# Representations of bicircular lift matroids

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**Abstract**

Bicircular lift matroids are a class of matroids defined on the edge set of a graph. For a given graph $G$, the circuits of its bicircular lift matroid are the edge sets of those subgraphs of $G$ that contain at least two cycles, and are minimal with respect to this property. The main result of this paper is a characterization of when two graphs give rise to the same bicircular lift matroid, which answers a question proposed by Irene Pivotto. In particular, aside from some appropriately defined "small" graphs, two graphs have the same bicircular lift matroid if and only if they are 2-isomorphic in the sense of Whitney.

**Keywords:** bicircular lift matroids, representation

## 1 Introduction

We assume the reader is familiar with fundamental definitions in matroid and graph theory. For a graph $G$, a set $X \subseteq E(G)$ is a *cycle* if $G|X$ is a connected 2-regular graph. Bicircular lift matroids are a class of matroids defined on the edge set of a graph. For a given graph $G$, the circuits of its bicircular lift matroid $L(G)$ are the edge sets of those subgraphs of $G$ that contain at least two cycles, and are minimal with respect to this property. That is, the circuits of $L(G)$ consists of the edge sets of two edge-disjoint cycles with at most one common vertex, or three internally disjoint paths between a pair of distinct vertices. Bicircular lift matroids are a special class of lift matroids that arises from biased graphs. Biased graphs and lift matroids were introduced by Zaslavsky in [8, 9].

Whitney [6] characterized which graphs have isomorphic graphic matroids. Chen, DeVos, Funk and Pivotto [2] generalized Whitney's result and characterized which biased

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graphs have isomorphic graphic frame matroids. Matthews [4] characterized which graphs give rise to isomorphic bicircular matroids that are graphic. Coullard, del Greco and Wagner [3, 7] characterized which graphs give rise to isomorphic bicircular matroids. In this paper, we characterize which graphs give rise to isomorphic bicircular lift matroids, which answers a question proposed by Pivotto in the Matroid Union blog [5]. In particular, except for some special graphs, each of which is a subdivision of a graph on at most four vertices, two graphs have the same bicircular lift matroid if and only if they are 2-isomorphic in the sense of Whitney [6]. The main result is used in [1] to prove that the class of matroids that are graphic or bicircular lift has a finite list of excluded minors.

To state our result completely we need more definitions. Let $k, l, m$ be positive integers. We denote by $K_m$ the complete graph with $m$ vertices. We denote by $K_2^m$ the graph obtained from $K_2$ with its unique edge replaced by $m$ parallel edges. And we denote by $K_3^{k,l,m}$ the graph obtained from $K_3$ with its three edges replaced by $k, l, m$ parallel edges respectively. A graph obtained from graph $G$ by replacing some edges of $G$ with internally disjoint paths is a *subdivision* of $G$. Note that $G$ is a subdivision of itself. A path $P$ of a connected graph $G$ is an *ear* if each internal vertex of $P$ has degree two and each end-vertex has degree at least three in $G$, and $P$ is contained in a cycle. A graph $G$ is *2-edge-connected* if each edge of $G$ is contained in some cycle. Let $M(G)$ denote the graphic matroid of a graph $G$.

Given a set $X$ of edges, we let $G|X$ denote the subgraph of $G$ with edge set $X$ and no isolated vertices. Let $(X_1, X_2)$ be a partition of $E(G)$ such that $V(G|X_1) \cap V(G|X_2) = \{u_1, u_2\}$. We say that $G'$ is obtained by a *Whitney Switching* on $G$ on $\{u_1, u_2\}$ if $G'$ is a graph obtained by identifying vertices $u_1, u_2$ of $G|X_1$ with vertices $u_2, u_1$ of $G|X_2$, respectively. A graph $G'$ is *2-isomorphic* to $G$ if $G'$ is obtained from $G$ by a sequence of the operations: Whitney switchings, identifying two vertices from distinct components of a graph, or partitioning a graph into components each of which is a block of the original graph.

**Theorem 1.** *(Whitney's 2-Isomorphism Theorem) Let $G_1$ and $G_2$ be graphs. Then $M(G_1) \cong M(G_2)$ if and only if $G_1$ and $G_2$ are 2-isomorphic.*

It follows from Theorem 1 that if $G_1$ and $G_2$ are 2-isomorphic, then $L(G_1) = L(G_2)$. The converse, however, is not true. This can be seen by choosing $G_1$ and $G_2$ to be isomorphic to $K_4$, but not to each other. Much of the remainder of the paper is aimed at characterizing when the converse to this statement is not true.

Let $G_1$ and $G_2$ be graphs with $L(G_1) = L(G_2)$. Since $E(G_i)$ is independent in $L(G_i)$ if and only if $G_i$ has at most one cycle, we may assume that $G_1$ and $G_2$ have at least two cycles. Moreover, since $e$ is a cut-edge of $G_1$ if and only if $e$ is a cut-edge of $G_2$ or $G_2 \backslash e$ is a forest, an edge is a cut-edge of $G_1$ if and only if it is a cut-edge of $G_2$. Hence, to simplify the analysis below, it will be assumed for the remainder of the paper that $G_1$ and $G_2$ are 2-edge-connected. Observe that when $L(G_1)$ has only one circuit, it is straightforward to characterize the structure of both $G_1$ and $G_2$. Thus, the remainder of the paper will further restrict the analysis to the case that $L(G_1)$ has at least two circuits. In the paper, we prove

**Theorem 2.** *Let $G_1$ be a 2-edge-connected graph such that $L(G_1)$ contains at least two circuits. Let $G_2$ be a graph with $L(G_1) = L(G_2)$. Then at least one of the following holds.*

*(1) $G_1$ and $G_2$ are 2-isomorphic.*

*(2) $G_1$ and $G_2$ are 2-isomorphic to subdivisions of $K_4$, where the edge set of an ear of $G_1$ is also the edge set of an ear of $G_2$.*

*(3) $G_1$ and $G_2$ are 2-isomorphic to subdivisions of $K_3^{m,2,n}$ for some $m \in \{1, 2\}$ and $n \geqslant 2$, where the edge set of an ear of $G_1$ is also the edge set of an ear of $G_2$. Moreover, when $n \geqslant 3$, the $n$ ears in $G_1$ having the same ends also have the same ends in $G_2$.*

*(4) $G_1$ and $G_2$ are 2-isomorphic to the graphs pictured in Figure 1.*

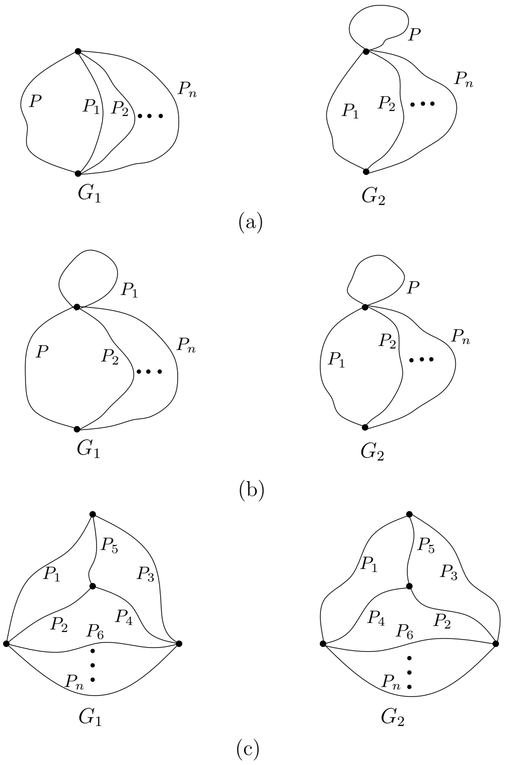


Figure 1: In (a) and (b), $n \geqslant 3$; and in (c), $n \geqslant 2$

The following result, which is an easy consequences of Theorem 2, is used in [1] to prove that the class of matroids that are graphic or bicircular lift matroids has a finite list of excluded minors.

Two elements are a *series pair* of a graph $G$ if and only if each cycle can not intersect them in exactly one element. A *series class* is a maximal set $X \subseteq E(G)$ such that every two edges of $X$ form a series pair. Let $\text{co}(G)$ denote a graph obtained from $G$ by contracting all cut-edges from $G$ and then, for each series class $X$, contracting all but one distinguished element of $X$.

**Corollary 3.** *Let $G_1$ and $G_2$ be connected graphs with $L(G_1) = L(G_2)$ and such that $L(G_1)$ has at least two circuits. If $|V(\mathrm{co}(G_1))| \geqslant 5$ then $G_1$ and $G_2$ are 2-isomorphic.*

## 2 Proof of Theorem 2

Let $G$ be a graph, and $e, f \in E(G)$. We say that $e$ is a *link* if it has distinct end-vertices; otherwise $e$ is a *loop*. If $\{e, f\}$ is a cycle, then $e$ and $f$ are *parallel*. A *parallel class* of $G$ is a maximal subset $P$ of $E(G)$ such that any two members of $P$ are parallel and no member is a loop. Moreover, if $|P| \geqslant 2$ then $P$ is *non-trivial*; otherwise $P$ is *trivial*. Let $\mathrm{si}(G)$ denote the graph obtained from $G$ by deleting all loops and all but one distinguished element of each non-trivial parallel class. Obviously, the graph we obtain is uniquely determined up to a renaming of the distinguished elements. If $G$ has no loops and no non-trivial parallel class, then $G$ is *simple*.

The following result is implied in ([9], Theorem 3.6.).

**Lemma 4.** *Let $e$ be an edge of a graph $G$. Then we have*

*(1) $L(G\backslash e) = L(G)\backslash e$;*

*(2) when $e$ is a loop, $L(G)/e = M(G\backslash e)$;*

*(3) when $e$ is a link, $L(G)/e = L(G/e)$.*

Corollary 5 follows immediately from Lemma 4 (2) and Theorem 1.

**Corollary 5.** *Let $G_1, G_2$ be graphs with $L(G_1) = L(G_2)$, and $e$ a loop of both $G_1$ and $G_2$. Then $G_1$ and $G_2$ are 2-isomorphic.*

The idea used to prove the following Lemma was given by the referee.

**Lemma 6.** *Let $G_1$ and $G_2$ be connected graphs without loops and with $|V(G_1)| = |V(G_2)|$ and $E(G_1) = E(G_2)$. Assume that for each edge $e \in E(G_1)$ the graphs $G_1/e$ and $G_2/e$ are 2-isomorphic. Then $G_1$ and $G_2$ are 2-isomorphic.*

*Proof.* By Whitney's 2-Isomorphism Theorem, to prove the result it suffices to show that each spanning tree of $G_1$ is also a spanning tree of $G_2$. Let $T_1$ be a spanning tree of $G_1$, and let $T_2$ be the subgraph of $G_2$ induced by $E(T_1)$. Assume that $T_2$ is not a spanning tree of $G_2$. Since $|V(G_1)| = |V(G_2)|$, the subgraph $T_2$ contains a cycle $C$. Let $e$ be an edge in $E(T_1)$. Then $T_1/e$ is acyclic and $T_2/e$ is not, and so $G_1/e$ and $G_2/e$ are not 2-isomorphic; a contradiction. □

**Lemma 7.** *Let $G_1$ be a 2-edge-connected graph such that $L(G_1)$ contains at least two circuits. Let $G_2$ be a graph with $L(G_1) = L(G_2)$. Assume that $G_1$ has a link $e$ such that $e$ is a loop of $G_2$. Then $G_1$ and $G_2$ are 2-isomorphic to the graphs pictured in Figure 2.*

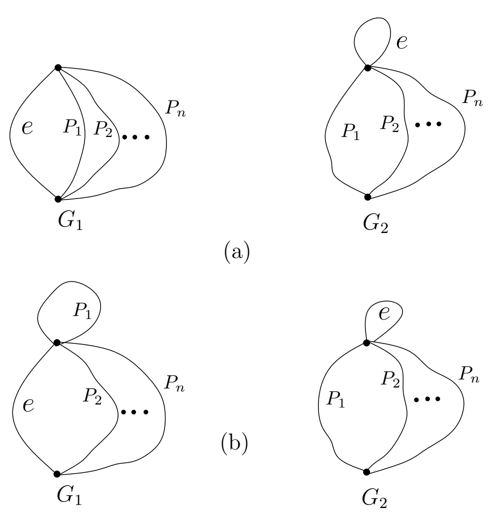


Figure 2: $n \geqslant 3$

*Proof.* Since $L(G_1)$ contains at least two circuits and $L(G_1) = L(G_2)$, the graph $G_2 - \{e\}$ has cycles $C_1$ and $C_2$ such that $C_1 \cup C_2$ is a circuit of $L(G_2)$. Since $e$ is a loop of $G_2$, for some integer $k \in \{2, 3\}$ there is a partition $(P_1, P_2, \cdots, P_k)$ of $E(C_1 \cup C_2)$ such that when $k = 2$ the sets $P_1 \cup \{e\}$ and $P_2 \cup \{e\}$ are circuits of $L(G_1)$, and when $k = 3$ the sets $P_1 \cup P_2 \cup \{e\}$, $P_2 \cup P_3 \cup \{e\}$ and $P_1 \cup P_3 \cup \{e\}$ are circuits of $L(G_1)$. Since $E(C_1 \cup C_2)$ is also a circuit of $L(G_1)$ and $e$ is a link of $G_1$, it is easy to verify that $k = 3$ (that is, $C_1 \cup C_2$ is a theta-subgraph of $G_2$.) and **(1)** $G_1|C_1 \cup C_2 \cup \{e\}$ is 2-isomorphic to graphs pictured in Figure 3. Hence, by the arbitrary choice of $C_1$ and $C_2$, **(2)** no two cycles in $G_2$ have at most one common vertex; and **(3)** each ear of a theta-subgraph of $G_2$ is a cycle in $G_1$ or a path connecting the end-vertices of $e$ in $G_1$.

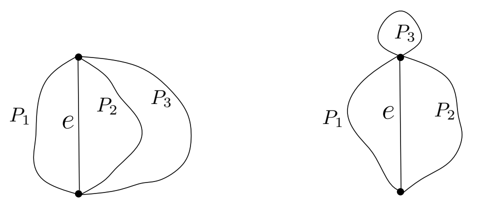


Figure 3: Possible structures of graph $G_1|C_1 \cup C_2 \cup \{e\}$.

For each edge $f \in E(G_2) - (C_1 \cup C_2 \cup \{e\})$, there is a set $X$ with $f \in X \subseteq E(G_2) - (C_1 \cup C_2 \cup \{e\})$ such that $G_2|C_1 \cup C_2 \cup X$ is 2-edge-connected. By (2) $G_2|C_1 \cup C_2 \cup X$ is a subdivision of $K_4$ or $K_2^4$. (1) and (3) imply that $G_2|C_1 \cup C_2 \cup X$ is a subdivision of $K_2^4$. Repeating the process several times, we have that $G_2 - \{e\}$ is a $K_2^n$-subdivision for some integer $n \geqslant 3$. Hence, $G_1$ and $G_2$ are 2-isomorphic to the graphs pictured in Figure 2. □

By Lemma 7, to prove Theorem 2 we only need to consider the case that an edge is a link in $G_1$ if and only if it is a link in $G_2$.

**Lemma 8.** *Let $G_1$ and $G_2$ be connected and 2-edge-connected graphs with $L(G_1) = L(G_2)$ such that $L(G_1)$ has at least two circuits and such that each series class of $G_i$ is an ear of $G_i$ for each $i \in \{1, 2\}$. Then a set of edges is the edge set of an ear of $G_1$ if and only if it is the edge set of an ear of $G_2$.*

*Proof.* Assume otherwise. Without loss of generality assume that $e$ and $f$ are contained in some ear of $G_1$, but not in the some ear of $G_2$. Evidently, $e$ is not in any cycle of $G_1 - \{f\}$ and $L(G_1 - \{f\})$ has a circuit as $L(G_1)$ has at least two circuits. Moreover, since $L(G_1 - \{f\}) = L(G_2 - \{f\})$, the edge $e$ is a coloop of $G_2 - \{f\}$; so $\{e, f\}$ is a bond of $G_2$. Then $e$ and $f$ are contained in the some ear of $G_2$ as each series class of $G_2$ is an ear of $G_2$, a contradiction. □

By possibly applying a sequence of Whitney's switching we can assume that each series class in a graph $G$ is an ear of $G$. Furthermore, by Lemma 8 we can further assume that a set of edges is the edge set of an ear of $G_1$ if and only if it is the edge set of an ear of $G_2$. Hence, we only need consider cosimple graphs, where a graph is *cosimple* if it has no cut-edges or non-trivial series classes.

Let $loop(G)$ be the set consisting of loops of $G$.

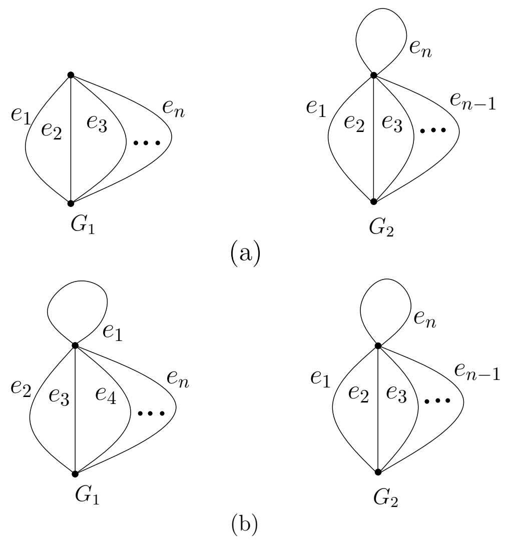


Figure 4: $n \geqslant 3$.

**Lemma 9.** *Let $G_1$ and $G_2$ be cosimple 2-edge-connected graphs with $2 \leqslant |V(G_1)| = |V(G_2)| \leqslant 3$. Assume that $L(G_1) = L(G_2)$ and $L(G_1)$ contains at least two circuits. Then exactly one of the following holds.*

*(1) $G_1$ and $G_2$ are 2-isomorphic.*

(2) $|V(G_1)| = 2$, *the graphs* $G_1$ *and* $G_2$ *are isomorphic to the graphs pictured in Figure 4.*

(3) $G_1$ *and* $G_2$ *are* 2*-isomorphic to* $K_3^{m,2,n}$ *for some integers* $m \in \{1, 2\}$ *and* $n \geqslant 2$, *moreover, the* $n$ *parallel edges in* $G_1$ *are also the* $n$ *parallel edges in* $G_2$ *when* $n \geqslant 3$.

*Proof.* By Lemma 7 we may assume that $loop(G_1) = loop(G_2)$. Then the lemma holds when $|V(G_1)| = 2$. So assume that $|V(G_1)| = 3$. Since $loop(G_1) = loop(G_2)$, each non-trivial parallel class of $G_1$ with at least three edges must be also a non-trivial parallel class of $G_2$. Hence, when $G_1$ has two parallel classes with at least three edges, (1) holds. So we may assume that $G_1$ has at most one parallel class with at least three edges. On the other hand, since $G_1$ and $G_2$ are cosimple, $G_1$ and $G_2$ have three parallel classes and at least two of them are non-trivial. Hence, when $G_1$ has no loops, (3) obviously holds; when $G_1$ has a loop, since $loop(G_1) = loop(G_2)$, Corollary 5 implies that $G_1$ and $G_2$ are 2-isomorphic, that is, (1) holds. □

The *star* of a vertex $v$ in a graph $G$, denoted by $st_G(v)$, is the set of edges of $G$ incident with $v$.

**Lemma 10.** *Let* $G_1$ *and* $G_2$ *be* 2*-edge-connected cosimple graphs with exactly four vertices and without loops. Assume that* $L(G_1) = L(G_2)$ *and* $L(G_1)$ *has at least two circuits. Then at least one of the following holds.*

(1) $G_1$ *and* $G_2$ *are* 2*-isomorphic;*

(2) $G_1$ *and* $G_2$ *are isomorphic to* $K_4$;

(3) $G_1$ *and* $G_2$ *are* 2*-isomorphic to the graphs pictured in Figure 5.*

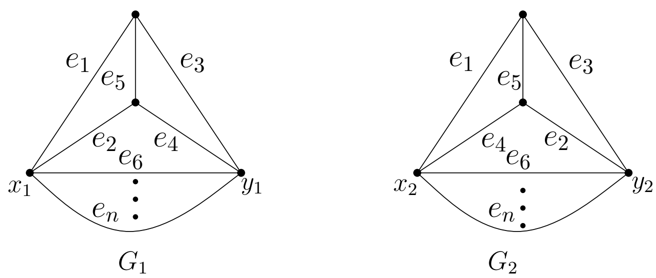


Figure 5: $n \geqslant 7$.

*Proof.* By Lemma 4 (3), for each edge $e \in E(G_1)$ we have $L(G_1/e) = L(G_2/e)$. If $G_1/e$ and $G_2/e$ are 2-isomorphic for each edge $e \in E(G_1)$, then Lemma 6 implies that $G_1$ and $G_2$ are 2-isomorphic. So we may assume that there is some edge $f \in E(G_1)$ such that $G_1/f$ and $G_2/f$ are not 2-isomorphic. Since $L(G_1)$ has at least two circuits, $L(G_1/f)$ also has at least two circuits. Moreover, since $G_1/f$ and $G_2/f$ are cosimple graphs with exactly three vertices, by Lemma 9 we have that **(a)** the graphs $G_1/f$ and $G_2/f$ are isomorphic to $K_3^{m,2,n}$ for some integers $m \in \{1,2\}$ and $n \geqslant 2$; moreover, when $n \geqslant 3$ the $n$ parallel edges in $G_1$ are also the $n$ parallel edges in $G_2$.

**10.1.** *Two edges are parallel in $G_1$ if and only if they are parallel in $G_2$.*

*Subproof.* If two edges are parallel in $G_1$ but not $G_2$, then contracting one of the edges produces a counterexample to Lemma 7. □

The simple proof of 10.1 is given by the referee. Since no non-trivial parallel classes in $G_1$ or $G_2$ contains $f$ by (a), 10.1 implies

**10.2.** *Each 2-edge path joining the end-vertices of a non-trivial parallel class of $G_1$ is also a 2-edge path joining the end-vertices of the non-trivial parallel class of $G_2$.*

**10.3.** *Let $P_1, P_2$ be non-trivial parallel classes of $G_1$. Then $\mathrm{si}(G_1|P_1 \cup P_2 \cup f)$ is a triangle.*

*Subproof.* Since $G_1/f$ has no loop, neither $P_1$ nor $P_2$ contains $f$. If $P_1$ and $P_2$ are not contained in a parallel class of $G_1/f$, then $P_1$ and $P_2$ are contained in two different non-trivial parallel classes of $G_1/f$. Moreover, since $P_1$ and $P_2$ are also non-trivial parallel classes of $G_2$ by 10.1, by (a) we have that $G_1/f$ and $G_2/f$ are isomorphic, a contradiction. So $P_1$ and $P_2$ are contained in a parallel class of $G_1/f$. Then $\mathrm{si}(G_1|P_1 \cup P_2 \cup f)$ is a triangle. □

First we consider the case that $G_1/f$ is isomorphic to $K_3^{2,2,n}$. By 10.3, $G_1$ is obtained from $G_1/f$ by splitting a degree-4 vertex. Since $G_1$ is cosimple, $G_1$ is isomorphic to the graph pictured in Figure 5 with $e_5$ relabelled by $f$. Let $P$ be the unique non-trivial parallel class of $G_1$ with $n$ edges. Since $P$ is a also non-trivial parallel class of $G_2$ by 10.1 and the fact that $G_2/f$ is isomorphic to $K_3^{2,2,n}$, the graph $G_2$ is isomorphic to the graph pictured in Figure 5 with $e_5$ relabelled by $f$. So (3) holds.

Secondly we consider the case that $G_1/f$ is isomorphic to $K_3^{1,2,n}$. Let $e_i$ be the edge of $G_i/f$ that is not in a parallel class for $1 \leqslant i \leqslant 2$. Evidently, when $n \geqslant 3$, since $G_1/f$ and $G_2/f$ are not 2-isomorphic, $e_1 \neq e_2$. Since each vertex of $G_1$ has degree at least three, by 10.3 the graph $G_1$ is obtained from $G_1/f$ by splitting the vertex $v$ incident with two non-trivial parallel classes. When $|st_{G_1/f}(v)| = 4$, since $G_1$ is cosimple $G_1$ is isomorphic to $K_4$. By symmetry $G_2$ is also isomorphic to $K_4$. So (2) holds.

Assume that $|st_{G_1/f}(v)| \geqslant 5$, that is, a non-trivial parallel class $P$ incident with $v$ in $G_1/f$ has at least three edges. Then some proper subset $P'$ of $P$ is a non-trivial parallel class in $G_1$ as $G_1$ is cosmiple. Let $\{f_1, f_2\}$ be the 2-edge parallel class in $G_1/f$. Since $\{f, f_1, f_2\}$ is a cycle in $G_1$ and $\{e_1, f_1, f_2\}$ is the neighbourhood of a degree-3 vertex in $G_1/f$ and $G_1$, by symmetry we may assume that $e_1, f_1$ is a 2-edge path joining the end-vertices of $P'$ in $G_1$ and $f_2$ is not incident with $P'$. On the other hand, by symmetry, $e_2$ is also contained in a 2-edge path joining the end-vertices of $P'$ in $G_2$. So $f_1 = e_2$ as $e_2 \in \{f_1, f_2\}$, consequently, $|P - P'| = 1$, for otherwise there are two such $P'$, which is not possible. Therefore, (3) holds. □

**Lemma 11.** *Let $G_1$ and $G_2$ be 2-edge-connected cosimple graphs with five vertices and without loops. Assume that $L(G_1) = L(G_2)$ and $L(G_1)$ has at least two circuits. Then $G_1$ and $G_2$ are 2-isomorphic.*

*Proof.* By Lemma 4 (3), for each edge $e \in E(G_1)$ we have $L(G_1/e) = L(G_2/e)$. If $G_1/e$ and $G_2/e$ are 2-isomorphic for each edge $e \in E(G_1)$, then Lemma 6 implies that $G_1$ and $G_2$ are 2-isomorphic. So we may assume that for some edge $f \in E(G_1)$ we have $L(G_1/f) = L(G_2/f)$ but $G_1/f$ and $G_2/f$ are not 2-isomorphic.

We claim that $G_1/f$ and $G_2/f$ have no loops. Since $L(G_1/f)$ has at least two circuits, Lemma 7 implies that $loop(G_1/f) = loop(G_2/f)$. If $loop(G_1/f) \neq \emptyset$, then Corollary 5 implies that $G_1/f$ and $G_2/f$ are 2-isomorphic, a contradiction.

Since $G_1/f$ and $G_2/f$ are cosimple with four vertices and without loops, Lemma 10 implies that $G_1/f$ and $G_2/f$ are either 2-isomorphic to $K_4$ or to the graphs pictured in Figure 5. Since each vertex in $K_4$ has degree three and $G_1$ and $G_2$ are cosimple, neither $G_1/f$ nor $G_2/f$ is 2-isomorphic to $K_4$. So $G_1/f$ and $G_2/f$ are 2-isomorphic to the graphs pictured in Figure 5 with $G_i$ replaced by $G_i/f$ and all other labeling the same. Let $P$ be the non-trivial parallel class in $G_1/f$ and $G_2/f$. For each $i \in \{1, 2\}$, let $u_i$ and $v_i$ be the end-vertices of $f$ in $G_i$, let $x_i$ be the vertex of degree at least four in $G_i/f$ incident with $e_1$, and $y_i$ be the vertex of degree at least four in $G_i/f$ incident with $e_3$. Since $|st_{G_i}(u_i)|, |st_{G_i}(v_i)| \geqslant 3$, the graph $G_i$ is obtained from $G_i/f$ by splitting $x_i$ or $y_i$. Without loss of generality we may assume that $G_i$ is obtained from $G_i/f$ by splitting $x_i$ for each $i \in \{1, 2\}$.

We claim that $|E(G_1/f)| = 7$, that is, $|P| = 2$. Assume otherwise. Then there is a subset $P'$ of $P$ with $|P'| \geqslant 2$ such that $P'$ is also a parallel class in $G_1$. Using a similar analysis to the one in the proof of 10.1 we have that $P'$ is also a parallel class in $G_2$. Assume that $e_1, e_2$ are adjacent in $G_1$. Since a union of any two edges in $P'$ and $\{e_1, e_2, e_5\}$ or $\{e_3, e_4, e_5\}$ is a circuit of $L(G_1)$, we deduce that $\{e_1, e_4, f\} \cup P'$ are contained in $st_{G_2}(u_2)$ or $st_{G_2}(v_2)$. Hence, $|P - P'| \leqslant 1$, implying that $(P - P') \cup \{f\}$ is a bond of $G_2$ with at most two edges, a contradiction as $G_2$ is cosimple. So $e_1, e_2$ are not adjacent in $G_1$. By symmetry we may assume that $st_{G_1}(v_1) = \{e_2, f\} \cup P'$. Since $P'$ is a parallel class of $G_2$ and the union of $\{e_3, e_4, e_5\}$ and any two edges in $P'$ is a circuit of $L(G_1)$, by symmetry we may assume that $\{e_4, f\} \cup P'$ are incident with $v_2$. Hence, $|P - P'| = 1$ and $st_{G_1}(u_1) = st_{G_2}(u_2) = (P - P') \cup \{e_1, f\}$. Set $\{e_6\} = P - P'$. See Figure 6. Then $\{e_1, e_2, e_3, e_5, e_6, f\}$ is a circuit of $L(G_1)$ but is not a circuit of $L(G_2)$, a contradiction. So $|E(G_1/f)| = 7$. Set $E(G_1/f) := \{e_1, e_2, \cdots, e_7\}$.

Since $G_1$ and $G_2$ are cosimple and $|E(G_1/f)| = 7$, we have $|st_{G_i}(u_i)| = |st_{G_i}(v_i)| = 3$ for each $i \in \{1, 2\}$. By symmetry, there are two cases to consider. First we consider the case $st_{G_1}(u_1) = \{f, e_1, e_2\}$. Since $\{e_1, e_2, e_3, e_4, e_5\}$ is a circuit of $L(G_1)$, by symmetry we can assume $st_{G_2}(u_2) = \{f, e_1, e_4\}$. Then $\{e_2, e_3, e_5, e_6, e_7, f\}$ is a circuit of $L(G_1)$ but is not a circuit of $L(G_2)$, a contradiction.

Secondly consider the case $st_{G_1}(u_1) = \{f, e_1, e_6\}$. Then $\{e_1, e_3, e_4, e_5, e_6\}$ is a circuit of $L(G_1)$. On the other hand, by symmetry and the analysis in the last paragraph we have $\{f, e_1, e_4\} \neq \{N_{G_2}(u_2), N_{G_2}(v_2)\}$. So $\{e_1, e_3, e_4, e_5, e_6\}$ is not a circuit of $L(G_2)$, a contradiction. □

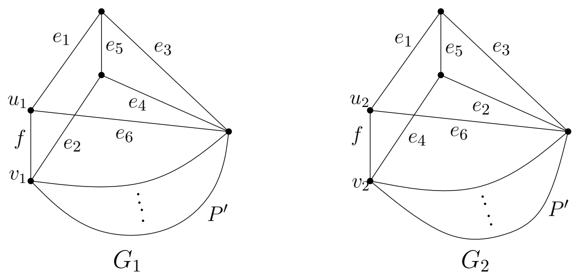


Figure 6:

For convenience, Theorem 2 is restated here.

**Theorem 12.** *Let $G_1$ be a 2-edge-connected graph such that $L(G_1)$ contains at least two circuits. Let $G_2$ be a graph with $L(G_1) = L(G_2)$. Then at least one of the following holds.*

*(1) $G_1$ and $G_2$ are 2-isomorphic.*

*(2) $G_1$ and $G_2$ are 2-isomorphic to subdivisions of $K_4$, where the edge set of an ear of $G_1$ is also the edge set of an ear of $G_2$.*

*(3) $G_1$ and $G_2$ are 2-isomorphic to subdivisions of $K_3^{m,2,n}$ for some $m \in \{1, 2\}$ and $n \geqslant 2$, where the edge set of an ear of $G_1$ is also the edge set of an ear of $G_2$. Moreover, when $n \geqslant 3$, the $n$ ears in $G_1$ having the same ends also have the same ends in $G_2$.*

*(4) $G_1$ and $G_2$ are 2-isomorphic to the graphs pictured in Figure 1.*

*Proof.* If some loop $e$ of $G_1$ is also a loop of $G_2$, then by Corollary 5 we have that $G_1 \backslash e$ and $G_2 \backslash e$ are 2-isomorphic. So $G_1$ and $G_2$ are 2-isomorphic. Moreover, when some link of $G_1$ is a loop of $G_2$, Lemma 7 implies that (4) holds. Therefore, we may assume that neither $G_1$ nor $G_2$ has loops. By Whitney's 2-Isomorphism Theorem we can further assume that $G_1$ and $G_2$ are connected, and each series class of $G_i$ is an ear of $G_i$ for each $i \in \{1, 2\}$. Using Lemma 8 we may assume that a subset of $E(G_1)$ is the edge set of an ear of $G_1$ if and only if it is the edge set of an ear of $G_2$. Therefore, we may assume that $G_1$ and $G_2$ are cosimple.

Since the rank of $L(G_i)$ is equal to $|V(G_i)|$, we have $|V(G_1)| = |V(G_2)|$. When $|V(G_1)| \leqslant 4$, Lemmas 9 and 10 imply that the result holds. We claim that when $|V(G_1)| \geqslant 5$ we have that $G_1$ and $G_2$ are 2-isomorphic. When $|V(G_1)| = 5$, the claim follows from Lemma 11. So we may assume that $|V(G_1)| \geqslant 6$. For each edge $e \in E(G_1)$, by Lemma 4 (3) we have $L(G_1/e) = L(G_2/e)$. By induction $G_1/e$ and $G_2/e$ are 2-isomorphic. So $G_1$ and $G_2$ are 2-isomorphic by Lemma 6. □

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