{1}$, from (2) we deduce that $h_3 = 0$. Since J is artinian and the field K is algebraically closed of characteristic $\neq 2$, by Lemma 10(iii) we get that $\bar{R}$, hence also R, are G-quadratic.

  Testing with Singular ([16]) the possible candidates for J, it is easy to check that all of them have a quadratic Gröbner basis with respect to revlex (usually induced by $x_2 > x_3 > x_4 > x_5$). All of them possess a Koszul filtration and in all but one situation presented in Remark 14 there exists a Gröbner flag with basis $\{x_2, x_3, x_4, x_5\}$.

- If $h_2 = 3 = \binom{3}{2}$, then $h_3 \leqslant 1$ and $h_4 = 0$. Note that J has at least two linearly independent squares of linear forms, namely x_2^2 and x_5^2. Under the assumption that K is algebraically closed and of characteristic $\neq 2$, by Lemma 10(iv) we infer that $\bar{R}$, hence also R, is G-quadratic.

  Scanning the possible candidates for J by the method described above it turns out that there always exists a Koszul filtration for $\bar{R}$, without any restriction on the field K. In most cases this filtration is a Gröbner flag and the ideal J has a quadratic Gröbner basis with respect to revlex (usually induced by $x_2 > x_3 > x_4 > x_5$). There are though, up to a permutation of the variables, a couple of candidates for J which do not admit a quadratic Gröbner basis with respect to any term order. We present these exceptions in Remark 15.

- If $h_2 = 4 = \binom{3}{2} + \binom{1}{1}$, then $h_3 \leqslant 1$ and $h_4 = 0$. We scanned the possible candidates for J and we eliminated those ideals where the resolution of K over $\bar{R}$ (computed with Singular [16]) is becoming nonlinear after at most 5 steps. All the other candidates had a quadratic Gröbner bases with respect to revlex (usually induced by $x_2 > x_3 > x_4 > x_5$) and even a Gröbner flag. All the non-Koszul ideals were among those with $h_3 = 0$, hence with $e(H) = 9$.

- If $h_2 = 5 = \binom{3}{2} + \binom{2}{1}$, then by (2) we get $h_3 \leqslant 2$ and $h_4 = 0$. It is well known and easy to see that if $\bar{R}$ is Koszul, then its Poincaré series equals $1/H(-t)$. It is routine to check that if $h_3 = 0$ then $1/H(-t) = 1 + 4t + \cdots - 29x^6 + \cdots$, and if $h_3 = 1$ then $1/H(-t) = 1 + 4t + \cdots - 174x^8 + \cdots$. Therefore, in either case R is not Koszul. However, if $h_3 = 2$ then $e(H) = 12 = 2^4 - 2^2$ and we may apply Theorem 1 to conclude that H is G-quadratic.

- If $h_2 = 6$, then I_H^* is a complete intersection, hence H is G-quadratic.

This finishes the proof of the theorem. □

Corollary 13. *Let H be a numerical semigroup with* $\operatorname{emb}\dim(H) = 5$ *and* $e(H)$ *different from* 9, 10 *and* 11*. Then H is quadratic if and only if it is G-quadratic.*

We next present the situations of Koszul ideals without Gröbner flags that occurred when discussing the cases $h_2 = 2$ and $h_2 = 3$ in the proof of Theorem 12.

Remark 14. The ideal J_1 has a quadratic Gröbner basis with respect to revlex induced by $x_2 > x_3 > x_4 > x_5$ and the h-vector of $\bar{R} = K[x_2, x_3, x_4, x_5]/J_1$ is $(1, 4, 2)$:

$$J_1 = (x_2^2, x_5^2, p_1 = x_3^2 - x_2x_4, x_4^2, x_2x_3, x_2x_5, x_3x_4, x_4x_5).$$

The following computations show that

$$\mathcal{J}_1 = \{0, (x_2), (x_2, x_3), (x_2, x_3, x_5), (x_2, x_4, x_5), (x_2, x_3, x_4, x_5)\}$$

is a Koszul filtration for $\bar{R}$:

$$\begin{aligned} 0 : (x_2) = (x_2, x_3, x_5), \quad (x_2) : (x_2, x_5) = (x_2, x_4, x_5), \\ (x_2, x_5) : (x_2, x_3, x_5) = (x_2, x_5) : (x_2, x_4, x_5) = (x_2, x_3, x_4, x_5), \\ (x_2, x_3, x_5) : (x_2, x_3, x_4, x_5) = (x_2, x_4, x_5) : (x_2, x_3, x_4, x_5) = (x_2, x_3, x_4, x_5). \end{aligned}$$

All the computations in these equations are made in $\bar{R}$. We hope there is no risk of confusion. It is also easy to check that there is no Gröbner flag for $\bar{R}$ with basis (the residue classes of) x_2, x_3, x_4, x_5.

Remark 15. The quotient of $\bar{S} = K[x_2, \ldots, x_5]$ modulo either one of the following two ideals has h-vector $(1, 4, 3)$:

$$\begin{aligned} J_2 = (x_2^2, x_5^2, p_1 = x_3^2 - x_2x_4, p_3 = x_4^2 - x_2x_5, x_2x_3, x_3x_4, x_3x_5), \\ J_3 = (x_2^2, x_5^2, p_1 = x_3^2 - x_2x_4, x_4^2, x_2x_3, x_3x_4, x_3x_5). \end{aligned}$$

We claim that none of them has a quadratic Gröbner basis with respect to any term order $<$.

Indeed, regarding J_2: if $\operatorname{in}_<(p_1) = x_2x_4$ then the S-polynomial $S(p_1, x_2x_3) = x_3^3$. Else, in case $\operatorname{in}_<(p_3) = x_4^2$ we obtain $S(p_1, x_3x_5) = x_2x_4x_5$ and in case $\operatorname{in}_<(p_3) = x_2x_5$ we compute $S(p_1, x_3x_4) = x_2x_4^2$.

Regarding J_3: if $\operatorname{in}_<(p_1) = x_3^2$, then $S(p_1, x_3x_5) = x_2x_4x_5$. Similarly, if $\operatorname{in}_<(p_1) = x_2x_4$ then $S(p_1, x_2x_3) = x_3^3$. It is easy to observe that in any of these cases the computed S-polynomial does not reduce to zero using the remaining quadrics that generate J_2, respectively J_3. Therefore J_2 and J_3 do not have a quadratic Gröbner basis with respect to any term order.

The following computations performed in $\bar{R} = \bar{S}/J_2$, respectively in $\bar{R} = \bar{S}/J_3$, show that

$$\mathcal{J} = \{0, (x_5), (x_3, x_5), (x_2, x_5), (x_2, x_4, x_5), (x_2, x_3, x_5), (x_2, x_3, x_4, x_5)\}$$

is a Koszul filtration for $\bar{R}$:

$$\begin{aligned} 0 : (x_5) = (x_3, x_5), (x_5) : (x_3, x_5) = (x_2, x_4, x_5), (x_5) : (x_2, x_5) = (x_2, x_3, x_5), \\ (x_3, x_5) : (x_2, x_3, x_5) = (x_2, x_5) : (x_2, x_4, x_5) = (x_2, x_3, x_4, x_5), \\ (x_2, x_3, x_5) : (x_2, x_3, x_4, x_5) = (x_2, x_4, x_5) : (x_2, x_3, x_4, x_5) = (x_2, x_3, x_4, x_5). \end{aligned}$$

Remark 16. In practice, we were not able to find quadratic numerical semigroups producing the ideals J_1, J_2, J_3 in Remarks 14 and 15. If such semigroups do not exist, we could drop the restrictions on the field K in Theorem 12.

Based on the proof of Theorem 12 and on the numerical experiments detailed before the proof, in Table 1 we summarize with examples our knowledge of the possible h-vectors of $\operatorname{gr}_{\mathfrak{m}} K[H]$, grouped by the multiplicity, when H is a 5-generated quadratic numerical semigroup. For two of these h-vectors we could not find examples of semigroups, hence we ask if this list should be further reduced. The abbreviation quad GB indicates that I_H^* has a quadratic Gröbner basis.

Table 1: h-vectors for quadratic 5-semigroups

$e(H)$	h-vector	Remarks on I_H^*	Example
5	$(1,4)$	quad GB	$\langle 5,6,7,8,9\rangle$
6	$(1,4,1)$	quad GB	$\langle 6,7,8,9,10\rangle$
7	$(1,4,2)$	quad GB	$\langle 7,8,9,10,11\rangle$
8	$(1,4,2,1)$	not CM, quad GB	$\langle 8,12,13,18,55\rangle$
	$(1,4,3)$	quad GB	$\langle 8,9,10,11,12\rangle$
9	$(1,4,3,1)$	quad GB	?
	$(1,4,4)$	quad GB	$\langle 9,10,11,12,15\rangle$
		not Koszul	$\langle 9,17,20,23,25\rangle$
10	$(1,4,4,1)$	quad GB	$\langle 10,16,19,22,25\rangle$
	$(1,4,5)$	not Koszul	?
11	$(1,4,5,1)$	not Koszul	$\langle 11,13,14,15,19\rangle$
12	$(1,4,5,2)$	quad GB	$\langle 12,14,16,18,27\rangle$
16	$(1,4,6,4,1)$	quad GB	$\langle 16,17,18,20,24\rangle$

Remark 17. In recent work ([11]) D'Anna et al. study the numerical semigroups H for which the Hilbert function of $\operatorname{gr}_{\mathfrak{m}} K[H]$ is non-decreasing. We observe that this is also the case when H is quadratic and $\operatorname{emb}\dim(H) \leqslant 5$.

Indeed, by Proposition 5 and the Table 1 we have that the h-vector of $\operatorname{gr}_{\mathfrak{m}} K[H]$ has nonnegative entries, hence the Hilbert function of $\operatorname{gr}_{\mathfrak{m}} K[H]$ is non-decreasing.

3 A long proof

In this section we prove the direct implication of Theorem 8.

Let H be a quadratic numerical semigroup minimally generated by $a_1 < \cdots < a_5$. Assume that $\operatorname{gr}_{\mathfrak{m}} K[H]$ is not Cohen-Macaulay.

By Theorem 1 we see that

$$4 < a_1 < 16. \tag{3}$$

By Proposition 4 and Lemma 3 the lack of the Cohen-Macaulay property is equivalent to say that there exist $\nu_i \in \{0,1\}$, $i = 2, \dots, 5$, such that $\sum_{i=2}^{5} \nu_i a_i \in a_1 + H$ and whenever

$$\sum_{i=2}^{5} \nu_i a_i = \sum_{i=1}^{5} \mu_i a_i, \text{ with integers } \mu_1 > 0, \mu_2, \dots, \mu_5 \geqslant 0, \tag{4}$$

one has $\sum_{i=2}^{5} \nu_i > \sum_{i=1}^{5} \mu_i$.

Without loss of generality we may assume that in any equation like (4) one has $\nu_i \mu_i = 0$ for all $i > 1$. Since $\operatorname{emb\,dim}(H) = 5$, at least two of the ν_i's are positive. If exactly two of the ν_i's are equal to 1, then $\sum_{i=1}^{5} \mu_i = 1$, $\mu_1 = 1$, and $a_1 \in \langle a_2, \dots, a_5 \rangle$, which is false. If all ν_i are positive, then $a_2 + \cdots + a_5 = \mu_1 a_1$ and since $a_1 = e(H)$ we get $\mu_1 > 4 = \sum_{i=2}^{5} \nu_i$, which contradicts the failure of the Cohen-Macaulay property.

Hence we have to consider only expressions where exactly one ν_i is zero. If $\nu_2 = 0$, then (4) is of the form $a_3 + a_4 + a_5 = a_1 + a_2$ or $a_3 + a_4 + a_5 = 2a_1$. If $\nu_3 = 0$, then $a_2 + a_4 + a_5 = a_1 + a_3$ or $a_2 + a_4 + a_5 = 2a_1$. If $\nu_4 = 0$, then $a_2 + a_3 + a_5 = a_1 + a_4$ or $a_2 + a_3 + a_5 = 2a_1$. If $\nu_5 = 0$, then $a_2 + a_3 + a_4 = a_1 + a_5$ or $a_2 + a_3 + a_4 = 2a_1$. Since $a_1 < a_2 < \cdots < a_5$, the only possibility for (4) is

$$a_2 + a_3 + a_4 = a_1 + a_5. \tag{5}$$

By Lemma 3(ii)

$$2a_2 = ua_1 + va_3 + wa_4 + \lambda a_5, \tag{6}$$

$$2a_3 = u'a_1 + v'a_2 + w'a_4 + \lambda' a_5, \tag{7}$$

$$2a_4 = u''a_1 + v''a_2 + w''a_3 + \lambda'' a_5, \tag{8}$$

for $u, v, w, \lambda, u', \dots, \lambda''$ nonnegative integers. Moreover, since all $a_i > 0$ we may assume, without loss of generality, that $v, w, \lambda, v', w', \lambda', v'', w'', \lambda'' \in \{0,1\}$. We later refer to these equations as *normalized expressions* for $2a_2$, $2a_3$ and $2a_4$, respectively.

We observe that due to the ordering of the a_i's and to (5) we have $a_5 > a_3 + a_4 > 2a_3 > 2a_2$, hence $\lambda = \lambda' = 0$. Also, (5) implies $u''\lambda'' = 0$, otherwise $a_4 \in \langle a_1, a_2, a_3 \rangle$, which is false. Similarly, $v + w < 2$.

The rest of the proof treats the remaining two possibilities: $2a_2 = ua_1 + a_3$, or $2a_2 = ua_1 + a_4$, where we must have $u \geqslant 1$. The rather long discussion depends on the coefficients that occur in the normalized expressions (7) and (8). We identify six situations when the tangent cone $\operatorname{gr}_{\mathfrak{m}} K[H]$ is not Cohen-Macaulay, but, after reordering, all of them fit into the two families (*i*) and (*ii*) in the text of the theorem.

3.1 Case (A)

Assume

$$2a_2 = ua_1 + a_3 \text{ with } u \geqslant 1. \tag{9}$$

3.1.1 Case $w' = 0$

Then

$$2a_3 = u'a_1 + v'a_2 \text{ with } v' \in \{0, 1\}. \tag{10}$$

If $v' = 1$, then $2a_3 = u'a_1 + a_2$, with $u' \geqslant 1$. Adding to this the equation (9), after obvious simplifications we obtain $a_2 + a_3 = (u + u')a_1$, hence

$$a_5 = (a_2 + a_3 + a_4) - a_1 = (u + u' - 1)a_1 + a_4 \in \langle a_1, a_4\rangle,$$

which is false.

Thus $v' = 0$ and $2a_3 = u'a_1$ where $u' > 1$ need to be odd and a_1 even. Together with (9) this gives $2(2a_2 - ua_1) = u'a_1$, i.e.

$$4a_2 = (u' + 2u) \cdot a_1.$$

Since u' is odd we get $4|a_1$, hence $a_1 \in \{8, 12\}$.

1. If $\underline{a_1 = 8}$, then

$$\begin{aligned} a_2 &= 2(u' + 2u), \\ a_3 &= 2a_2 - ua_1 = 4u'. \end{aligned}$$

Consider the normalized expression (8):

$$2a_4 = u''a_1 + v''a_2 + w''a_3 + \lambda''a_5.$$

If $\lambda'' = 1$, since a_1, a_2, a_3 are even, then a_5 is even. Using (5) we infer that a_4 is even, which is false.

Therefore $\lambda'' = 0$ and

$$2a_4 = u''a_1 + v''a_2 + w''a_3.$$

Since $a_2, a_3 < a_4$ we should have $u'' > 0$. Because $4|a_1$ and $4|a_3$, we can not have $v'' = 0$, otherwise a_4 is even, and by (5) also a_5 is even, which gives a contradiction. Hence $v'' = 1$. We distinguish two situations.

1.1. If $\underline{w'' = 1}$, then $2a_4 = u''a_1 + a_2 + a_3$. This gives

$$\begin{aligned} a_1 &= 8, \\ a_3 &= 4u', \\ a_2 &= 4u + 2u', \\ a_4 &= 4u'' + 2u + 3u', \\ a_5 &= 9u' + 6u + 4u'' - 8, \end{aligned} \tag{11}$$

which is of the desired form.

1.2. If $\underline{w'' = 0}$, then $2a_4 = u''a_1 + a_2$. This gives

$$\begin{aligned} a_1 &= 8, \\ a_3 &= 4u', \\ a_2 &= 4u + 2u', \\ a_4 &= 4u'' + 2u + u', \\ a_5 &= 7u' + 6u + 4u'' - 8, \end{aligned} \tag{12}$$

which is of the desired form.

2. If $\underline{a_1 = 12}$, then

$$\begin{aligned} a_2 &= 3u' + 6u, \\ a_3 &= 6u'. \end{aligned}$$

In the normalized expression

$$2a_4 = u''a_1 + v''a_2 + w''a_3 + \lambda''a_5$$

we add $a_4 + (1 - v'')a_2 + (1 - w'')a_3$ to both sides and using (5) we get

$$3a_4 + (1 - v'')a_2 + (1 - w'')a_3 = (1 + u'')a_1 + (1 + \lambda'')a_5,$$

hence $3|a_5$, and by (5) also $3|a_4$, hence $\gcd(a_1, \ldots, a_5) > 1$, a contradiction.

3.1.2 Case $w' = 1$

Then

$$2a_3 = u'a_1 + v'a_2 + a_4 \text{ with } v' \in \{0, 1\}.$$

If $v' = 1$, then $2a_1 + 2a_5 = 2a_2 + 2a_3 + 2a_4 = (ua_1 + a_3) + (u'a_1 + a_2 + a_4) + 2a_4 = (u' + u)a_1 + (a_2 + a_3 + a_4) + 2a_4$. This gives $a_5 = (u' + u - 1)a_1 + 2a_4$, which is false. Therefore $v' = 0$ and

$$2a_3 = u'a_1 + a_4 \text{ with } u' > 0. \tag{13}$$

Consider the normalized expression (8)

$$2a_4 = u''a_1 + v''a_2 + w''a_3 + \lambda''a_5.$$

If $\lambda'' = 1$, since $u''\lambda'' = 0$, we get $u'' = 0$. Equation (8) becomes

$$2a_4 = v''a_2 + w''a_3 + a_5.$$

To this we add (13) and $2a_2 = ua_1 + a_3$ from (9) and after using (5) we get that

$$a_5 = (u' + u - 2)a_1 + v''a_2 + (w'' + 1)a_3 + a_4 \in \langle a_1, a_2, a_3, a_4 \rangle,$$

which is false.

Thus $\lambda'' = 0$ and $2a_4 = u''a_1 + v''a_2 + w''a_3$. To this we add (13) and $2a_2 = ua_1 + a_3$ and, after using (5), we see that

$$2a_5 = (u'' + u' + u - 2)a_1 + v''a_2 + (w'' + 1)a_3 + a_4.$$

If $v'' > 0$, by (5) we get $a_5 \in \langle a_1, a_2, a_3, a_4 \rangle$, which is false.
Hence $v'' = 0$ and (8) becomes

$$2a_4 = u''a_1 + w''a_3 = u''a_1 + w''(2a_2 - ua_1) = (u'' - w''u)a_1 + 2w''a_2 \text{ with } w'' \in \{0, 1\}.$$

From (13) we extract

$$a_4 = 2a_3 - u'a_1 = 2(2a_2 - ua_1) - u'a_1 = 4a_2 - (2u + u')a_1$$

which we replace in the previous equation for $2a_4$. Routine manipulation gives

$$(8 - 2w'')a_2 = ((4 - w'')u + 2u' + u'')a_1. \tag{14}$$

1. If $\underline{w'' = 0}$, the equations (14), (9) and (8) together with (5) yield

$$\begin{aligned} a_2 &= \frac{4u + 2u' + u''}{8} \cdot a_1, \\ a_3 &= \frac{2u' + u''}{4} \cdot a_1, \\ a_4 &= \frac{u''}{2} \cdot a_1, \\ a_5 &= \frac{4u + 6u' + 7u'' - 8}{8} \cdot a_1. \end{aligned}$$

   From here we infer that u'' is odd (otherwise a_1 divides a_4, which is false) and consequently $8|a_1$. Hence $a_1 = 8$ and the generators of the semigroup are

$$\begin{aligned} a_1 &= 8, \\ a_4 &= 4u'', \\ a_3 &= 2u'' + 4u', \\ a_2 &= u'' + 2u' + 4u, \\ a_5 &= 4u + 6u' + 7u'' - 8, \end{aligned} \tag{15}$$

   as desired.

2. If $\underline{w'' = 1}$, equation (14) together with (9), (8) and (5) give after routine computations

$$\begin{aligned} a_2 &= \frac{3u + 2u' + u''}{6} \cdot a_1, \\ a_3 &= \frac{2u' + u''}{3} \cdot a_1, \\ a_4 &= \frac{u' + 2u''}{3} \cdot a_1, \\ a_5 &= \frac{3u + 8u' + 7u'' - 6}{6} \cdot a_1. \end{aligned}$$

   We note that $a_5 = a_2 + (u' + u'' - 1)a_1 \in \langle a_1, a_2 \rangle$, which is false.

3.2 Case (B)

Assume

$$2a_2 = ua_1 + a_4, \text{ with } u \geqslant 1. \tag{16}$$

If in the normalized expression

$$2a_4 = u''a_1 + v''a_2 + w''a_3 + \lambda''a_5 \quad \text{with } v'', w'', \lambda'' \in \{0, 1\}$$

we had $\lambda'' = 1$, then $u'' = 0$. Combined with (16), we get

$$2a_2 + a_4 = ua_1 + 2a_4 = ua_1 + v''a_2 + w''a_3 + a_5.$$

Using (5) and the latter equation we obtain

$$(1 - v'')a_2 = (u - 1)a_1 + (w'' + 1)a_3 > 0,$$

which is a contradiction since $v'' \leqslant 1$. Therefore $\lambda'' = 0$.

3.2.1 Case $w' = v' = 0$

Then $u' > 0$ and (7) becomes

$$2a_3 = u'a_1. \tag{17}$$

From this and (16) we get $a_3 = u'a_1/2$, $a_4 = 2a_2 - ua_1$ and that a_1 is even and u' is odd.

We substitute in the normalized expression above the values for a_3 and a_4 in terms of a_1 and a_2 and we get

$$(4 - v'') \cdot a_2 = \left(2u + u'' + w''\frac{u'}{2}\right) \cdot a_1.$$

1. If $\underline{v'' = 1}$, the previous equation becomes

$$3a_2 = \left(2u + u'' + w''\frac{u'}{2}\right) \cdot a_1.$$

We consider the possible even values of a_1.

1.1. If $\underline{a_1 = 6}$ we obtain the other generators

$$\begin{aligned} a_2 &= 4u + 2u'' + w''u', \\ a_3 &= 3u', \\ a_4 &= 2u + 4u'' + 2w''u', \\ a_5 &= 6u + 6u'' + (3w'' + 3)u' - 6. \end{aligned}$$

Note that $a_5 = (u + u'' - 1)a_1 + (w'' + 1)a_3 \in \langle a_1, a_3 \rangle$, which is false.

1.2. If $\underline{a_1 = 8 \text{ or } 12}$, then a_2, a_3, a_4 are even, hence a_5 is even as well, a contradiction.

1.3. If $\underline{a_1 = 10}$, then it easy to see that a_2, a_3, a_4, a_5 are divisible by 5, which is false.

1.4. If $\underline{a_1 = 14}$, then all the generators are divisible by 7, which is false.

2. If $\underline{v'' = 0}$, then
$$4a_2 = \left(2u + u'' + w''\frac{u'}{2}\right) \cdot a_1,$$
which forces a_1 to be even.

2.1. If $\underline{w'' = 0}$ we get $2a_4 = u''a_1$ and $4a_2 = (2u + u'')a_1$. Therefore u'' is odd, a_1 is divisible by 4, hence $a_1 \in \{4, 8\}$, and the other generators are
$$\begin{aligned} a_2 &= \frac{2u + u''}{4} \cdot a_1, \\ a_3 &= \frac{u'}{2} \cdot a_1, \\ a_5 &= \frac{3u'' + 2u + 2u' - 4}{4} \cdot a_1. \end{aligned}$$
It is immediate to note that if $a_1 = 8$ all generators are even, while if $a_1 = 12$ all of them are divisible by 3. None of these situations may hold.

2.2. If $\underline{w'' = 1}$ we obtain
$$8a_2 = (4u + 2u'' + u') \cdot a_1.$$
Since u' is odd we get $a_1 = 8$. From the other equations we compute the other generators
$$\begin{aligned} a_3 &= 4u', \\ a_4 &= 4u'' + 2u', \\ a_2 &= 4u + 2u'' + u', \\ a_5 &= 4u + 6u'' + 7u' - 8, \end{aligned} \tag{18}$$
which turn out to be of the desired form.

3.2.2 Case $w' = 0$ and $v' = 1$

Then $2a_3 = u'a_1 + a_2$ with $u' > 0$.

Using (16) and the normalized equation (8) we get
$$2a_1 + 2a_5 = 2a_2 + 2a_3 + 2a_4 = (u' + u'' + u)a_1 + (v'' + 1)a_2 + w''a_3 + a_4,$$
hence $w'' = 0$.

1. If $\underline{v'' = 1}$, by adding the equations $2a_2 = ua_1 + a_4$ and $2a_4 = u''a_1 + a_2$ we get $a_2 + a_4 = (u + u'')a_1$. Therefore $a_5 = (a_2 + a_4) + a_3 - a_1 = (u + u'' - 1)a_1 + a_3$, which is false.

2. If $\underline{v''=0}$, then $2a_4 = u''a_1$ with u'' odd and a_1 even. Since $2a_2 = ua_1 + a_4$ we get $4a_2 = (2u+u'')\cdot a_1$. This implies that $4|a_1$, hence $a_1 \in \{8, 12\}$.

   If $\underline{a_1 = 12}$, then $a_2 = 3(2u+u'')$. Since $2a_3 = u'a_1 + a_2$ and $a_4 = ua_1 - 2a_2$ we derive that a_3 and a_4 are divisible by 3, hence also $3|a_5$, which is false.

   If $\underline{a_1 = 8}$, the rest of the generators are

$$\begin{aligned} a_4 &= 4u'', \\ a_2 &= 4u + 2u'', \\ a_3 &= 2u + u'' + 4u', \\ a_5 &= 6u + 7u'' + 4u' - 8, \end{aligned} \tag{19}$$

   which are of the desired format.

3.2.3 Case $w' = 1$

Then

$$2a_3 = u'a_1 + v'a_2 + a_4. \tag{20}$$

Since in the normalized expression (8) we have $\lambda'' = 0$ and $v'', w'' \leqslant 1$, then $u'' > 0$. By (16) and (20) we may write

$$\begin{aligned} 2a_5 &= 2a_2 + 2a_3 + 2a_4 - 2a_1 \\ &= (ua_1 + a_4) + (u'a_1 + v'a_2 + a_4) + (u''a_1 + v''a_2 + w''a_3) - 2a_1 \\ &= (u + u' + u'' - 2)a_1 + (v' + v'')a_2 + w''a_3 + 2a_4. \end{aligned}$$

1. If $\underline{w''=1 \text{ and } v'+v''>0}$, we get $a_5 \in \langle a_1, a_2, a_3, a_4\rangle$, which is false.

2. If $\underline{w''=1 \text{ and } v'=v''=0}$, summing the equations

$$\begin{aligned} 2a_4 &= u''a_1 + a_3 \\ 2a_3 &= u'a_1 + a_4 \end{aligned}$$

   we obtain that $a_3 + a_4 = (u' + u'')a_1$, which, together with (5) yields $a_5 = (u' + u'' - 1)a_1 + a_2 \in \langle a_1, a_2\rangle$, a contradiction.

3. If $\underline{w''=0}$, then $2a_4 = u''a_1 + v''a_2$, and after substituting in here $a_4 = 2a_2 - ua_1$ (from (16)) we get

$$(4 - v'')a_2 = (2u + u'')a_1.$$

   3.1. If $\underline{v''=0}$, then $a_4 = u''a_1/2$ and u'' is odd. Other generators are obtained immediately:

$$\begin{aligned} a_2 &= \frac{2u + u''}{4} \cdot a_1, \\ a_3 &= \frac{4u' + 2u'' + v'(2u + u'')}{8} \cdot a_1. \end{aligned}$$

Since u'' is odd we deduce that $4|a_1$, hence $a_1 \in \{8, 12\}$.
If $\underline{a_1 = 12}$, because the denominators of a_2, a_3 and a_4 are powers of 2 we see that a_2, a_3, a_4 are divisible by 3, and the same holds for a_5, a contradiction.
Assume $\underline{a_1 = 8}$. If $v' = 0$ then we note that a_2, a_3, a_4 are even, hence a_5 is even, too. This is false. Hence $v' = 1$, and the generators of the semigroup are

$$\begin{aligned} a_1 &= 8, \\ a_4 &= 4u'', \\ a_2 &= 4u + 2u'', \\ a_3 &= 4u' + 2u + 3u'', \\ a_5 &= 4u' + 6u + 9u'' - 8, \end{aligned} \tag{21}$$

which is on our list.

3.2. If $\underline{v'' = 1}$, then $3a_2 = (2u + u'')a_1$ which implies $3 \nmid 2u + u''$ and $3|a_1$. We get

$$a_2 = \frac{2u + u''}{3} \cdot a_1, \qquad a_4 = 2a_2 - u = \frac{2u'' + u}{3} \cdot a_1.$$

By (5) we see that $a_5 = a_2 + a_3 + a_4 - a_1 = (u + u'' - 1)a_1 + a_3$, which is false.

The proof of Theorem 8 is now complete. $\square$

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