Cubic non-Cayley vertex-transitive bi-Cayley graphs over a regular p-group

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Abstract

A bi-Cayley graph is a graph which admits a semiregular group of automorphisms with two orbits of equal size. In this paper, we give a characterization of cubic non-Cayley vertex-transitive bi-Cayley graphs over a regular p-group, where $p > 5$ is a prime. As an application, a classification of cubic non-Cayley vertex-transitive graphs of order $2p^3$ is given for each prime p.

Keywords: bi-Cayley graph; vertex-transitive; automorphism group; Cayley graph

1 Introduction

All groups considered in this paper are finite, and all graphs are finite, connected, simple and undirected. For the group-theoretic and graph-theoretic terminology not defined here we refer the reader to [2, 18].

A graph is said to be a *bi-Cayley graph* over a group H if it admits H as a semiregular automorphism group with two orbits of equal size. (Some authors have used the term semi-Cayley instead [9, 8, 4, 13]. In this paper, we follow [11] to use the term bi-Cayley.) Note that every bi-Cayley graph admits the following concrete realization. Let R, L and S be subsets of a group H such that $R = R^{-1}$, $L = L^{-1}$ and $R \cup L$ does not contain the identity element of H. Define the graph $\mathrm{BiCay}(H, R, L, S)$ to have vertex set the union of the *right part* $H_0 = \{h_0 \mid h \in H\}$ and the *left part* $H_1 = \{h_1 \mid h \in H\}$, and edge set the union of the *right edges* $\{\{h_0, g_0\} \mid gh^{-1} \in R\}$, the *left edges* $\{\{h_1, g_1\} \mid gh^{-1} \in L\}$ and the *spokes* $\{\{h_0, g_1\} \mid gh^{-1} \in S\}$ (Note that some authors label the vertices of a bi-Cayley graph for a group H with ordered pairs (h, i) for $h \in H$ and $i \in \{0, 1\}$, while we are using h_i to denote (h, i)). For the case when $|S| = 1$, the bi-Cayley graph $\mathrm{BiCay}(H, R, L, S)$ is also called *one-matching bi-Cayley graph* (see [11]). Also, if $|R| = |L| = s$, then $\mathrm{BiCay}(H, R, L, S)$, is said to be an *s-type bi-Cayley graph*, and if H is abelian, then $\mathrm{BiCay}(H, R, L, S)$ is simply called an *abelian bi-Cayley graph*.

In the study of bi-Cayley graphs, a natural problem is to characterize or classify bi-Cayley graphs over a given group with certain valency and specific symmetric property. Some partial answers for this problem have been obtained. For example, in [16] Pisanski classified cubic bi-Cayley graphs over cyclic groups, and in [11], Kovács et al. gave a description of arc-transitive one-matching bi-Cayley graphs over abelian groups, from which one can obtain the classification of cubic arc-transitive one-matching bi-Cayley graphs over abelian groups. In [23], the automorphisms of the bi-Cayley graphs were investigated. In particular, some sufficient conditions for a bi-Cayley graph being vertex-transitive or Cayley were given, and moreover, for a one-matching bi-Cayley graph Γ over a group H, the normalizer of the group H in $\mathrm{Aut}(\Gamma)$ was determined. By using this, a classification of cubic vertex-transitive bi-Cayley graphs over abelian groups was given. The facts listed above provide the motivation for us to consider the following problem.

Problem 1. Characterize cubic non-Cayley vertex-transitive bi-Cayley graphs over a p-group for an odd prime p.

Another motivation for us to consider this problem is: it is also related to the study of non-Cayley vertex-transitive graphs which is very active in 1980's. Let $p > 3$ be a prime. It is easy to prove that every connected cubic non-Cayley vertex-transitive graph of order $2p^n (n \geqslant 1)$ is a bi-Cayley graph over a p-group (see Lemma 9). So the above problem is equivalent to the problem of characterizing cubic non-Cayley vertex-transitive graphs of order $2p^n$. By [5], every cubic symmetric graph of order $2p^n$ ($p > 5$ is a prime) is a Cayley graph (see Proposition 4). Clearly, this is not true for the case when $p = 5$ because the Petersen graph is non-Cayley. In fact, one may construct infinitely many cubic non-Cayley symmetric graphs of order $2 \cdot 5^n$ by considering the regular coverings of the Petersen graph.

It is known that for a prime p, every cubic non-Cayley vertex-transitive graph of order $2p$ or $2p^2$ is a generalized Petersen graph (see [14, 22]). In [12], the authors proved that every cubic non-Cayley vertex-transitive graph of order $2p^n$, where $p > 7$ is a prime and $n \leqslant p$, is a bi-Cayley graph over a p-group P generated by two elements a and b of the same order and admitting an automorphism $\alpha \in \mathrm{Aut}(P)$ of order 4 such that $a^\alpha = b$ and $b^\alpha = a^{-1}$.

In this paper, we solve the above problem for the case when P is a regular p-group where $p > 5$ is a prime. It is proved that a connected cubic vertex-transitive bi-Cayley graph over a regular p-group P, where $p > 5$ is a prime, is non-Cayley if and only if $\Gamma = \mathrm{BiCay}(P, R, L, \{1\})$ is 2-type, and $\mathrm{Cay}(P, R \cup L)$ is a tetravalent normal arc-transitive Cayley graph with $\mathrm{Aut}(P, R \cup L) \cong \mathbb{Z}_4$. As an application, a classification of cubic non-Cayley vertex-transitive graphs of order $2p^3$ is given for each prime p.

2 Preliminaries

In this section, we shall introduce some notations, terminology and preliminary results. Let n be a positive integer. Denote by $\mathbb{Z}_n$ the cyclic group of order n, by $\mathbb{Z}_n^*$ the multiplicative group of $\mathbb{Z}_n$ consisting of numbers coprime to n, and by D_{2n} the dihedral group of order $2n$, respectively.

For a finite, simple and undirected graph X, we use $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. For $u, v \in V(X)$, $u \sim v$ means that u is adjacent to v and denote by $\{u, v\}$ the edge incident to u and v in X. For any subset B of $V(X)$, the subgraph of X induced by B will be denoted by $X[B]$. A graph X is said to be *vertex-transitive*, and *arc-transitive* (or *symmetric*) if $\mathrm{Aut}(X)$ acts transitively on $V(X)$ and $A(X)$, respectively.

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. Given a finite group G and an inverse closed subset $S \subseteq G \setminus \{1\}$, the *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. A Cayley graph $\mathrm{Cay}(G, S)$ is connected if and only if S generates G. Given a $g \in G$, define the permutation $R(g)$ on G by $x \mapsto xg, x \in G$. Then $R(G) = \{R(g) \mid g \in G\}$, called the *right regular representation* of G, is a permutation group isomorphic to G. It is well-known that $R(G) \leqslant \mathrm{Aut}(\mathrm{Cay}(G, S))$. So, $\mathrm{Cay}(G, S)$ is vertex-transitive. In general, a vertex-transitive graph X is isomorphic to a Cayley graph on a group G if and only if its automorphism group has a subgroup isomorphic to G, acting regularly on the vertex set of X (see [1, Lemma 16.3]).

A Cayley graph $\mathrm{Cay}(G, S)$ is said to be *normal* if $R(G)$ is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$ (see [19]). Set $A = \mathrm{Aut}(\mathrm{Cay}(G, S))$ and $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$.

Proposition 2. [19, Proposition 1.5] *The Cayley graph* $\mathrm{Cay}(G, S)$ *is normal if and only if* $A_1 = \mathrm{Aut}(G, S)$*, where* A_1 *is the stabilizer of the identity* 1 *of* G *in* A.

Let p be a prime. A finite p-group P is called a *regular p-group* if for any two elements x and y in P, there exist $c_1, c_2, \cdots, c_r$ in the derived group of $\langle x, y\rangle$ such that $(xy)^p = x^p y^p c_1^p c_2^p \cdots c_r^p$.

Proposition 3. [6, Theorem 3.1] *Let p be a prime and G a regular p-group with $p \neq 2, 5$. Let $X = \mathrm{Cay}(G, S)$ be a connected tetravalent Cayley graph on G. Then we have* $\mathrm{Aut}(\mathrm{Cay}(G, S)) = R(G) \rtimes \mathrm{Aut}(G, S)$.

Proposition 4. [5, Corollary 3.4] *Let $p > 5$ be a prime. Then every connected cubic symmetric graph of order $2p^n$ is a Cayley graph.*

The following proposition lists all of the tetravalent connected arc-transitive Cayley graphs of order p^3 for each prime p.

Proposition 5. [7, Theorem 4.1] *Let p be a prime and let $X = \mathrm{Cay}(G, S)$ be a tetravalent connected arc-transitive Cayley graph of order p^3. Then one of the following holds.*

(1) $G = \mathbb{Z}_{p^3} = \langle a \rangle$, $S = \{a, a^{-1}, a^{\lambda}, a^{-\lambda}\}$ ($\lambda^2 \equiv -1 \pmod{p^3}$).

(2) $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p = \langle a \rangle \times \langle b \rangle$, $S = \{a, a^{-1}, a^{\lambda}b, (a^{\lambda}b)^{-1}\}$ ($\lambda^2 \equiv -1 \pmod{p}$).

(3) $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p = \langle a \rangle \times \langle b \rangle$, $S = \{a, a^{-1}, ab, (ab)^{-1}\}$.

(4) $G = \langle a, b, c \mid a^p = b^p = c^p = 1, c = [a, b], [a, c] = [b, c] = 1 \rangle$, $S = \{a, a^{-1}, b, b^{-1}\}$.

To end this section, we give some results regarding the bi-Cayley graphs. For the proof of these results, one may see [23]. Let $\Gamma = \mathrm{BiCay}(H, R, L, S)$ be a connected bi-Cayley graph over a group H. The following proposition gives some basic properties of Γ.

Proposition 6. *The following hold.*

(1) *H is generated by $R \cup L \cup S$.*

(2) *If $S \neq \emptyset$, then S can be chosen to contain the identity element of H.*

(3) *For any automorphism α of H,* $\mathrm{BiCay}(H, R, L, S) \cong \mathrm{BiCay}(H, R^{\alpha}, L^{\alpha}, S^{\alpha})$.

Let $R(H)$ denote the right regular representation of H. Then $R(H)$ can be regarded as a group of automorphisms of $\mathrm{BiCay}(H, R, L, S)$ acting on its vertices by the rule

$$h_i^{R(g)} = (hg)_i, \forall i \in \mathbb{Z}_2, h, g \in H.$$

For an automorphism α of H, define two permutations on $V(\Gamma) = H_0 \cup H_1$ as follows:

$$\begin{array}{ll} \delta_{\alpha}: & h_0 \mapsto (h^{\alpha})_1, h_1 \mapsto (h^{\alpha})_0, \forall h \in H, \\ \sigma_{\alpha}: & h_0 \mapsto (h^{\alpha})_0, h_1 \mapsto (h^{\alpha})_1, \forall h \in H. \end{array} \tag{1}$$

Set

$$\begin{array}{lll} \mathrm{I} & = & \{\delta_{\alpha} \mid \alpha \in \mathrm{Aut}(H) \ s.t. \ R^{\alpha} = L, L^{\alpha} = R, S^{\alpha} = S^{-1}\}, \\ \mathrm{F} & = & \langle \sigma_{\alpha} \mid \alpha \in \mathrm{Aut}(H) \ s.t. \ R^{\alpha} = R, L^{\alpha} = L, S^{\alpha} = S \rangle. \end{array} \tag{2}$$

Proposition 7. *Each element in $\mathrm{I} \cup \mathrm{F}$ is an automorphism of Γ. Furthermore, for any $\delta_{\alpha} \in \mathrm{I}$, if α has order 2, then $\langle R(H), \delta_{\alpha} \rangle = R(H) \rtimes \langle \delta_{\alpha} \rangle$ acts regularly on $V(\Gamma)$.*

Proposition 8. *Let $\Gamma = \mathrm{BiCay}(H, R, L, \{1\})$ be a connected one-matching bi-Cayley graph over the group H. Then $\mathrm{Aut}(\Gamma)$ contains a regular subgroup containing $R(H)$ if and only if there exists an automorphism $\alpha \in \mathrm{Aut}(H)$ of order at most 2 such that $R^{\alpha} = L$.*

3 Characterization

Lemma 9. *Let $p > 3$ be a prime. Then every connected cubic non-Cayley vertex-transitive graph of order $2p^n (n \geqslant 1)$ is a bi-Cayley graph over a p-group.*

Proof. Let Γ be a cubic non-Cayley vertex-transitive graph of order $2p^n$ with $p > 3$ a prime. Set $A = \text{Aut}(\Gamma)$. By Proposition 4, either Γ is non-symmetric or Γ is symmetric and $p = 5$. Let P be a Sylow p-subgroup of A. If Γ is non-symmetric, then the vertex-stabilizer A_v of $v \in V(\Gamma)$ is a 2-group, and so P is semiregular on $V(\Gamma)$. If Γ is symmetric and $p = 5$, then A_v is a $\{2, 3\}$-group, and so P is also semiregular on $V(\Gamma)$. Consequently, P is always semiregular and has two orbits of equal size. This implies that Γ must be a bi-Cayley graph over P. □

Now we introduce the concept of quotient graph which will be used in the proof of the following lemma. Let Γ be a connected vertex-transitive graph, and let $G \leqslant \text{Aut}(\Gamma)$ be vertex-transitive on Γ. For a G-invariant partition $\mathcal{B}$ of $V(\Gamma)$, the *quotient graph* $\Gamma_{\mathcal{B}}$ is defined as the graph with vertex set $\mathcal{B}$ such that, for any two vertices $B, C \in \mathcal{B}$, B is adjacent to C if and only if there exist $u \in B$ and $v \in C$ which are adjacent in Γ. Let N be a normal subgroup of G. Then the set $\mathcal{B}$ of orbits of N in $V(\Gamma)$ is a G-invariant partition of $V(\Gamma)$. In this case, the symbol $\Gamma_{\mathcal{B}}$ will be replaced by Γ_N.

Lemma 10. *Let $\Gamma = \text{BiCay}(P, R, L, S)$ be a connected cubic non-Cayley vertex-transitive bi-Cayley graph over a regular p-group P, where $p > 5$ is a prime. Let $A = \text{Aut}(\Gamma)$. Then $R(P)$ is a normal Sylow p-subgroup of A.*

Proof. By Proposition 4, Γ must be non-symmetric. It follows that the vertex-stabilizer A_v of any $v \in V(\Gamma)$ in A is a 2-group. This implies that $A_v/A_v^* \leqslant \mathbb{Z}_2$, where A_v^* is the kernel of A_v acting on the neighborhood of v. Since Γ is non-Cayley, one has $A_v > 1$. If $A_v/A_v^* = 1$, then $A_v(= A_v^*)$ fixes all neighbors of v, and by the vertex-transitivity and connectedness of Γ, we get that A_v fixes all vertices of Γ, forcing that $A_v = 1$, a contradiction. Thus, $A_v/A_v^* \cong \mathbb{Z}_2$, and so there is one and only one neighbor, say u, of v such that $A_u = A_v$. By the arbitrariness of v, the following set

$$\mathcal{B} = \{\{u, v\} \mid u, v \in V(\Gamma) \text{ such that } A_u = A_v\}.$$

is a 1-factor of Γ. Clearly, for any $g \in A$, $A_{u^g} = A_u^g = A_v^g = A_{v^g}$. It follows that $\mathcal{B}$ is also an A-invariant partition of $V(\Gamma)$. Consider the quotient graph $\Gamma_{\mathcal{B}}$ of Γ relative to $\mathcal{B}$, and let K be the kernel of A acting on $V(\Gamma_{\mathcal{B}})$. Then A/K is vertex-transitive on $\Gamma_{\mathcal{B}}$ and so $\Gamma_{\mathcal{B}}$ has regular valency. Since Γ is cubic, $\Gamma_{\mathcal{B}}$ has valency at most 4, and since $|\Gamma_{\mathcal{B}}| = |P|$ is odd, the valency of $\Gamma_{\mathcal{B}}$ is 2 or 4. Since p is odd and K is a 2-group, $R(P) \cong R(P)K/K$ must be regular on $V(\Gamma_{\mathcal{B}})$.

Suppose $K \neq 1$. Then, $\mathcal{B}$ is the set of orbits of K. Since each orbit of K is just an edge of Γ, the quotient graph $\Gamma_{\mathcal{B}}$ must be a cycle of length p^n, where $p^n = |P|$, and moreover, since Γ has valency 3, the edges between any two adjacent orbits of K form a perfect matching. It follows that the neighbors of any $v \in V(\Gamma)$ are in three different orbits of K. Thus, K_v fixes each neighbor of v. By the connectedness of Γ, we obtain that $K_v = 1$ and hence $K \cong \mathbb{Z}_2$. As $R(P)K/K$ is regular on $V(\Gamma_{\mathcal{B}})$, $R(P)K = R(P) \times K$ is regular on $V(\Gamma)$, implying that Γ is a Cayley graph, a contradiction.

Now we know that $K = 1$. Then A acts faithfully on $\mathcal{B}$, and so $A \leqslant \text{Aut}(\Gamma_{\mathcal{B}})$. Since $R(P) = R(P)K/K$ acts regularly on $V(\Gamma_{\mathcal{B}})$, $\Gamma_{\mathcal{B}}$ can be viewed as a Cayley graph on P.

If $\Gamma_{\mathcal{B}}$ has valency 2, then $\Gamma_{\mathcal{B}}$ is a cycle of length p^n and $\mathrm{Aut}(\Gamma_{\mathcal{B}}) \cong D_{2p^n}$. It follows that $|A| = 2p^n = 2|P|$, contradicting that A is not regular on $V(\Gamma)$. Thus, $\Gamma_{\mathcal{B}}$ has valency 4. Since P is a regular p-group with $p \neq 2, 5$, by Proposition 3, $\Gamma_{\mathcal{B}}$ is a normal Cayley graph on P. It follows that $R(P) \trianglelefteq \mathrm{Aut}(\Gamma_{\mathcal{B}})$, and since $A \leqslant \mathrm{Aut}(\Gamma_{\mathcal{B}})$, one has $R(P) \trianglelefteq A$, as required. □

By Huppert [10, III, Theorem 10.2], a p-group of order p^n with $n \leqslant p$ is regular. By Lemma 9, the following corollary is straightforward.

Corollary 11. [12, Lemma 4.2] *Let Γ be a connected cubic vertex-transitive graph of order $2p^n$, where $p > 7$ is a prime and $n \leqslant p$. Then a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$ is normal.*

The following is the main result of this section.

Theorem 12. *Let $\Gamma = \mathrm{BiCay}(P, R, L, S)$ be a connected cubic vertex-transitive bi-Cayley graph over a regular p-group P, where $p > 5$ is a prime. Then Γ is non-Cayley if and only if $\Gamma = \mathrm{BiCay}(P, R, L, \{1\})$ is 2-type, and $\mathrm{Cay}(P, R \cup L)$ is a tetravalent normal arc-transitive Cayley graph with $\mathrm{Aut}(P, R \cup L) \cong \mathbb{Z}_4$.*

Proof. We first prove the sufficiency. Since $R = R^{-1}$ and $L = L^{-1}$, we may assume that $R = \{a, a^{-1}\}$ and $L = \{b, b^{-1}\}$. Since $\mathrm{Aut}(P, R \cup L) \cong \mathbb{Z}_4$, we may assume that $\mathrm{Aut}(P, R \cup L) = \langle \alpha \rangle$ such that $a^\alpha = b, b^\alpha = a^{-1}$. In particular, α interchanges R and L. By the definition of δ_α (see Eq 1)), δ_α interchanges the two orbits of $R(P)$. It follows from Proposition 7 that Γ is vertex-transitive. Suppose that Γ is Cayley. Since $\Gamma = \mathrm{BiCay}(P, R, L, \{1\})$ is 2-type, by Proposition 8 there is an automorphism $\beta \in \mathrm{Aut}(P)$ of order at most 2 such that $R^\beta = L$. This implies that $\beta \in \mathrm{Aut}(P, R \cup L) = \langle \alpha \rangle$. Clearly, $R \neq L$, so β has order 2. It follows that $\beta = \alpha^2$. However, $R^{\alpha^2} = R$ and $L^{\alpha^2} = L$, a contradiction. Thus, Γ is non-Cayley.

For the necessity, since p is odd, the subgraph induced by each orbit of $R(P)$ must have even valency. It follows that Γ is 0- or 2-type.

Case 1 Γ is 0-type.

By Lemma 10, $R(P) \trianglelefteq \mathrm{Aut}(\Gamma)$. Since $R(P)$ has two orbits on $V(\Gamma)$, the quotient graph $\Gamma_{R(P)}$ of Γ relative to $R(P)$ is the 3-dipole Dip_3 (see Fig. (1)). This implies that

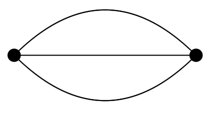

Figure 1: The 3-dipole Dip_3

Γ is a regular cover of Dip_3, and so $\mathrm{Aut}(\Gamma)$ can project to a subgroup of $\mathrm{Aut}(\Gamma_{R(P)})$. Since $\mathrm{Aut}(\mathrm{Dip}_3) \cong S_3 \times \mathbb{Z}_2$, $\mathrm{Aut}(\Gamma)/R(P) \leqslant \mathbb{Z}_2 \times \mathbb{Z}_2$ because Γ is not symmetric by Proposition 4. Since $p > 2$, $\mathrm{Aut}(\Gamma) = R(P) \rtimes Q$, where $1 < Q \leqslant \mathbb{Z}_2 \times \mathbb{Z}_2$ is a Sylow

2-subgroup of $\mathrm{Aut}(\Gamma)$. As Γ is vertex-transitive, there exists a 2-element, say g, such that g interchanges the two orbits of $R(P)$. Since $Q \leqslant \mathbb{Z}_2 \times \mathbb{Z}_2$, g must be an involution. Therefore, $R(P) \rtimes \langle g \rangle$ acts regularly on $V(\Gamma)$, and so Γ is Cayley. A contradiction occurs.

Case 2 Γ is 2-type.

Recall that $V(\Gamma) = P_0 \cup P_1$ with $P_0 = \{g_0 \mid g \in P\}$ and $P_1 = \{g_1 \mid g \in P\}$. By Proposition 6 (2), we may assume that $S = \{1\}$. It follows that $\{g_0, g_1\} \in E(\Gamma)$ for each $g \in P$. Set $A = \mathrm{Aut}(\Gamma)$. Then A is not regular on $V(\Gamma)$. So, for each $g \in P$, we have $A_{g_0} \neq 1$. Since $R(P) \trianglelefteq A$, A fixes the partition $V(\Gamma) = P_0 \cup P_1$, and since g_1 is the unique neighbor of g_0 in P_1, it follows that A_{g_0} fixes g_1. This implies that for any $\alpha \in A$, either $\{g_0, g_1\}^\alpha = \{g_0, g_1\}$ or $\{g_0, g_1\}^\alpha \cap \{g_0, g_1\} = \emptyset$. Set $\mathcal{B} = \{\{g_0, g_1\} \mid g \in P\}$. Then A acts transitively on $\mathcal{B}$. Let K be the kernel of A acting on $\mathcal{B}$.

Suppose $K \neq 1$. Clearly, $\mathcal{B}$ is the set of orbits of K. Since each orbit of K contains exactly one edge, the quotient graph $\Gamma_{\mathcal{B}}$ of Γ relative to $\mathcal{B}$ must be a cycle of length p^n, where $p^n = |P|$. It follows that the neighbors of any $v \in V(\Gamma)$ are in three different orbits of K. Thus, K_v fixes each neighbor of v. By the connectedness of Γ, we obtain that $K_v = 1$ and hence $K \cong \mathbb{Z}_2$. Then $R(P) \times K$ is regular on $V(\Gamma)$, and so Γ can be viewed as a Cayley graph on $R(P) \times K$. This is impossible.

Now assume that $K = 1$. Then A acts faithfully on $\mathcal{B}$. It follows that $A \leqslant \mathrm{Aut}(\Gamma_{\mathcal{B}})$. It is easy to see that $R(P)$ is regular on $\mathcal{B}$, and so $\Gamma_{\mathcal{B}}$ can be viewed as a Cayley graph on P. Recall that $\Gamma = \mathrm{BiCay}(P, R, L, \{1\})$. Set $R = \{a, b\}$ and $L = \{x, y\}$.

Suppose $|R \cap L| = 2$. Then $R = L$. In this case, it is easy to see that the permutation $\alpha = \prod_{g \in P}(g_0\ g_1)$ is an automorphism of Γ. Furthermore, α commutes with $R(P)$. So, $R(P) \times \langle \alpha \rangle$ acts regularly on $V(\Gamma)$, a contradiction.

Suppose $|R \cap L| = 1$. Without loss of generality, assume that $a = x$. Then $P = \langle a, b, y \rangle$. Since $R^{-1} = R$ and $L^{-1} = L$, all a, b, x, y are involutions. This is clearly impossible because P is a p-group with $p > 2$.

Suppose $|R \cap L| = 0$. In this case, the neighbors of $\{1_0, 1_1\}$ in $\Gamma_{\mathcal{B}}$ are $\{a_0, a_1\}$, $\{b_0, b_1\}$, $\{x_0, x_1\}$ and $\{y_0, y_1\}$. So, $\Gamma_{\mathcal{B}}$ is a tetravalent Cayley graph on P. Note that $A_{1_0} = A_{1_1}$. Since A is not regular on $V(\Gamma)$, A_{1_0} interchanges a_0 and b_0, and also interchanges x_1 and y_1. Since $\{1_0, 1_1\}$ is a block of A, A contains an element interchanging 1_0 and 1_1. This implies that $A_{\{1_0, 1_1\}}$ is transitive on the neighborhood of $\{1_0, 1_1\}$ in $\Gamma_{\mathcal{B}}$. So, A is an arc-transitive automorphism group of $\Gamma_{\mathcal{B}}$. Let $X = \mathrm{Cay}(P, R \cup L)$. It is easy to verify that the following map

$$f : g \mapsto \{g_0, g_1\}, \forall g \in P$$

is an isomorphism from X to $\Gamma_{\mathcal{B}}$. So, $X \cong \Gamma_{\mathcal{B}}$. Since $\Gamma_{\mathcal{B}}$ is arc-transitive, X is also arc-transitive. Since P is a p-group with $p > 2$, we may assume that $R = \{a, a^{-1}\}$ and $L = \{x, x^{-1}\}$. By Proposition 3, X is a normal Cayley graph, and so $\mathrm{Aut}(\Gamma_{\mathcal{B}}) = R(P) \rtimes \mathrm{Aut}(P, R \cup L)$. Clearly, $\mathrm{Aut}(P, R \cup L)$ acts faithfully on $R \cup L$. Since $p > 2$, it is easy to see that R, L are blocks of $\mathrm{Aut}(P, R \cup L)$ on $R \cup L$. It follows that $\mathrm{Aut}(P, R \cup L) \leqslant D_8$, and so $A \leqslant \mathrm{Aut}(\Gamma_{\mathcal{B}}) \leqslant R(P) \rtimes D_8$. Also, as R, L are blocks of $\mathrm{Aut}(P, R \cup L)$ on $R \cup L$, by Proposition 7, each element in $\mathrm{Aut}(P, R \cup L)$ can also induce an automorphism of Γ. This implies that $A = \mathrm{Aut}(\Gamma_{\mathcal{B}})$. Since Γ is arc-transitive, $\mathrm{Aut}(P, R \cup L)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4$, or D_8.

If $\mathrm{Aut}(P, R\cup L) \cong \mathbb{Z}_2\times\mathbb{Z}_2$ or D_8, then there exists an involution, say α, in $\mathrm{Aut}(P, R\cup L)$ which interchanges R and L. By Proposition 7, α can induce an automorphism, say δ_α, of Γ of order 2 such that $R(P) \rtimes \langle\delta_\alpha\rangle$ is regular on $V(\Gamma)$, and so Γ is Cayley, a contradiction. Thus, $\mathrm{Aut}(P, R \cup L) \cong \mathbb{Z}_4$. □

4 Cubic non-Cayley vertex-transitive graphs of order $2p^3$

Let p be an odd prime. We first introduce some cubic connected non-Cayley vertex-transitive graphs of order $2p^3$. It is well known that $\mathbb{Z}_{p^n}^*$ is cyclic and has order $p^{n-1}(p-1)$. So, if $4 \mid (p-1)$ then $\mathbb{Z}_{p^n}^*$ has a unique subgroup of order 4. Clearly, if λ is an element of order 4 in $\mathbb{Z}_{p^n}^*$, then $\{1, -1, \lambda, -\lambda\}$ is the unique subgroup of order 4 in the cyclic group $\mathbb{Z}_{p^n}^*$.

Example 13. Let p be a prime such that $p-1$ is divisible by 4 and let λ be an element of order 4 in $\mathbb{Z}_{p^3}^*$. The graph $\mathcal{NC}_{2p^3}^0$ is defined to be the bi-Cayley graph $\mathrm{BiCay}(\mathbb{Z}_{p^3}, R, L, \{1\})$, where $\mathbb{Z}_{p^3} = \langle a\rangle$, $R = \{a, a^{-1}\}$ and $L = \{a^\lambda, a^{-\lambda}\}$.

By the uniqueness of the subgroup of order 4 in $\mathbb{Z}_{p^3}$, the graph $\mathcal{NC}_{2p^3}^0$ is independent of the choice of λ. Let α be the automorphism of $\mathbb{Z}_{p^3}$ induced by the map $a \mapsto a^\lambda$. Then α swaps R and L, and by Proposition 7, $\delta_\alpha \in \mathrm{Aut}(\mathcal{NC}_{2p^3}^0)$ (see Equations (1)-(2) for the definition of δ_α) and so $\mathcal{NC}_{2p^3}^0$ is vertex-transitive because δ_α swaps the two orbits of $\mathbb{Z}_{p^3}$ on $V(\mathcal{NC}_{2p^3}^0)$.

In view of [20, Theorem 1], we have $\mathrm{Cay}(\mathbb{Z}_{p^3}, R \cup L)$ is a tetravalent normal arc-transitive Cayley graph and $\mathrm{Aut}(\mathbb{Z}_{p^3}, R\cup L) \cong \mathbb{Z}_4$. If $p = 5$, then by Magma [3], $\mathcal{NC}_{2p^3}^0$ is non-Cayley vertex-transitive and $|\mathrm{Aut}(\mathcal{NC}_{2p^3}^0)| = 4p^3$, and if $p > 5$, then by Theorem 12, again we have $\mathcal{NC}_{2p^3}^0$ is non-Cayley vertex-transitive.

Example 14. Let p be a prime such that $p-1$ is divisible by 4 and let λ be an element of order 4 in $\mathbb{Z}_p^*$. The graph $\mathcal{NC}_{2p^3}^1$ is defined to be the bi-Cayley graph $\mathrm{BiCay}(\mathbb{Z}_{p^2} \times \mathbb{Z}_p, R, L, \{1\})$, where $\mathbb{Z}_{p^2} \times \mathbb{Z}_p = \langle a\rangle \times \langle b\rangle$, $R = \{a, a^{-1}\}$ and $L = \{(ab)^\lambda, (ab)^{-\lambda}\}$.

By the uniqueness of the subgroup of order 4 in $\mathbb{Z}_p$, the graph $\mathcal{NC}_{2p^3}^1$ is independent of the choice of λ. Let β be the automorphism of $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$ induced by the map $a \mapsto (ab)^\lambda, b \mapsto a^{\lambda^3+\lambda}b^{-\lambda}$. Then β swaps R and L, and by Proposition 7, $\delta_\beta \in \mathrm{Aut}(\mathcal{NC}_{2p^3}^1)$ and so $\mathcal{NC}_{2p^3}^1$ is vertex-transitive because δ_β swaps the two orbits of $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ on $V(\mathcal{NC}_{2p^3}^1)$.

In view of [21, Proposition 3.3], we have $\mathrm{Cay}(\mathbb{Z}_{p^2} \times \mathbb{Z}_p, R \cup L)$ is a tetravalent normal arc-transitive Cayley graph and $\mathrm{Aut}(\mathbb{Z}_{p^2} \times \mathbb{Z}_p, R\cup L) \cong \mathbb{Z}_4$. If $p = 5$, then by Magma [3], $\mathcal{NC}_{2p^3}^1$ is non-Cayley vertex-transitive and $|\mathrm{Aut}(\mathcal{NC}_{2p^3}^1)| = 4p^3$, and if $p > 5$, then by Theorem 12, again we have $\mathcal{NC}_{2p^3}^1$ is non-Cayley vertex-transitive.

Also, note that $\mathrm{Aut}(\mathcal{NC}_{2p^3}^0) \cong \mathbb{Z}_{p^3} \rtimes \mathbb{Z}_4$ and $\mathrm{Aut}(\mathcal{NC}_{2p^3}^1) \cong (\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_4$. It follows that $\mathcal{NC}_{2p^3}^0$ and $\mathcal{NC}_{2p^3}^1$ are not isomorphic to each other.

Theorem 15. *Let p be a prime. Then a cubic vertex-transitive graph of order $2p^3$ is non-Cayley if and only if it is isomorphic to $\mathcal{NC}_{2p^3}^0$ or $\mathcal{NC}_{2p^3}^1$.*

Proof. By [15], all connected cubic vertex-transitive graphs of order 16 are Cayley. By [17], if $p = 3$, then all connected cubic vertex-transitive graphs of order 54 are Cayley, and if $p = 5$, then up to isomorphism, there are exactly two non-Cayley vertex-transitive graphs of order $2 \cdot 5^3$, and so $\Gamma \cong \mathcal{NC}^0_{2\cdot 5^3}$ or $\mathcal{NC}^1_{2\cdot 5^3}$.

In what follows, assume that $p > 5$. By Lemma 9, Γ is a bi-Cayley graph over a group P, where P is a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$. By Theorem 12, $\Gamma = \mathrm{BiCay}(P, R, L, \{1\})$ and $\mathrm{Cay}(P, R \cup L)$ is a tetravalent normal arc-transitive Cayley graph such that $\mathrm{Aut}(P, R \cup L) \cong \mathbb{Z}_4$. Since $|\Gamma| = 2p^3$, one has $|P| = p^3$. Noting that $R = R^{-1}$ and $L = L^{-1}$, by Proposition 5, one of the following happens:

(1) $P = \mathbb{Z}_{p^3} = \langle a\rangle, R = \{a, a^{-1}\}, L = \{a^\lambda, a^{-\lambda}\}(\lambda^2 \equiv -1 \pmod{p^3})$;
(2) $P = \mathbb{Z}_{p^2} \times \mathbb{Z}_p = \langle a\rangle \times \langle b\rangle, R = \{a, a^{-1}\}, L = \{(ab)^\lambda, (ab)^{-\lambda}\}(\lambda^2 \equiv -1 \pmod{p})$;
(3) $P = \mathbb{Z}_{p^2} \times \mathbb{Z}_p = \langle a\rangle \times \langle b\rangle, R = \{a, a^{-1}\}, L = \{ab, (ab)^{-1}\}$;
(4) $P = \langle a, b, c \mid a^p = b^p = c^p = 1, c = [a, b], [a, c] = [b, c] = 1\rangle$,
$R = \{a, a^{-1}\}, L = \{b, b^{-1}\}$.

If (1) happens, then $\Gamma \cong \mathcal{NC}^0_{2p^3}$, and if (2) happens, then $\Gamma \cong \mathcal{NC}^1_{2p^3}$. If (3) happens, then in view of [21, Proposition 3.3], we have $\mathrm{Aut}(P, R \cup L) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. This is impossible by Theorem 12. If (3) happens, then $P = \langle a, b, c \mid a^p = b^p = c^p = 1, c = [a, b], [a, c] = [b, c] = 1\rangle$, and any two elements generating P have the same relation as a and b. It follows that $\mathrm{Aut}(P, R \cup L) \cong D_8$, and by Theorem 12, Γ is not non-Cayley, a contradiction. □

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