An improved bound on $(A+A)/(A+A)$

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Abstract

We show that, for a finite set A of real numbers, the size of the set

$$\frac{A+A}{A+A} = \left\{ \frac{a+b}{c+d} : a,b,c,d \in A, c+d \neq 0 \right\}$$

is bounded from below by

$$\left| \frac{A+A}{A+A} \right| \gg \frac{|A|^{2+1/4}}{|A/A|^{1/8} \log |A|}.$$

This improves a result of Roche-Newton (2016).

1 Introduction

Given a finite set A of real numbers, we define its *sum set* to be

$$A + A = \{a + b : a, b \in A\},$$

its *product set* to be

$$AA = \{ab : a, b \in A\},$$

and its *quotient set* to be

$$A/A = \{a/b : a, b \in A\}.$$

We write the notation $f(n) \gg g(n)$ to indicate that there exists a constant $c > 0$ such that $f(n) \geqslant cg(n)$ for all $n \geqslant 1$.

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Erdős and Szemerédi [3] conjectured that, for any $\epsilon > 0$,

$$\max(|AA|, |A+A|) \gg |A|^{2-\epsilon}.$$

In this paper we study the related question of establishing lower bounds on the size of the set

$$\frac{A+A}{A+A} = \left\{ \frac{a+b}{c+d} : a, b, c, d \in A, c+d \neq 0 \right\}.$$

In [2], Balog and Roche-Newton showed that, if A is a set of strictly positive reals,

$$\left| \frac{A+A}{A+A} \right| \geqslant 2|A|^2 - 1.$$

This is completely sharp, as shown by the set $A = \{1, 2, 3\}$.

In [8], Roche-Newton and Zhelezov conjectured that

$$\left| \frac{A+A}{A+A} \right| \ll |A|^2 \Rightarrow |A+A| \ll |A|. \tag{1}$$

Shkredov [9] has made some progress in this direction, showing that

$$\left| \frac{A-A}{A-A} \right| \ll |A|^2 \Rightarrow |A-A| \ll |A|^{2-1/5} \log^{3/10} |A|. \tag{2}$$

In [7], Roche-Newton proved that

$$\left| \frac{A+A}{A+A} \right| \gg \frac{|A|^{2+2/25}}{|A/A|^{1/25} \log |A|}, \tag{3}$$

which implies that

$$\left| \frac{A+A}{A+A} \right| \ll |A|^2 \Rightarrow |A/A| \gg \frac{|A|^2}{\log^{25} |A|}. \tag{4}$$

It is known that (4) is implied by (1), for example by work of Li and Shen [5], but the reverse implication does not hold (even if the extra $\log^{-25} |A|$ term were removed).

This paper gives the following improvement to (3) and (4).

Theorem 1. *Let A be a finite set of real numbers. Then*

$$\left| \frac{A+A}{A+A} \right| \gg \frac{|A|^{2+1/4}}{|A/A|^{1/8} \log |A|}. \tag{5}$$

Consequently,

$$\left| \frac{A+A}{A+A} \right| \ll |A|^2 \Rightarrow |A/A| \gg \frac{|A|^2}{\log^{8} |A|}. \tag{6}$$

In broad outline, the proof of Theorem 1 is similar to the proof by Roche-Newton of (3). We combine ideas of Solymosi [10] and Konyagin and Shkredov [4], developed in work on the Erdős-Szemerédi sum-product conjecture, with an incidence bound and a probabilitistc argument. The key difference between this proof and that of Roche-Newton is that we apply the probabilistic method in a more flexible way, which leads to a simpler proof of a stronger result.

2 Proof of Theorem 1

The remainder of the paper is dedicated to the proof of inequality (5) Theorem 1; given (5), it is trivial to derive (6).

In section 2.1 we describe the general setup for the proof and fix notation. In section 2.2, we give the general idea of the proof, and briefly elaborate on the key difference between this proof and that of Roche-Newton [7]. Sections 2.3, 2.4, and 2.5 comprise the main body of the proof.

2.1 Setup

We assume that all of the elements of A are strictly positive. This is without loss of generality; if at least half of the elements of A are positive, we consider these elements; otherwise, we multiply by -1 and then consider the positive elements.

Using a dyadic pigeonholing argument, we find a set $P \subset A \times A$ and a number $|A|^2/\left(2|A/A|\right) \leqslant \tau \leqslant |A|$ such that P is contained in the union of lines through the origin (in $\mathbb{R}^2$) that each contain exactly τ points, and

$$|P| \gg |A|^2/\log|A|. \tag{7}$$

In more detail, $A \times A$ is contained in the union of $|A/A|$ lines through the origin, one for each ratio in A/A. Note that no more than $|A|^2/2$ points in $A \times A$ are contained in lines through the origin that each contain fewer than $\tau_0 = |A|^2/\left(2|A/A|\right)$ points. Note also that no line contains more than $|A|$ points of $A \times A$. We partition the numbers $[\tau_0, |A|]$ into $O(\log|A|)$ dyadic intervals of the form $[2^i, 2^{i+1})$, and use the pigeonhole principle to show that there is a τ such that there are at least $\Omega(|A|^2/\log|A|)$ points $(a,b) \in A \times A$ that lie on lines through the origin that each contain between τ and 2τ points. We then choose τ arbitrary points on each of these lines, and define the resulting point set to be P.

Note that, if a, b, c, d are elements of A, then the point $(a+b, c+d) \in (A\times A)+(A\times A)$ is contained in the line through the origin with slope $(c+d)/(a+b)$. Hence, $(A\times A)+(A\times A)$ is contained in the union of $(A+A)/(A+A)$ lines through the origin. Since $P+P$ is a subset of $(A \times A) + (A \times A)$, it suffices to show that, for the set S of lines through the origin that each contain at least one point of $P+P$, we have

$$|S| \geqslant \frac{|A|^{2+1/4}}{|A/A|^{1/8}\log|A|}. \tag{8}$$

Once (8) is demonstrated, the proof will be complete.

Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{|\Lambda|}\}$ with $\lambda_1 < \lambda_2 < \ldots \lambda_{|\Lambda|}$ be the set of slopes of lines through the origin that each contain τ points of P. Note that

$$|P| = \tau|\Lambda|. \tag{9}$$

Let M be an integer parameter that we will set later. For each $1 \leqslant i \leqslant \lfloor |\Lambda|/2M \rfloor$, let

$$\begin{aligned}
f_i &= 2M(i-1),\\
T_i &= \{\lambda_{f_i+1}, \lambda_{f_i+2}, \ldots, \lambda_{f_i+M}\},\\
U_i &= \{\lambda_{f_i+M+1}, \lambda_{f_i+M+2}, \ldots, \lambda_{f_i+2M}\}.
\end{aligned}$$

For the remainder of the proof, we work with an arbitrary i and set $T = T_i$ and $U = U_i$. We relabel λ_{f_i+j} as λ_j, so that $\lambda_1, \lambda_2, \ldots, \lambda_M \in T$ and $\lambda_{M+1}, \lambda_{M+2}, \ldots, \lambda_{2M} \in U$. With this relabeling, let P_i be the set of points of P contained in the line with slope λ_i.

2.2 Brief overview

The basic idea of the proof is to show that we can select points $a_{ij} \in P_i$ for $1 \leqslant i \leqslant M$ and $M+1 \leqslant j \leqslant 2M$ so that the union of the sets $a_{ij} + P_j$ determines many slopes. There are two basic parts of the proof. First, we use a proof based on a geometric incidence bound to show that, for fixed i, j, k, ℓ with $(i,j) \neq (k,\ell)$, there are many selections of $a_{ij}, a_{k\ell}$ such that $a_{ij} + P_j$ shares few slopes with $a_{k\ell} + P_\ell$. Second, we combine this with a probabilistic argument to show that we can select the a_{ij} so that the total number of slopes is large.

The key difference between this proof and the proof of Roche-Newton in [7] is that we choose a point a_{ij} for each pair in $T \times U$, where Roche-Newton chose a single representative for each line in T.

2.3 Bounding shared slopes

For a point $p \in \mathbb{R}^2$, denote by $r(p)$ the slope of the line passing through p and the origin. For $a_i \in P_i$ and $a_k \in P_k$, let

$$\mathcal{E}(a_i, P_j, a_k, P_\ell) = |\{(b_j, b_\ell) \in P_j \times P_\ell : r(a_i + b_j) = r(a_k + b_\ell)\}|. \tag{10}$$

The proof of the following lemma is based closely on the work of Roche-Newton [7].

Lemma 2. *Let $1 \leqslant i, k \leqslant M$ and $M+1 \leqslant j, \ell \leqslant 2M$, with at least one of $i \neq k$ and $j \neq \ell$. The number of pairs $(a_i, a_k) \in P_i \times P_k$ such that*

$$\mathcal{E}(a_i, P_j, a_k, P_\ell) \geqslant K$$

is bounded from above by

$$O(\tau^4/K^3 + \tau^2/K).$$

Proof. For each pair of points $(a, \lambda_i a), (b, \lambda_k b)$ in $P_i \times P_k$, we define the curve

$$l_{ab} = \{(x,y) : (\lambda_i a + \lambda_j x)(b + y) = (\lambda_k b + \lambda_\ell y)(a + x)\}.$$

For any set Q of points, denote by $\pi_x(Q)$ the projection of Q onto the x-axis; in other words,

$$\pi_x(Q) = \{a : \exists y((a,y) \in Q)\}.$$

Let

$$\mathcal{L} = \{l_{ab} : ((a, \lambda_i a), (b, \lambda_k b)) \in P_i \times P_k\},$$
$$\mathcal{P} = \pi_x(P_j) \times \pi_x(P_\ell).$$

Note that $(x, y) \in l_{ab}$ is equivalent to $r((a + x, \lambda_i a + \lambda_j x)) = r((b + y, \lambda_k b + \lambda_\ell y))$. Hence, to prove the lemma, it suffices to show that there are no more than $\tau^4/K^3 + \tau^2/K$ curves from $\mathcal{L}$ that each contain at least K points of $\mathcal{P}$. To this end, we use the following result of Pach and Sharir [6].

Theorem 3. *Let $\mathcal{L}$ be a family of curves and let $\mathcal{P}$ be a set of points in the plane such that*

1. *any two distinct curves from $\mathcal{L}$ intersect in at most two points of $\mathcal{P}$,*
2. *for any two distinct points $p, q \in \mathcal{P}$, there exist at most two curves $\mathcal{L}$ which pass through both p and q. Then, for any $k \geqslant 2$, the set $\mathcal{L}_k \subset \mathcal{L}$ of curves that contain at least k points of $\mathcal{P}$ satisfies the bound*

$$|\mathcal{L}_k| \ll |\mathcal{P}|^2 k^{-3} + |\mathcal{P}| k^{-1}.$$

Since $|\mathcal{P}| = \tau^2$, Lemma 2 will follow directly from this theorem. It remains to show that $\mathcal{L}$ satisfies the hypotheses of Theorem 3.

We will first check that two distinct curves of $\mathcal{L}$ intersect in at most two points.

Let l_{ab} and $l_{a'b'}$ be two distinct curves in $\mathcal{L}$. Their intersection is the set of all (x, y) such that

$$(\lambda_i a + \lambda_j x)(b + y) = (\lambda_k b + \lambda_\ell y)(a + x), \tag{11}$$
$$(\lambda_i a' + \lambda_j x)(b' + y) = (\lambda_k b' + \lambda_\ell y)(a' + x). \tag{12}$$

Let

$$\delta_1 = \lambda_j - \lambda_\ell, \qquad \delta_2 = \lambda_j - \lambda_k, \qquad \delta_3 = \lambda_\ell - \lambda_i, \qquad \delta_4 = \lambda_k - \lambda_i. \tag{13}$$

Note that $\delta_2, \delta_3 \neq 0$, and at least one of $\delta_1 \neq 0$ or $\delta_4 \neq 0$ holds. In addition, some simple algebra combined with the observation that $(\lambda_j - \lambda_i)(\lambda_\ell - \lambda_k) \neq 0$ shows that

$$\delta_1 \delta_4 - \delta_2 \delta_3 \neq 0. \tag{14}$$

Rearrange (11) and (12) into the form

$$x(\delta_1 y + \delta_2 b) = a(\delta_3 y + \delta_4 b), \tag{15}$$
$$x(\delta_1 y + \delta_2 b') = a'(\delta_3 y + \delta_4 b'). \tag{16}$$

Suppose, for contradiction, that $\delta_1 y + \delta_2 b = 0$. This implies that $\delta_1 \neq 0$, since $\delta_2, b \neq 0$. Then we have $y = -\delta_2 \delta_1^{-1} b$, and from the right side of (15), we have $\delta_3 y + \delta_4 b = 0$. From this, we conclude that $-\delta_3 \delta_2 \delta_1^{-1} + \delta_4 = 0$, which contradicts (14). Hence, $\delta_1 y + \delta_2 b \neq 0$, and by a similar argument, we conclude that $\delta_1 y + \delta_2 b' \neq 0$.

Hence, we conclude

$$a(\delta_3 y + \delta_4 b)(\delta_1 y + \delta_2 b)^{-1} = x = a'(\delta_3 y + \delta_4 b')(\delta_1 y + \delta_2 b')^{-1}. \tag{17}$$

From equation (17), we get the quadratic equation

$$\delta_1\delta_3(a-a')y^2 + (\delta_2\delta_3(ab'-a'b) + \delta_1\delta_4(ab-a'b'))\,y + \delta_2\delta_4 bb'(a'-a) = 0. \tag{18}$$

Either there are at most two values of y which give a solution to this quadratic, or all of the coefficients are zero.

Suppose that we are in this degenerate case. If $\delta_1 \neq 0$, then the fact that the coefficient of y^2 is zero implies that $a = a'$. Otherwise, $\delta_4 \neq 0$, and so the fact that the constant term is zero implies that $a = a'$. Combining the fact that $a' = a$ with the fact that the linear term is zero, we conclude that $\delta_2\delta_3 - \delta_1\delta_4 = 0$, which contradicts (14).

Hence, l_{ab} and $l_{a'b'}$ intersect in at most two points. Now, we show that there are at most two curves of $\mathcal{L}$ that pass through any pair of points in $\mathcal{P}$.

If $l_{xy} \in \mathcal{L}$ is a curve that that passes through (a,b) and (a',b'), then

$$(\lambda_i x + \lambda_j a)(y+b) = (\lambda_k y + \lambda_\ell b)(x+a), \tag{19}$$
$$(\lambda_i x + \lambda_j a')(y+b') = (\lambda_k y + \lambda_\ell b')(x+a'). \tag{20}$$

Rearrange these equations into the form

$$x(\delta_4 y + \delta_3 b) = a(\delta_2 y + \delta_1 b), \tag{21}$$
$$x(\delta_4 y + \delta_3 b') = a'(\delta_2 y + \delta_1 b'). \tag{22}$$

These equations are the same as equations (15) and (16) with δ_4 interchanged with δ_1, and δ_3 interchanged with δ_2. Since the conditions on these quantities are symmetric with regard to the pairs δ_1, δ_4 and δ_2, δ_3, we apply the previous argument to show that there are at most two solutions to this system of equations. $\square$

2.4 Choosing representatives

For each pair (i,j) with $1 \leqslant i \leqslant M$ and $M+1 \leqslant j \leqslant 2M$, choose $a_{ij} \in P_i$ uniformly at random. Note that we have chosen M^2 points. For each pair $a_{ij}, a_{k\ell}$ of chosen points, let $X(i,j,k,\ell)$ be the event that

$$\mathcal{E}(a_{ij}, P_j, a_{k\ell}, P_k) > B,$$

where B is a parameter that we will fix later. Applying Lemma 2, we find that, for each quadruple (i,j,k,ℓ) with at least one of $i \neq j$ and $k \neq \ell$, we have

$$Pr[X(i,j,k,\ell)] \ll \tau^2 B^{-3} + B^{-1}. \tag{23}$$

In addition, note that $X(i,j,k,\ell)$ depends on $X(i',j',k',\ell')$ only if either $(i,j) = (i',j')$ or $(k,\ell) = (k',\ell')$. Hence, $X(i,j,k,\ell)$ is independent of all but at most $2M^2$ other events.

We apply the following version of the Lovász Local Lemma to bound the probability that at least one of the events occurs - see Corollary 5.1.2 in [1].

Theorem 4. *Let $X_1, X_2, \ldots, X_n$ be events in an arbitrary probability space. Suppose that each event X_i is mutually independent from all but at most d of the events X_j, with $i \neq j$. Suppose also that the probability of the event X_i occuring is at most p, for all $1 \leqslant i \leqslant n$. Finally suppose that*

$$ep(d+1) \leqslant 1.$$

Then, with positive probability, none of the events $X_1, X_2, \ldots, X_n$ occur.

Using (23), we apply this lemma with $p = c_1(\tau^2 B^{-3} + B^{-1})$ and $d = 2M^2$, and conclude that, as long as

$$ec_1(\tau^2 B^{-3} + B^{-1})(2M^2+1) \leqslant 1, \tag{24}$$

there is a positive probability that none of the events $X(i,j,k,\ell)$ occur.

2.5 Combining the bounds

Let Q be the union of $a_{ij} + P_j$ over all the a_{ij}. Note that Q is the union of M^2 distinct sets, each containing τ points. Let R be the slopes of lines through points of Q. By inclusion-exclusion,

$$|R| \geqslant M^2\tau - \sum \mathcal{E}(a_{ij}, P_j, a_{k\ell}, P_k),$$

where the sum is over all $1 \leqslant i,k \leqslant M$ and $M+1 \leqslant j,\ell \leqslant 2M$ such that at least one of $i \neq k$ and $j \neq \ell$ holds. Since each $\mathcal{E}(a_{ij}, P_j, a_{k\ell}, P_\ell) < B$, we have

$$|R| \geqslant M^2\tau - BM^4. \tag{25}$$

We now set $B = \tau/(2M^2)$. Returning to the constraint (24), we require that

$$ec_1(8M^6\tau^{-1} + 2M^2\tau^{-1})(2M^2+1) \leqslant 1.$$

It is possible to satisfy this constraint with $M = c_2\tau^{1/8}$, for an appropriate choice of c_2.

Note that the sets $R \subset S$ obtained for different choices of T, U are disjoint. Since there are $\lfloor |\Lambda|/2M \rfloor$ choices for T and U, we have by equation 25 that the number of lines through the origin that contain some point of $P+P$ is at least

$$|S| = |R| \lfloor |\Lambda|/2M \rfloor \gg M\tau|\Lambda| \gg \tau^{1+1/8}|\Lambda|.$$

Applying equation (9), we have

$$|S| \gg |P|\tau^{1/8}.$$

Combining this with equation (7) and the observation that $\tau \geqslant |A|^2/|A/A|$, we have inequality (8), and the proof is complete.

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