

Guaranteed Scoring Games

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Submitted: Jul 17, 2015; Accepted: Jul 30, 2016; Published: Aug 19, 2016
Mathematics Subject Classifications: 91A46

Abstract

The class of Guaranteed Scoring Games (GS) is the class of all two-player scoring combinatorial games with the property that Normal-play games (Conway et. al.) are order embedded into GS. They include, as subclasses, the scoring games considered by Milnor (1953), Ettinger (1996) and Johnson (2014). We present the structure of GS and the techniques needed to analyze a sum of guaranteed games. Firstly, GS forms a partially ordered monoid, via defined outcomes over the reals, and with disjunctive sum as the operation. In fact, the structure is a quotient monoid with partially ordered congruence classes. We show that there are four reductions that when applied, in any order, give a unique representative for each congruence class. The monoid is not a group, but in this paper we prove that if a game has an inverse it is obtained by ‘switching the players’. The order relation between two games is defined by comparing their outcomes in *any* disjunctive sum. Here, we demonstrate how to compare the games via a finite algorithm instead, extending ideas of Ettinger, and also Siegel (2013).

*Supported by the Killam Trust.

†Partially supported by NSERC

‡Work supported by centre grant (to BioISI, Centre Reference: UID/MULTI/04046/2013), from FCT/MCTES/PIDDAC, Portugal.

1 Introduction

Combinatorial Game Theory (CGT) studies two-player games, (the players are called Left and Right) with perfect information and no chance device. A common, almost defining feature, is that these games often decompose into sub-components and a player is only allowed to move in one of these at each stage of play. This situation is called a *disjunctive sum* of games. It is also commonplace to allow addition of games with similar and well defined properties; games in such a family do not necessarily have the same rule sets.

The convention we wish to study, has the winner as the player with the best score. This convention includes rule sets such as DOTS-&-BOXES, GO and MANCALA. A general, useful, theory has been elusive and, to our current knowledge, only four approaches appear in the literature. Milnor [14], see also Hanner [11], considers dicot games (for all positions, either both or none of the players can move) with nonnegative incentive. In games with a nonnegative incentive, a move never worsens the player's score; that is, zugzwang games, where neither player wishes to move, do not appear. Ettinger [7, 6] considers all dicot games. Stewart [18] defines a comprehensive class but it has few useful algebraic properties. Johnson [12] considers another subclass of dicot games, for which, for any position, the lengths of every branch of the game tree has the same parity.

We study the class of *Guaranteed Scoring Games*, \mathbb{GS} [13]. This class has a partial order relation, \succcurlyeq , which together with the disjunctive sum operation induces a congruence relation $(\sim, +)$. The resulting quotient monoid inherits partially ordered congruence classes, and it is the purpose of this paper to continue the study of these classes. In [13], it was shown that Normal-play games (see Remark 1) can be ordered embedded in a natural way and that a positive incentive for games without Right or Left options is an obstacle to the order embedding. It was also demonstrated how to compare games with numbers using waiting moves (images of Normal-play integers) and pass-allowed outcomes. Intuitively, this class of games has the property that the players want the component games to continue; every game in which at least one player cannot move has non-positive incentive.

Here we show that \mathbb{GS} has the properties:

1. There is a constructive way to give the order relation between games G and H . It only requires G , H and a special type of simplistic games that we call 'waiting moves', games with the sole purpose of giving one of the player an extra number of moves, but with no change in score.
2. There are four reduction theorems, and we find a unique representative game for each congruence class. Of these, 'Eliminating Dominated Options' and 'Bypassing Reversible Moves' with a non-empty set of options are analogous to those found in the theory of Normal-play games. Bypassing a reversible move by just replacing it with an empty-set of options leads to a non-equivalent game. In this case, the appropriate reduction requires consideration of the pass-allowed outcomes. This has no corresponding concept in Normal-play theory.

3. In \mathbb{GS} the Conjugate Property holds: if $G + H$ is equivalent to 0 then H is the game obtained by interchanging the roles of Left and Right in G . In Normal-play, this is called the negative of G ; however in \mathbb{GS} ‘negatives’ do not always exist.
4. We solve each of these problem via a finite algorithm, which is also implemented in a Scoring Games Calculator.

The organization of the paper is as follows: Section 2 introduces the main concepts for Combinatorial Games. In Section 2.1, the class of Guaranteed Scoring Games, together with the order relations and congruence classes, is presented. Section 2.2 contains results concerning the order embedding of Normal-play games in \mathbb{GS} . Section 2.3 presents results on pass-allowed stops and waiting moves. Section 3.1 proves four reductions that simplify games. Section 3.3 proves that applying these reductions leads to a unique game. The proofs require extending Siegel’s ‘linked’ concept for misère games to scoring games which is in Section 3.2. Section 4 shows that the Conjugate Property holds in \mathbb{GS} . In Section 5 we give a brief intro to the Scoring Games Calculator.

Remark 1. Other famous winning conditions in CGT are considering who moves last. *Normal-play* games, the first player who cannot move loses, find their origins with the analysis of NIM [4]; see also [10, 17]. Conway developed the first encompassing theory; see [2, 5]. A comprehensive *Misère* theory, the first player who cannot move wins, has not yet been developed but large strides have been made for impartial games, see [15]. A related winning convention arises in the Maker-Breaker (or Maker-Maker) games usually played on a graph—one player wishes to create a structure and the opponent wants to stop this (or both want to create a structure) such as HEX or generalized TIC-TAC-TOE. See Beck [1] for more details.

2 Background

In a combinatorial game (regardless of the winning condition) there are two players who, by convention, are called *Left* (female) and *Right* (male)¹. A position that some player can move to (in a single move) is an *option*. The *left* options are those to which Left can move and the corresponding set is denoted by $G^{\mathcal{L}}$ if the game is G . An element of $G^{\mathcal{L}}$ is often denoted by $G^{\mathcal{L}}$. Similarly, there is a set of *right* options denoted by $G^{\mathcal{R}}$, with a typical game $G^{\mathcal{R}}$. There is no requirement that $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ be disjoint. A game can be recursively defined in terms of its options. We will use the representation $G = \langle G^{\mathcal{L}} \mid G^{\mathcal{R}} \rangle$ (so as to distinguish them from Normal-play games where the convention is $\{G^{\mathcal{L}} \mid G^{\mathcal{R}}\}$). The *followers* of G are defined recursively: G and all its options are followers of G and each follower of a follower of G is a follower of G . The set of *proper* followers of G are the followers except for G itself. The *game tree* of a position G would then consist of all the followers of G drawn recursively: i.e. the options of a follower H of G are the children of H in the tree.

¹Remember, Louise and Richard Guy who have contributed much to combinatorial games.

Positions fall into two kinds: *atomic* positions in which at most one player can move, and *non-atomic* positions in which both players can move. A position with no Left options is called *left-atomic*, and in case of no Right options it is *right-atomic*. A game with no options at all is called *purely-atomic*, that is, such games are both left-atomic and right-atomic.

2.1 Introduction to Guaranteed scoring games

In a combinatorial game, the result of a game is determined at the end, that is when the player to move has no option. In scoring games, this result is a score, a real number.² We adapt the following notation.

Definition 2 (Game termination). Let G be a left-atomic game. Then $G^{\mathcal{L}} = \emptyset^{\ell}$, indicates that, if Left to move, the game is over and the result is the *score* $\ell \in \mathbb{R}$. Similarly, if G is right-atomic then $G^{\mathcal{R}} = \emptyset^r$ and, if it is Right to move, the score is $r \in \mathbb{R}$. Left wins if the score is positive, Right wins if the score is negative, and it is a tie if the score is zero.

Since the game $\langle \emptyset^s \mid \emptyset^s \rangle$ results in a score of s regardless of whose turn it is, we call this game (the number) s . We refer to the adorned empty set, \emptyset^s , $s \in \mathbb{R}$, as an *atom* or, if needed for specificity, the *s-atom*. By an *atom in a game G* , we mean an atom of some atomic follower of G . By an atom in a set of games we mean an atom in one of the games in that set. In the general scoring universe, \mathbb{S} [13, 18], there is no restriction to the form of the atomic games.

Definition 3. Let \mathbb{S}_0 be the set of scoring games of birthday 0, i.e. the set $\{\langle \emptyset^{\ell} \mid \emptyset^r \rangle : \ell, r \in \mathbb{R}\}$, and for $i > 0$, let \mathbb{S}_i be the set of non-atomic games of the form $\langle \mathcal{G} \mid \mathcal{H} \rangle$, where \mathcal{G} and \mathcal{H} are sets of games of birthday less than i (including the possibility of atomic games of the forms $\langle \emptyset^{\ell} \mid \emptyset^r \rangle$, $\langle \emptyset^{\ell} \mid \mathcal{H} \rangle$ and $\langle \mathcal{G} \mid \emptyset^r \rangle$). For $i > 0$, if $G \in \mathbb{S}_i \setminus \mathbb{S}_{i-1}$ then G is said to have *birthday i* and we write $b(G) = i$. Let $\mathbb{S} = \cup_{i \geq 0} \mathbb{S}_i$.

We impose the following restriction to the set \mathbb{S} . (This is a top down approach instead of the bottom up approach in Definition 3; see also Lemma 5.)

Definition 4. A scoring game $H \in \mathbb{S}$ is *guaranteed* if, for every atomic follower G of H (which includes the possibility that $G = H$),

1. if $G = \langle \emptyset^{\ell} \mid \emptyset^r \rangle$ then $\ell \leq r$;
2. if $G = \langle \emptyset^{\ell} \mid G^{\mathcal{R}} \rangle$ then $\ell \leq s$, for every s -atom in G ;
3. if $G = \langle G^{\mathcal{L}} \mid \emptyset^r \rangle$ then $s \leq r$, for every s -atom in G .

²Loosely speaking our scoring games would correspond to a zero-sum game in economic game theory, but the concept is not useful here, since our objective concerns the algebra on the disjunctive sum of games. Still, if we imagine a rule set, where the players gain or lose point during play, then, it makes sense to call our games zero-sum, because, if Left gains 5 points by a certain move, then the total score is increased by 5, and this means that Right is 5 points worse off than before that move.

Let $\mathbb{GS} \subset \mathbb{S}$ be the set of all guaranteed scoring games.

Note that this definition is formally equivalent to: a game $H \in \mathbb{S}$ is *guaranteed* if, for every atomic follower G of H , $\ell \leq r$, for all ℓ -atoms in $G^\mathcal{L}$ and all r -atoms in $G^\mathcal{R}$. Moreover, observe that if H is guaranteed then every follower of H is also guaranteed. For example, $\langle \langle \emptyset^1 \mid \emptyset^1 \rangle \mid \langle \emptyset^0 \mid \emptyset^0 \rangle \rangle = \langle 1 \mid 0 \rangle$ is guaranteed but $H = \langle \emptyset^1 \mid 4, \langle \emptyset^3 \mid 3, \langle \emptyset^5 \mid 4 \rangle \rangle$ is not (by item d):

- (a) H is left-atomic and 1 is less than the other scores;
- (b) both $3 = \langle \emptyset^3 \mid \emptyset^3 \rangle$ and $4 = \langle \emptyset^4 \mid \emptyset^4 \rangle$ are atomic and each satisfies (i) in Definition 4;
- (c) $\langle \emptyset^3 \mid 3, \langle \emptyset^5 \mid 4 \rangle \rangle$ is left-atomic and 3 is less than or equal to 3, 5 and 4;
- (d) $\langle \emptyset^5 \mid 4 \rangle = \langle \emptyset^5 \mid \langle \emptyset^4 \mid \emptyset^4 \rangle \rangle$ does not satisfy Definition 4 (2) and thus H is not guaranteed.

Similar to the general class \mathbb{S} , the class of guaranteed scoring games can be defined recursively. The birthday of a guaranteed game corresponds to the depth of its game tree, and this stratification into birthdays is very useful for proofs by induction.

Let \mathbb{T}_0 be the set of birthday 0 guaranteed games, so these are of the form $\{\langle \emptyset^\ell \mid \emptyset^r \rangle : \ell, r \in \mathbb{R}, \ell \leq r\}$. Let $\mathbb{T}_i, i > 0$, be the set consisting of the games:

- (i) $\langle \mathcal{G} \mid \mathcal{H} \rangle$ where \mathcal{G} and \mathcal{H} are non-empty subsets of T_{i-1} ;
- (ii) $\langle \emptyset^\ell \mid \emptyset^r \rangle, \ell \leq r$;
- (iii) $\langle \emptyset^\ell \mid \mathcal{H} \rangle$ where $\ell \leq r$, for all r -atoms in \mathcal{H} , a non-empty subset of T_{i-1} ;
- (iv) $\langle \mathcal{G} \mid \emptyset^r \rangle$, where $\ell \leq r$, for all ℓ -atoms in \mathcal{G} , a non-empty set of T_{i-1} .

Finally, let $\mathbb{T} = \cup_{i \geq 0} \mathbb{T}_i$

Lemma 5. *The two sets of games \mathbb{GS} and \mathbb{T} are identical.*

Proof. This is immediate by Definition 4. □

A player may be faced with several component games/positions, and if there is at least one in which he can move then he has an option and the game is not over yet. A move in a disjunctive sum of positions is a move in exactly one of the component positions, and the other ones remain unchanged. It is then the other player's turn to move. We formalize this in the next definition by listing all the possible cases. The distinction between the two uses of $+$, the disjunctive sum of games and the addition of real numbers, will be clear from the context. If $\mathcal{G} = \{G_1, \dots, G_m\}$ is a set of games and H is a single game then $\mathcal{G} + H = \{G_1 + H, \dots, G_m + H\}$ if \mathcal{G} is non-empty; otherwise $\mathcal{G} + H$ is not defined and will be removed from any list of games.

An intuitively obvious fact that is worthwhile highlighting at this point: if Left has no move in $G + H$ then Left has no move in either of G and H (and reverse), that is:

$$G + H \text{ is left-atomic if and only if both } G \text{ and } H \text{ are left-atomic,}$$

and analogously for right-atomic games.

Definition 6 (Disjunctive Sum). The disjunctive sum of two guaranteed scoring games G and H is given by:

$$G + H = \begin{cases} \langle \emptyset^{\ell_1+\ell_2} \mid \emptyset^{r_1+r_2} \rangle, & \text{if } G = \langle \emptyset^{\ell_1} \mid \emptyset^{r_1} \rangle \text{ and } H = \langle \emptyset^{\ell_2} \mid \emptyset^{r_2} \rangle; \\ \langle \emptyset^{\ell_1+\ell_2} \mid G^{\mathcal{R}} + H, G + H^{\mathcal{R}} \rangle, & \text{if } G = \langle \emptyset^{\ell_1} \mid G^{\mathcal{R}} \rangle \text{ and } H = \langle \emptyset^{\ell_2} \mid H^{\mathcal{R}} \rangle, \\ & \text{and at least one of } G^{\mathcal{R}} \text{ and } H^{\mathcal{R}} \text{ is non-empty;} \\ \langle G^{\mathcal{L}} + H, G + H^{\mathcal{L}} \mid \emptyset^{r_1+r_2} \rangle, & \text{if } G = \langle G^{\mathcal{L}} \mid \emptyset^{r_1} \rangle \text{ and } H = \langle G^{\mathcal{L}} \mid \emptyset^{r_2} \rangle, \\ & \text{and at least one of } G^{\mathcal{L}} \text{ and } H^{\mathcal{L}} \text{ is non-empty;} \\ \langle G^{\mathcal{L}} + H, G + H^{\mathcal{L}} \mid G^{\mathcal{R}} + H, G + H^{\mathcal{R}} \rangle, & \text{otherwise.} \end{cases}$$

Note that in the last equality, if there are no left options in G , then $G^{\mathcal{L}} + H$ gets removed, unless both $G^{\mathcal{L}}$ and $H^{\mathcal{L}}$ are atoms, in which case some earlier item applies. It is an easy exercise to show that if $G, H \in \mathbb{GS}$ then $G + H \in \mathbb{GS}$. For completeness, we include a proof of this and some other observations.

Theorem 7. $(\mathbb{GS}, +)$ is a commutative monoid.

Proof. In all cases, the proof is by induction on the sum of the birthdays of the positions.

1. \mathbb{GS} is closed, that is, $G, H \in \mathbb{GS} \Rightarrow G + H \in \mathbb{GS}$.

Suppose that $G + H$ is left-atomic. Then both $G = \langle \emptyset^g \mid G^{\mathcal{R}} \rangle$ and $H = \langle \emptyset^h \mid H^{\mathcal{R}} \rangle$ are left-atomic. Since both games are guaranteed, then each s -atom in G satisfies $g \leq s$ and each t -atom in H satisfies $h \leq t$. Therefore $g + h \leq \min\{s + t\}$, where the minimum runs over all s -atoms in G and all t -atoms in H , and so $G + H = \langle \emptyset^{g+h} \mid (G + H)^{\mathcal{R}} \rangle$ is also guaranteed. this case includes the possibility that $(G + H)^{\mathcal{R}}$ is the $(s + t)$ -atom. Finally, suppose that both $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ are non-empty sets of games of \mathbb{GS} . Both players have moves in $G + H$ that, by induction, are games of \mathbb{GS} . So, $G + H \in \mathbb{GS}$.

2. Disjunctive sum is commutative.

If $G = \langle \emptyset^{\ell_1} \mid \emptyset^{r_1} \rangle$ and $H = \langle \emptyset^{\ell_2} \mid \emptyset^{r_2} \rangle$ then $G + H = \langle \emptyset^{\ell_1+\ell_2} \mid \emptyset^{r_1+r_2} \rangle = \langle \emptyset^{\ell_2+\ell_1} \mid \emptyset^{r_2+r_1} \rangle = H + G$.

If $G = \langle \emptyset^{\ell_1} \mid G^{\mathcal{R}} \rangle$ and $H = \langle \emptyset^{\ell_2} \mid H^{\mathcal{R}} \rangle$ then

$$G + H = \langle \emptyset^{\ell_1+\ell_2} \mid G^{\mathcal{R}} + H, G + H^{\mathcal{R}} \rangle \underset{\text{induction}}{=} \langle \emptyset^{\ell_2+\ell_1} \mid H + G^{\mathcal{R}}, H^{\mathcal{R}} + G \rangle = H + G.$$

The other cases are analogous using induction and the fact that the addition of real numbers is commutative.

3. Disjunctive sum is associative.

If $G = \langle \emptyset^{\ell_1} \mid \emptyset^{r_1} \rangle$, $H = \langle \emptyset^{\ell_2} \mid \emptyset^{r_2} \rangle$ and $J = \langle \emptyset^{\ell_3} \mid \emptyset^{r_3} \rangle$ then $G + (H + J) = (G + H) + J$ is just a consequence of that the addition of real numbers is associative.

If $G = \langle \emptyset^{\ell_1} \mid \emptyset^{r_1} \rangle$, $H = \langle \emptyset^{\ell_2} \mid \emptyset^{r_2} \rangle$ and $J = \langle \emptyset^{\ell_3} \mid J^{\mathcal{R}} \rangle$ then

$$\begin{aligned} G + (H + J) &= \langle \emptyset^{\ell_1} \mid \emptyset^{r_1} \rangle + \langle \emptyset^{\ell_2+\ell_3} \mid H + J^{\mathcal{R}} \rangle \\ &= \langle \emptyset^{\ell_1+(\ell_2+\ell_3)} \mid G + (H + J^{\mathcal{R}}) \rangle \\ &\stackrel{\text{induction}}{=} \langle \emptyset^{(\ell_1+\ell_2)+\ell_3} \mid (G + H) + J^{\mathcal{R}} \rangle = (G + H) + J. \end{aligned}$$

The other cases are analogous using induction and the fact that the addition of real numbers is associative.

4. It follows directly from the definition of disjunctive sum that $G + 0 = 0 + G = G$ so the zero of $(\mathbb{GS}, +)$ is $0 = \langle \emptyset^0 \mid \emptyset^0 \rangle$. \square

When analyzing games, the following observation, which follows from the definition of the disjunctive sum, is useful for human players.

Observation 8 (Number Translation). Let $G \in \mathbb{GS}$ and $x \in \mathbb{R}$ then

$$G + x = \begin{cases} \langle \emptyset^{\ell+x} \mid \emptyset^{r+x} \rangle & \text{if } G = \langle \emptyset^\ell \mid \emptyset^r \rangle, \\ \langle \emptyset^{\ell+x} \mid G^{\mathcal{R}} + x \rangle & \text{if } G = \langle \emptyset^\ell \mid G^{\mathcal{R}} \rangle, \\ \langle G^{\mathcal{L}} + x \mid \emptyset^{r+x} \rangle & \text{if } G = \langle G^{\mathcal{L}} \mid \emptyset^r \rangle, \\ \langle G^{\mathcal{L}} + x \mid G^{\mathcal{R}} + x \rangle & \text{if } G = \langle G^{\mathcal{L}} \mid G^{\mathcal{R}} \rangle. \end{cases}$$

Next, we give the fundamental definitions for comparing games.

Definition 9. For a game $G \in \mathbb{GS}$, the outcomes are:

$$Ls(G) = \begin{cases} \ell & \text{if } G = \langle \emptyset^\ell \mid G^{\mathcal{R}} \rangle \\ \max\{Rs(G^{\mathcal{L}}) : G^{\mathcal{L}} \in G^{\mathcal{L}}\} & \text{otherwise;} \end{cases}$$

and

$$Rs(G) = \begin{cases} r & \text{if } G = \langle G^{\mathcal{L}} \mid \emptyset^r \rangle \\ \min\{Ls(G^{\mathcal{R}}) : G^{\mathcal{R}} \in G^{\mathcal{R}}\} & \text{otherwise.} \end{cases}$$

We call the real numbers $Ls(G)$ and $Rs(G)$ the *Left-* and *Right-score* of G , respectively.

The motivation for the following (standard CGT) definition of inequality is that Left prefers a game G over another game H if and only if the outcome is at least as good in a disjunctive sum, with any game in the same class. Take for example the guaranteed scoring games $G = \langle \langle \emptyset^{-1} \mid \emptyset^{1000} \rangle \mid 1000 \rangle$ and $H = 0$. Then $G \not\preceq H$, because, for example $-1 = Rs(G + X) \not\preceq Rs(0 + X) = 0$, if $X = \langle 0 \mid \langle \emptyset^0 \mid 0 \rangle \rangle$.

Definition 10 (Inequalities for games). Let $G, H \in \mathbb{GS}$. Then $G \succcurlyeq H$ if for all $X \in \mathbb{GS}$ we have $Ls(G + X) \geq Ls(H + X)$ and $Rs(G + X) \geq Rs(H + X)$. The games G and H are *equivalent*, denoted by $G \sim H$, if $G \succcurlyeq H$ and $H \succcurlyeq G$.

Lemma 11. Let $G, H \in \mathbb{GS}$. If $G \succcurlyeq H$ then $G + J \succcurlyeq H + J$ for any $J \in \mathbb{GS}$.

Proof. Consider any game $J \in \mathbb{GS}$. Since $G \succcurlyeq H$, it follows that, $Ls(G + (J + X)) \geq Ls(H + (J + X))$, for any $X \in \mathbb{GS}$. Since disjunctive sum is associative this inequality is the same as $Ls((G + J) + X) \geq Ls((H + J) + X)$. The same argument gives $Rs((G + J) + X) \geq Rs((H + J) + X)$ and thus, since X is arbitrary, this gives that $G + J \succcurlyeq H + J$. \square

Lemma 11 confirms that \mathbb{GS} is a partially ordered monoid (a monoid endowed with a partial order that is compatible with the addition). Hence, Theorem 12, Corollaries 13, 14 and Lemma 15 are standard results, holding for partially ordered monoids (see [3, 8, 9]).

Theorem 12. *The relation \succcurlyeq is a partial order and \sim is an equivalence relation.*

Corollary 13. *Let $G, H, J, W \in \mathbb{GS}$. If $G \succcurlyeq H$ and $J \succcurlyeq W$ then $G + J \succcurlyeq H + W$.*

Corollary 14. *Let $G, H, J, W \in \mathbb{GS}$. If $G \sim H$ and $J \sim W$ then $G + J \sim H + W$.*

Lemma 15. *Let $G, H \in \mathbb{GS}$. Let $J \in \mathbb{GS}$ be invertible, then $G + J \succcurlyeq H + J$ if and only if $G \succcurlyeq H$.*

The *conjugate* of a game G , \overleftrightarrow{G} , is defined recursively: $\overleftrightarrow{G} = \langle \overleftrightarrow{G^R} \mid \overleftrightarrow{G^L} \rangle$, where $\overleftrightarrow{G^R}$ means $\overleftrightarrow{G^R}$, for each $G^R \in G^R$ (and similarly for G^L), unless $G^R = \emptyset^r$, in which case $\overleftrightarrow{G^R} = \emptyset^{-r}$. It is easy to see that if a game is guaranteed, then its conjugate is also. As mentioned early, this is equivalent to interchanging Left and Right. In Normal-play $G + \overleftrightarrow{G} \sim 0$, but in \mathbb{GS} this is not necessarily true. For example, if $G = \langle \emptyset^\ell \mid \emptyset^r \rangle$, then conjugate is $\overleftrightarrow{G} = \langle \emptyset^{-r} \mid \emptyset^{-\ell} \rangle$ and $G + \overleftrightarrow{G} \sim 0$ if and only if $\ell = r$.

The next result will be useful in proving the Conjugate Property in Section 4.

Lemma 16. *Let $G, H \in \mathbb{GS}$. If $G \succ 0$ and $H \succcurlyeq 0$ then $G + H \succ 0$.*

Proof. By Corollary 13, we already know that $G + H \succcurlyeq 0$. So, it is enough to show that $G + H \not\sim 0$. Since $G \succ 0$ then, without loss of generality, we may assume that $Ls(G + X) > Ls(X)$ for some X . Because $H \succcurlyeq 0$, we have $Ls(G + X + H) \geq Ls(G + X + 0) = Ls(G + X) > Ls(X)$, and therefore $G + H \not\sim 0$. \square

Lemma 16 shows that the somewhat bizarre event, that the sum of two strictly positive elements is zero, does not happen in \mathbb{GS} . Still \mathbb{GS} is not archimedean. It is well known that Normal-play games have hierarchies of infinitesimal elements, and these structures are inherited in guaranteed scoring play.

Observation 17. If a partially ordered monoid is *archimedean* (that is $\forall a > 0, \forall b > 0, \exists n \in \mathbb{N}$ such that $n.a > b$) then it has no *infinitesimal elements* (given $b > a > 0$, then a is *infinitesimal* with respect to b if, $\forall n \in \mathbb{N}, n.a < b$). A natural question comes up: is \mathbb{GS} archimedean? Because Normal-play games are order-embedded in \mathbb{GS} (Theorem 19), $G = \langle \langle \emptyset^0 \mid \emptyset^0 \rangle \mid \langle \langle \emptyset^0 \mid \emptyset^0 \rangle \mid \langle \emptyset^0 \mid \emptyset^0 \rangle \rangle \rangle > 0$ is infinitesimal with respect to $H = \langle \langle \emptyset^0 \mid \emptyset^0 \rangle \mid \emptyset^0 \rangle > 0$. That happens because G is the image of $\uparrow = \{0 \mid *\}$ and H is the image of $1 = \{0 \mid \}$ via order embedding, and \uparrow is infinitesimal with respect to 1 in the Normal-play structure. Hence \mathbb{GS} is not archimedean.

2.2 Relation between Normal-play and Guaranteed games

One of the main results in [13] is that Normal-play games are order-embedded in \mathbb{GS} .

Definition 18. For a Normal-play game G , let \widehat{G} be the scoring game obtained by replacing each empty set, \emptyset , in G by the atom \emptyset^0 .

This operation retains the game tree structure. For example, the leaves of a Normal-play game tree are labelled $0 = \{\emptyset \mid \emptyset\}$ which is replaced by $0 = \langle \emptyset^0 \mid \emptyset^0 \rangle$ for the scoring game.

Theorem 19 (Normal-play order embedding [13]). *Let \mathbb{Np} be the set of Normal-play games. The set $\{\widehat{G} : G \in \mathbb{Np}\}$ induces an order-embedding of \mathbb{Np} in \mathbb{GS} .*

That is, $G \succcurlyeq H$ in Normal-play if and only if $\widehat{G} \succcurlyeq \widehat{H}$ in guaranteed games.

Let n be an integer. The games \widehat{n} are called *waiting moves*³. For example, $\widehat{0} = \langle \emptyset^0 \mid \emptyset^0 \rangle = 0$ and $\widehat{1} = \langle 0 \mid \emptyset^0 \rangle$ and $\widehat{2} = \langle \widehat{1} \mid \emptyset^0 \rangle$. Regardless, the score of a waiting move will be 0, but in a game $G + \widehat{1}$, Left has the ability to force Right to play consecutive moves in the G component.

The ability to pass may appear as something beneficial for a player. This is true in \mathbb{GS} but not necessarily in the general universe of scoring games. For example, let $G = \langle \emptyset^1 \mid \langle \emptyset^{-9} \mid \emptyset^9 \rangle \rangle$ and note $G \notin \mathbb{GS}$. Clearly Left wins playing first. In $G + \widehat{1}$, Left has no move in G and she must play her waiting move, $\widehat{1}$. Right then plays to $\langle \emptyset^{-9} \mid \emptyset^9 \rangle$. Now Left has no move and the score is -9 , a Right win.

There are useful inequalities relating Normal-play and Scoring games.

Definition 20. Let $G \in \mathbb{GS}$, and let G_x be as G , but with each atom replaced by \emptyset^x . Let $\max(G) = \max\{s \mid \emptyset^s \text{ is an atom in } G\}$ and $\min(G) = \min\{s \mid \emptyset^s \text{ is an atom in } G\}$. Set $G_{\min} = G_{\min(G)}$ and $G_{\max} = G_{\max(G)}$.

Theorem 21 (Projection Theorem). *Let $G \in \mathbb{GS}$. If $n = b(G)$ then*

1. $G_{\min} \preccurlyeq G \preccurlyeq G_{\max}$
2. $\min(G) - \widehat{n} \preccurlyeq G \preccurlyeq \max(G) + \widehat{n}$
3. $b(\min(G) - \widehat{n}) = b(\max(G) + \widehat{n}) = n$.

Proof. For part 1, for any X , we establish the inequalities $Ls(G_{\min} + X) \leq Ls(G + X)$ and $Rs(G_{\min} + X) \leq Rs(G + X)$. First, if the game $G + X$ is purely atomic, then, so is $G_{\min} + X$, and the inequalities are trivial, given Definition 20.

Consider the game, $(G_{\min} + X)^L$, obtained after an optimal move by Left. Ignoring the scores, Left can make exactly the same move in the game $G + X$, to say $(G + X)^L$. Because, we maintain an identical tree structure of the respective games, we get

$$Ls(G_{\min} + X) = Rs((G_{\min} + X)^L) \leq Rs((G + X)^L) \leq Ls(G + X),$$

³These games correspond to the ‘integers’ in Conway’s Normal-play theory, but in our context the ‘numbers’ are atomic games of the form $\langle \emptyset^s \mid \emptyset^s \rangle$.

by induction.

To prove the inequality for the Right scores, we consider the game $(G + X)^R$, obtained after an optimal move by Right. Ignoring the scores, Right can make exactly the same move in the game $G_{min} + X$, to say $(G_{min} + X)^R$. Therefore

$$Rs(G_{min} + X) \leq Ls((G_{min} + X)^R) \leq Ls((G + X)^R) = Rs(G + X),$$

by induction.

For part 2, it suffices to prove that $\min(G) - \widehat{N} \preceq G$ (and the proof of second inequality is similar). It is easy to see that $\widehat{N} - \widehat{N} \sim 0$. Therefore, it suffices to prove that $\min(G) \preceq G + \widehat{N}$, which holds if and only if $\min(G) \leq \underline{Ls}(G + \widehat{N})$ and the latter is easy to see. Part 3 follows by definition of waiting-moves. \square

2.3 Pass-allowed outcomes and Waiting moves

The following three points about the outcomes are immediate from the definitions but we state them explicitly since they will appear in many proofs.

Observation 22. Given a game $G \in \mathbb{GS}$,

- (i) $Ls(G) \geq Rs(G^L)$ for all G^L , and there is some G^L for which $Ls(G) = Rs(G^L)$;
- (ii) $Rs(G) \leq Ls(G^R)$ for all G^R , and there is some G^R for which $Rs(G) = Ls(G^R)$;
- (iii) $Ls(G + s) = Ls(G) + s$ for any number s .

The next result indicates that we only need to consider one of Ls and Rs for game comparison in \mathbb{GS} . However, in the sequel, the proofs that use induction on the birthdays need the inequalities for both the Left- and Right-scores, because we must consider games with a *fixed* birthday. However, Theorem 23 enables a simple proof of Lemma 44.

Theorem 23. *Let $G, H \in \mathbb{GS}$. Then $Ls(G + X) \geq Ls(H + X)$ for all $X \in \mathbb{GS}$ if and only if $Rs(G + Y) \geq Rs(H + Y)$ for all $Y \in \mathbb{GS}$.*

Proof. The proof depends on the following result.

Claim 1: Given G and H in \mathbb{GS} , then there exists $X \in \mathbb{GS}$ such that $Ls(G + X) > Ls(H + X)$ iff there exists a game $Y \in \mathbb{GS}$ such that $Rs(G + Y) > Rs(H + Y)$.

Proof of Claim 1. Suppose that there is some X such that $Ls(G + X) > Ls(H + X)$. Let $M = \max\{Ls(G + X) - Rs(G^R) : G^R \in G^{\mathcal{R}}\}$ and let $G^{R'}$ be an option where $M = Ls(G + X) - Rs(G^{R'})$. Put $Y = \langle M \mid X \rangle$.

Now, $Ls(G^R + Y) \geq Rs(G^R + M)$, since M is a Left option of Y . For any $G^R \in G^{\mathcal{R}}$,

$$\begin{aligned} Rs(G^R + M) &= Rs(G^R) + M, \text{ by Observation 22 (iii),} \\ &= Rs(G^R) + Ls(G + X) - Rs(G^{R'}) \\ &= Ls(G + X) + Rs(G^R) - Rs(G^{R'}) \\ &\geq Ls(G + X). \end{aligned}$$

Therefore, $Ls(G^R + Y) \geq Ls(G + X)$. Thus

$$Rs(G + Y) = \min\{Ls(G + X), Ls(G^R + Y) : G^R \in G^{\mathcal{R}}\} = Ls(G + X)$$

Now, $Rs(G + Y) = Ls(G + X) > Ls(H + X) \geq Rs(H + Y)$ where the first inequality follows from the assumption about X , and, since X is a Right option of Y , the second inequality follows from Observation 22 (ii).

End of the proof of Claim 1.

Suppose $Ls(G + X) \geq Ls(H + X)$ for all X . By Claim 1, there is no game Y for which $Rs(G + Y) < Rs(H + Y)$, in other words, $Rs(G + Y) \geq Rs(H + Y)$ for all Y . \square

In the next definition, “pass-allowed” typically means that one player has an arbitrary number of waiting moves in another component.

Definition 24 (Pass-allowed Outcomes [13]). Let $G \in \mathbb{GS}$. Then $\underline{Ls}(G) = \min\{Ls(G - \hat{n}) : n \in \mathbb{N}_0\}$ is *Right’s pass-allowed Left-score* of G . *Left’s pass-allowed Right-score* is defined analogously, $\overline{Rs}(G) = \max\{Rs(G + \hat{n}) : n \in \mathbb{N}_0\}$. We also define $\overline{Ls}(G) = \max\{Ls(G + \hat{n}) : n \in \mathbb{N}_0\}$ and $\underline{Rs}(G) = \min\{Rs(G + \hat{n}) : n \in \mathbb{N}_0\}$.

The ‘overline’ indicates that Left can pass and the ‘underline’ that Right can pass. Note that, in $\overline{Ls}(G)$, Left can even start by passing.

Lemma 25. *Let $G \in \mathbb{GS}$. If $n \geq b(G)$ then $\underline{Ls}(G) = Ls(G - \hat{n})$ and $\overline{Rs}(G) = Rs(G + \hat{n})$.*

Proof. Suppose that $n \geq b(G)$. By Theorem 19, we have $G - \widehat{b(G)} \succcurlyeq G - \hat{n}$ which gives $Ls(G - \widehat{b(G)}) \geq Ls(G - \hat{n}) \geq \min\{Ls(G - \hat{m})\} = \underline{Ls}(G)$. Since Left begins, Right does not require more than $b(G)$ waiting-moves, until Left has run out of moves in G . Hence $\underline{Ls}(G) = \min\{Ls(G - \hat{m})\} \geq Ls(G - \widehat{b(G)})$. This proves the first claim, and the claim for the Right-score is analogous. \square

From this result, it follows that the first part of Definition 24 is equivalent to: for $G \in \mathbb{GS}$ and $n = b(G)$, $\underline{Ls}(G) = Ls(G - \hat{n})$ and $\overline{Rs}(G) = Rs(G + \hat{n})$. The pass-allowed Left- and Right-scores of a disjunctive sum of games can be bounded by pass-allowed outcomes of the respective game components.

Theorem 26 (Pass-allowed Outcomes of Disjunctive Sums). *For all $G, H \in \mathbb{GS}$ we have*

$$\underline{Ls}(G) + \underline{Rs}(H) \leq \underline{Ls}(G + H) \leq \underline{Ls}(G) + \underline{Ls}(H).$$

Symmetrically

$$\overline{Rs}(G) + \overline{Rs}(H) \leq \overline{Rs}(G + H) \leq \overline{Rs}(G) + \overline{Ls}(H).$$

Proof. Let $n = b(G)$ and $m = b(H)$ and let $N = n + m$.

Right plays second in $G + H - \widehat{N}$ which he can regard as $(G - \hat{n}) + (H - \hat{m})$. He can restrict his moves to responding only in the component in which Left has just played

and he has enough waiting moves to force Left to start both components. Thus he can achieve $\underline{Ls}(G) + \underline{Ls}(H)$, i.e., $\underline{Ls}(G + H) \leq \underline{Ls}(G) + \underline{Ls}(H)$.

In the global game $G + H$, suppose that Right responds in H to Left's first move in G , then, for the rest of the game, Left can copy each local move in the global setting and has enough waiting moves to achieve a score of $\underline{Ls}(G) + \underline{Rs}(H)$. Since she has other strategies, we have $\underline{Ls}(G) + \underline{Rs}(H) \leq \underline{Ls}(G + H)$. The other inequality is proved analogously. \square

The results for the rest of the paper are sometimes stated only for Left. The proofs for Right are the same with the roles of Left and Right interchanged.

Corollary 27. *Let $G, H \in \mathbb{GS}$. If $H = \langle \emptyset^h \mid H^R \rangle$ then $Ls(G + H) \geq \underline{Ls}(G + H) = \underline{Ls}(G) + h$.*

Proof. By Theorem 26 it suffices to show that $\underline{Ls}(H) = \underline{Rs}(H) = h$. Since Left starts, $\underline{Ls}(H) = h$. Now $\underline{Rs}(H) \leq h$, since Right can achieve the score h by passing. Since $H \in \mathbb{GS}$ then $h = \min\{x : \emptyset^x \text{ is an atom in } H\}$. Hence $\underline{Rs}(H) = h$. That $Ls(G + H) \geq \underline{Ls}(G + H)$ is by definition. \square

Definition 28. Let $s \in \mathbb{R}$ and $G \in \mathbb{GS}$. The game G is *left- s -protected* if $\underline{Ls}(G) \geq s$ and either G is right-atomic or for all G^R , there exists G^{RL} such that G^{RL} is left- s -protected. Similarly, G is *right- s -protected* if $\overline{Rs}(G) \leq s$ and, for all G^L , there exists G^{LR} such that G^{LR} is right- s -protected.

In [13] we prove a necessary and sufficient condition for a game to be greater than or equal to a number.

Theorem 29 (A Generalized Ettinger's Theorem [13]). *Let $s \in \mathbb{R}$ and $G \in \mathbb{GS}$. Then $G \succcurlyeq s$ if and only if G is left- s -protected.*

3 Reductions and Canonical form

The reduction results, Theorems 32, 34, and 36, give conditions under which the options of a game can be modified resulting in a game in the same equivalence class. In all cases, it is easy to check that the new game is also in \mathbb{GS} . Theorem 37 requires an explicit check that the modified game is a guaranteed game. In Normal-play games, the reduction procedures result in a unique game, which also has minimum birthday, called the 'canonical form'. It is noted by Johnson that both the scoring games he studied and those studied by Ettinger there may be many equivalent games with the minimum birthday. The same is true for guaranteed games. However, Theorem 37 gives a reduction that while it does not necessarily reduce the birthday does lead to a unique reduced game.

The results in this section will often involve showing that $G \succcurlyeq H$ or $G \sim H$ for some games G, H where both have the same right options and they differ only slightly in the left options. Strategically, one would believe that only the non-common left options need to be considered in inductive proofs, that is, the positions of $(G^L \setminus H^L) \cup (H^L \setminus G^L)$. The next lemma shows that this is true.

Lemma 30. *Let F and K be guaranteed games with the same sets of right options, and in case this set is empty, the atoms are identical. Let X be a guaranteed game.*

1. *If $Ls(F + X^R) = Ls(K + X^R)$ for all $X^R \in X^R$ then $Rs(F + X) = Rs(K + X)$.*
2. *If $Rs(F + X^L) \geq Rs(K + X^L)$, for all $X^L \in X^L$, and $Rs(F^L + X) = Ls(F + X)$, for some $F^L \in F^L \cap K^L$, then $Ls(F + X) \geq Ls(K + X)$.*

Proof. Part 1: We prove the ‘ \geq ’ inequality and then ‘ \leq ’ follows by symmetry. If Right’s best move in $F + X$ is obtained in the X component, then $Rs(F + X) = Ls(F + X^R) \geq Ls(K + X^R) \geq \min\{Ls((K + X)^R)\} = Rs(K + X)$. Otherwise, if Right’s best move is in the F component, then he achieve a score at least as good in $K + X$ by mimicking. If there are no right-options in $F + X$ then neither are there any in $K + X$. Then, by assumption, the right-atom in $F + X$ is identical to the right-atom in $K + X$, and hence the Right-scores are identical.

The proof of part 2 is very similar to that of part 1, since the respective Right-scores are obtained via a common option. □

For example, in part 2 of Lemma 30, if $Rs(F^L + X) = Ls(F + X)$, for some $F^L \in F^L \setminus K^L$, then the inequality $Ls(F + X) \geq Ls(K + X)$ does not follow directly. As we will see later in this section, when it holds, it is by some other property of the games F and K .

The next result re-affirms that provided a player has at least one option then adding another option cannot do any harm. This is not true if the player has no options. For example, consider $G = \langle \emptyset^1 \mid 2 \rangle$, now adding the left option -1 to G gives the game $H = \langle -1 \mid 2 \rangle$. But, since $Ls(G) = 1$ and $Ls(H) = 0$ then $H \not\geq G$.

Lemma 31 (Monotone Principle). *Let $G \in \mathbb{GS}$. If $|G^L| \geq 1$ then for any $A \in \mathbb{GS}$, $\langle G^L \cup A \mid G^R \rangle \geq G$.*

Proof. The proof is clear since Left never has to use the new option. □

3.1 Reductions

We first consider the most straightforward reduction, that of removing dominated options. For this to be possible we require at least two left options.

Theorem 32 (Domination). *Let $G \in \mathbb{GS}$ and suppose $A, B \in G^L$ with $A \preceq B$. Let $H = \langle G^L \setminus \{A\} \mid G^R \rangle$. Then $H \in \mathbb{GS}$ and $G \sim H$.*

Proof. Note that $H \in \mathbb{GS}$, because H is not atomic (at least B is a left option) and $G \in \mathbb{GS}$. By the monotone principle, Lemma 31, $G \geq H$. Therefore we only have to prove that $H \geq G$. For this, we need to show that $Ls(H + X) \geq Ls(G + X)$ and $Rs(H + X) \geq Rs(G + X)$ for all X . We will proceed by induction on the birthday of X . Fix $X \in \mathbb{GS}$.

By induction, for each $X^R \in X^R$, we know that $Ls(H + X^R) \geq Ls(G + X^R)$. Thus from Lemma 30(1), it follows that $Rs(H + X) \geq Rs(G + X)$.

Now consider the Left-scores. By induction, for each $X^L \in X^L$, we know that $Rs(H + X^L) \geq Rs(G + X^L)$, that is the first condition of Lemma 30(2) is satisfied. By assumption, the only non-common option is $A \in G \setminus H$. Therefore, by Lemma 30(2), it suffices to study the case $Rs(G + X) = Rs(A + X)$. Since $A \preceq B$, we get $Rs(H + X) \geq Rs(B + X) \geq Rs(A + X) = Rs(G + X)$. Hence $H \succeq G$, and so $H \sim G$. \square

We remind the reader that while we only define the following concepts from Left's perspective, the corresponding Right concepts are defined analogously.

Definition 33. For a game G , suppose there are followers $A \in G^L$ and $B \in A^R$ with $B \preceq G$. Then the Left option A is *reversible*, and sometimes, to be specific, A is said to be *reversible through* its right option B . In addition, B is called a *reversing* option for A and, if B^L is non-empty then B^L is a *replacement set* for A . In this case, A is said to be *non-atomic-reversible*. If the reversing option is left-atomic, that is, if $B^L = \emptyset^L$, then A is said to be *atomic-reversible*.

If Left were to play a reversible option then Right has a move that retains or improves his situation. Indeed, it is the basis for the second reduction. In Normal-play games, *bypassing a reversible option* is to replace a reversible option by its replacement set, even if the replacement set is empty. This results in a simpler game equal to the original. In \mathbb{CS} , there are more cases to consider. We begin by showing that, if the replacement set is non-empty, then bypassing a reversible option does result in a new but equal game. In Theorem 36, we then treat the case of an atomic-reversible option.

Theorem 34 (Reversibility 1). *Let $G \in \mathbb{CS}$ and suppose that A is a left option of G reversible through B . If B^L is non-empty, then $G \sim \langle G^L \setminus \{A\}, B^L \mid G^R \rangle$.*

Proof. Consider G, A, B as in the statement of the theorem, and recall that, since B is a reversing right option, $G \succeq B$. Moreover, there is a replacement set B^L , so we let $H = \langle G^L \setminus \{A\}, B^L \mid G^R \rangle$. We need to prove that $H \sim G$, i.e., $Rs(G + X) = Rs(H + X)$ and $Rs(G + X) = Rs(H + X)$ for all X . We proceed by induction on the birthday of X .

Fix X . Note that B^L, G^L and H^L are non-empty so that $B + X, G + X$ and $H + X$ all have Left options. Moreover $A + X$ has Right options.

For the Right-scores: by induction we have that $Rs(G + X^R) = Rs(H + X^R)$ for any $X^R \in X^R$. Thus by Lemma 30(1), we have $Rs(G + X) = Rs(H + X)$.

For the Left-scores, and within the induction, we first prove a necessary inequality.

Claim 1: $H \succeq B$.

Proof of Claim 1: For the Left-scores: if $C \in B^L$ then $C \in H^L$ and thus $Rs(H + X) \geq Rs(C + X)$. If $Rs(B + X) = Rs(C + X)$ for some $C \in B^L$ then it follows that $Rs(H + X) \geq Rs(B + X)$. Otherwise, $Rs(B + X) = Rs(B + X^L)$. By induction, $Rs(B + X^L) \leq Rs(H + X^L)$ and since $Rs(H + X^L) \leq Rs(H + X)$, we get $Rs(H + X) \geq Rs(B + X)$.

For the Right-scores: by the argument before the claim, $Rs(H + X) = Rs(G + X)$. Since $G \succeq B$ then $Rs(G + X) \geq Rs(B + X)$ and thus $Rs(H + X) \geq Rs(B + X)$. This concludes the proof of Claim 1.

By induction we have that $Rs(G + X^L) = Rs(H + X^L)$ for any $X^L \in X^\mathcal{L}$, which gives the first assumptions of Lemma 30(2). It remains to consider the cases where the second assumption does not hold.

First, we consider $Ls(G + X)$. By Lemma 30(2), the remaining case to consider is $Ls(G + X) = Rs(A + X)$. Since $B \in A^\mathcal{R}$, we have $Rs(A + X) \leq Ls(B + X)$. By Claim 1, we know that $Ls(H + X) \geq Ls(B + X)$. By combining these inequalities we obtain $Ls(G + X) \leq Ls(H + X)$.

Secondly, we consider $Ls(H + X)$. The only possibly non-common option is $C \in B^\mathcal{L}$, with $C \in H^\mathcal{L} \setminus G^\mathcal{L}$, and where we, by Lemma 30(2), may assume that $Ls(H + X) = Rs(C + X)$. Moreover, $G \succ B$, and thus $Ls(H + X) = Rs(C + X) \leq Ls(B + X) \leq Ls(G + X)$. \square

For the next reduction theorem, there is no replacement set, because the reversing option is left-atomic. We first prove a strategic fact about atomic reversible options—nobody wants to play to one!

Lemma 35 (Weak Avoidance Property). *Let $G \in \mathbb{GS}$ and let A be an atomic-reversible Left option of G . For any game X , if $X^\mathcal{L} \neq \emptyset$ then there is an X^L such that $Rs(A + X) \leq Rs(G + X^L)$.*

Proof. Let A be an atomic-reversible Left option of G and let $B \in A^\mathcal{R}$ be a reversing option for A . Assume that X has a left option.

By definition, $G \succ B$ and $B = \langle \emptyset^\ell \mid B^\mathcal{R} \rangle$. Since B is a right option of A then $A + X \neq \langle (A + X)^\mathcal{L} \mid \emptyset^r \rangle$. Consequently,

$$\begin{aligned} Rs(A + X) &\leq Ls(B + X) \\ &= Rs(B + X^L), \text{ for some } X^L, \\ &\leq Rs(G + X^L), \text{ since } G \succ B. \end{aligned} \quad \square$$

The next reduction is about replacing a left atomic-reversible option A in a game G . There are two cases. If Left has a ‘good’ move other than A then A can be eliminated. Otherwise, we can only simplify A .

Theorem 36 (Atomic Reversibility). *Let $G \in \mathbb{GS}$ and suppose that $A \in G^\mathcal{L}$ is reversible through $B = \langle \emptyset^\ell \mid B^\mathcal{R} \rangle$.*

1. *If $\underline{Ls}(G) = \underline{Rs}(G^L)$ for some $G^L \in G^\mathcal{L} \setminus \{A\}$, then $G \sim \langle G^\mathcal{L} \setminus \{A\} \mid G^\mathcal{R} \rangle$;*
2. *If $\underline{Ls}(G) = \underline{Rs}(A) > \underline{Rs}(G^L)$ for all $G^L \in G^\mathcal{L} \setminus \{A\}$, then $G \sim \langle \langle \emptyset^\ell \mid B \rangle, G^\mathcal{L} \setminus \{A\} \mid G^\mathcal{R} \rangle$.*

Proof. Let $A \in G^\mathcal{L}$ and $B \in A^\mathcal{R}$ be as in the statement of the theorem, with $G \succ B$. First an observation:

Claim 1: $\underline{Ls}(G) \geq \ell$.

Let n be the birthday of G and since B is a proper follower of G , the birthday of B is less than n . Since $G \succ B$, from Lemma 25 we have

$$Ls(G - \hat{n}) \geq Ls(B - \hat{n}) = Ls(\langle \emptyset^\ell \mid B^\mathcal{R} \rangle - \hat{n}) = \ell,$$

where n is the birthday of G . This proves the claim.

The proof of the equality in both parts will proceed by induction on the birthday of X . Again, in both parts, let H be the game that we wish to show is equal to G . We have, by induction, that $Ls(G + X^R) = Ls(H + X^R)$, and by $G^R = H^R$, from Lemma 30(1), it then follows that $Rs(G + X) = Rs(H + X)$.

It remains to show that $Ls(G + X) = Ls(H + X)$ in both parts.

Part 1. The assumption is that there exists $C \in G^L \setminus \{A\}$ with $\underline{Ls}(G) = \underline{Rs}(C)$. Let $H = \langle G^L \setminus \{A\} \mid G^R \rangle$. Note that both $G + X$ and $H + X$ have left options since C is in both G^L and H^L . From Lemma 31 we have $G \succcurlyeq H$, and thus it remains to show that $Ls(H + X) \geq Ls(G + X)$.

By Lemma 30(2), we need only consider the case $Ls(G + X) = Rs(A + X)$. Note that X must be left-atomic; else, by Lemma 35, there would exist $X^L \in X^L$ with $Rs(A + X) \leq Rs(G + X^L)$. Therefore, we may assume that $X = \langle \emptyset^x \mid X^R \rangle$. In this case, since $C \neq A$ is the best pass-allowed Left move in G then this is also true for H . We now have the string of inequalities,

$$Ls(H + X) \geq \underline{Ls}(H + X) = \underline{Ls}(H) + x = \underline{Rs}(C) + x = \underline{Ls}(G) + x \geq \ell + x,$$

where the first inequalities are from Corollary 27, and the last inequality is by Claim 1. Since B is a right option of A , we also have that

$$Ls(G + X) = Rs(A + X) \leq Ls(B + X) = \ell + x.$$

Thus $Ls(G + X) \leq Ls(H + X)$ and this completes the proof of part 1 of the theorem.

Part 2. In this case, the Right's-pass-allowed Left-score of G is obtained only through A . Let $H = \langle \langle \emptyset^\ell \mid B \rangle, G^L \setminus \{A\} \mid G^R \rangle$. Recall that it only remains to show that $Ls(G + X) = Ls(H + X)$, and that, by Lemma 30, we only need to consider the non-common options in the respective games.

First, suppose $Ls(H + X) = Rs(\langle \emptyset^\ell \mid B \rangle + X)$. Since $G \succcurlyeq B$ and B is a right option of $\langle \emptyset^\ell \mid B \rangle$, we have the inequalities

$$Ls(H + X) = Rs(\langle \emptyset^\ell \mid B \rangle + X) \leq Ls(B + X) \leq Ls(G + X).$$

Thus, by Lemma 30(2), $Ls(H + X) \leq Ls(G + X)$.

Secondly, suppose that $Ls(G + X) = Rs(A + X)$. Note that if X has a left option then, by Lemma 35, there exists some $X^L \in X^L$ such that $Ls(G + X) = Rs(G + X^L)$. By induction, then $Rs(G + X^L) = Rs(H + X^L) \leq Ls(H + X)$. Therefore, we may assume that $X = \langle \emptyset^\ell \mid X^R \rangle$. Since B is a right option of A , the only Left option in G , we have the string of inequalities

$$Ls(G + X) = Rs(A + X) \leq Ls(B + X) = \ell + x.$$

To show that $Ls(H + X) \geq \ell + x$, we note that it suffices for Left to move in the H component to $\langle \ell \mid B \rangle \in H^L$, since all scores in $B = \langle \ell \mid B^R \rangle$ are at least ℓ . Thus, by Lemma 30(2), we now have $Ls(G + X) \leq Ls(H + X)$.

From this, together with the conclusion of the previous paragraph, we have $Ls(G + X) = Ls(H + X)$. \square

Suppose that $G \in \mathbb{GS}$ has an atomic-reversible option, $A \in G^{\mathcal{L}}$, with the reversing option $B = \langle \emptyset^{\ell} \mid B^{\mathcal{R}} \rangle$. Given the reduction in Theorem 36(2), a remaining problem of atomic reducibility is to find a simplest substitution for B . In Section 3.3, we will show that the following result solves this problem.

Theorem 37 (Substitution Theorem). *Let A be an atomic-reversible Left option of $G \in \mathbb{GS}$ and let $B = \langle \emptyset^{\ell} \mid B^{\mathcal{R}} \rangle$ be a reversing Right option of A . Suppose also that $\underline{Ls}(G) = \underline{Rs}(A) > \underline{Rs}(G^{\mathcal{L}})$ for all $G^{\mathcal{L}} \in G^{\mathcal{L}} \setminus \{A\}$.*

1. *There exists a smallest nonnegative integer n such that $G \succcurlyeq \ell - \widehat{n}$ and $G \sim \langle \ell - (\widehat{n+1}), G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}} \rangle$.*

2. *If A is the only Left option of G and $\langle \emptyset^{\ell} \mid G^{\mathcal{R}} \rangle \in \mathbb{GS}$, then $G \sim \langle \emptyset^{\ell} \mid G^{\mathcal{R}} \rangle$.*

Proof. Case 1: Let $m = b(B)$. By assumption $G \succcurlyeq B$ and, by Theorem 21(2), $B \succcurlyeq \ell - \widehat{m}$, and thus $G \succcurlyeq \ell - \widehat{m}$. Since m is a nonnegative integer, the existence part is clear. Let n be the minimum nonnegative integer such that $G \succcurlyeq \ell - \widehat{n}$.

Let $K = \ell - (\widehat{n+1})$, which upon expanding is $\langle \emptyset^{\ell} \mid \ell - \widehat{n} \rangle$, let $H = \langle K, G^{\mathcal{L}} \setminus \{A\} \mid G^{\mathcal{R}} \rangle$, and let $G' = \langle K, G^{\mathcal{L}} \mid G^{\mathcal{R}} \rangle$. By Lemma 31 and the definition of n , we have $G' \succcurlyeq G \succcurlyeq \ell - \widehat{n}$. Hence $\ell - \widehat{n}$ is a reversing game in both G and G' , and both A and K are atomic-reversible Left options in G' .

Since G satisfies part 2 of Theorem 36. Then Claim 1 in Theorem 36 can be strengthened.

Claim 1: $\underline{Ls}(G) = \ell$.

Proof of Claim 1: This is true because $\underline{Ls}(G) = \underline{Rs}(A) \leq \underline{Ls}(B) = \ell$.

Hence, $\ell = \underline{Ls}(G) = \underline{Rs}(A)$. We also have that $\underline{Rs}(K) = \ell$. It is now easy to see that $\underline{Ls}(G') = \ell$. Thus we have two atomic-reversible Left options in G' , and so we can apply part 1 in Theorem 36. We get that $G' \sim G$ since K is an atomic-reversible Left option in G' . Moreover, $G' \sim H$, since A is also atomic-reversible. This finishes the proof of Case 1.

Case 2: This is the case where $G^{\mathcal{L}} = \{A\}$. We put $H = \langle \emptyset^{\ell} \mid G^{\mathcal{R}} \rangle \in \mathbb{GS}$. To prove $G \sim H$ we proceed by induction on the birthday of the distinguishing game X .

From Lemma 30(1) and induction, we have that $Rs(G + X) = Rs(H + X)$, for any $X \in \mathbb{GS}$.

For the Left-scores, from Case 1, we know that $G \sim \langle \ell - (\widehat{n+1}) \mid G^{\mathcal{R}} \rangle$. Therefore, in the case $X = \langle \emptyset^x \mid \emptyset^y \rangle$ it is easy to see that $Ls(H + X) = \ell + x \leq Ls(G + X)$, since $y \geq x$. Moreover, we also have $Ls(G + X) = Rs(A + X) \leq Ls(B + X) = \ell + x$, which thus proves equality.

If $X^{\mathcal{L}} = \emptyset^x$ and $X^{\mathcal{R}} \neq \emptyset$, then, $Ls(G + X) = Rs(\ell - (\widehat{n+1}) + X)$ and it is clear that Right can obtain the score $\ell + x$ by playing to $\ell - \widehat{n} + X$. Since both games are left-atomic and in \mathbb{GS} , then $Rs(\ell - (\widehat{n+1}) + X) \geq \ell + x$, so in fact, equality holds. Hence, in this case, we get $Ls(G + X) = \ell + x = Ls(H + X)$.

If $X^{\mathcal{L}} \neq \emptyset$, then by Lemma 35 (weak avoidance), there is some $X^{\mathcal{L}}$ such that $Rs(A + X) \leq Rs(G + X^{\mathcal{L}})$. Therefore, $Ls(G + X) = \max\{Rs(G + X^{\mathcal{L}}) : X^{\mathcal{L}} \in X^{\mathcal{L}}\}$. Also,

$Ls(H + X) = \max\{Rs(H + X^L) : X^L \in X^{\mathcal{L}}\}$ since there is no Left move in H . By induction, $Rs(H + X^L) = Rs(G + X^L)$ and consequently, $Ls(G + X) = Ls(H + X)$. \square

In summary, there are four types of reductions for Left:

1. Erase dominated options;
2. Reverse non-atomic-reversible options;
3. Replace atomic-reversible options by $\ell - \widehat{n+1}$;
4. If possible, when an atomic-reversible is the only left option, substitute \emptyset^ℓ for $\ell - \widehat{n+1}$ (which could give a game of lower rank).

Here ℓ is a real number and $n \geq 0$ is an integer (as given in Theorem 37) providing a number of waiting moves for Right. We have the following definition.

Definition 38. A game $G \in \mathbb{GS}$ is said to be *reduced* if none of Theorems 32, 34, 36, or 37 can be applied to G to obtain an equivalent game with different sets of options.

3.2 Constructive game comparison

We wish to prove that, for a given guaranteed scoring game, there is one unique reduced game representing the full congruence class, a canonical form. To this purpose, in this subsection, we first develop another major tool (also to be used in Section 4) of constructive game comparison. The existence of a canonical form is far from obvious, as the order of reduction can vary. In Normal-play, the proof of uniqueness uses the fact that if $G \sim H$ then $G - H \sim 0$. However, in (guaranteed) scoring play, $G \sim H$ does *not* imply $G + \overleftrightarrow{H} \sim 0$. We use an idea, ‘linked’, adapted from Siegel[16], which only uses the partial order. To fully adapt it for guaranteed games, we require a generalization of Theorem 29 (which in its turn is a generalization of Ettinger’s [7] theorem for dicot games).

Recall that $\overleftrightarrow{G} = \langle \overleftrightarrow{G^{\mathcal{R}}} \mid \overleftrightarrow{G^{\mathcal{L}}} \rangle$, where the conjugate is applied to the respective options, and if, for example, $G^{\mathcal{R}} = \emptyset^r$, then $\overleftrightarrow{G^{\mathcal{R}}} = \emptyset^{-r}$.

Definition 39. Let $G \in \mathbb{GS}$ and let $m(G) = \max\{|t| : \emptyset^t \text{ is an atom in } G\}$. Let r, s be two nonnegative real numbers. The (r, s) -adjoint of G (or just *adjoint*) is $G_{r,s}^\circ = \overleftrightarrow{G} + \langle \emptyset^{-m(G)-r-1} \mid \emptyset^{m(G)+s+1} \rangle$.

Since $-m(G) - r - 1 \leq m(G) + s + 1$, it follows that $G_{r,s}^\circ \in \mathbb{GS}$.

Theorem 40. Given $G \in \mathbb{GS}$ and two nonnegative real numbers r, s then $Ls(G + G_{r,s}^\circ) < -r$ and $Rs(G + G_{r,s}^\circ) > s$.

Proof. In the game $G + \overset{\leftrightarrow}{G} + \langle \emptyset^{-m(G)-r-1} \mid \emptyset^{m(G)+s+1} \rangle$, the second player can mirror each move in the $G + \overset{\leftrightarrow}{G}$ component, and there are no other moves since the remaining component is purely-atomic. Therefore,

$$Ls(G + G_{r,s}^\circ) = Ls(G + \overset{\leftrightarrow}{G}) - m(G) - r - 1 \leq m(G) - m(G) - r - 1 < -r.$$

The bound for the Right-score is obtained similarly. □

Observation 41. If $r = s = 0$ in Definition 39, then Theorem 40 corresponds to the particular case where $Ls(G + G_{0,0}^\circ) < 0$ and $Rs(G + G_{0,0}^\circ) > 0$. This will suffice in the below proof of Lemma 45. Thus we will use the somewhat simpler notation G° for the $(0, 0)$ -adjoint of G .

Definition 42. Let $G, H \in \mathbb{GS}$. We say that H is linked to G (by T) if there exists some $T \in \mathbb{GS}$ such that $Ls(H + T) < 0 < Rs(G + T)$.

Note that, if H is linked to G , it is not necessarily true that G is linked to H .

Lemma 43. Let $G, H \in \mathbb{GS}$. If $H \succ G$ then H is linked to no G^L and no H^R is linked to G .

Proof. Consider $T \in \mathbb{GS}$ such that $Ls(H + T) < 0$. Because $H \succ G$, we have $Ls(H + T) \geq Ls(G + T)$. Therefore, $0 > Ls(H + T) \geq Ls(G + T) \geq Rs(G^L + T)$, for any G^L . Analogously, consider $T \in \mathbb{GS}$ such that $0 < Rs(G + T)$, we have $0 < Rs(G + T) \leq Rs(H + T) \leq Ls(H^R + T)$, for any H^R . □

Lemma 44. Let $G, H \in \mathbb{GS}$. Suppose that $G \not\asymp H$.

1. There exists $X \in \mathbb{GS}$ such that $Ls(G + X) < 0 < Ls(H + X)$
2. There exists $Y \in \mathbb{GS}$ such that $Rs(G + Y) < 0 < Rs(H + Y)$.

Proof. By assumption, there exists X such that $Ls(G + X) < Ls(H + X)$ or there exists Y such that $Rs(G + Y) < Rs(H + Y)$. By Theorem 23 (the claim in its proof), we have that

$$\exists X : Ls(G + X) < Ls(H + X) \Leftrightarrow \exists Y : Rs(G + Y) < Rs(H + Y).$$

Suppose that there exists Z such that $\alpha = Ls(G + Z) < Ls(H + Z) = \beta$. Let $X = Z - (\alpha + \beta)/2$. Then $Ls(G + X) = Ls(G + Z) - (\alpha + \beta)/2 = (\alpha - \beta)/2 < 0$ and $0 < (\beta - \alpha)/2 = Ls(H + Z) - (\alpha + \beta)/2 = Ls(H + X)$. Hence the first part holds. The proof of the other part is analogous. □

Lemma 45. Let $G, H \in \mathbb{GS}$. Then G is linked to H if and only if no $G^L \succ H$ and no $H^R \preccurlyeq G$.

Proof. (\Rightarrow): Consider G linked to H by T , that is $Ls(G + T) < 0 < Rs(H + T)$. It follows

1. $Rs(G^L + T) \leq Ls(G + T) < 0 < Rs(H + T)$, for any G^L

2. $Ls(G + T) < 0 < Rs(H + T) \leq Ls(H^R + T)$, for any H^R .

The two items contradict both $G^L \succcurlyeq H$ and $H^R \preccurlyeq G$.

(\Leftarrow): Suppose no $G^L \succcurlyeq H$ and no $H^R \preccurlyeq G$. Consider $G^{\mathcal{L}} = \{G^{L_1}, \dots, G^{L_k}\}$ and $H^{\mathcal{R}} = \{H^{R_1}, \dots, H^{R_\ell}\}$, including the case that either or both are atoms. By Lemma 44, for each i , $1 \leq i \leq k$, we can define X_i such that $Ls(G^{L_i} + X_i) < 0 < Ls(H + X_i)$, and, for each j , $1 \leq j \leq \ell$, we can define Y_j such that $Rs(G + Y_j) < 0 < Rs(H^{R_j} + Y_j)$. Let $T = \langle T^{\mathcal{L}} \mid T^{\mathcal{R}} \rangle$ where

$$T^{\mathcal{L}} = \begin{cases} \{-g - 1\}, & \text{if } G = \langle G^{\mathcal{L}} \mid \emptyset^g \rangle \text{ and also } H \text{ is right-atomic;} \\ G^{\mathcal{R}^\circ} \cup \bigcup_{j=1}^{\ell} \{Y_j\}, & \text{otherwise.} \end{cases}$$

$$T^{\mathcal{R}} = \begin{cases} \{-h + 1\}, & \text{if } H = \langle \emptyset^h \mid H^{\mathcal{R}} \rangle \text{ and also } G \text{ is left-atomic;} \\ H^{\mathcal{L}^\circ} \cup \bigcup_{i=1}^k \{X_i\}, & \text{otherwise.} \end{cases}$$

Here $G^{\mathcal{R}^\circ}$ denotes the set of $(0, 0)$ -adjoints of the Right options of G , and if there is no Right option of G , then it is defined as the empty set. Note that, in this case, if also $H^{\mathcal{R}}$ is empty, then the first line of the definition of $T^{\mathcal{L}}$ applies, so $T^{\mathcal{L}}$ (and symmetrically for $T^{\mathcal{R}}$) is never empty. Thus $T \in \mathbb{GS}$, because each option is a guaranteed game. (For example, if both $G = \langle \emptyset^a \mid \emptyset^b \rangle$ and $H = \langle \emptyset^c \mid \emptyset^d \rangle$ are purely-atomic guaranteed games, then $T = \langle -b - 1 \mid -c + 1 \rangle$ is trivially guaranteed, because each player has an option to a number. Note also that the scores a and d become irrelevant in this construction.)

Consider first $G + T$ with $T^{\mathcal{L}}$ as in the second line of the definition. It follows that $Ls(G + T) < 0$ because:

1. if Left plays to $G^{L_i} + T$, then, because there is a Left option, the second line applies also to $T^{\mathcal{R}}$. Right answers with $G^{L_i} + X_i$, and $Ls(G^{L_i} + X_i) < 0$, by definition of X_i ;
2. if Left plays to $G + G^{R^\circ}$, Right answers in G to the corresponding G^R and $Ls(G^R + G^{R^\circ}) < 0$, by Observation 41;
3. if Left plays to $G + Y_i$, then by construction, $Rs(G + Y_i) < 0$.

Consider next $G + T$ with $T^{\mathcal{L}}$ in the first line of the definition. We get that $Ls(G + T) < 0$ because, either

1. $Ls(G + T) = Rs(G + T^{\mathcal{L}}) = Rs(G - g - 1) = g - g - 1 = -1 < 0$; or
2. $Ls(G + T) = Rs(G^{L_i} + T) \leq Ls(G^{L_i} + X_i) < 0$.

The last case follows because there are left options in G , so the second line of the definition of $T^{\mathcal{R}}$ applies. In every case, $Ls(G + T) < 0$. The argument for $Rs(H + T) > 0$ is analogous. Therefore, $Ls(G + T) < 0 < Rs(H + T)$ and G is linked to H by T . \square

In the following result we extend Theorem 29 by using the linked results. From an algorithmic point of view, when comparing games G and H , it ultimately removes the need to consider $G + X$ and $H + X$ for all X .⁴

Theorem 46 (Constructive Comparison). *Let $G, H \in \mathbb{GS}$. Then, $G \succcurlyeq H$ if and only if*

1. $\underline{Ls}(G) \geq \underline{Ls}(H)$ and $\overline{Rs}(G) \geq \overline{Rs}(H)$;
2. For all $H^L \in H^\mathcal{L}$, either $\exists G^L \in G^\mathcal{L} : G^L \succcurlyeq H^L$ or $\exists H^{LR} \in H^{LR} : G \succcurlyeq H^{LR}$;
3. For all $G^R \in G^\mathcal{R}$, either $\exists H^R \in H^\mathcal{R} : G^R \succcurlyeq H^R$ or $\exists G^{RL} \in G^{RL} : G^{RL} \succcurlyeq H$.

Proof. (\Rightarrow) Suppose that $\underline{Ls}(G) < \underline{Ls}(H)$. Then, for some n , $Ls(G - \hat{n}) < Ls(H - \hat{n})$. This, however, contradicts $G \succcurlyeq H$ and so part 1 holds.

Consider $H^L \in H^\mathcal{L}$. Because $G \succcurlyeq H$, by Lemma 43, G is not linked to H^L . Therefore, by Lemma 45, we have $\exists G^L \in G^\mathcal{L} : G^L \succcurlyeq H^L$ or $\exists H^{LR} \in H^{LR} : G \succcurlyeq H^{LR}$. The proof of part 3 is similar.

(\Leftarrow) Assume 1, 2 and 3, and also suppose that $G \not\succeq H$. By the definition of the partial order, there is a distinguishing game X such that either $Ls(G + X) < Ls(H + X)$ or $Rs(G + X) < Rs(H + X)$. Choose X to be of the smallest birthday such that $Ls(G + X) < Ls(H + X)$. There are three cases:

(a) $H + X = \langle \emptyset^h \mid H^\mathcal{R} \rangle + \langle \emptyset^x \mid X^\mathcal{R} \rangle$.

In this case, $Ls(H + X) = h + x$. On the other hand, $Ls(G + X) \geq \underline{Ls}(G + X) \geq \underline{Ls}(G) + \underline{Rs}(X)$ (this last inequality holds by Theorem 26). Also, $\underline{Ls}(G) + \underline{Rs}(X) \geq \underline{Ls}(H) + x$, because $\underline{Ls}(G) \geq \underline{Ls}(H)$ and by $X \in \mathbb{GS}$, Definition 4(2). Finally, $\underline{Ls}(H) + x = h + x$ because $\underline{Ls}(H)$ is trivially equal to h . This contradicts $Ls(G + X) < Ls(H + X)$.

(b) $Ls(H + X) = Rs(H^L + X)$, for some $H^L \in H^\mathcal{L}$.

In this case, because of part 2, we have either $G^L \succcurlyeq H^L$ or $G \succcurlyeq H^{LR}$. If the first holds, then $Ls(G + X) \geq Rs(G^L + X) \geq Rs(H^L + X) = Ls(H + X)$. If the second holds, then $Ls(G + X) \geq Ls(H^{LR} + X) \geq Rs(H^L + X) = Ls(H + X)$. Both contradict the assumption $Ls(G + X) < Ls(H + X)$.

(c) $Ls(H + X) = Rs(H + X^L)$, for some $X^L \in X^\mathcal{L}$.

By the “smallest birthday” assumption, $Rs(G + X^L) \geq Rs(H + X^L)$. Therefore, $Ls(G + X) \geq Rs(G + X^L) \geq Rs(H + X^L) = Ls(H + X)$. Once more, we contradict $Ls(G + X) < Ls(H + X)$.

For the Right-scores $Rs(G + X) < Rs(H + X)$ the argument is similar. Hence, we have shown that $G \succcurlyeq H$. \square

Note that we can derive the known result, Theorem 29, as a simple corollary of Theorem 46, by letting $H = s$ be a number.

⁴This is as close as guaranteed games get to the Normal-play constructive comparison—Left wins playing second in $G - H$ iff $G \succcurlyeq H$. For not-necessarily-guaranteed scoring games, no efficient method for game comparison is known.

3.3 Uniqueness of Reduced forms

We are now able to prove the existence of a unique reduced form for a congruence class of games. We let \approx denote “identical to”, that is if $G, H \in \mathbb{GS}$, then $G \approx H$ if they have identical game tree structure and, given this structure, each atom in G corresponds to an identical atom, in precisely the same position, in the game H .

Theorem 47. *Let $G, H \in \mathbb{GS}$. If $G \sim H$ and both are reduced games, then $G \approx H$.*

Proof. We will proceed by induction on the sum of the birthdays of G and H . We will exhibit a correspondence $G^{L_i} \sim H^{L_i}$ and $G^{R_j} \sim H^{R_j}$ between the options of G and H . By induction, it will follow that $G^{L_i} \approx H^{L_i}$, for all i , and $G^{R_j} \approx H^{R_j}$, for all j , and consequently $G \approx H$.

Part 1. For the base case, if $G = \langle \emptyset^a \mid \emptyset^b \rangle$ and $H = \langle \emptyset^c \mid \emptyset^d \rangle$ then, since $G \sim H$, we must in particular have $a = Ls(G) = Ls(H) = c$ and $b = Rs(G) = Rs(H) = d$. Hence $G \approx H$.

Without loss of generality, we may assume that there is a Left option H^L . We also assume that if H^L is reversible, then, since H is reduced, it has to be atomic-reversible of the form in Theorem 37.

Part 2. Assume that H^L is not atomic-reversible.

Since $G \sim H$, of course, $G \succcurlyeq H$. From Theorem 46, there exists a G^L with $G^L \succcurlyeq H^L$ or there exists a $H^{LR} \preccurlyeq G$. Now $H^{LR} \preccurlyeq G \sim H$ would contradict that H^L is not reversible. Thus, there is some G^L with $G^L \succcurlyeq H^L$.

Suppose that G^L is atomic-reversible, that is, $G^L \sim \langle \emptyset^\ell \mid \ell - \widehat{n} \rangle \sim \ell - \widehat{n+1}$ for some nonnegative integer n and with $G \succcurlyeq \ell - \widehat{n}$. Since $G \sim H$ we also have $H \succcurlyeq \ell - \widehat{n}$. (For any real number s and nonnegative integer m , $s - \widehat{m}$ is invertible since $s - \widehat{m} + (-s + \widehat{m}) = 0$.) Therefore

$$G^L \succcurlyeq H^L \Leftrightarrow \ell - \widehat{n+1} \succcurlyeq H^L \Leftrightarrow 0 \succcurlyeq H^L - \ell + \widehat{n+1},$$

where the last equivalence is by Lemma 15. From Theorem 29, after a Left move from $H^L - \ell + \widehat{n+1}$ to $H^L - \ell + \widehat{n}$, Right must have a move to a position less than or equal to zero, say $H^{LR} - \ell + \widehat{n} \preccurlyeq 0$. The inequalities $H \succcurlyeq \ell - \widehat{n}$ and $H^{LR} - \ell + \widehat{n} \preccurlyeq 0$ give that $H \succcurlyeq H^{LR}$, which contradicts that H^L is not atomic-reversible. It follows therefore, that G^L is not atomic-reversible.

A similar argument for G^L gives a left option $H^{L'}$ such that $H^{L'} \succcurlyeq G^L$. Therefore, $H^{L'} \succcurlyeq G^L \succcurlyeq H^L$. Since there is no domination, $H^{L'} \sim H^L \sim G^L$. By induction, $H^L \approx G^L$.

The symmetric argument gives that each non-atomic option H^R is identical to some G^R . In conclusion, we have a *pairwise correspondence* between options of G and H that are not atomic-reversible.

Part 3. Assume that $A = H^L$ is atomic-reversible.

The proof is divided into two cases.

Case 1: $|H^L| > 1$.

Observe that part 2 of Theorem 36 (the atomic-reversibility theorem) applies, because if A would have been as in part 1 of that theorem, then it would have reversed out

(contradicting the assumptions on G and H). Therefore, A is the only Left option with $\underline{Ls}(H) = \underline{Rs}(A)$.

If, for every $G^L \in G^{\mathcal{L}}$ we have $\underline{Ls}(H) \neq \underline{Rs}(G^L)$, then $\underline{Ls}(G) \neq \underline{Ls}(H)$, which contradicts $G \sim H$. Thus, there is some $A' \in G^{\mathcal{L}}$ with $\underline{Ls}(H) = \underline{Rs}(A')$ and, from the pairwise correspondence for non-atomic-reversible options, it also follows that A' is atomic-reversible. Therefore, we may assume that $A = a - \widehat{n + 1}$ and that $A' = a' - \widehat{m + 1}$ for some real numbers a, a' , and some nonnegative integers, n, m . Since $\underline{Rs}(A') = \underline{Rs}(A)$ then $a = a'$. That $m = n$ follows from (Theorem 37(1)), the definition of minimal nonnegative integer, since $A^R = a - \widehat{n}$ and $A'^R = a' - \widehat{m}$ are reversing options. Therefore $A \approx A'$, and again, if there was another Left option, $G^L \in G^{\mathcal{L}}$ with $\underline{Ls}(G) = \underline{Rs}(G^L)$, then it must have been reversed out, because of the assumption of reduced form. Hence A' is the only such Left option in G .

Case 2: The only left option of H is $A = \langle \emptyset^h \mid h - \widehat{n} \rangle$, for some real number h and nonnegative integer n , that is $H = \langle \langle \emptyset^h \mid h - \widehat{n} \rangle \mid H^R \rangle$. Since H cannot be reduced further, by the second part of Theorem 37, it follows that $\langle \emptyset^h \mid H^R \rangle \notin \mathbb{GS}$. Thus there must exist an s -atom, with $s < h$, in an atomic follower of H^R .

Consider the Left options of G . By the pairwise correspondence of non-atomic-reversible options, since $H^{\mathcal{L}}$ has none then neither has $G^{\mathcal{L}}$. So, if $G^{\mathcal{L}}$ has options they are atomic-reversible.

First, suppose that $G = \langle \emptyset^h \mid G^R \rangle$.

The non-atomic-reversible right options of G and H are paired (the conclusion of Part 2 of this proof). Since $G \in \mathbb{GS}$ then \emptyset^s is not in any non-atomic-reversible right option of G and hence \emptyset^s is not in any non-atomic-reversible right option of H . Thus, either $H^R = \emptyset^s$ or H has a right atomic-reversible option $\langle s - \widehat{m} \mid \emptyset^s \rangle$. In the latter case, by Theorem 36(2) (with Left and Right interchanged) $Rs(H) = s$. Thus, in both cases, $Rs(H) = s$, from which it follows that $Rs(G) = s$ which, in turn, implies that \emptyset^s is in G^R . This again contradicts $G \in \mathbb{GS}$. Therefore, $G = \langle \emptyset^h \mid G^R \rangle$ is impossible.

Therefore $G = \langle \langle \emptyset^\ell \mid \ell - \widehat{m} \rangle \mid G^R \rangle$ for some ℓ and m . Since $Ls(G) = Ls(H)$ it follows that $\ell = h$. By Theorem 37, since $G \sim H$, the number of waiting moves (for Right), is given by exactly the same definition as for H . Hence, $m = n$ and $G^{\mathcal{L}} = \{A\}$.

In all cases, we have shown that $H^{\mathcal{L}}$ is identical to $G^{\mathcal{L}}$. The proof for H^R and G^R is similar. Consequently $G \approx H$. \square

The next result is immediate. It allows us to talk about *the canonical form* of a game/congruence class.

Corollary 48 (Canonical Form). *Let $G \in \mathbb{GS}$. There is a unique reduced form of G .*

Finally, the canonical form can be used for induction proofs since it has the minimum birthday of all games in its congruence class. Incidentally, it has the least width (number of leaves) of all such trees. However, minimum birthday and minimum width is not a characterization of canonical form. Let $G = \langle -1, 1 - \widehat{1} \mid \langle 2 \mid 2 \rangle \rangle$ and $H = \langle -1, \langle \emptyset^1 \mid \langle \emptyset^1 \mid \emptyset^2 \rangle \rangle \mid \langle 2 \mid 2 \rangle \rangle$. Then G is the canonical form of H but the game trees of the two games have the same depth and width.

4 Additive Inverses and Conjugates

From the work in Misère games comes the following concept.

Definition 49. Let \mathcal{X} be a class of combinatorial games with defined disjunctive sum and game comparison. It has the *Conjugate Property* if for each game $G \in \mathcal{X}$ for which there exists an inverse, that is a game $H \in \mathcal{X}$ such that $G + H \sim 0$, then $H = \overset{\leftrightarrow}{G}$.

Theorem 50. \mathbb{GS} has the *Conjugate Property*.

Proof. Consider $G, H \in \mathbb{GS}$ in their reduced forms, such that $G + H \sim 0$. We will prove, by induction on the birthday of $G + H$, that we must have $H = \overset{\leftrightarrow}{G}$.

Case 1: The game $G + H$ is purely-atomic. Let $G = \langle \emptyset^\ell | \emptyset^r \rangle$ where $\ell \leq r$. Then, by definition, $\overset{\leftrightarrow}{G} = \langle \emptyset^{-r} | \emptyset^{-\ell} \rangle$ and $\overset{\leftrightarrow}{G} \in \mathbb{GS}$. Let $H = \langle \emptyset^{-\ell} | \emptyset^{-r} \rangle$. Then $G + H \approx \langle \emptyset^0 | \emptyset^0 \rangle = 0$, but $H \in \mathbb{GS}$ if and only if $\ell = r$. Hence, the game $\overset{\leftrightarrow}{G}$ is the inverse to G if and only if $\ell = r$. Thus, a purely atomic game is invertible if and only if it is a number.

In the below proof, because of numerous algebraic manipulations, we will revert to the short hand notation $-G = \overset{\leftrightarrow}{G}$, if the existence of a negative is given by induction.

Case 2: The game $G + H$ has at least one option. We may assume that G and H are in their respective canonical forms. Let $J = G + H \sim 0$. It is a consequence of Theorem 29 that for all Left moves J^L , there exists J^{LR} such that $J^{LR} \preceq 0$. Without loss of generality, we will assume that $J^L = G^L + H$. There are two cases:

Case 2a: Suppose there exists a non-atomic-reversible $G^L \in G^{\mathcal{L}}$.

(This option is not reversible, because only atomic-reversible options may exist in the reduced form.) We prove two claims.

Claim (i): There exists H^R such that $G^L + H^R \preceq 0$.

If there is a good Right reply $G^{LR} + H \preceq 0$, after adding G to both sides (Theorem 14), we would have $G^{LR} \preceq G$. This is a contradiction, since G^L is not a reversible option. Therefore, there exists H^R with $G^L + H^R \preceq 0$.

Claim (ii): With H^R as in (i), $G^L + H^R \sim 0$.

We have that $G^L + H^R \preceq 0$. Suppose that $G^{L_1} + H^{R_1} \prec 0$, where we index the options starting with $G^L = G^{L_1}$ and $H^R = H^{R_1}$.

Consider the Right move in $G + H$ to $G + H^{R_1}$. Since $G + H \sim 0$, i.e., $G + H \succcurlyeq 0$, then there exists a Left option such that $(G + H^{R_1})^L \succcurlyeq 0$.

Suppose that $G + H^{R_1 L} \succcurlyeq 0$. Then, by adding H to both sides, we get $H^{R_1 L} \succcurlyeq H$. Therefore, since H is in canonical form, H^{R_1} is an atomic-reversible option and, by Theorem 37, $H^{R_1} = r + \widehat{n + 1}$ for some real r and nonnegative integer n . The two inequalities $G^{L_1} + H^{R_1} \prec 0$ and $G + H^{R_1 L} \succcurlyeq 0$ become $G^{L_1} + r + \widehat{n} \prec 0$ and $G + r + \widehat{n} \succcurlyeq 0$, respectively. In $G^{L_1} + r + \widehat{n + 1}$, against the Left move to $G^{L_1} + r + \widehat{n}$, Right must have a move of the form $G^{L_1 R} + r + \widehat{n} \preceq 0$. Since $r + \widehat{n}$ is invertible, $G^{L_1 R} + r + \widehat{n} \preceq 0$

and $G + r + \widehat{n} \succ 0$ leads to $G^{L_1 R} \preceq G$ (Lemma 15), i.e., G^{L_1} is reversible, which is a contradiction.

Consequently, we may assume that there is a non-reversible option, G^{L_2} , such that $G^{L_2} + H^{R_1} \succ 0$. If $G^{L_2} + H^{R_1} \sim 0$ then, by induction, $H^{R_1} = -G^{L_2}$ (since G and H are in canonical form). Since $0 \succ G^{L_1} + H^{R_1} = G^{L_1} - G^{L_2}$, then $G^{L_2} \succ G^{L_1}$, which is a contradiction, because $G^{\mathcal{L}}$ has no dominated options. Therefore, $G^{L_2} + H^{R_1} \succ 0$.

By Claim (i), there must exist a Right option in H , H^{R_2} , corresponding to G^{L_2} , and so on. If we repeat the process, we now have the following inequalities:

$$\begin{aligned} G^{L_1} + H^{R_1} &< 0, & G^{L_2} + H^{R_1} &> 0 \\ G^{L_2} + H^{R_2} &< 0, & G^{L_3} + H^{R_2} &> 0 \\ & & \vdots & \end{aligned}$$

but the number of options is finite. Thus, without loss of generality, we may assume that there is some m such that $G^{L_1} + H^{R_m} \succ 0$ (re-indexing if necessary).

Because the inequalities are strict, summing the left-hand and the right-hand inequality gives, respectively,

$$\sum_{i=1}^m G^{L_i} + \sum_{i=1}^m H^{R_i} < 0 \quad \text{and} \quad \sum_{i=1}^m G^{L_i} + \sum_{i=1}^m H^{R_i} > 0$$

which is a contradiction.

Therefore, we conclude that $G^L + H^R \sim 0$ and, by induction, that $H^R = -G^L$.

Case 2b: Suppose there exists an atomic-reversible option, $A \in G^{\mathcal{L}}$.

Since A is atomic-reversible, it follows, by Theorem 37, that $A = \ell - \widehat{(n+1)}$, where n is the minimum nonnegative integer such that $G \succ \ell - \widehat{n}$, and where $\ell = Ls(B)$ is a real number (where B is the reversing option).

(i) Suppose first that there is some Right option in H . We prove four claims.

(a) $\overline{Rs}(G) \geq \ell$.

Since $G \succ \ell - \widehat{n}$, we get $G + \widehat{n} \succ \ell$. Hence, $\overline{Rs}(G) \geq Rs(G + \widehat{n}) \geq \ell$, where the first inequality holds because Left can pass.

(b) There exists an atomic-reversible option $H^R \in H^{\mathcal{R}}$.

Suppose not; we will argue that this implies $\overline{Rs}(G + H) > 0$, contradicting $G + H \sim 0$ (Theorem 29). Because H has no atomic-reversible Right option, we saw in Case 2a that for all H^R there exists non-atomic-reversible G^L such that $G^L + H^R \sim 0$. By induction, $G^L \sim -H^R$. Because $A = \ell - \widehat{n+1}$ is an atomic-reversible option in $G^{\mathcal{L}}$, by Theorem 36(2), $\underline{Rs}(G^L) < \underline{Rs}(A) = \ell$. Hence,

$$\overline{Ls}(H^R) = -\underline{Rs}(-H^R) = -\underline{Rs}(G^L) > -\ell, \tag{1}$$

where the first equality is by definition of the conjugate of a game. This holds for all $H^R \in H^R$ and so, $\overline{Rs}(H) > -\ell$. Therefore, by Theorem 26,

$$\overline{Rs}(G + H) \geq \overline{Rs}(G) + \overline{Rs}(H) \geq \ell + \overline{Rs}(H) > \ell - \ell = 0, \quad (2)$$

and the claim is proved.

- (c) The atomic-reversible Right option of H is $-\ell + \widehat{m+1}$ (where m is minimum such that $H \preceq -\ell + \widehat{m}$).

We have seen in the inequality (1) that for all non-atomic-reversible H^R , we have $\overline{Ls}(H^R) > -\ell$. If the only atomic-reversible Right option of H was $-s + \widehat{m+1}$ and $\ell > s$, we would have $\overline{Rs}(H) > -\ell$, leading to the same contradiction as obtained in the inequality (2). Suppose, instead, that the only atomic-reversible Right option of H were $-s + \widehat{m+1}$ with $\ell < s$. By definition of a reversing option (for an atomic-reversible Right option), we have that $H \preceq -s + \widehat{m}$. Altogether, $\underline{Ls}(H) \leq Ls(H - \widehat{m}) \leq -s < -\ell$. Therefore, by Theorem 26, $\underline{Ls}(G + H) \leq \underline{Rs}(G) + \underline{Ls}(H) \leq \ell - s < \ell - \ell = 0$. The two contradictions together imply $s = \ell$.

- (d) Finally, $m = n$.

Consider the integers, n and m as previously defined. They are minimal such that $G \succ \ell - \widehat{n}$ and $H \preceq -\ell + \widehat{m}$, respectively. If $n \neq m$, say $n < m$, from $G \succ \ell - \widehat{n}$, adding H to both sides gives $0 \succ H + \ell - \widehat{n} \Rightarrow H \preceq -\ell + \widehat{n}$. This is a contradiction (m is not minimal). Hence, we must have $m = n$.

Thus, we have proved that if $A = \ell - \widehat{n}$ (in reduced form) is a Left atomic-reversible option of G , then there is an $H^R \in H^R$ with $H^R = -\ell + \widehat{n} = -A$.

- (ii) Since $A \in G^L$ is an atomic-reversible option, then H^R is not an atom.

First, if it were true that $H^R = \emptyset^{-s}$, for some real number s , then this would force $s = \ell$. This follows by an argument similar to that in 2b(i.c). $\overline{Rs}(G) \geq \ell$ holds by 2a (i.a). Thus, if $\overline{Rs}(H) = -s > -\ell$, then $\overline{Rs}(G + H) \geq \overline{Rs}(G) + \overline{Rs}(H) > 0$. Also, if $\overline{Rs}(H) = -s < -\ell$, then, because of the guaranteed property, $\underline{Ls}(H) \leq -s < -\ell$. So, by $\underline{Ls}(G) = \ell$, we have $\underline{Ls}(G + H) \leq \underline{Ls}(G) + \underline{Ls}(H) < 0$. The inequalities are contradictory, and so $s = \ell$.

Suppose therefore that $H^R = \emptyset^\ell$. In this case, $A = \ell - \widehat{n+1}$ is the only Left option of G ; any other options would be non-atomic-reversibles (by domination) paired in H^R (by Case 2a), but there are none. Now, the non-atomic-reversible options of H^L and G^R are paired and since $G \in \mathbb{GS}$ then ℓ is less than or equal to all the scores in the games of G^R . Since $n \geq 0$ then, by Theorem 37, G^L could be replaced by \emptyset^ℓ contradicting that G was in reduced form.

We have seen that each G^L has a corresponding $-G^L$ in the set of Right options of H . This finishes the proof. \square

As a final comment, not every game is invertible and we do not have a full characterization of invertible games. We do know that zugzwang games do not have inverses.

Theorem 51. *Let G be a game with $Ls(G) < Rs(G)$. Then G is not invertible.*

Proof. Suppose G is a game with $Ls(G) < Rs(G)$. If G is invertible then $G + \overset{\leftrightarrow}{G} = 0$, which by Theorem 29 implies that $\underline{Ls}(G + \overset{\leftrightarrow}{G}) = 0$.

Now,

$$\begin{aligned} \underline{Ls}(G + \overset{\leftrightarrow}{G}) &\leq \underline{Ls}(G) + \underline{Ls}(\overset{\leftrightarrow}{G}) && \text{(by Theorem 26)} \\ &\leq Ls(G) + Ls(\overset{\leftrightarrow}{G}) \\ &= Ls(G) - Rs(G) \\ &< 0 \end{aligned}$$

which contradicts $\underline{Ls}(G + \overset{\leftrightarrow}{G}) = 0$ and finishes the proof. □

The converse is not true. For example, $G = \langle \langle -1 \mid 1 \rangle \mid 0 \rangle$ is not invertible, since $\underline{Ls}(G + \overset{\leftrightarrow}{G}) = -1 \neq 0$, and is not zugzwang since $Ls(G) > Rs(G)$.

5 A Guaranteed scoring games' calculator

The translation of a guaranteed game position to its canonical scoring value is not a trivial computation task and cannot be done manually except for very simple examples. A computer program is required for more complex positions. The Scoring Games Calculator (SGC) is such a program. It is implemented as a set of Haskell modules that run on an interpreter available in any Haskell distribution or embedded in a program that imports these modules.

The SGC has two main modules, **Scoring** and **Position**, that act as containers of two data types: **Game** and **Position**. The first module deals with scoring game values and the second with board positions given a ruleset.

Game values represent values from set \mathbb{S} like $\langle 1 \mid \emptyset^3 \rangle$. This type includes an extensive list of Haskell functions that mirror the mathematical functions presented in this article. One simple example is predicate **guaranteed** that checks if a game value in \mathbb{S} is also in \mathbb{GS} . Another operation is the sum of games that takes two values in \mathbb{GS} and computes their disjunctive sum.

Position values represent board positions. Type **Position** is an abstract type. It encloses a set of services useful for all games, like reading a position from file or converting a position to its scoring value. These functions are only able to work when a concrete ruleset is implemented. Given a game, say **Diskonnnect**, there should be a module **Diskonnnect** that imports module **Position**, and is required to implement the **Diskonnnect** ruleset. Almost all effort to define a new game focus in the implementation of function **moves** that, given a board position and the next player, returns the list of all possible

next positions. With this, `Position` is able to construct a game tree for a given board position and to translate that position into its scoring value.

The scoring universe together with its main theorems concerning reductions and comparisons all have a strong recursive structure that fits quite well into a functional programming language like Haskell. Not all mathematical definitions are simply translations to functions, but some are. For example, the implementation of left-r-protected mirrors quite closely its definition,

```
lrp :: NumberData -> Game -> Bool
lrp r g =
  ls_d g >= r &&
  for_all [
    for_any [lrp r gRL | gRL <- leftOp gR] | gR <- rightOp g]
```

where `ls_d` is `LS` and syntax `[f x|x<-list]` defines list comprehensions.

The SGC includes too many functions to be described here⁵. Currently, the following guaranteed rulesets are implemented: Diskonnect, Kobber, TakeSmall and TakeTall.

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⁵The source code and a user guide presenting all functionalities are available at <https://github.com/jpneto/ScoringGames>

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