A new construction of non-extendable intersecting families of sets

Kaushik Majumder
R C Bose Centre for Cryptology and Security
Indian Statistical Institute
202 Barrackpore Trunk Road, Kolkata - 700108, India.
kaushikbnmajumder@gmail.com

Submitted: Mar 1, 2016; Accepted: Aug 1, 2016; Published: Aug 19, 2016
Mathematics Subject Classifications: 05D05, 05A16.

Abstract

In 1975, Lovász conjectured that any maximal intersecting family of k-sets has at most $\lfloor (e-1)k! \rfloor$ blocks, where e is the base of the natural logarithm. This conjecture was disproved in 1996 by Frankl and his co-authors. In this short note, we reprove the result of Frankl et al. using a vastly simplified construction of maximal intersecting families with many blocks. This construction yields a maximal intersecting family $\mathbb{G}_k$ of k-sets whose number of blocks is asymptotic to $e^2(\frac{k}{2})^{k-1}$ as $k \to \infty$.

Keywords: Intersecting family of k-sets, Maximal k-cliques.

1 Introduction

For positive integers k, by a k-set we mean a set of size k. The members of a family $\mathcal{F}$ of k-sets are usually called the blocks of $\mathcal{F}$. Such a family is said to be an intersecting family if any two of its blocks have a non-empty intersection. A family $\mathcal{F}$ of k-sets is called a *maximal intersecting family of k-sets* if (a) $\mathcal{F}$ is an intersecting family of k-sets, and (b) there is no intersecting family $\mathcal{G}$ of k-sets such that $\mathcal{G} \supsetneqq \mathcal{F}$. Maximal intersecting families of k-sets are also known as *k-uniform maximal cliques.*

This notion was introduced by Erdős and Lovász in [1]. In this paper they proved the amazing result that any maximal intersecting family of k-sets has at most k^k blocks, and hence for any given k, there are only finitely many maximal intersecting families of k-sets. (This result may be viewed as a special case of [4, Theorem 2.3].) Therefore, Erdős and Lovász initiated the problem of finding or estimating the function $\mathrm{M}(k)$ defined by

$$\mathrm{M}(k) := \max \{|\mathcal{F}| : \mathcal{F} \text{ is a maximal intersecting family of } k-\text{sets}\}.$$

In [1], Erdős and Lovász also proved:

Lemma 1. *For $k \geqslant 2$, $\mathrm{M}(k) \geqslant 1 + k \cdot \mathrm{M}(k-1)$.*

Proof. Let $\mathcal{F}$ be a maximal intersecting family of $(k-1)$-sets with $\mathrm{M}(k-1)$ blocks. Choose a k-set B disjoint from all the blocks of $\mathcal{F}$, and consider the family

$$\widehat{\mathcal{F}} := \{B\} \cup \{A \sqcup \{x\} : A \in \mathcal{F},\ x \in B\}$$

Then it is easy to verify that $\widehat{\mathcal{F}}$ is a maximal intersecting family of k-sets with $1+k\cdot\mathrm{M}(k-1)$ blocks. □

Since $\mathrm{M}(1) = 1$, using Lemma 1 and an easy induction on k, one deduces that

$$\mathrm{M}(k) \geqslant k! \left(\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!}\right) = \lfloor (e-1)k! \rfloor.$$

Thus Erdős and Lovász showed

$$\lfloor (e-1)k! \rfloor \leqslant \mathrm{M}(k) \leqslant k^k. \tag{1}$$

In [3], Lovász conjectured that the lower bound in (1) is sharp, i.e. $\mathrm{M}(k) = \lfloor (e-1)k! \rfloor$ for all k. In [2], Frankl et al. disproved this conjecture (for all $k \geqslant 4$) by an extremely elegant but complicated family of counterexamples. Indeed, it is hard to verify that their construction actually yields a maximal intersecting family of k-sets. (We addressed this question in a recent paper [5] with Mukherjee.) There appears to be a gap in the proof sketched in [2]. Specifically, the Claim 2 in [2] seems to be incorrect. Therefore, it seems desirable to present a simpler construction (with short and complete proof) reproving this result.

In this note we prove:

Theorem 2.

$$\mathrm{M}(k) \geqslant \begin{cases} (\frac{k}{2}+1)^{k-1} + (k-1)(\frac{k}{2}+1)^{\frac{k}{2}-1} & \text{for even } k, \\ 1 + k(\frac{k+1}{2})^{k-2} + k(k-2)(\frac{k+1}{2})^{\frac{k-3}{2}} & \text{for odd } k. \end{cases}$$

Note that this lower bound is asymptotic to $e^2(\frac{k}{2})^{k-1}$ through even k and $2e(\frac{k}{2})^{k-1}$ through odd k. Since $k!$ grows roughly like $(\frac{k}{e})^k$, the bound in Theorem 2 beats the lower bound of (1) by the exponential factor $(\frac{e}{2})^k$ (roughly). This bound is comparable to the explicit bound of [2], but the latter is somewhat better. Our bound beats the lower bound of (1) for $k \geqslant 8$, while the bound in [2] is better than (1) for $k \geqslant 4$. In a final remark in section 2, we indicate how the lower bound of Theorem 2 may be sharpened.

2 The construction

In this section, k is an even number. We construct a family $\mathbb{G}_k$ as follows.

Construction. Let X_n, $0 \leqslant n \leqslant k-2$ be pairwise disjoint sets, where each X_n is of size $1+\frac{k}{2}$. Let α be a new symbol which does not belong to any of the X_n's. The blocks of $\mathbb{G}_k$ are of two types. The blocks of type 1 are the sets $X_n \sqcup \{x_i : 1 \leqslant i \leqslant \frac{k}{2}-1\}$, where $0 \leqslant n \leqslant k-2$, and $x_i \in X_{n+i}$ for $1 \leqslant i \leqslant \frac{k}{2}-1$. Here the addition in the suffix is modulo $k-1$. The blocks of type 2 are the sets $\{\alpha\} \sqcup \{x_i : 0 \leqslant i \leqslant k-2\}$, where $x_i \in X_i$ for $0 \leqslant i \leqslant k-2$.

To appreciate the following proof, it may be helpful to visualize the sets X_n as $k-1$ equally spaced blobs arranged on a circle.

Theorem 3. *For each even number k, $\mathbb{G}_k$ is a maximal intersecting family of k-sets.*

Proof. Clearly $\mathbb{G}_k$ is a family of k-sets. For indices m, n in the range $0 \leqslant m, n \leqslant k-2$, let us write $m \to n$ to denote that $n \equiv m+i \pmod{k-1}$, for some i in the range $1 \leqslant i \leqslant \frac{k}{2}-1$. Notice that, for $m \neq n$, exactly one of the relations $m \to n$ and $n \to m$ holds true.

Clearly any block of type 2 intersects all the blocks of $\mathbb{G}_k$. Let B_1 and B_2 be two blocks of type 1. Then there are two indices m, n such that $B_1 \supseteq X_m$ and $B_2 \supseteq X_n$. If $m = n$, then B_1 and B_2 intersect at least in X_m. Otherwise, we may assume without loss of generality that $m \to n$. Then every block containing X_m intersects X_n. Therefore B_1 and B_2 intersect in this case also. Thus, $\mathbb{G}_k$ is an intersecting family of k-sets.

Let C be a set of size k which intersects all the blocks of $\mathbb{G}_k$. To prove that $\mathbb{G}_k$ is maximal, it suffices to show that C must be a block.
Case A : $\alpha \notin C$. Since C intersects all the blocks of type 2, it follows that there is an index m such that $C \supseteq X_m$. Since C is a k-set and the X's are pairwise disjoint sets of size $\frac{k}{2}+1$, this index m is unique. Suppose, if possible, that there is an index n such that $C \cap X_n = \emptyset$ and $m \to n$. Since C intersects all the blocks containing X_n, it follows there must exist an index l such that $X_l \subset C$ and $n \to l$. By the uniqueness of the index m, we get $l = m$. Therefore, $m \to n$ and $n \to m$, a contradiction. Thus, C intersects all the $(\frac{k}{2}-1)$ sets X_n such that $m \to n$. Since, also, $C \supseteq X_m$ it follows that there is a block $B \supseteq X_m$ such that $C \supseteq B$. Since $|C| = k = |B|$, we get that $C = B$ is a block of type 1 in this case.
Case B : $\alpha \in C$. Let $T = C \setminus \{\alpha\}$. Thus T is a set of size $k-1$ which intersects all the type 1 blocks. Suppose, if possible, that there is an index n such that $T \cap X_n = \emptyset$. Then arguing as in the previous case, we see that there is a unique index m such that $T \supseteq X_m$. Also, $n \to m$ for all indices n for which $T \cap X_n = \emptyset$. Contrapositively, $X_n \cap T \neq \emptyset$ for all the $\frac{k}{2}-1$ indices n such that $m \to n$. Since $T \supseteq X_m$ and $|X_m| = \frac{k}{2}+1$, it follows that $|T| \geqslant \frac{k}{2}+1+\frac{k}{2}-1 = k$, and hence $|C| > k$, a contradiction. Thus, $T \cap X_n \neq \emptyset$ for all n. Since there are $k-1 = |T|$ pairwise disjoint sets X_n, it follows that $|T \cap X_n| = 1$ for all n. Hence C is a block of type 2 in this case. □

Proof of Theorem 2. First let k be an even positive integer. Note that $\mathbb{G}_k$ has $(\frac{k}{2}+1)^{k-1}$ blocks of type 2 and $(k-1)(\frac{k}{2}+1)^{\frac{k}{2}-1}$ blocks of type 1. Therefore the total number of blocks in $\mathbb{G}_k$ is $(\frac{k}{2}+1)^{k-1} + (k-1)(\frac{k}{2}+1)^{\frac{k}{2}-1}$. Since by Theorem 3, $\mathbb{G}_k$ is a maximal intersecting family of k-sets, this number is a lower bound on $\mathrm{M}(k)$.

Next let $k > 1$ be an odd integer. (The result is trivial for $k = 1$.) Using Lemma 1 and the above bound (with $k-1$ in place of k) we get $\mathrm{M}(k) \geqslant 1 + k \cdot \mathrm{M}(k-1) \geqslant 1 + k(\frac{k+1}{2})^{k-2} + k(k-2)(\frac{k+1}{2})^{\frac{k-3}{2}}$. □

Remark. When k is an odd positive integer, we may modify the above construction by putting

$$|X_n| = \begin{cases} \frac{k+1}{2} & \text{if } 0 \leqslant n \leqslant \frac{k-1}{2} - 1 \\ \frac{k+1}{2} + 1 & \text{if } \frac{k-1}{2} \leqslant n \leqslant k-2 \end{cases}$$

and taking the type 1 blocks of $\mathbb{G}_k$ to be the sets $X_m \sqcup \{x_i : 1 \leqslant i \leqslant k - |X_m|\}$, where $0 \leqslant m \leqslant k-2$ and $x_i \in X_{m+i}$ for all i. Here addition in the suffix is modulo $k-1$. The type 2 blocks are as before. Then it can be shown that the resulting family $\mathbb{G}_k$ is again a maximal intersecting family of k-sets. The proof is similar, but a little more complicated. Using this construction (together with the preceding construction for even positive integer k) we can prove the following estimate, which improves upon Theorem 2.

Theorem 4.

$$\mathrm{M}(k) \geqslant |\mathbb{G}_k| = \begin{cases} (k-1)(\frac{k}{2}+1)^{\frac{k}{2}-1} + (\frac{k}{2}+1)^{k-1} & \text{if } k \text{ is an even integer} \\ \frac{k+5}{2}\left\{(\frac{k+3}{2})^{\frac{k-1}{2}} - (\frac{k+1}{2})^{\frac{k-1}{2}}\right\} + (\frac{k+1}{2})^{\frac{k-1}{2}}(\frac{k+3}{2})^{\frac{k-1}{2}} & \text{if } k \text{ is an odd integer.} \end{cases}$$

This lower bound is asymptotic to $e^2(\frac{k}{2})^{k-1}$ as $k \to \infty$. It seems safe to propose:

Conjecture. $\mathrm{M}(k)$ is asymptotic to $e^2(\frac{k}{2})^{k-1}$ as $k \to \infty$.

Acknowledgement

The author's sincere thanks go to Professor Bhaskar Bagchi for his constant help in preparing this note.

References

[1] Paul Erdős and László Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, *Infinite and finite sets* Vol.-II, North-Holland, Amsterdam, 1975, pp. 609–627. Colloquia Mathematica Societatis János Bolyai **10**.

[2] Péter Frankl, Katsuhiro Ota and Norihide Tokushige, Covers in uniform intersecting families and a counterexample to a conjecture of Lovász, *J. Combin. Theory Ser. A* **74** (1996), No. 1, 33–42.

[3] László Lovász, On minimax theorems of combinatorics, *Matematikai Lapok* **26** (1975), No. 3-4, 209–264.

[4] Kaushik Majumder, On the maximum number of points in a maximal intersecting family of finite sets, *Combinatorica*, to appear. http://dx.doi.org/10.1007/s00493-015-3275-8

[5] Kaushik Majumder and Satyaki Mukherjee, A note on the transversal size of a series of families constructed over cycle graph, arXiv:1501.02178 (2015).