Every triangulated 3-polytope of minimum degree 4 has a 4-path of weight at most 27

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Abstract

By δ and w_k denote the minimum degree and minimum degree-sum (weight) of a k-vertex path in a given graph, respectively. For every 3-polytope, $w_2 \leqslant 13$ (Kotzig, 1955) and $w_3 \leqslant 21$ (Ando, Iwasaki, Kaneko, 1993), where both bounds are sharp. For every 3-polytope with $\delta \geqslant 4$, we have sharp bounds $w_2 \leqslant 11$ (Lebesgue, 1940) and $w_3 \leqslant 17$ (Borodin, 1997).

Madaras (2000) proved that every triangulated 3-polytope with $\delta \geqslant 4$ satisfies $w_4 \leqslant 31$ and constructed such a 3-polytope with $w_4 = 27$.

We improve the Madaras bound $w_4 \leqslant 31$ to the sharp bound $w_4 \leqslant 27$.

Keywords: Plane graph, structural property, normal plane map, 4-path.

1 Introduction

The degree of a vertex or face x in a graph G, that is the number of edges incident with x, is denoted by $d(x)$. A *k-vertex* is a vertex v with $d(v) = k$. By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-face f satisfies $d(f) \geqslant k$, etc.

Let $\delta(G)$ be the minimum vertex degree, and $w_k(G)$ be the minimum degree sum (called *weight*) of a path on k vertices (*k-path*) in a graph G.

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By $\mathbf{G}_\delta$ denote the class of plane graphs G with $\delta(G) \geqslant \delta$; it is trivial that $\delta \leqslant 5$. The subset of 3-connected graphs in $\mathbf{G}_\delta$ is denoted by $\mathbf{P}_\delta$. Given k, by w_k denote the maximal weight of k-paths over $G \in \mathbf{G}_\delta$ or $P \in \mathbf{P}_\delta$.

Already in 1904, Wernicke [40] proved that every $P \in \mathbf{P_5}$ satisfies $w_2(P) \leqslant 11$, which is tight. It follows from Lebesgue's [31] results of 1940 that $P \in \mathbf{P_3}$ implies $w_2(G) \leqslant 14$, which was improved in 1955 by Kotzig [30] to the tight bound $w_2 \leqslant 13$. In 1972, Erdős (see [20]) conjectured that Kotzig's bound $w_2 \leqslant 13$ holds also in $\mathbf{G_3}$. Barnette (see [20]) announced to have proved this conjecture, but the proof has never appeared in print. The first published proof of Erdős' conjecture is due to Borodin [4].

A number of sharp upper bounds on w_2 have been obtained as lemmas in numerous papers on coloring *sparse* planar graphs (for a survey see Borodin [10]). A traditional measure of sparseness of a planar graph G is its *girth* $g(G)$, which is the length of the shortest cycle in G. Another measure, suggested by Erdős (see [20]), is the absence of cycles of length from 4 to a certain constant. More generally, given a set S of integers, a graph is S-*free* if it has no cycle with length from S.

The first results of this type were known already to Lebesgue's [31] for $\mathbf{P_3}$: if $g \geqslant 4$, then $w_2 \leqslant 8$, and if $g \geqslant 5$, then $w_2 = 6$.

As for $\mathbf{G_2}$, we note that $\delta(K_{2,t}) = 2$ and $w_2(K_{2,t}) = t + 2$, so w_2 is unbounded in $\mathbf{G_2}$ if $g \leqslant 4$. In addition to forbidding certain collections S of cycle lengths, another way to find subclasses of $\mathbf{G_2}$ with bounded w_2 is to impose restrictions on the set of 2-vertices in a graph. For example, forbidding 2-*alternating cycles*, which are cycles $v_1 \dots v_{2k}$ with $d(v_1) = d(v_3) = \dots = d(v_{2k-1}) = 2$, we have $w_2 \leqslant 15$ (Borodin [4]).

The first application of this fact was to show that the total choosability of planar graphs with maximum degree Δ at least 14 equals $\Delta+1$ ([4]). The notion of 2-alternating cycles, along with its more sophisticated analogues, have appeared in dozens of papers, since it sometimes provides crucial reducible configurations in coloring and partition problems (more often, on sparse graphs).

Some other results concerning the structure of edge neighborhoods in plane graphs can be found in [2, 5–7, 10, 14, 15, 18, 28, 29].

So far, not many precise results have been obtained on w_k with $k \geqslant 3$. Back in 1922, Franklin [19] strengthened Wernicke's bound $w_2 \leqslant 11$ for $\mathbf{P_5}$ in [40] to $w_3(G) \leqslant 17$. Both bounds 11 and 17 are sharp, as shown by putting a vertex inside each face of the dodecahedron and joining it with the five boundary vertices.

In 1993, Ando, Iwasaki, and Kaneko [3] proved that $w_3 \leqslant 21$ holds in $\mathbf{P_3}$, which is sharp due to the Jendrol' construction in [23]. In 1997, Borodin [8] showed that $\mathbf{P_3}$ restricted to the graphs with $w_2 \geqslant 8$ (in particular, to $\mathbf{P_4}$) satisfies $w_3 \leqslant 17$, which extends Franklin's bound [19].

In 1996, Jendrol' and Madaras [27] proved $w_4 \leqslant 23$ in $\mathbf{P_5}$, which bound was extended in 2003 by Mohar, Škrekovski, and Voss [37] to the part of $\mathbf{P_4}$ satisfying $w_2 \geqslant 9$. For the triangulations with $\delta = 5$, Madaras [33] proved in 2000 that $w_5 \leqslant 29$. Both bounds 23 and 29 are sharp.

For 4-connected planar graphs with at least k vertices and any natural number k, Mohar [36] nicely proved a sharp bound $w_k \leqslant 6k-1$, using Tutte's theorem [39] of 1956 that such graphs are hamiltonian.

We now consider sharp upper bounds on w_3 for graphs in $\mathbf{P_2}$ with given girth g. It is an old folkloric fact that $g(G) \geqslant 16$ implies $w_3(G) = 6$, which probably first appeared in print in Nešetřil, Raspaud, and Sopena [38].

Recently, Jendrol' and Maceková [25] proved $w_3 \leqslant 7$ if $g \geqslant 10$ and $w_3 \leqslant 9$ if $g \geqslant 8$. Aksenov, Borodin, and Ivanova [1] proved $w_3 \leqslant 9$ for $g = 7$.

As observed in [25], $w_3 = \infty$ if $g \leqslant 6$, but if we forbid vertices with two neighbors of degree 2, that is consider $(2, \infty, 2)$-*free* planar graphs, then $w_3 \leqslant 10$ for $g \geqslant 5$. The sharpness of this bound 10 was later on confirmed in [1], where it was also proved that $w_3 \leqslant 9$ for $g = 6$. Jendrol', Maceková, and Soták [26] soon after proved $w_3 \leqslant 12$ if $g = 4$ (also for $(2, \infty, 2)$-free graphs in $\mathbf{P_2}$).

Some other structural results on 3^+-paths in plane graphs can be found in [1, 3, 8, 9, 11–13, 16–18, 21–28, 32–38] and the informative survey by Jendrol' and Voss [29].

In 2000, Madaras [33] proved for triangulations with $\delta = 4$ that $w_4 \leqslant 31$ and gave a construction with $w_4 = 27$ by putting a 3-cycle $A_1A_3A_3$ into each face $Z_1Z_2Z_3$ of the icosahedron followed by adding the edges A_iZ_j whenever $1 \leqslant i, j \leqslant 3$ and $i \neq j$.

The purpose of our paper is to improve the bound $w_4 \leqslant 31$ by Madaras [33] to the sharp bound $w_4 \leqslant 27$.

Theorem 1. *Every plane triangulation with $\delta \geqslant 4$ has a 4-path of weight at most 27, which is tight.*

2 Proving Theorem 1

Proof of Theorem 1. Suppose that G is a counterexample to Theorem 1. In the course of the proof we should take care that a hypothetic 4-path $P_4 = v_1v_2v_3v_4$ with $w(P_4) \leqslant 27$ in G would not degenerate into a 3-cycle, which happens when v_1 coincides with v_4.

A *k-component* c_k is a maximal connected subgraph of G that consists of k vertices of degree at most 5. Clearly, any c_1 is simply a 5^--vertex, c_2 consists of two adjacent vertices, and $c_3 = uvw$ is either a 3-path (when there is no edge uw in G) or a 3-cycle. Furthermore, the following lemma shows, in particular, that if a 3-component c_3 is a cycle, then it is the boundary of a 3-face.

Lemma 2. *No separating 3-cycle in G consists of three 5^--vertices or has two 4-vertices.*

Proof of Lemma 2. Suppose G has a separating 3-cycle $S = uvw$, which means that at least one vertex of G lies inside S and at least one outside S. Note that u has at least one neighbor inside S and at least one outside S, for otherwise $\{v, w\}$ is a separating set in G, contrary to the 3-connectedness of G. The same is true for v and w.

First suppose that S consists of 5^--vertices, which implies that at most nine edges join S to vertices not on S. By symmetry, we can assume that at most four edges lead from S inside (rather than outside) S. Again by symmetry, we can assume that each of

u and v has precisely one neighbor inside S. Since G is a triangulation, there are 3-faces uvx, vwx, and wxu inside S. This implies that there is precisely one vertex x, necessarily with $d(x) = 3$, inside S; a contradiction.

Now if $d(u) = d(v) = 4$, then the same argument works, which proves the second part of Lemma 2. $\square$

Lemma 3. *G has no k-component with $k \geqslant 4$.*

Proof of Lemma 3. Suppose on the contrary that there is a connected subgraph H of G on four 5^{-}-vertices. Since G has no 4-path P_4 with $w(P_4) \leqslant 4 \times 5$, it follows that H does not contain a 4-path. In turn, this means that H is a star, with three rays and a 5^{-} vertex as a center. However, the center forms a 3-face with two consecutive neighbors in our triangulation G, which produces a 4-path on H, a contradiction. $\square$

2.1 Discharging

Euler's formula $|V| - |E| + |F| = 2$ for G may be written as

$$\sum_{v \in V} (d(v) - 6) = -12, \tag{1}$$

where V, E, and F are sets of vertices, edges, and faces of G, respectively. By V_{6^+} and V_{5^-} denote the sets of 6^{+}-vertices and 5^{-}-vertices in V, respectively, so (1) can be written as follows.

$$\sum_{v \in V_{6^+}} (d(v) - 6) + \sum_{v \in V_{5^-}} (d(v) - 6) = -12, \tag{2}$$

By MC denote the set of minor components $[mc]$ of G. It follows from Lemma 3 that V_{5^-} is split into minor components, each of which is a k^{-}-component with $k \leqslant 3$.

Every vertex v contributes the *initial charge* $\mu(v) = d(v) - 6$ to (2), so only the charges of 5^{-}-vertices are negative. The *initial charge* of a minor component $[mc]$ is the total initial charge of its vertices. Thus (2) can be rewritten as follows.

$$\sum_{v \in V_{6^+}} \mu(v) + \sum_{[mc] \in MC} \mu([mc]) = -12, \tag{3}$$

Using the properties of M as a counterexample, we define a local redistribution of μ's, preserving their sum, such that the *new charge* μ' is non-negative for all $v \in V_{6^+}$ and $[mc] \in MC$. This will contradict the fact that the sum of the new charges is, by (3), equal to -12.

For $v \in V_{6^+}$, we put $\xi(v) = \frac{d(v)-6}{d(v)}$ and $\overline{\xi}(v) = \min\{\xi(v), \frac{1}{2}\}$. For an integer k with $k \geqslant 6$, we put $\psi(k) = \frac{k-6}{k}$.

Our rules of discharging are as follows (see Fig. 1):

R1. *For each* 3*-face* $f = xuv$ *with* $d(u) \leqslant 5$*,* $d(v) \geqslant 6$*, and* $d(x) \geqslant 6$*, the vertex* u *receives* $\overline{\xi}(v)$ *and* $\overline{\xi}(x)$ *through* f *from* v *and* x*, respectively.*

R2. *For each* 3*-face* $f = xuv$ *with* $d(u) \leqslant 5$*,* $d(v) \leqslant 5$*, and* $d(x) \geqslant 6$*, each of* u *and* v *receives* $\frac{\overline{\xi}(x)}{2}$ *from* x *through* f*, with the following exception:*

(E) if $d(u) = d(v) = 4$ *and there is a* 3*-face* uvw *with* $d(w) = 4$*, then (the* 16^{+}*-vertex)* x *gives* $\frac{1}{2}$ *to each of* u *and* v *through* f*.*

R3. *For each* 3*-face* $f = xuv$ *with* $d(u) = d(v) = 4$ *and* $6 \leqslant d(x) \leqslant 7$*, each of* u *and* v *receives* $\frac{1}{4}$ *from* x *through* f*. In turn,* x *receives* $\frac{1}{12}$ *from each adjacent* 13^{+}*-vertex* y *through each* 3*-face* xyz *such that* $d(z) \geqslant 13$*.*

Concerning the second part of R3, we note that there are at least three 3-faces incident with x and two 13^{+}-vertices, so x receives at least $6 \times \frac{1}{12}$ through such faces.

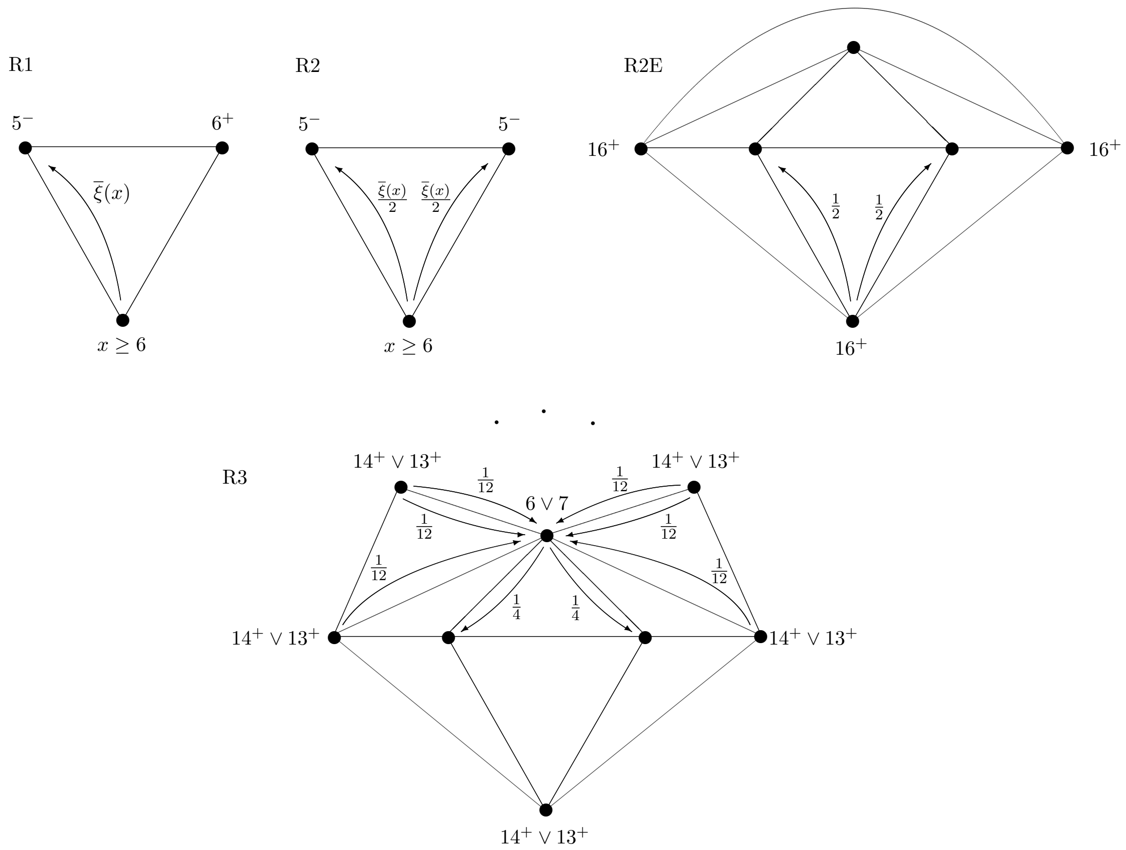

Figure 1: Rules of discharging.

2.2 Checking $\mu'(v) \geqslant 0$ if $d(v) \leqslant 5$ and v forms a 1-component

By definition, v is surrounded by 6^{+}-neighbors, so only R1 is in action. Let $S(v) = \{d_1, \ldots, d_{d(v)}\}$ be the multi-set of degrees of the neighbors of v, where $d_1 \leqslant \ldots \leqslant d_{d(v)}$.

For example, if v has one 6-neighbor, two 7-neighbors, and two 8-neighbors, then $S(v) = \{6, 7, 7, 8, 8\}$. We note that $d_1 + d_2 + d_3 \geqslant 28 - d(v)$, for otherwise G would have a 4-path of weight at most 27, which is impossible.

Case 1. $d(v) = 4$. If $d_3 \geqslant 12$, then $\mu'(v) \geqslant 4 - 6 + 4 \times \psi(12) = 0$ by R1.

If $d_3 = 11$, then $d_2 \geqslant 7$, since otherwise v has two 6-neighbors and an 11-neighbor, which implies a 4-path of weight at most $6 + 6 + 11 < 28 - 4$, a contradiction. So, $\mu'(v) \geqslant -2 + 2\psi(7) + 4 \times \psi(11) = -2 + \frac{2}{7} + \frac{4\times5}{11} > -2 + \frac{22}{11} = 0$, as desired.

If $d_3 = 10$, then $d_1 + d_2 \geqslant 14$, so v receives at least $2\min\{\psi(6) + \psi(8), \psi(7) + \psi(7)\} = \frac{1}{2}$ from two neighbors of smallest degree and at least $4\psi(10) = \frac{8}{5}$ from the other two neighbors. This implies that $\mu'(v) \geqslant -2 + \frac{1}{2} + \frac{8}{5} > 0$.

If $d_3 = 9$, then $d_1 + d_2 \geqslant 15$, so v receives at least $2\min\{\psi(6) + \psi(9), \psi(7) + \psi(8)\} = \frac{2}{3}$ plus at least $4\psi(9) = \frac{4}{3}$ by R1, whence $\mu'(v) \geqslant 0$.

Finally, $d_3 \leqslant 8$ is possible only if $d_1 = \ldots = d_3 = 8$, in which case $\mu(v) \geqslant -2 + 8\psi(8) = 0$, as desired.

Case 2. $d(v) = 5$. Note that $d_3 \geqslant 8$, for otherwise $d_1 + d_2 + d_3 \leqslant 3 \times 7 < 28 - 5$, a contradiction. Hence $\mu'(v) \geqslant 5 - 6 + 6 \times \psi(8) > 0$ by R1, and we are done.

2.3 Checking $\mu'(c_2) \geqslant 0$ for a 2-component $c_2 = uv$

Case 1. $d(u) = d(v) = 4$. Denote the external neighbors of u and v by $z_1, \ldots, z_4$ as shown in Fig. 2a. Due to Lemma 3 applied to z_2 and z_4 combined with the 3-connectedness of G, all z_i are distinct. Let z_k have the smallest degree in $Z = \{z_1, \ldots, z_4\}$, where $d(z_k) \geqslant 6$ by the definition of 2-component. Note that each other vertex in Z has degree at least $20 - d(z_k)$ since $w(P_4) \geqslant 28$ by assumption.

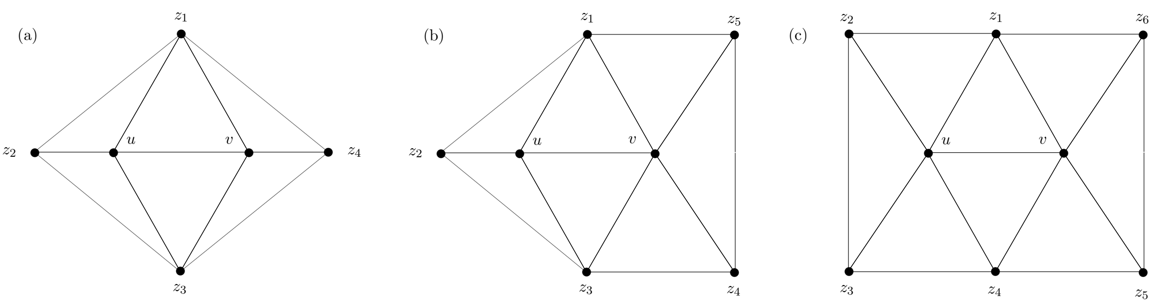

Figure 2: Notation for 2-components.

Subcase 1.1. $k \in \{1, 3\}$. If $d(z_k) \leqslant 7$, then the three 13^+-vertices in $Z - z_k$ together give c_2, through the incident faces, the charge $6 \times \frac{1}{2}$ by R1 and $2 \times \frac{1}{4}$ by R2, while z_k gives $2\xi(v)$ by R1, $2 \times \frac{\xi(v)}{2}$ by R2, and $2 \times \frac{1}{4}$ by R3, so $\mu'(c_2) \geqslant 2(4 - 6) + 6 \times \frac{1}{2} + 4 \times \frac{1}{4} = 0$.

If $d(z_k) = 8$, then c_2 receives the following charges by R1 and R2: $2 \times \frac{1}{4} + 2 \times \frac{1}{8}$ from z_k, $2 \times \frac{1}{2} + 2 \times \frac{1}{4}$ from z_{4-k}, and $4 \times \frac{1}{2}$ from z_2 and z_4 together, which yields $\mu'(c_2) > 0$.

If $d(z_k) = 9$, then we similarly have $\mu'(c_2) \geqslant -4 + 6 \times \frac{5}{11} + 2 \times \frac{5}{22} + 2 \times \frac{1}{3} + 2 \times \frac{1}{6} > 0$. Finally, if $d(z_k) \geqslant 10$, then $\mu'(c_2) \geqslant -4 + 8 \times \frac{2}{5} + 4 \times \frac{1}{5} = 0$.

Subcase 1.2. $k \in \{2, 4\}$. If $d(z_k) \leqslant 8$, then the three 12^+-vertices in Z together give $6 \times \frac{1}{2} + 4 \times \frac{1}{4}$ to c_2 by R1 and R2, whence $\mu'(c_2) \geqslant 0$.

If $d(z_k) = 9$, then we have $\mu'(c_2) \geqslant -4 + 6 \times \frac{5}{11} + 2 \times \frac{5}{22} + 2 \times \frac{1}{3} > 0$, and if $d(z_k) \geqslant 10$, then again $\mu'(c_2) \geqslant -4 + 8 \times \frac{2}{5} + 4 \times \frac{1}{5} = 0$.

Case 2. $d(u) = 4$, $d(v) = 5$. Here, u and v have neighbors $z_1, \ldots, z_5$ shown in Fig. 2b, and c_2 receives charge from Z by R1 and R2 through twelve faces. However, now Lemma 2 is not applied, thus the edge uv can belong to a separating 3-cycle z_2uv, which is only possible when $z_2 \in \{z_4, z_5\}$. If so, then z_2 appears twice in Z and can have a relatively small degree, say 6 or 7, with no contradiction (since we are looking for a light 4-path rather than a light 3-cycle). Still, c_2 has at least three neighbors other than z_k.

This time, we can ignore the donations from z_k, since each neighbor other than z_k has degree at least $19 - d(z_k) \geqslant 10$. Indeed, otherwise we would have a 4-path of weight at most $9 + 4 + 5 + 9$ on $\{z_k, u, v, z_l\}$ with $z_l \neq z_k$, which is impossible. Therefore, c_2 has at least three 10^+-neighbors, and each of them gives at least $\frac{2}{5}$ to c_2 through each incident face by R1 and R2. Altogether, such donations happen at least eight times, which results in $\mu'(c_2) \geqslant 4 - 6 + 5 - 6 + 8 \times \frac{2}{5} > 0$.

Case 3. $d(u) = d(v) = 5$. Now u and v have six neighbors $z_1, \ldots, z_6$ shown in Fig. 2c. Again Lemma 2 is not applied, and it can happen that $\{z_2, z_3\} \cap \{z_5, z_6\} = z_k$. Since z_k can appear in the multi-set Z at most twice, it follows that c_2 has at least four neighbors different from z_k and receives the charge by R1 and R2 from them at least ten times through the incident faces. Indeed, there are fourteen faces incident with $\{u, v\}$ and at most four of them may be incident with z_k. Since each vertex in $Z - z_k$ this time has degree at least $18 - d(z_k) \geqslant 9$, it follows that $\mu'(c_2) = 5 - 6 + 5 - 6 + 10 \times \frac{1}{3} > 0$, as desired.

2.4 Checking $\mu'(c_3) \geqslant 0$ for a 3-component $c_3 = xuv$

By definition, c_3 is surrounded by 6^+-neighbors, say z_i.

Case 1. $d(x) = d(u) = d(v) = 4$. Now all z_i's are 16^+-vertices, regardless of the presence or absence of the edge xv in G. If $xv \in E(G)$, then c_3 is a 3-face due to Lemma 2 and receives $12 \times \frac{1}{2}$ by R1 and R2E, whence $\mu'(c_3) = 3(4 - 6) + 6 = 0$. Otherwise, c_3 is a 3-path and hence receives $12 \times \frac{1}{2}$ by R2, so again $\mu'(c_3) = -6 + 6 = 0$.

Case 2. $d(x) = 5$. Now all z_i's are 13^+-vertices, for otherwise we would have a 4-path of weight at most $5 + 5 + 5 + 12$ on $\{u, v, x, z_i\}$. Hence c_3 receives at least $11 \times \frac{1}{2}$ by R1 and R2, whence $\mu'(c_3) \geqslant -1 - 2 - 2 + \frac{11}{2} > 0$.

2.5 Checking $\mu'(v) \geqslant 0$ for a 6^+-vertex v

Case 1. $d(v) = 6$. If v does not participate in R3, then $\mu'(v) = \mu(v) = 0$. Otherwise, v has four 14^+-neighbors and receives $6 \times \frac{1}{12}$ from them by R3. Also, v gives $\frac{1}{2}$ to the only 2-component by R3, so again $\mu'(v) = 0$.

Case 2. $d(v) = 7$. If v does not participate in R3, then $\mu'(v) \geqslant 1 - 7 \times \frac{1}{7} = 0$ by R1 and R2. Otherwise, v has five 13^+-neighbors, receives $8 \times \frac{1}{12}$ from them by R3, and gives $\frac{1}{2}$ to the 2-component by R3 and $3 \times \frac{1}{7}$ by R1 and R2, whence $\mu'(v) > 1 - \frac{1}{2} - \frac{3}{7} > 0$.

Case 3. $8 \leqslant d(v) \leqslant 15$. Now v can give charge away by R1, R2 and, if $d(v) \geqslant 13$, by R3. Since the donation through each incident face is at most $\xi(v)$, it follows that $\mu'(v) \geqslant d(v) - 6 - d(v) \times \xi(v) = 0$.

Case 4. $d(v) \geqslant 16$. Now v can participate in all the rules R1–R3. Since the donation through each incident face is at most $\frac{1}{2}$ except for R2E, and the donation of $4 \times \frac{1}{2}$ through the three consecutive faces in R2E combined with R1 can be regarded as giving $\frac{2}{3}$ per face, we have $\mu'(v) \geqslant d(v) - 6 - d(v) \times \frac{2}{3} = \frac{d(v)-18}{3}$. Thus we are already done for $d(v) \geqslant 18$.

Suppose $16 \leqslant d(v) \leqslant 17$. As we know, the (averaged) donation of $\frac{2}{3}$ through a face appears only in R2E, while the other, "individual," donations by R1–R3 are of at most $\frac{1}{2}$. Furthermore, the faces receiving $\frac{2}{3}$ on the average form disjoint triples.

If $d(v) = 17$, then at most fifteen faces conduct $\frac{2}{3}$ from v each after averaging, so $\mu'(v) \geqslant 17 - 6 - 15 \times \frac{2}{3} - 2 \times \frac{1}{2} = 0$.

Finally, suppose $d(v) = 16$. If at most twelve faces conduct $\frac{2}{3}$ each after averaging, then $\mu'(v) \geqslant 16 - 6 - 12 \times \frac{2}{3} - 4 \times \frac{1}{2} = 0$. If each of fifteen faces conducts $\frac{2}{3}$, then the sixteenth face consists of 16^+-vertices and does not receive any charge from v by R1–R3. This implies that $\mu'(v) \geqslant 16 - 6 - 15 \times \frac{2}{3} = 0$.

Thus we have proved that $\mu'(v) \geqslant 0$ for every $v \in V_{6+}$ and $\mu'([mc]) \geqslant 0$ for every $[mc] \in MC$, which contradicts (3):

$$0 \leqslant \sum_{v \in V_{6+}} \mu'(v) + \sum_{[mc] \in MC} \mu'([mc]) = \sum_{v \in V_{6+}} \mu(v) + \sum_{[mc] \in MC} \mu([mc]) = -12.$$

This completes the proof of Theorem 1. □

References

[1] V.A. Aksenov, O.V. Borodin, A.O. Ivanova, Weight of 3-paths in sparse plane graphs, Electronic J. Combin., 22, 3 (2015) Paper #P3.28.

[2] V.A. Aksenov, O.V. Borodin, L.S. Mel'nikov, G. Sabidussi, B. Toft, M. Stiebitz, On deeply asymmetric planar graphs, J. Combin. Theory B, 95 (2005) 68–78.

[3] K. Ando, S. Iwasaki, A. Kaneko, Every 3-connected planar graph has a connected subgraph with small degree sum (Japanese), Annual Meeting of Mathematical Society of Japan (1993).

[4] O.V. Borodin, On the total coloring of planar graphs, J. Reine Angew. Math., 394 (1989) 180–185.

[5] O.V. Borodin, Joint generalization of the theorems of Lebesgue and Kotzig on the combinatorics of planar maps (Russian), Diskret. Mat., 3, no. 4 (1991) 24–27.

[6] O.V. Borodin, Joint extension of two Kotzig's theorems on 3-polytopes, Combinatorica, 13, no. 1 (1993) 121–125.

[7] O.V. Borodin, More about the weight of edges in planar graphs, Tatra Mountains Math. Publ., 9 (1996) 11–14.

[8] O.V. Borodin, Minimal vertex degree sum of a 3-path in plane maps, Discuss. Math. Graph Theory, 17, no. 2 (1997) 279–284.

[9] O.V. Borodin, Triangulated 3-polytopes without faces of low weight, Discrete Math., 186 (1998) 281–285.

[10] O.V. Borodin, Colorings of plane graphs: a survey, Discrete Math., 313, no. 4 (2013) 517–539.

[11] O.V. Borodin, A.O. Ivanova, Describing tight descriptions of 3-paths in triangle-free normal plane maps, Discrete Math., 338 (2015) 1947–1952.

[12] O.V. Borodin, A.O. Ivanova, T.R. Jensen, A.V. Kostochka, M.P. Yancey, Describing 3-paths in normal plane maps, Discrete Math., 313, no. 23 (2013) 2702–2711.

[13] O.V. Borodin, A.O. Ivanova, A. V. Kostochka, Describing faces in plane triangulations, Discrete Math., 319 (2014) 47–61.

[14] O.V. Borodin, A.V. Kostochka, N.N. Sheikh, Gexin Yu, M-degrees of quadrangle-free planar graphs, J. Graph Theory, 60, no. 1 (2009) 80–85.

[15] O.V. Borodin, A.V. Kostochka, D.R. Woodall, List edge and list total colourings of multigraphs, J. Combin. Theory Ser. B, 71, no. 2 (1997) 184–204.

[16] I. Fabrici, E. Hexel, S. Jendrol', H. Walther, On vertex-degree restricted paths in polyhedral graphs, Discrete Math., 212 (2000) 61–73.

[17] I. Fabrici, J. Harant, S. Jendrol', Paths of low weight in planar graph, Discuss. Math. Graph Theory, 28 (2008) 121–135.

[18] B. Ferencová, T. Madaras, Light graphs in families of polyhedral graphs with prescribed minimum degree, face size, edge and dual edge weight, Discrete Math., 310 (2010) 1661–1675.

[19] Ph. Franklin, The four-color problem, Amer. J. Math., 44 (1922) 225–236.

[20] B. Grünbaum, New views on some old questions of combinatorial geometry, Int. Teorie Combinatorie, Rome, (1973), 1 (1976) 451–468.

[21] J. Harant, S. Jendrol', M. Tkáč, On 3-connected plane graphs without triangular faces, J. Combin. Theory Ser. B, 77 (1999) 150–161.

[22] E. Hexel, H. Walther, On vertex-degree restricted paths in 4-connected planar graphs, Tatra Mt. Math. Publ., 18 (1999) 1–13.

[23] S. Jendrol', Paths with restricted degrees of their vertices in planar graphs, Czechoslovak Math. J., 49, 124 (1999) 481–490.

[24] S. Jendrol', A structural property of convex 3-polytopes, Geom. Dedicata, 68 (1997) 91–99.

[25] S. Jendrol', M. Maceková, Describing short paths in plane graphs of girth at least 5, Discrete Math., 338 (2015) 149–158.

[26] S. Jendrol', M. Maceková, R. Soták, Note on 3-paths in plane graphs of girth 4, Discrete Math., 338 (2015) 1643–1648.

[27] S. Jendrol', T. Madaras, On light subgraphs in plane graphs of minimum degree five, Discuss. Math. Graph Theory, 16 (1996) 207–217.

[28] S. Jendrol', H.-J. Voss, Light paths in large polyhedral maps with prescribed minimum degree, Australas. J. Combin., 25 (2002) 79–102.

[29] S. Jendrol', H.-J. Voss, Light subgraphs of graphs embedded in the plane – a survey, Discrete Math., 313 (2013) 406–421.

[30] A. Kotzig, Contribution to the theory of Eulerian polyhedra (Slovak), Mat. Čas., 5 (1955) 101–113.

[31] H. Lebesgue, Quelques conséquences simples de la formule d'Euler, J. Math. Pures Appl., 19 (1940) 27–43.

[32] T. Madaras, Note on weights of paths in polyhedral graphs, Discrete Math., 203 (1999) 267–269.

[33] T. Madaras, Note on the weight of paths in plane triangulations of minimum degree 4 and 5, Discuss. Math. Graph Theory, 20, no. 2 (2000) 173–180.

[34] T. Madaras, On the structure of plane graphs of minimum face size 5, Discuss. Math. Graph Theory, 24, no. 3 (2004) 403–411.

[35] T. Madaras, Two variations of Franklin's theorem, Tatra Mt. Math. Publ., 36 (2007) 61–70.

[36] B. Mohar, Light paths in 4-connected graphs in the plane and other surfaces, J. Graph Theory, 34 (2000) 170–179.

[37] B. Mohar, R. Škrekovski, H.-J. Voss, Light subgraphs in planar graphs of minimum degree 4 and edge-degree 9, J. Graph Theory, 44 (2003) 261–295.

[38] J. Nešetřil, A. Raspaud, E. Sopena, Colorings and girth of oriented planar graphs, Discrete Math., 165–166 (1997) 519–530.

[39] W.T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc., 82 (1956) 99–116.

[40] P. Wernicke, Über den kartographischen Vierfarbensatz, Math. Ann., 58 (1904) 413–426.