A generalized Alon-Boppana bound and weak Ramanujan graphs

Fan Chung*
Department of Mathematics
University of California, San Diego
La Jolla, CA, U.S.A.
fan@ucsd.edu

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Abstract

A basic eigenvalue bound due to Alon and Boppana holds only for regular graphs. In this paper we give a generalized Alon-Boppana bound for eigenvalues of graphs that are not required to be regular. We show that a graph G with diameter k and vertex set V, the smallest nontrivial eigenvalue λ_1 of the normalized Laplacian $\mathcal{L}$ satisfies

$$\lambda_1 \leqslant 1 - \sigma\big(1 - \frac{c}{k}\big)$$

for some constant c where $\sigma = 2\sum_v d_v\sqrt{d_v - 1}/\sum_v d_v^2$ and d_v denotes the degree of the vertex v.

We consider weak Ramanujan graphs defined as graphs satisfying $\lambda_1 \geqslant 1 - \sigma$. We examine the vertex expansion and edge expansion of weak Ramanujan graphs and then use the expansion properties among other methods to derive the above Alon-Boppana bound.

1 Introduction

The well-known Alon-Boppana bound [8] states that for any d-regular graph with diameter k, the second largest eigenvalue ρ of the adjacency matrix satisfies

$$\rho \geqslant 2\sqrt{d-1}\Big(1 - \frac{2}{k}\Big) - \frac{2}{k}. \qquad (1)$$

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A natural question is to extend Alon-Boppana bounds for graphs that are irregular. Hoory [6] showed that for an irregular graph, the second largest eigenvalue ρ of the adjacency matrix satisfies

$$\rho \geqslant 2\sqrt{d-1}\Big(1 - \frac{c \log r}{r}\Big)$$

if the average degree of the graph after deleting a ball of radius r is at least d where $r, d > 2$.

For irregular graphs, it is often advantageous to consider eigenvalues of the normalized Laplacian for deriving various graph properties. For a graph G, the normalized Laplacian $\mathcal{L}$, defined by

$$\mathcal{L} = I - D^{-1/2} A D^{-1/2}$$

where D is the diagonal degree matrix and A denotes the adjacency matrix of G. One of the main tools for dealing with general graphs is the Cheeger inequality which relates the least nontrivial eigenvalue λ_1 to the Cheeger constant h_G:

$$2h_G \geqslant \lambda_1 \geqslant \frac{h_G^2}{2} \tag{2}$$

where $h_G = \min_S |\partial(S)|/\text{vol}(S)$ for S ranging over all vertex subsets with volume $\text{vol}(S) = \sum_{u \in S} d_u$ no more than half of $\sum_{u \in V} d_u$ and $\partial(S)$ denotes the set of edges leaving S. For k-regular graphs, we have $\lambda_1 = 1 - \rho/k$ where ρ denotes the second largest eigenvalue of the adjacency matrix. In general,

$$\frac{\rho}{\max_v d_v} \leqslant 1 - \lambda_1 \leqslant \frac{\rho}{\min_v d_v}$$

which can be used to derive a version of the Cheeger inequality involving ρ which is less effective than (2) for irregular graphs.

In this paper, we will show that for a connected graph G with diameter k, λ_1 is upper bounded by

$$\lambda_1 \leqslant 1 - \sigma(1 - \frac{c}{k}) \tag{3}$$

for a constant c where $\sigma = 2\sum_v d_v\sqrt{d_v - 1}/\sum_v d_v^2$. The above inequality will be proved in Section 6.

The above bound of Alon-Boppana type improves a result of Young [10] who derived a similar eigenvalue bound using a different method. In [10] the notion of (r, d, δ)-robust graphs was considered and it was shown that for a (r, d, δ)-robust graph, the least nontrivial eigenvalue λ_1 satisfies

$$\lambda_1 \leqslant 1 - \frac{2d\sqrt{d-1}}{\delta}\Big(1 - \frac{c}{r}\Big). \tag{4}$$

Here (r, d, δ)-robustness means for every vertex v and the ball $B_r(v)$ consisting of all vertices with distance at most r, the induced subgraph on the complement of $B_r(v)$ has average degree at least d and $\sum_{v \notin B_r(v)} d_v^2/|V \setminus B_r(v)| \leqslant \delta$. We remark that our result in (3) does not require the condition of robustness.

We define *weak Ramanujan* graphs to be graphs with eigenvalue λ_1 satisfying

$$\lambda_1 \geqslant 1 - \sigma \geqslant \frac{1}{2} \tag{5}$$

where $\sigma = 2\sum_v d_v\sqrt{d_v - 1}/\sum_v d_v^2$.

To prove the Alon-Boppana bound in (3), it suffices to consider only weak Ramanujan graphs. Weak Ramanujan graphs satisfy various expansion properties. We will describe several vertex-expansion and edge-expansion properties involving λ_1 in Section 3, which will be needed later for proving a diameter bound for weak Ramanujan graphs in Section 4. The diameter bound and related properties of weak Ramanujan graphs are useful in the proof of the Alon-Boppana bound for general graphs.

We will also show that the largest eigenvalue λ_{n-1} of the normalized Laplacian satisfies

$$\lambda_{n-1} \geqslant 1 + \sigma(1 - \frac{c}{k}). \tag{6}$$

The proof will be given in Section 7.

2 Preliminaries

For a graph $G = (V, E)$, we consider the normalized Laplacian

$$\mathcal{L} = I - D^{-1/2}AD^{-1/2}$$

where A denotes the adjacency matrix and D denotes the diagonal degree matrix with $D(v, v) = d_v$, the degree of v. We assume that there is no isolated vertex throughout this paper. For a vertex v and a positive integer l, let $B_l(v)$ denote the ball consisting of all vertices within distance l from v. For an edge $\{x, y\} \in E$ we say x is adjacent to y and write $x \sim y$.

Let $\lambda_0 \leqslant \lambda_1 \leqslant \ldots \leqslant \lambda_{n-1}$ denote eigenvalues of $\mathcal{L}$, where n denotes the number of vertices in G. It can be checked (see [2]) that $\lambda_1 > 0$ if G is connected. The Alon-Boppana bound obviously holds if $\lambda_1 = 0$. In the remainder of this paper, we assume G is connected.

Let φ_i denote the orthonormal eigenvector associated with eigenvalue λ_i. In particular, $\varphi_0 = D^{1/2}\mathbf{1}/\sqrt{\mathrm{vol}(G)}$ where $\mathbf{1}$ is the all 1's vector and $\mathrm{vol}(G) = \sum_{v \in V} d_v$. We can then write

$$\begin{aligned}
\lambda_1 &= \inf_{g \perp \varphi_0} \frac{\langle g, \mathcal{L}g \rangle}{\langle g, g \rangle} \\
&= \inf_{f \perp D\mathbf{1}} \frac{\sum_{x \sim y} \big(f(x) - f(y)\big)^2}{\sum_z f^2(z) d_z} \\
&= \inf_{f \perp D\mathbf{1}} R(f)
\end{aligned}$$

where f ranges over all functions satisfying $\sum_u f(u)d_u = 0$ and the sum $\sum_{x\sim y}$ ranges over all unordered pairs $\{x, y\}$ where x is adjacent to y. Here $R(f)$ denote the *Rayleigh quotient* of f, which can be written as follows:

$$R(f) = \frac{\int |\nabla f|}{\int \|f\|^2}$$
$$\text{where } \int \|f\|^2 = \sum_x f^2(x)d_x$$
$$\text{and } \int |\nabla f| = \sum_{x\sim y} \big(f(x) - f(y)\big)^2.$$

For eigenfunction φ_i, the function $f_i = D^{-1/2}\varphi_i$, called the combinatorial eigenfucntion associated with λ_i, satisfies

$$\lambda_i f(u)d_u = \sum_{v\sim u} \big(f(u) - f(v)\big) \tag{7}$$

for each vertex u. In particular, for f satisfying $\sum_u f(u)d_u = 0$, we have

$$\langle f, Af\rangle \leqslant (1 - \lambda_1)\langle f, Df\rangle \tag{8}$$

and

$$|\langle f, Af\rangle| \leqslant \max_{i\neq 0}(1 - \lambda_i)\langle f, Df\rangle. \tag{9}$$

3 Vertex and edge expansions

For any subset S of vertices, there are two types of boundaries. The *edge boundary* of S, denoted by $\partial(S)$ consists of all edges with exactly one endpoint in S. The *vertex boundary* of S, denoted by $\delta(S)$ consists of all vertices not in S but adjacent to vertices in S. Namely,

$$\partial(S) = \{\{u, v\} \in E \ : \ u \in S \text{ and } v \notin S\} = E(S, \bar{S})$$
$$\delta(S) = \{u \notin S \ : \ u \sim v \in S \text{ for some vertex } v\}$$

In this section, we will examine vertex expansion and edge expansion relying only on λ_1. These expansion properties will be needed for deriving diameter bounds for weak Ramanujan graphs which will be used in our proof of the general Alon-Boppana bound later in Section 6.

From the definition of the Cheeger constant, for all vertex subsets S, we have

$$\frac{|\partial(S)|}{\mathrm{vol}(S)} \geqslant h_G \geqslant \frac{\lambda_1}{2}$$

Later in the proofs, we will be interested in the case that $\mathrm{vol}(S)$ is small and therefore we will use the following version.

Lemma 1. *Let S be a subset of vertices in G. Then*

$$\frac{|\partial(S)|}{\mathrm{vol}(S)} \geqslant \lambda_1\Big(1 - \frac{\mathrm{vol}(S)}{\mathrm{vol}(G)}\Big).$$

Proof. Suppose f is defined by

$$f = \frac{\mathbf{1}_S}{\mathrm{vol}(S)} - \frac{\mathbf{1}_{\bar{S}}}{\mathrm{vol}(\bar{S})}$$

where $\mathbf{1}_S$ denotes the characteristic function defined by $\mathbf{1}_S(v) = 1$ if $v \in S$ and 0 otherwise.

The Rayleigh quotient $R(f)$ satisfies

$$\lambda_1 \leqslant R(f) = \frac{|\partial(S)|}{\mathrm{vol}(S)} \cdot \frac{\mathrm{vol}(G)}{\mathrm{vol}(\bar{S})}.$$ □

For the expansion of the vertex boundary, the Tanner bound [9] for regular graphs can be generalized as follows.

Lemma 2. *Let $\bar{\lambda} = \min_{i\neq 0} |1 - \lambda_i|$. Then for any vertex subset S in a graph,*

$$\frac{\mathrm{vol}(\delta(S))}{\mathrm{vol}(S)} \geqslant \frac{1-\bar{\lambda}^2}{\bar{\lambda}^2 + \frac{\mathrm{vol}(S)}{\mathrm{vol}(\bar{S})}} \tag{10}$$

The proof of the above inequality is by using the following discrepancy inequality (as seen in [2]).

Lemma 3. *In a graph G, for two subset X and Y of vertices, the number $e(X,Y) = |E(X,Y)|$ of edges between X and Y satisfies*

$$\left| e(X,Y) - \frac{\mathrm{vol}(X)\mathrm{vol}(Y)}{\mathrm{vol}(G)} \right| \leqslant \bar{\lambda}\frac{\sqrt{\mathrm{vol}(X)\mathrm{vol}(Y)\mathrm{vol}(\overline{X})\mathrm{vol}(\overline{Y})}}{\mathrm{vol}(G)} \tag{11}$$

where $\bar{\lambda} = \min_{i\neq 0} |1 - \lambda_i|$.

The proof of Lemma 3 follows from (9) and can be found in [2]. The proof of (12) results from (11) by setting $X = S$ and $Y = \overline{S \cup \delta(S)}$.

Here we will give a version of the vertex-expansion bounds for general graphs which only rely on λ_1 and are independent of other eigenvalues.

Lemma 4. *In a graph G with vertex set V and the first nontrivial eigenvalue λ_1, for a subset S of V with $\mathrm{vol}(S \cup \delta S) \leqslant \epsilon \mathrm{vol}(G) \leqslant \mathrm{vol}(G)/2$, the vertex boundary of S satisfies*

(i)
$$\frac{\mathrm{vol}(\delta(S))}{\mathrm{vol}(S)} \geqslant \frac{2\lambda_1}{1-\lambda_1+2\epsilon} \tag{12}$$

(ii) If $1/2 \leqslant \lambda_1 \leqslant 1 - 2\epsilon$, then

$$\frac{\mathrm{vol}(\delta(S))}{\mathrm{vol}(S)} \geqslant \frac{1}{(1-\lambda_1+2\epsilon)^2}. \tag{13}$$

Proof. The proof of (i) follows from Lemma 1 since

$$\begin{aligned}\frac{\mathrm{vol}(\delta(S))}{\mathrm{vol}(S)} &\geqslant \frac{|\partial(S \cup \delta(S))| + |\partial(S)|}{\mathrm{vol}(S)} \\ &\geqslant \frac{\lambda_1(1-\epsilon)\big(\mathrm{vol}(S) + \mathrm{vol}(\delta(S)\big) + \lambda_1(1-\epsilon)\mathrm{vol}(S)}{\mathrm{vol}(S)}\end{aligned}$$

Therefore

$$\frac{\mathrm{vol}(\delta(S))}{\mathrm{vol}(S)} \geqslant \frac{2\lambda_1(1-\epsilon)}{1-\lambda_1(1-\epsilon)} \geqslant \frac{2\lambda_1}{1-\lambda_1+2\epsilon}$$

To prove (ii), we set $f = \mathbf{1}_S + \gamma \mathbf{1}_{\delta(S)}$ where $\gamma = 1 - \lambda_1$. Consider $g = f - c\mathbf{1}_V$ where $c = \sum_u f(u)d_u \Big/ \mathrm{vol}(G)$. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}c^2 = \frac{1}{(\mathrm{vol}(G))^2}\Big(\sum_{u \in S \cup \delta(S)} f(u)d_u\Big)^2 &\leqslant \frac{\mathrm{vol}(S \cup \delta(S))}{(\mathrm{vol}(G))^2} \sum_u f^2(u)d_u \\ &\leqslant \frac{\epsilon}{\mathrm{vol}(G)} \sum_u f^2(u)d_u.\end{aligned}$$

Using the inequality in (8), we have

$$\begin{aligned}\langle f, Af\rangle &\leqslant \langle g, Ag\rangle + c^2\mathrm{vol}(G) \\ &\leqslant \gamma\langle g, Dg\rangle + c^2\mathrm{vol}(G) \\ &= \gamma\langle f, Df\rangle + (1-\gamma)c^2\mathrm{vol}(G) \\ &\leqslant (\gamma+\epsilon)\langle f, Df\rangle \\ &= (\gamma+\epsilon)\big(\mathrm{vol}(S) + \gamma^2\mathrm{vol}(\delta(S))\big).\end{aligned}$$

Let $e(S,T)$ denote the number of ordered pairs (u,v) where $u \in S, v \in T$ and $\{u,v\} \in E$. Since $\gamma = 1 - \lambda \leqslant 1/2$, we have

$$\begin{aligned}\langle f, Af\rangle &\geqslant e(S,S) + 2\gamma e(S, \delta(S)) \\ &\geqslant (1-2\gamma)e(S,S) + 2\gamma\mathrm{vol}(S) \\ &\geqslant 2\gamma\mathrm{vol}(S)\end{aligned}$$

Together we have

$$\begin{aligned}\frac{\mathrm{vol}(\delta(S))}{\mathrm{vol}(S)} &\geqslant \frac{\gamma-\epsilon}{\sigma^2(\gamma+\epsilon)} \\ &\geqslant \frac{1}{(\gamma+2\epsilon)^2}\end{aligned}$$

since $\gamma \geqslant 2\epsilon$. □

Recall that weak Ramanujan graphs have eigenvalue λ_1 satisfying

$$\lambda_1 \geqslant 1 - \sigma \tag{14}$$

where $\sigma = 2\sum_v d_v\sqrt{d_v - 1}/\sum_v d_v^2$. Lemma 1 implies that for S with $\mathrm{vol}(S \cup \delta(S)) \leqslant \epsilon\mathrm{vol}(G)$,

$$\frac{\mathrm{vol}(\delta(S))}{\mathrm{vol}(S)} \geqslant \frac{1}{(\sigma + 2\epsilon)^2}.$$

For k-regular Ramanujan graphs with eigenvalue $\lambda_1 = 1 - 2\sqrt{k-1}/k$, the above inequality is consistent with the bound

$$\frac{\mathrm{vol}(\delta(S))}{\mathrm{vol}(S)} = \frac{|\delta(S)|}{|S|} \geqslant \frac{1}{(\frac{2\sqrt{k-1}}{k} + 2\epsilon)^2}$$

which is about $k/4$ when $\mathrm{vol}(S)$ is small. The factor $k/4$ in the above inequality was improved by Kahale [4] to $k/2$. There are many applications (see [1]) that require graphs having expansion factor to be $(1 - \epsilon)k$. Such graphs are called *lossless* expanders. In [1], lossless graphs were constructed explicitly by using the zig-zag construction but the method for deriving the expansion bounds does not use eigenvalues. In this paper, the expansion factor as in Lemma 4 is enough for our proof later.

4 Weak Ramanujan graphs

We recall that a graph is said to be a weak Ramanujan graph as in (14) if

$$\lambda_1 \geqslant 1 - \sigma \geqslant \frac{1}{2}$$

where

$$\sigma = 2\frac{\sum_v d_v\sqrt{d_v - 1}}{\sum_v d_v^2}. \tag{15}$$

To prove the Alon-Boppana bound, it is enough to consider only weak Ramanujan graphs.

Lemma 5. *As defined in (15), σ satisfies*

$$\frac{2\sqrt{\bar{d} - 1}}{\breve{d}} \leqslant \sigma \leqslant \frac{2\sqrt{\bar{d} - 1}}{\bar{d}}$$

where $\bar{d}$ denotes the average degree in G and $\breve{d}$ denote the second order degree, i.e.,

$$\bar{d} = \frac{\sum_v d_v}{n} \quad \text{and} \quad \breve{d} = \frac{\sum_v d_v^2}{\sum_v d_v}.$$

Proof. The proof is mainly by using the Cauchy-Schwarz inequality. For the upper bound, we note that

$$\begin{aligned} \sigma = 2\frac{\sum_v d_v\sqrt{d_v - 1}}{\sum_v d_v^2} &\leqslant 2\frac{\sqrt{\sum_v d^2 \sum_v (d_v - 1)}}{\sum_v d_v^2} \\ &= 2\frac{\sqrt{\sum_v (d_v - 1)}}{\sqrt{\sum_v d_v^2}} \\ &\leqslant 2\frac{\sqrt{\sum_v (d_v - 1)}}{\sum_v d_v/\sqrt{n}} \\ &\leqslant 2\frac{\sqrt{\sum_v (d_v - 1)}}{\bar{d}\sqrt{n}} \leqslant \frac{2\sqrt{\bar{d} - 1}}{\bar{d}}. \end{aligned}$$

For the upper bound, we will use the fact that for $a, b > 1$ and $a + b = c$,

$$a\sqrt{a - 1} + b\sqrt{b - 1} \geqslant c\sqrt{\frac{c}{2} - 1}$$

and therefore

$$\sum_v d_v\sqrt{d_v - 1} \geqslant \sum_v d_v\sqrt{\frac{\sum_v d_v}{n} - 1}.$$

Consequently, we have

$$\sigma = 2\frac{\sum_v d_v\sqrt{d_v - 1}}{\sum_v d_v^2} \geqslant 2\frac{\sum_v d_v\sqrt{\frac{\sum_v d_v}{n} - 1}}{\frac{\sum_v d_v^2}{\sum_v d_v}\sum_v d_v} \geqslant 2\frac{\sqrt{\bar{d} - 1}}{\breve{d}}$$

as desired. ☐

We remark that for graphs with average degree at least 20, we have $\sigma < 1/2 < \lambda_1$.

Theorem 6. *Suppose a weak Ramanujan graph G has diameter k. Then for any $\epsilon > 0$, we have*

$$k \leqslant (1 + \epsilon)\frac{2\log \mathrm{vol}(G)}{\log \sigma^{-1}}$$

provided that the volume of G is large, i.e., $\mathrm{vol}(G) \geqslant c\sigma^{\log(\sigma)}/\epsilon$ for some small constant c.

Proof. We set

$$t = \left\lceil (1 + \epsilon)\frac{\log(\mathrm{vol}(G))}{\log \sigma^{-1}} \right\rceil.$$

It suffices to show that for every vertex v, the ball $B_t(v)$ has volume more than $\mathrm{vol}(G)/2$.

Suppose $\mathrm{vol}(B_t(v)) \leqslant \mathrm{vol}(G)/2$. Let

$$s_j = \frac{\mathrm{vol}(B_j(u))}{\mathrm{vol}(G)}.$$

By part (i) of Lemma 4, we have $\mathrm{vol}(\delta(B_u(j))) \geqslant 0.5\mathrm{vol}(B_u(j))$ for $j \leqslant t-1$ and therefore $s_{j+1} \geqslant 1.5 s_j$. Thus, if $j \leqslant t - c_1 \log(\sigma^{-1})$, then $s_j \leqslant \sigma^4$ where c_1 is some small constant satisfying $c_1 \leqslant 4(\log 1.5)^{-1}$.

Now we apply part (ii) of Lemma 4 and we have, for $j \leqslant t - c_1 \log(\sigma^{-1})$,

$$\frac{s_{j+1}}{s_j} = \frac{\mathrm{vol}(B_{j+1}(u))}{\mathrm{vol}(B_j(u))} \geqslant \frac{\mathrm{vol}(\delta(B_j(u)))}{\mathrm{vol}(B_j(u))} \geqslant \frac{1}{(\sigma + 2s_j)^2} \geqslant \frac{1}{(\sigma + 2\sigma^4)}.$$

This implies, for $l \leqslant t - c_1 \log(\sigma^{-1})$,

$$\begin{aligned} \frac{s_l}{s_0} \geqslant \prod_{0<j<l} \frac{1}{(\sigma + 2s_j)^2} &\geqslant \prod_{0<j<l} \frac{1}{(\sigma + 2\sigma^4)^2} \\ &\geqslant \frac{1}{\sigma^{2l}(1 + 2\sigma^4)^{2l}}. \end{aligned}$$

Since $s_0 \geqslant 1/\mathrm{vol}(G)$ and $s_l \leqslant s_t \leqslant 1/2$, we have

$$\mathrm{vol}(G) \geqslant \frac{1}{\sigma^{2l}(1 + 2\sigma^4)^{2l}}.$$

Hence

$$l \leqslant \frac{\log(\mathrm{vol}(G))}{\log(\sigma^{-1}) + 2\sigma^4}.$$

However,

$$(1+\epsilon)\frac{\log(\mathrm{vol}(G))}{\log(\sigma^{-1})} \leqslant t \leqslant c_1 \log(\sigma^{-1}) + \frac{\log(\mathrm{vol}(G))}{\log(\sigma^{-1}) + 2\sigma^4}$$

which is a contracdiction for G with $\mathrm{vol}(G)$ large, say, $\mathrm{vol}(G) \geqslant \sigma^{2c_1 \log \sigma}/\epsilon$. Thus we conclude that $s_t \geqslant 1/2$ and Theorem 6 is proved. □

Theorem 7. *For a weak Ramanujan graph with diameter k, for any vertex v and any $l \leqslant k/4$, the ball $B_u(l)$ has volume at most $\epsilon\mathrm{vol}(G)$ if $k \geqslant c \log \epsilon^{-1}$, for some constants c.*

Proof. We will prove by contradiction. Suppose that for $j_0 = \lceil k/4 \rceil$, there is a vertex u with $\mathrm{vol}(B_v(j_0)) > \epsilon\mathrm{vol}(G)$. Let r denote the largest integer such that

$$s_r = \frac{\mathrm{vol}(B_u(r))}{\mathrm{vol}(G)} > \frac{1}{2}.$$

By the assumption, we have $r > k/4$ and $s_{j_0} > \epsilon$. There are two possibilities:

Case 1: $r \geqslant k/2$.
By part (i) of Lemma 4, we have $\mathrm{vol}(\delta(B_u(j))) \geqslant 0.5\mathrm{vol}(B_u(j))$ for $j \leqslant k/2$ and therefore $s_{j+1} \geqslant 1.5 s_j$. Thus, for $j \leqslant k/2 - c_1 \log \epsilon^{-1}$, we have $s_j \leqslant \epsilon$ where $c_1 = 1/\log 1.5$. Since $k/4 \leqslant k/2 - c_1 \log \epsilon^{-1}$, we have a contradiction.

Case 2: $r < k/2$.
We define

$$\bar{s}_j = \frac{\mathrm{vol}(V \setminus B_u(j))}{\mathrm{vol}(G)}.$$

Thus $\bar{s}_j < 1/2$ for all $j \geqslant k/2$. We consider two subcases.

Subcase 2a: Suppose $\bar{s}_j \geqslant \epsilon$ for $j \geqslant k/2$.

Using Lemma 4, for j where $r \leqslant j \leqslant k/2$, we have $\bar{s}_j \geqslant 1.5\bar{s}_{j+1}$. Thus, for some $j_1 \geqslant k/2 - c_1 \log \epsilon^{-1}$, we have $\bar{s}_j \geqslant 1/2$ or equivalently, $s_j \leqslant 1/2$. By using Lemma 4 again, for $j \leqslant j_1$, we have $s_{j+1} \geqslant 1.5 s_j$ and therefore for any $j \leqslant j_1 - c_1 \log \epsilon^{-1}$ we have $s_j \leqslant \epsilon$. Since $j_1 - c_1 \log \epsilon^{-1} \geqslant k/2 - 2c_1 \log \epsilon^{-1} \geqslant k/4$, we again have a contradiction to the assumption $s_{j_0} \geqslant \epsilon$.

Subcase 2b: Suppose $\bar{s}_j < \epsilon$ for $j \geqslant k/2$

We apply part (ii) of Lemma 4 and we have, for $j \geqslant k/2$,

$$\frac{\bar{s}_j}{\bar{s}_{j+1}} \geqslant \frac{1}{(\sigma + 2\epsilon)^2}.$$

This implies, for $j_2 = \lceil k/2 \rceil$,

$$\frac{\bar{s}_{j_2}}{\bar{s}_k} \geqslant \prod_{k/2 < j \leqslant k} \frac{1}{(\sigma + 2s_j)^2} \geqslant \frac{1}{(\sigma + 2\epsilon)^k}.$$

Since $\bar{s}_k \geqslant 1/\mathrm{vol}(G)$, we have

$$\bar{s}_{j_1} \geqslant \frac{1}{\mathrm{vol}(G)(\sigma + 2\epsilon)^k}.$$

Since the assumption of this subcase is $\bar{s}_{j_1} < \epsilon$, we have

$$k \geqslant \frac{\log n + \log \epsilon^{-1}}{\log \sigma^{-1}}.$$

We now use Lemma 4 and we have, for $j = k/2 - j' \geqslant r$

$$\bar{s}_j \geqslant \frac{1}{\mathrm{vol}(G)(\sigma + 2\epsilon)^{k+2j'}}.$$

Therefore, for some $j \leqslant k/2 - \log \epsilon^{-1}/\log \sigma^{-1}$, we have $\bar{s}_j > 1/2$ which implies $r \geqslant k/2 - \log \epsilon^{-1}/\log \sigma^{-1}$.

Now we use the same argument as in Case 1 except shifting r by $\log \epsilon^{-1}/\log \sigma^{-1}$. For some $j \leqslant r - c_1 \log \epsilon^{-1} \leqslant k/2 - \log \epsilon^{-1}/\log \sigma^{-1} - c_1 \log \epsilon^{-1}$, we have $s_j < \epsilon$. Since $\log \epsilon^{-1}/\log \sigma^{-1} + c_1 \log \epsilon^{-1} < k/4$, this leads to a contradiction and Theorem 7 is proved. □

5 Non-backtracking random walks

Before we proceed to the proof of the Alon-Boppana bound, we will need some basic facts on non-backtracking random walks.

A non-backtracking walk is a sequence of vertices $\mathbf{p} = (v_0, v_1, \ldots, v_t)$ for some t such that $v_{i-1} \sim v_i$ and $v_{i+1} \neq v_{i-1}$ for $i = 1, \ldots, t-2$. The non-backtracking random walk can be described as follows: For $i \geqslant 1$, at the ith step on v_i, choose with equal probability a neighbor u of v_i where $u \neq v_{i-1}$, move to u and set $v_{i+1} = u$. To simplify notation, we call a non-backtracking walk an NB-walk. The modified transition probability matrix $\tilde{P}_k$, for $k = 0, 1, \ldots, t-1$, is defined by

$$\tilde{P}_k(u,v) = \begin{cases} P^k(u,v) & \text{if } k = 0 \\ \sum\limits_{\mathbf{p} \in \mathscr{P}_{u,v}^{(k)}} w(\mathbf{p}) & \text{if } k \geqslant 1 \end{cases} \tag{16}$$

where the weight $w(\mathbf{p})$ for an NB-walk $\mathbf{p} = (v_0, v_1, \ldots, v_t)$ with $t \geqslant 1$ is defined to be

$$w(\mathbf{p}) = \frac{1}{d_{v_0} \prod_{i=1}^{t-1} (d_{v_i} - 1)} \tag{17}$$

and $\mathscr{P}_{u,v}^{(k)}$ denotes the set of non-backtracking walks from u to v. For a walk $\mathbf{p} = (v_0)$ of length 0, we define $w(\mathbf{p}) = 1$.

Although a non-backtracking random walk is not a Markov chain, it is closely related to an associated Markov chain as we will describe below (also see [6]).

For each edge $\{u, v\}$ in E, we consider two directed edges (u, v) and (v, u). Let $\hat{E}$ denote the set consisting of all such directed edges, i.e. $\hat{E} = \{(u,v) \ : \ \{u,v\} \in E\}$. We consider a random walk on $\hat{E}$ with transition probability matrix $\boldsymbol{P}$ defined as follows:

$$\boldsymbol{P}((u,v),(u',v')) = \begin{cases} \frac{1}{d_v - 1} & \text{if } v = u' \text{and } u \neq v' \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{1}_E$ denote the all 1's function defined on the edge set E as a row vector. From the above definition, we have

$$\mathbf{1}_E \boldsymbol{P} = \mathbf{1}_E. \tag{18}$$

In addition, we define the vertex-edge incidence matrix B and B^* for $a \in V$ and $(b,c) \in \hat{E}$ by

$$B(a,(b,c)) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise} \end{cases}$$

$$B^*((b,c),a)) = \begin{cases} 1 & \text{if } c = a, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{1}_V$ denote all 1's vector defined on the vertex set V. Then

$$\mathbf{1}_V B = \mathbf{1}_E. \tag{19}$$

Although $\tilde{P}_k$ is not a Markov chain, it is related to the Markov chain determined by $\boldsymbol{P}$ on $\hat{E}$ as follows:

Fact 1: For $l \geqslant 1$.

$$\tilde{P}_l = D^{-1} B \boldsymbol{P}^l B^* \tag{20}$$

and for the case of $l = 0$, we have $\tilde{P}_0 = I$.

By combining (19) and (20), we have
Fact 2:

$$\mathbf{1}_V D \tilde{P}_l = \mathbf{1}_E B^* = \mathbf{1}_V D. \tag{21}$$

Note that $\mathbf{1}_V D$ is just the degree vector for the graph G. Therefore (21) states that the degree vector is an eigenvector of $\tilde{P}_l$. Using Fact 1 and 2, we have the following:

Lemma 8.
(i) For a fixed vertex x and any integer $j \geqslant 0$, we have

$$\sum_u d_u \sum_{\mathbf{p} \in \mathscr{P}_{u,x}^{(j)}} w(\mathbf{p}) = d_x \tag{22}$$

(ii) For a fixed vertex u, we have

$$\sum_x \sum_{\mathbf{p} \in \mathscr{P}_{u,x}^{(j)}} w(\mathbf{p}) = \mathbf{1}_u (I + \tilde{P}_1 + \ldots + \tilde{P}_l) \mathbf{1}^* = l + 1 \tag{23}$$

where $\mathbf{1}_u$ denotes the characteristic function which assumes value 1 at u and 0 else where.

Proof. The proof of (22) and (23) follows from the fact that

$$\mathbf{1}_V D \tilde{P}_j(x) = \mathbf{1}_V D (D^{-1} B \mathbf{P}^j B^*) = \mathbf{1}_E \mathbf{P}^j B^* = \mathbf{1}_E B^* = \mathbf{1}_V D(x)$$

and $\mathbf{1}_u \tilde{P}_j(x) = w(\mathbf{p})$ for $\mathbf{p} \in \mathscr{P}_{u,x}^{(j)}$. □

6 An Alon-Boppana bound for $\boldsymbol{\lambda_1}$

Theorem 9. *In a graph $G = (V, E)$ with diameter k, the first nontrivial eigenvalue λ_1 satisfies*

$$\lambda_1 \leqslant 1 - \sigma\Big(1 - \frac{c}{k}\Big)$$

where σ is as defined in (15), provided $k \geqslant c' \log \sigma^{-1}$ and $\mathrm{vol}(G) \geqslant c'' \sigma^{\log \sigma}$ for some absolute constants c's.

Proof. If G is not a weak Ramanujan graph, we have $\lambda_1 \leqslant 1-\sigma$ and we are done. We may assume that G is weak Ramanujan.

From the definition of λ_1, we have

$$\lambda_1 \leqslant \frac{\sum_{x\sim y}(f(x)-f(y))^2}{\sum_x f^2(x)d_x} = R(f) \tag{24}$$

where f satisfies $\sum_x f(x)d_x = 0$.

We will construct an appropriate f satisfying $R(f) \leqslant 1-\sigma(1-c/k)$ and therefore serve as an upper bound for λ_1. We set

$$t = \left\lfloor \frac{\log(\mathrm{vol}(G))}{\log \sigma^{-1}} \right\rfloor$$

and choose ϵ satisfying

$$\epsilon \leqslant \frac{\sigma}{t} \leqslant \frac{c\sigma}{k}$$

by using Theorem 6 where σ is as defined in (15).

We consider a family of functions defined as follows. For a specified vertex u and an integer $l = \lfloor k/4 \rfloor$, we consider a function $g_u : V \to \mathbb{R}^+$, defined by

$$\begin{aligned} g_u(x) &= \Big(\mathbf{1}_u(I+\tilde{P}_1+\ldots+\tilde{P}_l)(x)\Big)^{1/2} \\ &= \Big(\sum_{j=0}^{l}\sum_{\mathbf{p}\in\mathscr{P}_{u,x}^{(j)}} w(\mathbf{p})\Big)^{1/2} \end{aligned}$$

where $\tilde{P}_j$ is as defined in (20) and $\mathbf{1}_u$ is treated as a row vector. In other words, g_u denotes the square root of the sum of non-backtracking random walks starting from u taking i steps for i ranging from 0 to l.

Claim A:

$$\sum_u d_u \sum_x g_u^2(x)d_x = \sum_{j=0}^{l}\sum_x \sum_{\mathbf{p}\in\mathscr{P}_{u,x}^{(l)}} d_u w(\mathbf{p}) d_x = (l+1)\sum_x d_x^2$$

where the weight $w(\mathbf{p})$ of a walk $\mathbf{p}$ is as defined in (17).

Proof of Claim A: From the definition of g_u and (16), we have

$$\begin{aligned} \sum_u d_u \sum_x g_u^2(x)d_x &= \sum_{j=0}^{l}\sum_x \sum_{\mathbf{p}\in\mathscr{P}_{u,x}^{(l)}} d_u w(\mathbf{p}) \\ &= \sum_u d_u \mathbf{1}_u B(I+\tilde{P}_1+\ldots+\tilde{P}_l)(x) \end{aligned}$$

$$
\begin{aligned}
&= \sum_u d_u \sum_{i=1}^{l} \mathbf{1}_u D^{-1} B \mathbf{P}^i B^*(x) d_x + \sum_x d_x^2 \\
&= \sum_{i=1}^{l} \sum_u \mathbf{1}_u B \mathbf{P}^i B^*(x) d_x + \sum_x d_x^2 \\
&= \sum_{i=1}^{l} \mathbf{1}_E \mathbf{P}^i B^*(x) d_x + \sum_x d_x^2 \\
&= l \mathbf{1}_E B^*(x) d_x + \sum_x d_x^2 \\
&= (l+1) \sum_x d_x^2.
\end{aligned}
$$

Claim A is proved.

Claim B:

$$
\sum_u d_u \sum_{x \sim y} \big(g_u(x) - g_u(y)\big)^2 \leqslant (l + 1 - l\sigma) \sum_x d_x^2.
$$

where $\sum_{x \sim y}$ denotes the sum ranging over unordered pairs $\{x, y\}$ where x is adjacent to y.

Proof of Claim B:

We will use the following fact for $a_i, b_i > 0$.

$$
\Big(\sqrt{\sum_i a_i} - \sqrt{\sum_i b_i}\Big)^2 \leqslant \sum_i \Big(\sqrt{a_i} - \sqrt{b_i}\Big)^2 \tag{25}
$$

which can be easily checked.

For a fixed vertex u, we apply Claim B:

$$
\begin{aligned}
&\sum_{x \sim y} \big(g_u(x) - g_u(y)\big)^2 \\
&= \sum_{x \sim y} \left(\sqrt{\sum_{\substack{\mathbf{p} \in \mathscr{P}_{u,x}^{(t)} \\ t \leqslant l}} w(\mathbf{p})} - \sqrt{\sum_{\substack{\mathbf{p}' \in \mathscr{P}_{u,y}^{(t)} \\ t \leqslant l}} w(\mathbf{p}')} \right)^2 \\
&\leqslant \sum_{t \leqslant l-1} \sum_{r \in V} \sum_{\substack{\mathbf{p} \in \mathscr{P}_{u,r}^{(t)} \\ \mathbf{p}' = \mathbf{p} \cup s \in \mathscr{P}_{u,s}^{(t+1)}}} \Big(\sqrt{w(\mathbf{p})} - \sqrt{w(\mathbf{p}')}\Big)^2 + \sum_{\mathbf{p} \in \mathscr{P}_{u,x}^{(l)}} w(\mathbf{p})(d_x - 1) \\
&\leqslant \sum_{t \leqslant l-1} \sum_x \sum_{\mathbf{p} \in \mathscr{P}_{u,x}^{(t)}} \left(\sqrt{w(\mathbf{p})} - \sqrt{\frac{w(\mathbf{p})}{d_x - 1}} \right)^2 (d_x - 1) + \sum_{\mathbf{p} \in \mathscr{P}_{u,x}^{(l)}} \sqrt{w(\mathbf{p})}(d_x - 1)
\end{aligned}
$$

$$\leqslant \sum_{t\leqslant l-1}\sum_{x}\sum_{\mathbf{p}\in\mathscr{P}^{(t)}_{u,x}} w(\mathbf{p})\Big(1+\frac{1}{d_x-1}-\frac{2}{\sqrt{d_x-1}}\Big)(d_x-1)+\sum_{\mathbf{p}\in\mathscr{P}^{(l)}_{u,x}} w(\mathbf{p})(d_x-1)$$
$$\leqslant \sum_{t\leqslant l-1}\sum_{x}\sum_{\mathbf{p}\in\mathscr{P}^{(t)}_{u,x}} w(\mathbf{p})\Big(d_x-2\sqrt{d_x-1}\Big)+\sum_{\mathbf{p}\in\mathscr{P}^{(l)}_{u,x}} w(\mathbf{p})(d_x-1).$$

Using Fact 3, we have

$$\begin{aligned}
&\sum_u d_u \sum_{x\sim y}\big(g_u(x)-g_u(y)\big)^2\\
&\leqslant \sum_{t\leqslant l-1}\sum_u d_u \sum_{\mathbf{p}\in\mathscr{P}^{(t)}_{u,x}} w(\mathbf{p})\Big(d_x-2\sqrt{d_x-1}\Big)+\sum_u d_u\sum_{\mathbf{p}\in\mathscr{P}^{(l)}_{u,x}} w(\mathbf{p})(d_x-1)\\
&= l\sum_x d_x\big(d_x-2\sqrt{d_x-1}\big)+\sum_x d_x^2\\
&= l(1-\sigma)\sum_x d_x^2+\sum_x d_x^2\\
&= \big(l+1-l\sigma\big)\sum_x d_x^2
\end{aligned}$$

This proves Claim B.

Claim C: There is a vertex u satisfying

$$R(g_u)\leqslant 1-\sigma\big(1-\frac{1}{l+1}\big)$$

Proof of Claim C:

Combining Claim A and B, we have

$$\begin{aligned}
&\sum_u d_u \sum_{x\sim y}\big(g_u(x)-g_u(y)\big)^2\\
&\leqslant \big(l+1-l\sigma\big)\sum_x d_x^2\\
&\leqslant \big(l+1-l\sigma\big)\Big(\frac{1}{l+1}\Big)\sum_u d_u\sum_x g_u^2(x)d_x\\
&= \Big(1-\frac{l\sigma}{l+1}\Big)\sum_u d_u\sum_x g_u^2(x)d_x
\end{aligned}\tag{26}$$

Thus we deduce that there is a vertex u such that

$$\begin{aligned}
R(g_u) &= \frac{\sum_{x\sim y}\big(g_u(x)-g_u(y)\big)^2}{\sum_x g_u^2(x)d_x}\\
&\leqslant 1-\frac{l\sigma}{l+1}.
\end{aligned}\tag{27}$$

We define

$$\alpha_v = \frac{\sum_x g_v(x) d_x}{\sum_x d_x} = \frac{\sum_x g_v(x) d_x}{\mathrm{vol}(G)}$$

We consider the function g'_u defined by

$$g'_u(x) = g_u(x) - \alpha_u$$

Clearly, g'_u satisfies the condition that

$$\sum_x g'_u(x) d_x = 0$$

Hence, we have

$$\begin{aligned} \lambda_1 \leqslant R(g'_u) &= \frac{\sum_{x \sim y} \big(g'_u(x) - g'_u(y)\big)^2}{\sum_x g'^2_u(x) d_x} \\ &= \frac{\sum_{x \sim y} \big(g_u(x) - g_u(y)\big)^2}{\sum_x g_u^2(x) d_x - \alpha_u^2 \mathrm{vol}(G)}. \end{aligned} \tag{28}$$

Note that by the Cauchy-Schwarz inequality, we have

$$\Big(\sum_{x \in B_u(l)} g_u(x) d_x\Big)^2 \leqslant \mathrm{vol}(B_u(l)) \sum_{x \in B_u(l)} g_u^2(x) d_x.$$

and therefore

$$\alpha_u^2 \leqslant \frac{\mathrm{vol}(B_u(l))}{\mathrm{vol}(G)^2} \sum_x g_u^2(x) d_x.$$

By substitution into (28) and using (35), we have

$$\lambda_1 \leqslant R(g'_u) \leqslant \frac{R(g)}{1 - \frac{\mathrm{vol}(B_u(l))}{\mathrm{vol}(G)}} \leqslant \frac{1 - \sigma\big(1 - \frac{1}{l+1}\big)}{1 - \frac{\mathrm{vol}(B_u(l))}{\mathrm{vol}(G)}} \tag{29}$$

$$\leqslant 1 - \sigma\big(1 - \frac{1}{l+1}\big) + \frac{\mathrm{vol}(B_u(l))}{\mathrm{vol}(G)} \tag{30}$$

$$\leqslant 1 - \sigma\big(1 - \frac{c}{l+1}\big) \tag{31}$$

The last inequality follows from Theorem 7 and the choice of $\epsilon = \sigma/k$. This completes the proof of Theorem 9. □

7 A lower bound for λ_{n-1}

If a graph is bipartite, it is known (see [2]) that $\lambda_i = 2 - \lambda_{n-i-1}$ for all $0 \leqslant i \leqslant n-1$ and, in particular, $\lambda_{n-1} = 2 - \lambda_0 = 2$. If G is not bipartite, it is easy to derive the following lower bound:

$$\lambda_{n-1} \geqslant 1 + 1/(n-1)$$

by using the fact that the trace of $\mathcal{L}$ is n. This lower bound is sharp for the complete graph. However if G is not the complete graph, is it possible to derive a better lower bound? The answer is affirmative. Here we give an improved lower bound for λ_{n-1}.

Theorem 10. *In a connected graph $G = (V, E)$ with diameter k, the largest eigenvalue λ_{n-1} of the normalized Laplacian $\mathcal{L}$ of G satisfies*

$$\lambda_{n-1} \geqslant 1 + \sigma\Big(1 - \frac{c}{k}\Big) \tag{32}$$

where σ is as defined in (15), provided $k \geqslant c' \log \sigma^{-1}$ and $\mathrm{vol}(G) \geqslant c'' \sigma^{\log \sigma}$ for some absolute constants c's.

Proof. By definition, λ_{n-1} satisfies

$$\lambda_{n-1} \geqslant \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_x f^2(x) d_x} = R(f) \tag{33}$$

for any $f : V \to \mathbb{R}$.

We will construct an appropriate f such that $R(f) \geqslant 1 + \sigma(1 - c/\gamma)$ by considering the following function $f_u : V \to \mathbb{R}^+$, for a fixed vertex u, defined by

$$\eta_u(x) = \begin{cases} (-1)^t \chi_u\big(\tilde{P}_t(x)\big)^{-1/2} & \text{if } \mathrm{dist}(u, x) = t \leqslant l \\ 0 & \text{otherwise} \end{cases}$$

where $l \leqslant \gamma/2$. Note that $|\eta_u(x)| = g_u(x)$ since we assume that $l \leqslant \gamma/2$. Using the same proof in Claim A, we have

Claim A':

$$\sum_u d_u \sum_x \eta_u^2(x) d_x = \sum_{j=0}^{l} \sum_x \sum_{\mathbf{p} \in \mathscr{P}_{u,x}^{(l)}} d_u w(\mathbf{p}) d_x = (l+1) \sum_x d_x^2.$$

Claim B':

$$\sum_u d_u \sum_{x \sim y} \big(\eta_u(x) - \eta_u(y)\big)^2 \geqslant (l + 1 + l\sigma) \sum_x d_x^2.$$

Proof of Claim B': The proof is quite similar to that of Claim B. For a fixed vertex u, the sum over unordered pair $\{x, y\}$ where $x \sim y$,

$$
\begin{aligned}
&\sum_{x\sim y} \big(\eta_u(x) - \eta_u(y)\big)^2 \\
&\leqslant \sum_{t\leqslant l-1} \sum_{r\in V} \sum_{\substack{\mathbf{p}\in \mathscr{P}_{u,r}^{(t)} \\ \mathbf{p}'=\mathbf{p}\cup s\in \mathscr{P}_{u,s}^{(t+1)}}} \Big(\sqrt{w(\mathbf{p})} + \sqrt{w(\mathbf{p}')}\Big)^2 - \sum_{\mathbf{p}\in \mathscr{P}_{u,x}^{(l)}} w(\mathbf{p})(d_x - 1) \\
&\leqslant \sum_{t\leqslant l-1} \sum_{x} \sum_{\mathbf{p}\in \mathscr{P}_{u,x}^{(t)}} \left(\sqrt{w(\mathbf{p})} + \sqrt{\frac{w(\mathbf{p})}{d_x - 1}}\right)^2 (d_x - 1) - \sum_{\mathbf{p}\in \mathscr{P}_{u,x}^{(l)}} \sqrt{w(\mathbf{p})}(d_x - 1) \\
&\leqslant \sum_{t\leqslant l-1} \sum_{x} \sum_{\mathbf{p}\in \mathscr{P}_{u,x}^{(t)}} w(\mathbf{p})\Big(1 + \frac{1}{d_x - 1} + \frac{2}{\sqrt{d_x - 1}}\Big)(d_x - 1) - \sum_{\mathbf{p}\in \mathscr{P}_{u,x}^{(l)}} w(\mathbf{p})(d_x - 1) \\
&\leqslant \sum_{t\leqslant l-1} \sum_{x} \sum_{\mathbf{p}\in \mathscr{P}_{u,x}^{(t)}} w(\mathbf{p})\Big(d_x + 2\sqrt{d_x - 1}\Big) - \sum_{\mathbf{p}\in \mathscr{P}_{u,x}^{(l)}} w(\mathbf{p})(d_x - 1).
\end{aligned}
$$

Using Fact 3, we have

$$
\begin{aligned}
&\sum_{u} d_u \sum_{x\sim y} \big(\eta_u(x) - \eta_u(y)\big)^2 \\
&\geqslant \sum_{t\leqslant l-1} \sum_{u} d_u \sum_{\mathbf{p}\in \mathscr{P}_{u,x}^{(t)}} w(\mathbf{p})\Big(d_x + 2\sqrt{d_x - 1}\Big) - \sum_{u} d_u \sum_{\mathbf{p}\in \mathscr{P}_{u,x}^{(l)}} w(\mathbf{p})(d_x - 1) \\
&= l \sum_{x} d_x\big(d_x + 2\sqrt{d_x - 1}\big) - \sum_{x} d_x^2 \\
&= l(1+\sigma) \sum_{x} d_x^2 - \sum_{x} d_x^2 \\
&= \big(l - 1 + l\sigma\big) \sum_{x} d_x^2
\end{aligned}
$$

This proves Claim B'.

Combining Claims A' and B', we have

$$
\begin{aligned}
&\sum_{u} d_u \sum_{x\sim y} \big(\eta_u(x) - \eta_u(y)\big)^2 \\
&\geqslant \big(l - 1 + l\sigma\big) \sum_{x} d_x^2 \\
&\geqslant \big(l - 1 + l\sigma\big)\Big(\frac{1}{l+1}\Big) \sum_{u} d_u \sum_{x} \eta_u^2(x) d_x \\
&= \Big(1 + \frac{l\sigma}{l-1}\Big) \sum_{u} d_u \sum_{x} \eta_u^2(x) d_x \qquad (34)
\end{aligned}
$$

Thus we deduce that there is a vertex u such that

$$R(\eta_u) = \frac{\sum_{x\sim y}\left(\eta_u(x) - \eta_u(y)\right)^2}{\sum_x \eta_u^2(x) d_x} \leqslant 1 + \frac{l\sigma}{l-1}. \tag{35}$$

We consider the function η'_u defined by

$$\eta'_u(x) = \eta_u(x) - \alpha_u$$

where

$$\alpha_v = \frac{\sum_x \eta_v(x) d_x}{\sum_x d_x} = \frac{\sum_x \eta_v(x) d_x}{\mathrm{vol}(G)}$$

so that η'_u satisfies the condition that

$$\sum_x \eta'_u(x) d_x = 0$$

Hence, we have

$$\begin{aligned} \lambda_{n-1} \geqslant R(\eta'_u) &= \frac{\sum_{x\sim y}\left(\eta'_u(x) - \eta'_u(y)\right)^2}{\sum_x \eta'^2_u(x) d_x} \\ &= \frac{\sum_{x\sim y}\left(\eta_u(x) - \eta_u(y)\right)^2}{\sum_x \eta_u^2(x) d_x - \alpha_u^2 \mathrm{vol}(G)} \\ &\geqslant 1 + \sigma(1 + \frac{c}{l}) - \frac{\mathrm{vol}(B_u(l))}{\mathrm{vol}(G)}. \end{aligned}$$

This completes the proof of Theorem 10. □

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Corrigendum – added 3th November 2017

1. In the abstract, line 6-8, the statement of the main result should be replaced by

$$\lambda_1 \leqslant 1 - \sigma\big(1 - \frac{5}{k}\big)$$

provided $\sigma = 2\sum_v d_v\sqrt{d_v - 1}/\sum_v d_v^2 \leqslant 1/2$ and $k(1.5)^k \geqslant \sigma^{-1}$ where d_v denotes the degree of the vertex v with minimum degree at least 2.

Also, page 12, line -3 to -1, the statement of Theorem 9 should be similarly replaced as above.

2. Page 3, line 13, the constant c should be replaced by 5.

3. Page 9, line -9. "… for some constant c." should be replaced by "… for $c = 1/\log 1.5$."

4. Page 9, line -6. Replace "… largest …" by "… least …".

5. Page 10, line 3, $\bar{s}_j$ should be replaced by $\bar{s}_{j+1}$.

6. Page 3, line 7 to 11. Delete "We set … as defined in (15)." Note that ϵ was defined later near the end of the proof of Theorem 9.

7. Page 16, line -6, replace "… using (35), …" by "… using (27), …".

8. Page 16, line -3. Replace " $c/(l+1)$" by "$5/k$".

9. Page 16, line -2. Replace " … the choice of $\epsilon = \sigma/k$. " by " … the choice of $\epsilon = \sigma/k$ which satisfies $k \geqslant (\log \epsilon^{-1})/\log 1.5$."

10. Page 17, line 11 to line 13, the statement of Theorem 10 should be replaced by

$$\lambda_{n-1} \geqslant 1 + \sigma\big(1 - \frac{5}{k}\big)$$

provided $\sigma = 2\sum_v d_v\sqrt{d_v - 1}/\sum_v d_v^2 \leqslant 1/2$ and $k(1.5)^k \geqslant \sigma^{-1}$ where d_v denotes the degree of the vertex v with minimum degree at least 2.