All Ramsey numbers for brooms in graphs

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Abstract

For $k, \ell \geqslant 1$, a broom $B_{k,\ell}$ is a tree on $n = k + \ell$ vertices obtained by connecting the central vertex of a star $K_{1,k}$ with an end-vertex of a path on $\ell - 1$ vertices. As $B_{n-2,2}$ is a star and $B_{1,n-1}$ is a path, their Ramsey number have been determined among rarely known $R(T_n)$ of trees T_n of order n. Erdős, Faudree, Rousseau and Schelp determined the value of $R(B_{k,\ell})$ for $\ell \geqslant 2k \geqslant 2$. We shall determine all other $R(B_{k,\ell})$ in this paper, which says that, for fixed n, $R(B_{n-\ell,\ell})$ decreases first on $1 \leqslant \ell \leqslant 2n/3$ from $2n-2$ or $2n-3$ to $\lceil \frac{4n}{3} \rceil - 1$, and then it increases on $2n/3 < \ell \leqslant n$ from $\lceil \frac{4n}{3} \rceil - 1$ to $\lfloor \frac{3n}{2} \rfloor - 1$. Hence $R(B_{n-\ell,\ell})$ may attain the maximum and minimum values of $R(T_n)$ as ℓ varies.

Keywords: Ramsey number; Tree; Broom

1 Introduction

Given a graph G, the *Ramsey number* $R(G)$ is the smallest integer N such that every red-blue coloring of the edges of K_N contains a monochromatic G. Let T_n be a tree of order n. Finding $R(T_n)$ for an arbitrary T_n is a difficult unsolved problem in Ramsey theory. Most works focus on improving the known bounds, see [10]. Erdős and Sós conjectured that if a graph G has average degree greater than $n-1$, then G contains every tree of n edges, which implies that $R(T_n) \leqslant 2n-2$ for $n \geqslant 2$. A result of Erdős, Faudree, Rousseau and Schelp in [4] yields

$$r(T_n) \geqslant \left\lceil \frac{4n}{3} \right\rceil - 1, \tag{1}$$

under (2) by minimizing the lower bound with $b = 2a$, and the lower bound can be attained by some brooms. For $k, \ell \geqslant 1$, a broom $B_{k,\ell}$ is a tree on $k + \ell$ vertices obtained by connecting the central vertex of a star $K_{1,k}$ with an end-vertex of a path on $\ell - 1$

vertices. Thus $B_{k,1} = K_{1,k}$, $B_{k,2} = K_{1,k+1}$ and $B_{1,\ell} = P_{\ell+1}$, where $P_{\ell+1}$ is a path of order $\ell + 1$. They obtained the following result.

Theorem 1. *([4]) Let k and ℓ be integers with $\ell \geqslant 2k \geqslant 2$ and $n = k + \ell$. Then*

$$R(B_{k,\ell}) = n + \left\lceil \frac{\ell}{2} \right\rceil - 1.$$

Thus $R(B_{k,\ell}) = \lceil \frac{4n}{3} \rceil - 1$ for $\ell \in \{2k, 2k+1, 2k+2\}$ and $n = k+l$, which attain the lower bound in (1). In this paper, we shall determine the values of $R(B_{k,\ell})$ for $1 \leqslant \ell \leqslant 2k - 1$.

Note that when k is fixed and ℓ is sufficient large, $B_{k,\ell}$ is similar to a path P_n; when ℓ is fixed and k is sufficient large, $B_{k,\ell}$ is similar to a star $K_{1,n-1}$. Among few known results of $R(T_n)$, $R(P_n)$ and $R(K_{1,n-1})$ have been determined completely. For $n \geqslant 2$, the exact value of $R(P_n)$ was determined in [7] as

$$R(P_n) = \left\lfloor \frac{3n}{2} \right\rfloor - 1,$$

and $R(K_{1,n-1})$ was determined in [3] as

$$R(K_{1,n-1}) = \begin{cases} 2n - 3 & \text{if } n \text{ is odd,} \\ 2n - 2 & \text{otherwise.} \end{cases}$$

As $B_{1,\ell} = P_{\ell+1}$, $B_{k,1} = K_{1,k}$ and $B_{k,2} = K_{1,k+1}$, their Ramsey numbers can be determined by the above results. It was proved that $R(B_{k,3}) = R(K_{1,k+1})$ in [2]. Thus we shall consider the case $\ell \geqslant 4$ and $k \geqslant 2$.

Theorem 2. *Let k and ℓ be integers with $k \geqslant 2$ and $n = k + \ell$. Then*

$$R(B_{k,\ell}) = \begin{cases} n + \lceil \frac{\ell}{2} \rceil - 1 & \text{if } \ell \geqslant 2k - 1, \\ 2n - 2\lceil \frac{\ell}{2} \rceil - 1 & \text{if } 4 \leqslant \ell \leqslant 2k - 2. \end{cases}$$

Remark. Roughly speaking, for fixed n, $R(B_{n-\ell,\ell})$ decreases first on $2 \leqslant \ell \leqslant \frac{2n-1}{3}$ from $2n - 2$ or $2n - 3$ to $\lceil \frac{4n}{3} \rceil - 1$, and then increases on $\frac{2n-1}{3} < \ell \leqslant n$ from $\lceil \frac{4n}{3} \rceil - 1$ to $\lfloor \frac{3n}{2} \rfloor - 1$. Hence $R(B_{n-\ell,\ell})$ may attain the maximum and minimum values of $R(T_n)$ when ℓ varies, as it is believed that $R(K_{1,n-1})$ is the maximum value of $R(T_n)$.

2 Proofs

For any red-blue edge-coloring of K_N, denote R and B be the induced red and blue subgraph, respectively, and $N_R(x)$ and $N_B(x)$ be the red neighborhood and blue neighborhood of x, respectively. Let $N_R[x] = N_R(x) \cup \{x\}$, $N_B[x] = N_B(x) \cup \{x\}$, $\deg_R(x) = |N_R(x)|$, and $\deg_B = |N_B(x)|$. For a graph G and disjoint subset A and D, denote by $G(A)$ the subgraph of G induced by A, and $G(A, D)$ the bipartite subgraph of G induced by A and D. If G is the red-blue edge-colored K_N, we write $G_R(A) = G(A) \cap R$ and $G_R(A, D) = G(A, D) \cap R$. Notation not specifically mentioned will follow from [1]. We do not distinguish the vertex set and the graph when there is no danger of confusion.

Consider a tree T_n as a bipartite graph with two parts of size a and b, respectively, where $a \leqslant b$, $a + b = n$. Observing that a red-blue edge-colored K_{2a+b-2} with $R = K_{a-1} \cup K_{a+b-1}$ contains no monochromatic T_n, and a red-blue edge-colored K_{2b-2} with $R = K_{b-1} \cup K_{b-1}$ contains no monochromatic T_n. We see that

$$R(T_n) \geqslant \max\Big\{2a + b - 1,\ 2b - 1\Big\}, \tag{2}$$

where a and b are determined by T_n.

Note that $B_{k,\ell}$ is a bipartite graphs on parts of sizes $a = \lceil \frac{\ell}{2} \rceil$ and $b = k + \lfloor \frac{\ell}{2} \rfloor$, then

$$R(B_{k,\ell}) \geqslant \max\Big\{k + \Big\lceil \frac{3\ell}{2} \Big\rceil - 1,\ 2k + 2\Big\lfloor \frac{\ell}{2} \Big\rfloor - 1\Big\}. \tag{3}$$

We shall prove the cases $\ell = 2k - 1$ and $\ell = 4$ in Theorem 2 via the following two lemmas.

Lemma 3. *Let $k \geqslant 2$ be an integer. Then*

$$R(B_{k,2k-1}) = 4k - 2.$$

Lemma 4. *Let $k \geqslant 2$ be an integer. Then*

$$R(B_{k,4}) = 2k + 3$$

In order to prove Lemma 3, we need two results from [9] and [6], respectively.

Lemma 5. *([9]) Let $G(A, D)$ be a bipartite graph on parts A and D with $|A| = k$ and $|D| = 2k - 2$ such that*

$$\min\{d(x) :\ x \in A\} \geqslant k.$$

Then $G(A, D)$ contains a cycle C_{2k}.

Lemma 6. *([6]) $R(C_{2k}) = 3k - 1$ for integer $k \geqslant 3$*

Proof of Lemma 3. It is easy to see that $R(B_{2,3}) = 6$, we assume that $k \geqslant 3$. As the lower bound (3) implies $R(B_{k,2k-1}) \geqslant 4k - 2$, it suffices to show the opposite inequality.

Let G be a red-blue edge-colored K_{4k-2}. We shall show that G contains a monochromatic $B_{k,2k-1}$. By lemma 6, G contains a monochromatic cycle C_{2k}. Without loss of generality, we assume that this C_{2k} is blue and denote it by $C_{2k}^{(B)}$. Let $D = G \setminus C_{2k}^{(B)}$. Then $|D| = 2k - 2$. If there exists a vertex $x \in C_{2k}^{(B)}$ such that $|N_B(x) \cap D| \geqslant k - 1$, then G contains a blue $B_{k,2k-1}$. We then assume that $|N_B(x) \cap D| \leqslant k - 2$ for each $x \in C_{2k}^{(B)}$. The fact that

$$|N_B(x) \cap D| + |N_R(x) \cap D| = |D| = 2k - 2$$

implies that $|N_R(x) \cap D| \geqslant k$ for each $x \in C_{2k}^{(B)}$, hence the number of red edges between $C_{2k}^{(B)}$ and D is at least $2k^2$. So there exists a vertex $u \in D$ such that

$$|N_R(u) \cap C_{2k}^{(B)}| \geqslant \frac{2k^2}{|D|} = \frac{2k^2}{2k - 2} \geqslant k + 1.$$

Let $\{u_0, u_1, \ldots, u_k\} \subseteq N_R(u) \cap C_{2k}^{(B)}$, and $A = C_{2k}^{(B)} \setminus \{u_1, u_2, \ldots, u_k\}$. Then $|A| = k$. Consider the bipartite graph $G_R(A, D)$. By Lemma 5, there is a red cycle C_{2k} between A and D. Denote by $C_{2k}^{(R)}$ the red C_{2k}. Since $C_{2k}^{(R)}$ contains A hence u_0, so the graph on $\{u_1, u_2, \ldots, u_k\}, u, u_0, C_{2k}^{(R)}$ induced by red edges contains a red $B_{k,2k-1}$.

This completes the proof of Lemma 3. □

Proof of Lemma 4. The lower bound (3) implies $R(B_{k,4}) \geqslant 2k+3$, and we shall show $R(B_{k,4}) \leqslant 2k+3$. Let G be a red-blue edge-colored K_{2k+3}. Assume that G contains no monochromatic $B_{k,4}$.

If $k=2$, $2k+3=7$. As $R(P_5) = 6 < 7$, we suppose that G contain a red P_5. Label the vertices of the path in order as $\{x_1, x_2, \ldots, x_5\}$, denote another two vertices as y_1, y_2. Since there is no red $B_{2,4}$, all the edges between $\{x_2, x_4\}$ and $\{y_1, y_2\}$ are red. If there is red edge between $\{x_1, x_5\}$ and $\{y_1, y_2\}$, say edge x_5y_2 is red. Then edges x_5y_1, x_3y_1,x_3y_2 are blue. Now $\{x_2, x_3\}, y_2, x_4, y_1, x_5$ contain a blue $B_{2,4}$, a contradiction. If all the edges between $\{x_1, x_5\}$ and $\{y_1, y_2\}$ are blue, then $\{x_1, x_2\}, y_2, x_4, y_1, x_5$ contain a blue $B_{2,4}$, a contradiction.

Now we consider the case $k \geqslant 3$. As $R(K_{1,k+1}) < 2k+3$, we suppose that there is a blue star $K_{1,k+1}$, which is denoted by $K_{1,k+1}^{(B)}$. Let x be the center of $K_{1,k+1}^{(B)}$, $A = K_{1,k+1}^{(B)} \setminus \{x\}$ and $D = G \setminus K_{1,k+1}^{(B)}$. Then $|A| = |D| = k+1$.

Claim. D induces a red K_{k+1}.

Proof. Suppose to the contrary, there is a blue edge uv in D. Since G contains no blue $B_{k,4}$, the edges between $\{u, v\}$ and A are all red, and thus all the edges between $\{u, v\}$ and $D \setminus \{u, v\}$ are blue from the assumption that G contains no red $B_{k,4}$. Now consider the blue edges between $\{u, v\}$ and A. With a similar analysis, we get that D induces a blue K_{k+1} and all edges between D and A are red.

Now, consider the adjacency between x and a vertex of D, say xu, no matter what the color of xu is, we have a monochromatic $B_{k,4}$, leading to a contradiction and the claim is proved.

Now D is a red K_{k+1}. If there exists a red edge xw with $w \in D$, then $D \cup \{x\}$ induces a red $K_{1,k+1}$ with center w. As $A = V(G) \setminus (D \cup \{x\})$, a similar analysis for the above claim tells us that A is a blue K_{k+1}. If the number of blue edges between A and D is at least $k+2$, then there exists a vertex $y \in A$ such that $|N_B(y) \cap D| \geqslant 2$. Now choose two vertices $\{y_1, y_2\} \subseteq N_B(y) \cap D$ and two vertices $\{a_1, a_2\} \subseteq A \setminus y$, then $(A \setminus \{a_1, a_2, y\}) \cup \{y_1, y_2\}, y, a_1, a_2, x$ contains a blue $B_{k,4}$, a contradiction. Thus assume to the contrary, there exists a vertex $z \in D$ such that $|N_R(z) \cap A| \geqslant \frac{(k+1)^2-(k+1)}{|D|} = k \geqslant 2$. If w, z are the same vertex, we can choose two vertices $\{z_1, z_2\} \subseteq N_R(z) \cap A$ and three vertices $\{d_1, d_2, d_3\} \subseteq D \setminus z$ for $|D| = k+1 \geqslant 4$. Then $(D \setminus \{d_1, d_2, d_3, z\}) \cup \{z_1, z_2, x\}, z, d_1, d_2, d_3$ contain a red $B_{k,4}$. If w, z are different, choose a vertex $d_1 \in D \setminus \{z, w\}$, then $(D \setminus \{z, w, d_1\}) \cup \{z_1, z_2\}, z, d_1, w, x$ contain a red $B_{k,4}$, a contradiction.

Finally, assume that x is adjacent to D completely blue. Choose any set $F \subseteq A \cup D$ such that $|F| = k+1$ and denote $M = V(G) \setminus (F \cup x)$. A similar analysis for the claim says that M is a red K_{k+1}. The choice of F tells us that $A \cup D$ is a red K_{2k+2}, hence G contains a red $B_{k,4}$, which is a contradiction too.

This completes the proof of Lemma 4. $\square$

Lemma 7. *For integers k, ℓ, N with $5 \leqslant \ell \leqslant 2k-2$ and $N \geqslant 2k + 2\lfloor \frac{\ell}{2} \rfloor - 1$, let the edges of K_N be colored by two colors $i \equiv 0, 1 \pmod 2$. Suppose i is a color and x is a vertex such that*

$$\deg_i(x) = \max_v \max\{\deg_i(v),\ \deg_{i+1}(v)\}.$$

If there exist vertices $y, z \subseteq N_{i+1}(x)$, not necessarily distinct, satisfying

1. *$|N_{i+1}(y) \cap N_i(x)| \geqslant k$, and*

2. *$\deg_{i+1}(z) \geqslant N - \ell$*

then G contains a monochromatic $B_{k,\ell}$.

Proof of Lemma 7. Since $|N_{i+1}(y) \cap N_i(x)| \geqslant k$, we can choose a subset A in $N_{i+1}(y) \cap N_i(x)$ such that $|A| = k$. Let $H = G \setminus A$, then $|H| = N - k$. Since $R(C_{2t}) = 3t - 1$ for $t \geqslant 3$, H contains a monochromatic C_{2t} in color j, denoted by $C_{2t}^{(j)}$, where

$$2t \geqslant 2\Big\lfloor \frac{N-k+1}{3} \Big\rfloor \geqslant 2\Big\lfloor \frac{k + 2\lfloor \ell/2 \rfloor}{3} \Big\rfloor \geqslant \ell$$

for $5 \leqslant \ell \leqslant 2k-2$. The choice of vertex x implies $\deg_i(x) \geqslant \deg_{i+1}(z) \geqslant N - \ell$, and thus

$$\Big|N_i[x] \setminus A\Big| + \Big|C_{2t}^{(j)}\Big| \geqslant |H| + 1, \qquad \Big|N_{i+1}[z] \setminus A\Big| + \Big|C_{2t}^{(j)}\Big| \geqslant |H| + 1,$$

which implies that both $N_i[x] \setminus A$ and $N_{i+1}[z] \setminus A$ contain a vertex of $C_{2t}^{(j)}$.

Case 1. $y = z$. If $j = i$, namely, $C_{2t}^{(j)} = C_{2t}^{(i)}$ is in color i, there is a monochromatic $B_{k,\ell}$ in color i in $A \cup \{x\} \cup C_{2t}^{(i)}$, and otherwise $j = i+1$, there exists a monochromatic $B_{k,\ell}$ in color $i+1$ in $A \cup \{y\} \cup C_{2t}^{(i+1)}$.

Case 2. $y \neq z$. Similarly, we can find a monochromatic $B_{k,\ell}$ either in $A \cup \{x\} \cup C_{2t}^{(j)}$ or in $A \cup \{y, x, z\} \cup C_{2k}^{(j)}$.

This completes the proof of Lemma 7. $\square$

The next two lemmas are results about the extremal edges in graph that contains no path P_t.

Lemma 8. *([5]) Let $t \geqslant 2$ be an integer, and G a graph of order N that contains no P_t. Let $e(G)$ be the number of edges of G, then $e(G) \leqslant \frac{(t-2)N}{2}$.*

Lemma 9. *([8]) Let $G(X_B, X_R)$ be a bipartite graph on parts X_B and X_R with $|X_R| \leqslant |X_B|$. If $G(X_B, X_R)$ contains no P_{2t} with $2(t-1) \leqslant |X_R|$, then*

$$e(G(X_B, X_R)) \leqslant (t-1)\Big[|X_B| + |X_R| - 2(t-1)\Big].$$

Proof of Theorem 2. We may assume that $5 \leqslant \ell \leqslant 2k-2$ from Theorem 1, Lemma 3 and Lemma 4.

Set $N = 2n-2\lceil \frac{\ell}{2} \rceil - 1 = 2k+2\lfloor \frac{\ell}{2} \rfloor - 1$. Let G be a red-blue edge-colored K_N, and let R and B be the induced red and blue subgraph, respectively. Without loss of generality, we may assume that the maximal monochromatic degree of G is the maximum blue degree and x is a vertex such that

$$\deg_B(x) = \max_v \max\{\deg_B(v),\ \deg_R(v)\}.$$

To simplify the notation, we write $X_B = N_B(x)$, $X_R = N_R(x)$ and

$$t = \ell + k - |X_B| - 1.$$

The choice of x implies $N_B(u) \cap X_B \neq \emptyset$ for each vertex $u \in X_R$ as otherwise $N_B[x] \subseteq N_R(u)$ and thus $\deg_R(u) > \deg_B(x)$, which is impossible. We shall separate the proof into three cases depending on $|X_B|$.

Case 1. $|X_B| < k + \ell - 1$, and either $G(X_R)$ contains a blue P_t, denoted by $P_t^{(B)}$, or $G(X_B, X_R)$ contains a blue P_{2t}, denoted $P_{2t}^{(B)}$.

In $G(X_B \cup X_R)$, let $P^{(B)}$ be the longest blue path extended from $P_t^{(B)}$ such that one of its end-vertices is in X_B if $G(X_R)$ contains a blue P_t, or that from $P_{2t}^{(B)}$ otherwise. If $|P^{(B)}| \geqslant \ell - 1$, then there exists a blue $B_{k,\ell}$. Thus we assume that $|P^{(B)}| \leqslant \ell - 2$, then $P^{(B)}$ fails to contain at least $|X_B| - (\ell - 2 - t) = k + 1$ vertices of X_B. Let y be the other end-vertex of $P^{(B)}$. Then $|N_R(y) \cap X_B| \geqslant k + 1$. The maximality of $|P^{(B)}|$ implies $|N_R(y)| \geqslant N - 1 - (\ell - 2) = N - \ell + 1$, which and Lemma 7 imply that G contains a monochromatic $B_{k,\ell}$.

Case 2. $|X_B| \geqslant k + \ell - 1$.

Let $P^{(B)}$ be the longest blue path in $G(X_B \cup X_R)$ that has an end-vertex in X_B. A similar analysis in Case 1 implies that G contains a monochromatic $B_{k,\ell}$.

Case 3. $|X_B| < k + \ell - 1$, and neither $G(X_R)$ contains a blue P_t nor $G(X_B, X_R)$ contains a blue P_{2t}.

As $t = \ell + k - |X_B| - 1 \geqslant 2$, then $|X_B| \leqslant \ell + k - 3$ and $|X_R| = N - 1 - |X_B| \geqslant k$.

Since $G(X_R)$ contains no blue P_t, Lemma 8 implies $e(G_B(X_R)) \leqslant (t-2)|X_R|/2$. The choice of x implies that the $\min\{\deg_R(v),\ \deg_B(v)\} \geqslant |X_R|$ for each vertex v of G, and thus

$$e(G_B(X_B, X_R)) \geqslant |X_R| \cdot |X_R| - (t-2)|X_R| = (|X_R| - t + 2)|X_R|.$$

Since $G(X_B, X_R)$ contains no blue P_{2t}, Lemma 9 yields

$$e(G_B(X_B, X_R)) \leqslant M_B,$$

where

$$M_B = (t-1)\Big[|X_B| + |X_R| - 2(t-1)\Big].$$

Claim for Case 3. $G(X_B, X_R)$ has at most $|X_R|(|X_B - k|) - 1$ blue edges.

Proof. Suppose opposite, then

$$e(G_B(X_B, X_R)) \geqslant |X_R| \cdot \max\{|X_R| - t + 2, |X_B| - k\} \geqslant m_B.$$

where

$$m_B = k(|X_R| - t + 2) + (|X_R| - k)(|X_B| - k).$$

Note that M_B and m_B are upper and lower bound of number of blue edges in $G(X_B, X_R)$, respectively, and thus $M_B \geqslant m_B$.

Case 3.1. ℓ is even. In this subcase, $|X_B| + |X_R| = 2k + \ell - 2$ and

$$\begin{aligned} t &= \ell + k - |X_B| - 1 = |X_R| - k + 1, \\ m_B &= k(k+1) + (|X_R| - k)(|X_B| - k), \\ M_B &= (|X_R| - k)\Big[|X_B| + k - (|X_R| - k)\Big] \end{aligned}$$

we have

$$m_B - M_B = k^2 - 2k(|X_R| - k) + (|X_R| - k)^2 + k = (2k - |X_R|)^2 + k > 0,$$

which is a contradiction.

Case 3.2. ℓ is odd. In this subcase, $|X_R| + |X_B| = 2k + \ell - 3$ and

$$\begin{aligned} t &= \ell + k - |X_B| - 1 = |X_R| - k + 2, \\ m_B &= k^2 + (|X_R| - k)(|X_B| - k), \\ M_B &= (|X_R| - k + 1)\Big[|X_B| + k - (|X_R| - k) - 2\Big]. \end{aligned}$$

We have

$$\begin{aligned} m_B - M_B &= (2k - |X_R|)^2 + 3|X_R| - |X_B| - 4k + 2 \\ &= (2k - |X_R|)(2k - |X_R| - 4) + 2k - \ell + 5. \end{aligned}$$

For $\ell \leqslant 2k - 2$,is odd, we get $|X_R| \leqslant \lfloor \frac{N-1}{2} \rfloor \leqslant 2k - 3$. If $|X_R| \leqslant 2k - 4$, $m_B - M_B \geqslant 2k - \ell + 5 > 0$; if $\ell_2 = 2k - 3$, $m_B - M_B = 2k - \ell + 2 > 0$, a contradiction, hence the claim holds.

We now have

$$e(G_R(X_R)) \geqslant \binom{|X_R|}{2} - \frac{(t-2)|X_R|}{2} \geqslant \frac{(k-1)|X_R|}{2}.$$

and

$$e((G_R(X_B, X_R)) \geqslant |X_R| \cdot |X_B| - \big[|X_R|(|X_B| - k) - 1\big] = k|X_R| + 1,$$

Recall $X_R = N_R(x)$, and thus

$$\sum_{v \in X_R} |N_R(v)| \geqslant e(G_R(X_B, X_R)) + 2e(G_R(X_R)) + |X_R| = 2k|X_R| + 1.$$

Therefore, there exist $y, z \subseteq X_R = N_R(x)$,not necessarily distinct, such that $|N_R(y) \cap N_B(x)| \geqslant k + 1$ and $|N_R(z)| \geqslant 2k + 1 \geqslant N - \ell$. Then, Lemma 7 implies that G contains a monochromatic $B_{k,\ell}$.

This completes the proof of Theorem 2. □

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References

[1] B. Bollobás. *Modern Graph Theory.* Springer-Verlag, New York, 1998.

[2] P. Bahls and T. Spencer. On the Ramsey numbers of trees with small diameter. *Graphs Combin.*, 29:39–44, 2013.

[3] V. Chvátal and F. Harary. Generalized Ramsey theory for graphs II, Small diagonal numbers. *Proc. Amer. Math. Soc.*, 32:389–394, 1972.

[4] P. Erdős, R. Faudree, C. Rousseau and R. Schelp. Ramsey numbers for brooms. *Congr. Numer.*, 35:283–293, 1982.

[5] P. Erdős and T. Gallai. On maximal paths and circuits graphs. *Acta Math. Acad. Sci. Hung.*, 10: 337–356, 1959.

[6] R. Faudree and R. Schelp. All Ramsey numbers for cycles in graphs. *Discrete Math.*, 8:35–52,1974.

[7] L. Geréncser and A. Gyárfas. On Ramsey-type problems. *Ann. Univ. Sci. Budapest, Eötvös Sect. Math.*, 10:167–170, 1967.

[8] A. Gyárfás, C. Rousseau and R. Schelp. An extremal problem for paths in bipartite graphs, *J. Graph Theory.*, 8:83–95, 1984.

[9] B. Jackson. Cycles in bipartite graphs. *J. Combin. Theory Ser.B.*, 30:332–342, 1981.

[10] S. Radziszowski. Small Ramsey numbers. *Electronic J. Combin.*, #DS1 1994.