Recurrence relations for linear transformations preserving the strong q-log-convexity

Lily Li Liu[*] Ya-Nan Li

School of Mathematical Science
Qufu Normal University
Qufu 273165, P.R. China

liulily@mail.qfnu.edu.cn 471009811@qq.com

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Abstract

Let $[T(n,k)]_{n,k\geqslant 0}$ be a triangle of positive numbers satisfying the three-term recurrence relation

$$T(n,k) = (a_1n + a_2k + a_3)T(n-1,k) + (b_1n + b_2k + b_3)T(n-1,k-1).$$

In this paper, we give a new sufficient condition for linear transformations

$$Z_n(q) = \sum_{k=0}^{n} T(n,k)X_k(q)$$

that preserves the strong q-log-convexity of polynomials sequences. As applications, we show linear transformations, given by matrices of the binomial coefficients, the Stirling numbers of the first kind and second kind, the Whitney numbers of the first kind and second kind, preserving the strong q-log-convexity in a unified manner.

Keywords: Strong q-log-convexity; Linear transformation; Stirling number; Whitney number

1 Introduction

Let $(x_n)_{n\geqslant 0}$ be a sequence of nonnegative real numbers. It is said to be *log-convex log-concave* resp.) if $x_{n+1}x_m - x_nx_{m+1} \geqslant 0$ ($x_{n+1}x_m - x_nx_{m+1} \leqslant 0$ resp.) for $n \geqslant m \geqslant 0$. Log-concave and log-convex sequences arise often in combinatorics. We refer the reader to [3, 10, 14, 15] for log-concavity and [11, 16] for the log-convexity.

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For two real polynomials $f(q)$ and $g(q)$, denote $f(q) \geqslant_q g(q)$ if the difference $f(q)-g(q)$ has only nonnegative coefficients. A real polynomial sequence $(f_n(q))_{n\geqslant 0}$ is said to be *q-log-convex* if

$$f_{n+1}(q)f_{n-1}(q) \geqslant_q f_n^2(q)$$

for $n \geqslant 1$, and is called *strongly q-log-convex* if

$$f_{n+1}(q)f_m(q) \geqslant_q f_n(q)f_{m+1}(q)$$

for $n \geqslant m \geqslant 0$. Clearly, the strong q-log-convexity of polynomials sequence implies the q-log-convexity. It is known that many famous polynomial sequences, such as the Bell polynomials, the Eulerian polynomials, the Narayana polynomials and the Narayana polynomials of type B, are strongly q-log-convex [16]. In this paper, we study the strong q-log-convexity of polynomials sequences by using recurrence relations of their triangle.

Let $[T(n,k)]_{n,k\geqslant 0}$ be a triangle of positive numbers satisfying the three-term recurrence relation

$$T(n,k) = (a_1n + a_2k + a_3)T(n-1,k) + (b_1n + b_2k + b_3)T(n-1,k-1), \qquad (1)$$

with $T(n,k) = 0$ unless $0 \leqslant k \leqslant n$. It is natural to assume that $a_1n + a_2k + a_3 \geqslant 0$ for $0 \leqslant k < n$ and $b_1n+b_2k+b_3 \geqslant 0$ for $0 < k \leqslant n$. Note that the former is equivalent to $a_1 \geqslant 0, a_1+a_2 \geqslant 0, a_1+a_3 \geqslant 0$ and the latter is equivalent to $b_1 \geqslant 0, b_1+b_2 \geqslant 0, b_1+b_2+b_3 \geqslant 0$. It is known that for each fixed n, the sequence $(T(n,k))_{0\leqslant k\leqslant n}$ is log-concave (Kurtz [9]), and forms a Pólya frequency sequence if $a_2b_1 \geqslant a_1b_2$ and $a_2(b_1 + b_2 + b_3) \geqslant (a_1 + a_3)b_2$ (Wang and Yeh [15, Corollary 3]). Let

$$T_n(q) = \sum_{k=0}^{n} T(n,k)q^k$$

be the row generating functions of $[T(n,k)]_{n,k\geqslant 0}$. Liu and Wang [11] obtained a sufficient condition for the q-log-convexity of the sequence $(T_n(q))_{n\geqslant 0}$. Subsequently, Chen et al. [5] discussed the strong q-log-convexity of the sequence $(T_n(q))_{n\geqslant 0}$.

We say that the linear transformation

$$Z_n(q) = \sum_{k=0}^{n} T(n,k)X_k(q) \qquad (2)$$

preserves the strong q-log-convexity if the strong q-log-convexity of the polynomials sequence $(X_n(q))_{n\geqslant 0}$ implies the strong q-log-convexity of $(Z_n(q))_{n\geqslant 0}$. The corresponding triangle $[T(n,k)]_{n,k\geqslant 0}$ is also said to be have the property of *preserving the strong q-log-convexity*. Since the sequence $(q^k)_{k\geqslant 0}$ is strongly q-log-convex, the strong q-log-convexity of $(T_n(q))_{n\geqslant 0}$ can be deduced from the linear transformation (2) preserving the strong q-log-convexity.

As pointed out by Brenti [3], whenever we study a set of "objects" in mathematics, we look for linear transformations that preserve their structures. There have been many results in combinatorics obtained using this idea. For instance, Wang and Yeh [15] proved the log-concavity of combinatorial sequences by linear transformations. Liu and Wang [11] presented the log-convexity of combinatorial sequences by linear transformations. It is therefore natural to look for linear transformations that preserve the strong q-log-convexity. Recently, Zhu and Sun [17] obtained several linear transformations preserving the strong q-log-convexity. In this paper, we present some different linear transformations preserving the strong q-log-convexity.

In combinatorics, many classical numbers satisfy the three-term recurrence. For example, the binomial coefficients satisfy

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \tag{3}$$

The Stirling numbers $S(n,k)$ of the second kind satisfy

$$S(n,k) = S(n-1,k-1) + kS(n-1,k). \tag{4}$$

In this paper, using the three-term recurrence relation (1) of the triangle $[T(n,k)]_{n,k\geqslant 0}$, we give a new sufficient condition for linear transformations (2) to preserve the strong q-log-convexity.

The rest of the paper is organized as follows. In section 2, we obtain a sufficient condition for linear transformations (2) to preserve the strong q-log-convexity. As applications, we show some linear transformations, given by matrices of the binomial coefficients, the Stirling numbers of the second kind, the signless Stirling numbers of the first kind, the Whitney numbers of the second kind, the signless Whitney numbers of the first kind and the Bessel numbers, preserving the strong q-log-convexity in a unified manner. Moreover, we obtain the strong q-log-convexity of the r-Dowling polynomials.

2 Linear transformations preserving the strong q-log-convexity

In this section, we first recall the following result.

Lemma 1. ([5]) *Suppose that the triangle $[T(n,k)]_{n,k\geqslant 0}$ of positive numbers satisfies* (1) *with $a_2, b_2 \geqslant 0$. Then for any $k \geqslant j \geqslant 0$, $n \geqslant m \geqslant 0$, we have*

$$T(n,k)T(m,j) - T(n,j)T(m,k) \geqslant 0. \tag{5}$$

Then we present the main result of this paper.

Theorem 2. *Suppose that the triangle $[T(n,k)]_{n,k\geqslant 0}$ of positive numbers defined by* (1) *with $a_2, b_2 \geqslant 0$. Then the linear transformation (2) preserves the strong q-log-convexity.*

Proof. In order to prove the strong q-log-convexity of $(Z_n(q))_{n\geqslant 0}$, it suffices to prove that

$$Z_{n+1}(q)Z_m(q) - Z_n(q)Z_{m+1}(q) \geqslant_q 0 \tag{6}$$

for any $n \geqslant m \geqslant 0$.

Since the triangle $[T(n,k)]_{n,k\geqslant 0}$ satisfies the recurrence relation (1), we have

$$\begin{aligned}
& Z_{n+1}(q) = \sum_{k=0}^{n+1} T(n+1,k)X_k(q) \\
= & \sum_{k=0}^{n+1}(a_1n+a_1+a_2k+a_3)T(n,k)X_k(q) + \sum_{k=0}^{n+1}(b_1n+b_1+b_2k+b_3)T(n,k)X_k(q) \\
= & \sum_{k=0}^{n}\left((a_1n+a_1+a_2k+a_3)X_k(q)+(b_1n+b_1+b_2k+b_2+b_3)X_{k+1}(q)\right)T(n-1,k).
\end{aligned}$$

Substituting $Z_{n+1}(q)$ and $Z_{m+1}(q)$ into (6) yields

$$\begin{aligned}
& Z_{n+1}(q)Z_m(q) - Z_n(q)Z_{m+1}(q) \\
= & \sum_{k=0}^{n}\sum_{j=0}^{m}(a_1n+a_1+a_2k+a_3)T(n,k)T(m,j)X_k(q)X_j(q) \\
& +\sum_{k=0}^{n}\sum_{j=0}^{m}(b_1n+b_1+b_2k+b_2+b_3)T(n,k)T(m,j)X_{k+1}(q)X_j(q) \\
& -\sum_{k=0}^{n}\sum_{j=0}^{m}(a_1m+a_1+a_2j+a_3)T(m,j)T(n,k)X_j(q)X_k(q) \\
& -\sum_{k=0}^{n}\sum_{j=0}^{m}(b_1m+b_1+b_2j+b_2+b_3)T(m,j)T(n,k)X_{j+1}(q)X_k(q) \\
= & \sum_{k=0}^{n}\sum_{j=0}^{m}[a_1(n-m)+a_2(k-j)]T(n,k)T(m,j)X_k(q)X_j(q) \\
& +\sum_{k=0}^{n}\sum_{j=0}^{m}((b_1n+b_1+b_2k+b_2+b_3)X_{k+1}(q)X_j(q) \\
& -(b_1m+b_1+b_2j+b_2+b_3)X_k(q)X_{j+1}(q))T(n,k)T(m,j) \\
= & R_1(q)+R_2(q)+S_1(q)+S_2(q),
\end{aligned}$$

where

$$R_1(q) = \sum_{k=0}^{m}\sum_{j=0}^{m}(a_1(n-m)+a_2(k-j))T(n,k)T(m,j)X_k(q)X_j(q),$$

$$R_2(q) = \sum_{k=m+1}^{n}\sum_{j=0}^{m}(a_1(n-m)+a_2(k-j))T(n,k)T(m,j)X_k(q)X_j(q),$$

$$\begin{aligned} S_1(q) &= \sum_{k=0}^{m}\sum_{j=0}^{m}((b_1n+b_1+b_2k+b_2+b_3)X_{k+1}(q)X_j(q) \\ &\quad -(b_1m+b_1+b_2j+b_2+b_3)X_k(q)X_{j+1}(q))T(n,k)T(m,j), \\ S_2(q) &= \sum_{k=m+1}^{n}\sum_{j=0}^{m}((b_1n+b_1+b_2k+b_2+b_3)X_{k+1}(q)X_j(q) \\ &\quad -(b_1m+b_1+b_2j+b_2+b_3)X_k(q)X_{j+1}(q))T(n,k)T(m,j). \end{aligned}$$

Clearly, $R_2(q), S_2(q) \geqslant_q 0$, for $k > m \geqslant j$. So to prove (6), it suffices to prove $R_1(q), S_1(q) \geqslant_q 0$.

We first prove $R_1(q) \geqslant_q 0$. Note that

$$\begin{aligned} R_1(q) &= \sum_{k=0}^{m}\sum_{j=0}^{m} a_1(n-m)T(n,k)T(m,j)X_k(q)X_j(q) \\ &\quad + \sum_{0\leqslant j<k\leqslant m} a_2(k-j)(T(n,k)T(m,j)-T(n,j)T(m,k))X_k(q)X_j(q). \end{aligned}$$

Thus by Lemma 1, we have $R_1(q) \geqslant_q 0$.

Then we prove $S_1(q) \geqslant_q 0$. Note that

$$\begin{aligned} S_1(q) &= \sum_{0\leqslant j<k\leqslant m}((b_1n+b_1+b_2k+b_2+b_3)T(n,k)T(m,j) \\ &\quad -(b_1m+b_1+b_2k+b_2+b_3)T(n,j)T(m,k))X_{k+1}(q)X_j(q) \\ &\quad + \sum_{0\leqslant j<k\leqslant m}((b_1n+b_1+b_2j+b_2+b_3)T(n,j)T(m,k) \\ &\quad -(b_1m+b_1+b_2j+b_2+b_3)T(n,k)T(m,j))X_{j+1}(q)X_k(q) \\ &\quad + \sum_{k=0}^{m} b_1(n-m)X_{k+1}(q)X_k(q)T(n,k)T(m,k). \end{aligned}$$

Since $n \geqslant m \geqslant 0$, we have

$$\begin{aligned} S_1(q) &\geqslant_q \sum_{0\leqslant j<k\leqslant m}((b_1n+b_1+b_2k+b_2+b_3)T(n,k)T(m,j) \\ &\quad -(b_1n+b_1+b_2k+b_2+b_3)T(n,j)T(m,k))X_{k+1}(q)X_j(q) \\ &\quad + \sum_{0\leqslant j<k\leqslant m}((b_1n+b_1+b_2j+b_2+b_3)T(n,j)T(m,k) \\ &\quad -(b_1n+b_1+b_2j+b_2+b_3)T(n,k)T(m,j))X_k(q)X_{j+1}(q) \\ &\quad + \sum_{k=0}^{m} b_1(n-m)X_{k+1}(q)X_k(q)T(n,k)T(m,k) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{0\leqslant j<k\leqslant m} (b_1 n + b_1 + b_2 k + b_2 + b_3)(T(n,k)T(m,j) - T(n,j)T(m,k))X_{k+1}(q)X_j(q) \\
&- \sum_{0\leqslant j<k\leqslant m} (b_1 n + b_1 + b_2 j + b_2 + b_3)(T(n,k)T(m,j) - T(n,j)T(m,k))X_k(q)X_{j+1}(q) \\
&+ \sum_{k=0}^{m} b_1(n-m)X_{k+1}(q)X_k(q)T(n,k)T(m,k).
\end{aligned}
$$

On the other hand, for $k > j$, we have

$$
\begin{aligned}
S_1(q) \geqslant_q & \sum_{0\leqslant j<k\leqslant m} (b_1 n + b_1 + b_2 k + b_2 + b_3)(T(n,k)T(m,j) - T(n,j)T(m,k)) \\
& (X_{k+1}(q)X_j(q) - X_k(q)X_{j+1}(q)) \\
& + \sum_{k=0}^{m} b_1(n-m)X_{k+1}(q)X_k(q)T(n,k)T(m,k).
\end{aligned}
$$

Thus we obtain $S_1(q) \geqslant_q 0$ from Lemma 1 and the strong q-log-convexity of $(X_n(q))_{n\geqslant 0}$. The proof of the theorem is complete. □

3 Applications

In this section, as applications of Theorem 2, we present some linear transformations preserving the strong q-log-convexity.

3.1 Some classical linear transformations

Let $s(n,k)$ and $S(n,k)$ be the Stirling numbers of the first kind and second kind respectively. Denote by $c(n,k) = (-1)^{n+k}s(n,k)$ the signless Stirling numbers of the first kind, i.e., the number of permutations of $[n]$ with k cycles. It is known that the signless Stirling numbers of the first kind and the Stirling numbers of the second kind satisfy the recurrence relations

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1), \tag{7}$$

and

$$S(n,k) = S(n-1,k-1) + kS(n-1,k) \tag{8}$$

respectively, with $c(n,0) = c(0,k) = S(n,0) = S(0,k) = 0$, except for $c(0,0) = S(0,0) = 1$ (see [7] for instance).

Liu and Wang [11] showed that the Stirling transformations of the first kind and second kind preserve the log-convexity. Now we generalize it by showing that these two linear transformations also preserve the strong q-log-convexity by Theorem 2.

Corollary 3. *The Stirling transformations of the first kind and second kind*

$$Z_n(q) = \sum_{k=0}^{n} c(n,k)X_k(q), \quad \bar{Z}_n(q) = \sum_{k=0}^{n} S(n,k)X_k(q)$$

preserve the strong q-log-convexity respectively.

The Bell polynomials $B_n(q) = \sum_{k=0}^{n} S(n,k)q^k$ can be viewed as a q-analog of the Bell numbers and have many fascinating properties. For example, the Bell polynomials $B_n(q)$ have only real zeros (see [15] for instance). Liu and Wang [11] established that the Bell polynomials are q-log-convex. Moreover, the Bell polynomials are strongly q-log-convex obtained by Chen et al. [5] and Zhu [16] respectively. Now the strong q-log-convexity of $B_n(q)$ follows immediately from Corollary 3.

Corollary 4 ([5, 16])**.** *The Bell polynomials $B_n(q)$ form a strongly q-log-convex sequence.*

The polynomial $F_n(q) = \sum_{k=0}^{n} k!S(n,k)q^k$ also admits a variety of combinatorial properties, such as the strong q-log-convexity [5]. Thus we have the following result from Corollary 3 and the strong q-log-convexity of $(k!q^k)_{k\geqslant 0}$.

Corollary 5 ([5])**.** *The polynomials $F_n(q)$ form a strongly q-log-convex sequence.*

The Dowling lattice $Q_n(G)$ is a geometric lattice of rank n over a finite group G of order $m \geqslant 1$ [8]. When $m = 1$, that is, G is the trivial group, $Q_n(G)$ is isomorphic to the lattice $\prod_{n+1}$ of partitions of an $(n+1)$-element set. So the Dowling lattices can be viewed as group-theoretic analogs of the partition lattices. Let $w_m(n,k)$ and $W_m(n,k)$ be the kth Whitney numbers of the first kind and second kind of the Dowling lattice respectively. Denote by $t_m(n,k) = (-1)^{n+k}w_m(n,k)$ the signless Whitney numbers of the first kind. It is known that the signless Whitney numbers of the first kind and the Whitney numbers of the second kind satisfy the recurrence relations

$$t_m(n,k) = (1 + m(n-1))t_m(n-1,k) + t_m(n-1,k-1)$$

and

$$W_m(n,k) = (1 + mk)W_m(n-1,k) + W_m(n-1,k-1), \quad n \geqslant k \geqslant 1$$

respectively, with $t_m(n,n) = t_m(n,0) = W_m(n,n) = W_m(n,0) = 1$ for $n \geqslant 0$ and $t_m(n,k) = W_m(n,k) = 0$, except for $0 \leqslant k \leqslant n$. So we obtain the following result from Theorem 2.

Corollary 6. *The Whitney transformations of the first kind and second kind*

$$Z_n(q) = \sum_{k=0}^{n} t_m(n,k)X_k(q), \quad \bar{Z}_n(q) = \sum_{k=0}^{n} W_m(n,k)X_k(q)$$

preserve the strong q-log-convexity respectively for $m \geqslant 1$.

Benoumhani [1] introduced the Dowling polynomial $D_m(n,x) = \sum_{k=0}^{n} W_m(n,k)q^k$ and following two generalized polynomials $F_{m,1}(n;q) = \sum_{k=0}^{n} k!W_m(n,k)m^k q^k$, $F_{m,2}(n;q) = \sum_{k=0}^{n} k!W_m(n,k)q^k$. It is known that polynomials $D_m(n;q)$, $F_{m,1}(n;q)$ and $F_{m,2}(n;q)$ have only real zeros respectively (see [2, 10] for instance). Chen et al. [5] also prove the strong q-log-convexity of polynomials $D_m(n;q)$, $F_{m,1}(n;q)$ and $F_{m,2}(n;q)$. It is easy to prove that sequences $(k!q^k)_{k\geqslant 0}$ and $(k!m^k q^k)_{k\geqslant 0}$ are strongly q-log-convex respectively. Thus we have the following result from Corollary 6.

Corollary 7 ([5]). *The polynomials $D_m(n;q)$, $F_{m,1}(n;q)$ and $F_{m,2}(n;q)$ form strongly q-log-convex sequences respectively for $m \geqslant 1$.*

The Bessel polynomials are defined by

$$y_n(q) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \left(\frac{q}{2}\right)^k,$$

and they have been extensively studied. Let

$$Y(n,k) = \frac{(n+k)!}{(n-k)!k!}.$$

Chen et al. [5] deduced that $[Y(n,k)]_{n,k\geqslant 0}$ satisfies the following recurrence relation

$$Y(n,k) = Y(n-1,k) + (2n+2k-2)Y(n-1,k-1), \quad n \geqslant k \geqslant 1.$$

So we have the following result from Theorem 2.

Corollary 8. *The Bessel transformation*

$$Z_n(q) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} X_k(q)$$

preserves the strong q-log-convexity.

Since the sequence $((q/2)^k)_{k\geqslant 0}$ is strongly q-log-convex, we can also present the following result from Corollary 8.

Corollary 9 ([5]). *The Bessel polynomials $y_n(q)$ form a strongly q-log-convex sequence.*

3.2 Strong q-log-convexity of the r-Dowling polynomials

The r-Dowling polynomial $D_{m,r}(n,q)$ [6] is the generating function of r-Whitney numbers of the second kind [12], which is a common generalization of the Whitney numbers of the second kind of the Dowling lattice and the r-Stirling numbers [4]. Cheon and Jung [6] established that the r-Dowling polynomials $D_{m,r}(n,q)$ have only negative real zeros, for all $r,m,n \geqslant 1$. In this subsection, we give the strong q-log-convexity of the r-Dowling polynomials using the following relation established by Rahmani [13]

$$D_{m,r}(n,q) = \sum_{k=0}^{n} \binom{n}{k} m^k r^{n-k} B_k\left(\frac{q}{m}\right), \tag{9}$$

where $B_k(q)$ are the Bell polynomials.

From Theorem 2 and the recurrence relation (3), we also give the binomial transformation preserving the strong q-log-convexity directly, which is first obtained by Zhu and Sun [17].

Corollary 10 ([17])**.** *The binomial transformation*

$$Z_n(q) = \sum_{k=0}^{n} \binom{n}{k} X_k(q)$$

preserves the strong q-log-convexity.

To prove the strong q-log-convexity of r-Dowling polynomials, we need two lemmas.

Lemma 11. *If $(x_n)_{n\geqslant 0}$ is a log-convex sequence and $(X_n(q))_{n\geqslant 0}$ is a strongly q-log-convex sequence, then $(x_nX_n(q))_{n\geqslant 0}$ is also a strongly q-log-convex sequence.*

Proof. Note that

$$\begin{aligned} & x_{k+1}X_{k+1}(q)x_jX_j(q) - x_kX_k(q)x_{j+1}X_{j+1}(q) \\ = \; & (x_{k+1}x_j - x_kx_{j+1})X_{k+1}(q)X_j(q) + (X_{k+1}(q)X_j(q) - X_k(q)X_{j+1}(q))x_kx_{j+1} \\ \geqslant_q \; & 0. \end{aligned}$$

The inequality holds by the log-convexity of $(x_n)_{n\geqslant 0}$ and the strong q-log-convexity of $(X_n(q))_{n\geqslant 0}$. Thus the sequence $(x_nX_n(q))_{n\geqslant 0}$ is strongly q-log-convex. □

Lemma 12. *If $(X_n(q))$ is strongly q-log-convex, then so does $(X_n(mq))_{n\geqslant 0}$ for $m \geqslant 1$.*

Proof. Let $X_n(q) = \sum_{t=0}^{n} a(n,t)q^t$. Denote by $a(k,j;r,t)$ the following expression:

$a(k+1,t)a(j,r-t) + a(k+1,r-t)a(j,t) - a(k,t)a(j+1,r-t) - a(k,r-t)a(j+1,t).$

By the assumption,

$$X_{k+1}(q)X_j(q) - X_k(q)X_{j+1}(q) = \sum_{r=0}^{k+s+1} \sum_{t=0}^{r} a(k,j;r,t)q^r \geqslant_q 0,$$

for $k \geqslant j \geqslant 0$. Hence the coefficient $a(k,j;r,t) \geqslant 0$, for $k \geqslant j \geqslant 0$.

Thus we have

$$X_{k+1}(mq)X_j(mq) - X_k(mq)X_{j+1}(mq) = \sum_{r=0}^{k+s+1} \sum_{t=0}^{r} a(k,j;r,t)m^rq^r \geqslant_q 0,$$

i.e., $(X_n(mq))_{n\geqslant 0}$ is strongly q-log-convex, for $m \geqslant 1$. □

Now the strong q-log-convexity of the r-Dowling polynomials follows immediately from Corollary 10 and the relation (9).

Corollary 13. *The r-Dowling polynomials $D_{m,r}(n,q)$ form a strongly q-log-convex sequence for $m \geqslant 1$ and $r \geqslant 0$.*

Remark 14. An immediate consequence of Corollary 13 is the strong q-log-convexity of the Dowling polynomials $D_m(n,q) = D_{m,1}(n,q)$ (see [5]).

In particular, when $m = 1$, we show the strong q-log-convexity of the r-Bell polynomials [12].

Corollary 15. *The r-Bell polynomials $B_r(n,q)$ form a strongly q-log-convex sequence for $r \geqslant 0$.*

Remark 16. Note that the Bell polynomials $B_n(q) = B_0(n,q)$. Hence the strong q-log-convexity of the Bell polynomials $B_n(q)$ follows from Corollary 15 (see [5]).

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