Transversals in 4-Uniform Hypergraphs

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Abstract

Let H be a 4-uniform hypergraph on n vertices. The transversal number $\tau(H)$ of H is the minimum number of vertices that intersect every edge. The result in [J. Combin. Theory Ser. B 50 (1990), 129–133] by Lai and Chang implies that $\tau(H) \leqslant 7n/18$ when H is 3-regular. The main result in [Combinatorica 27 (2007), 473–487] by Thomassé and Yeo implies an improved bound of $\tau(H) \leqslant 8n/21$. We provide a further improvement and prove that $\tau(H) \leqslant 3n/8$, which is best possible due to a hypergraph of order eight. More generally, we show that if H is a 4-uniform hypergraph on n vertices and m edges with maximum degree $\Delta(H) \leqslant 3$, then $\tau(H) \leqslant n/4 + m/6$, which proves a known conjecture. We show that an easy corollary of our main result is that if H is a 4-uniform hypergraph with n vertices and n edges, then $\tau(H) \leqslant \frac{3}{7}n$, which was the main result of the Thomassé-Yeo paper [Combinatorica 27 (2007), 473–487].

Keywords: Transversal; Hypergraph.
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1 Notation and Definitions

In this paper we continue the study of transversals in hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V, E)$ is a finite set $V = V(H)$ of elements, called *vertices*, together with a finite multiset $E = E(H)$ of subsets of V, called *hyperedges* or simply *edges*. The *order* of H is $n(H) = |V|$ and the *size* of H is $m(H) = |E|$.

A k-*edge* in H is an edge of size k. The hypergraph H is said to be k-*uniform* if every edge of H is a k-edge. Every (simple) graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs. For $i \geqslant 2$, we denote the number of edges in H of size i by $e_i(H)$. The *degree* of a vertex v in H, denoted by $d_H(v)$ or simply by $d(v)$ if H is clear from the context, is the number of edges of H which contain v. The maximum degree among the vertices of H is denoted by $\Delta(H)$. We say that two edges in H *overlap* if they intersect in at least two vertices.

Two vertices x and y of H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq e$. The *neighborhood* of a vertex v in H, denoted $N_H(v)$ or simply $N(v)$ if H is clear from the context, is the set of all vertices different from v that are adjacent to v. Two vertices x and y of H are *connected* if there is a sequence $x = v_0, v_1, v_2 \ldots, v_k = y$ of vertices of H in which v_{i-1} is adjacent to v_i for $i = 1, 2, \ldots, k$. A *connected hypergraph* is a hypergraph in which every pair of vertices are connected. A maximal connected subhypergraph of H is a *component* of H. Thus, no edge in H contains vertices from different components.

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if T has a nonempty intersection with every edge of H. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. A transversal of size $\tau(H)$ is called a $\tau(H)$-set. Transversals in hypergraphs are well studied in the literature (see, for example, [1, 2, 3, 4, 5, 6, 7, 9, 11]).

Given a hypergraph H and subsets $X, Y \subseteq V(H)$ of vertices, we let $H(X, Y)$ denote the hypergraph obtained by deleting all vertices in $X \cup Y$ from H and removing all edges containing vertices from X and removing the vertices in Y from any remaining edges. When we use the definition $H(X, Y)$ we furthermore assume that no edges of size zero are created. That is, there is no edge $e \in E(H)$ such that $V(e) \subseteq Y \setminus X$. In this case we note that if add X to any $\tau(H(X, Y))$-set, then we get a transversal of H, implying that $\tau(H) \leqslant |X| + \tau(H(X, Y))$. We will often use this fact throughout the paper.

A *total dominating set*, also called a TD-set, of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G. Total domination in graphs is now well studied in graph theory. The literature on the subject has been surveyed and detailed in the recent book [8].

2 The Family, $\mathcal{B}$, of Hypergraphs

In this section, we define a family, $\mathcal{B}$, of "bad" hypergraphs as follows.

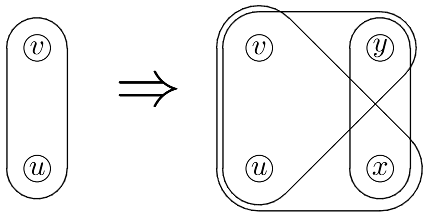


Figure 1: An illustration of Step (B) in Definition 1.

Definition 1. *Let $\mathcal{B}$ be the class of* ***bad hypergraphs*** *defined as exactly those that can be generated using the operations (A)-(D) below.*

(A): *Let H_2 be the hypergraph with two vertices $\{x, y\}$ and one edge $\{x, y\}$ and let H_2 belong to $\mathcal{B}$.*

(B): *Given any $B' \in \mathcal{B}$ containing a 2-edge $\{u, v\}$, define B as follows. Let $V(B) = V(B') \cup \{x, y\}$ and let $E(B) = E(B') \cup \{\{u, v, x\}, \{u, v, y\}, \{x, y\}\} \setminus \{u, v\}$. Now add B to $\mathcal{B}$.*

(C): *Given any $B' \in \mathcal{B}$ containing a 3-edge $\{u, v, w\}$, define B as follows. Let $V(B) = V(B') \cup \{x, y\}$ and let $E(B) = E(B') \cup \{\{u, v, w, x\}, \{u, v, w, y\}, \{x, y\}\} \setminus \{u, v, w\}$. Now add B to $\mathcal{B}$.*

(D): *Given any $B_1, B_2 \in \mathcal{B}$, such that B_i contains a 2-edge $\{u_i, v_i\}$, for $i = 1, 2$, define B as follows. Let $V(B) = V(B_1) \cup V(B_2) \cup \{x\}$ and let $E(B) = E(B_1) \cup E(B_2) \cup \{\{u_1, v_1, x\}, \{u_2, v_2, x\}, \{u_1, v_1, u_2, v_2\}\} \setminus \{\{u_1, v_1\}, \{u_2, v_2\}\}$. Now add B to $\mathcal{B}$.*

We call the two vertices, $\{x, y\}$, added in step (A) above an **(A)-*pair***. *Note that in operations (B) and (C), $\{a, b\}$ is an (A)-pair in B if and only if it is an (A)-pair in B'. Analogously in operation (D), $\{a, b\}$ is an (A)-pair in B if and only if it is an (A)-pair in B_1 or B_2.*

The hypergraph $B \in \mathcal{B}$ created by applying Step (B) in Definition 1 to the hypergraph H_2 is shown in Figure 1, while Figure 1 illustrates Step (C) and Step (D) in Definition 1.

We shall need the following definition.

Definition 2. *If H is a hypergraph, then let $b(H)$ denote the number of components in H that belong to $\mathcal{B}$. Further for $i \geqslant 0$, let $b^i(H)$ denote the maximum number of vertex disjoint subhypergraphs in H which are isomorphic to hypergraphs in $\mathcal{B}$ and which are intersected by exactly i other edges of H.*

3 Main Results

Let $\mathcal{H}$ denote the class of all hypergraphs where all edges have size at most four and at least two and with maximum degree at most three. We shall prove the following result a proof of which is presented in Section 5.

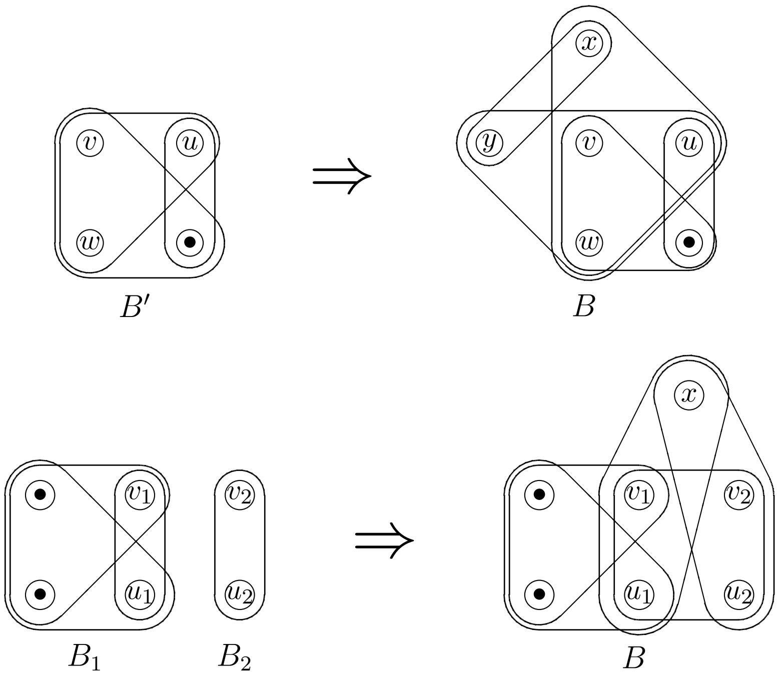


Figure 2: An illustration of Steps (C) and (D) in Definition 1.

Theorem 3. *If $H \in \mathcal{H}$, then*

$$24\tau(H) \leqslant 6n(H) + 4e_4(H) + 6e_3(H) + 10e_2(H) + 2b(H) + b^1(H).$$

Furthermore if $b^1(H)$ is odd, then the above inequality is strict.

Let H be a 4-uniform hypergraph with $\Delta(H) \leqslant 3$. Since every hypergraph in $\mathcal{B}$ contains a 2-edge or a 3-edge, we note that $b(H) = b^1(H) = 0$. By the 4-uniformity of H, we have that $e_2(H) = e_3(H) = 0$ and $e_4(H) = m(H)$. Therefore, Theorem 3 implies that $24\tau(H) \leqslant 6n(H) + 4m(H)$. Hence as an immediate consequence of Theorem 3 we have our two main results.

Theorem 4. *If H is a 4-uniform hypergraph with $\Delta(H) \leqslant 3$, then $\tau(H) \leqslant \frac{n(H)}{4} + \frac{m(H)}{6}$.*

Theorem 5. *If H is a 3-regular 4-uniform hypergraph, then $\tau(H) \leqslant \frac{3n(H)}{8}$.*

Theorem 4 and Theorem 5 are best possible due to the hypergraph, H_8, depicted in Figure 3, of order $n = 8$, size $m = 6$, satisfying $\tau(H_8) = 3 = \frac{3n}{8} = \frac{n}{4} + \frac{m}{6}$.

As an application of our main result, Theorem 4, we can easily prove the following strong theorem which was first proved in [11]. In Section 6 we provide a half page proof of this result (which uses a one page proof from [2]).

Theorem 6. ([11]) *If H is a 4-uniform hypergraph with n vertices and n edges, then $\tau(H) \leqslant \frac{3}{7}n$.*

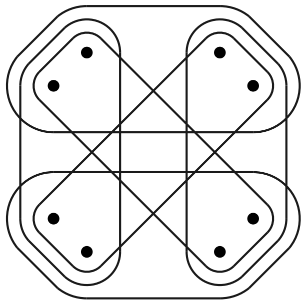
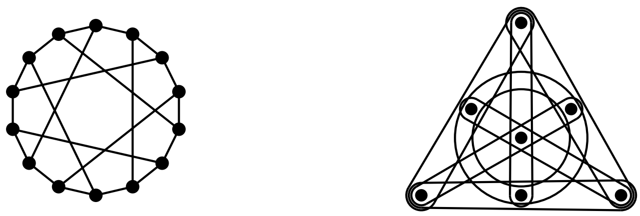


Figure 3: The 3-regular 4-uniform hypergraph, H_8.

The Heawood graph, shown in Figure 4(a), is the unique 6-cage. The bipartite complement of the Heawood graph is the bipartite graph formed by taking the two partite sets of the Heawood graph and joining a vertex from one partite set to a vertex from the other partite set by an edge whenever they are not joined in the Heawood graph. The bipartite complement of the Heawood graph can also be seen as the incidence bipartite graph of the complement of the Fano plane which is shown in Figure 4(b).



(a) (b)

Figure 4: The Heawood graph and the Fano Plane.

As an application of Theorem 6, we prove the following result. We remark that the first statement of Theorem 7 was first proved in [11]. Recall that $\delta(G)$ denotes the minimum degree of a graph G. A proof of Theorem 7 is discussed in Section 6.

Theorem 7. *If G is a connected graph of order n with $\delta(G) \geqslant 4$, then $\gamma_t(G) \leqslant 3n/7$, with equality if and only if G is the bipartite complement of the Heawood Graph.*

3.1 Motivation

There has been much interest in determining upper bounds on the transversal number of a 4-uniform hypergraph. In particular, as a consequence of more general results we have the Chvátal-McDiarmid bound, the improved Lai-Chang bound and the further improved Thomassé-Yeo bound. These bounds are summarized in Theorem 8.

Theorem 8. *Let H be a 3-regular 4-uniform hypergraph on n vertices. Then the following bounds on $\tau(H)$ have been established.*

(a) $\tau(H) \leqslant \frac{5}{12}n$ (Chvátal, McDiarmid [2]).

(b) $\tau(H) \leqslant \frac{7}{18}n$ (Lai, Chang [9]).
(c) $\tau(H) \leqslant \frac{8}{21}n$ (Thomassé, Yeo [11]).

In this paper, we provide a further improvement on the bounds in Theorem 8 as shown in our main result, Theorem 5, by proving that $\tau(H) \leqslant \frac{3}{8}n$. As mentioned above our bound is best possible, due to a hypergraph on eight vertices.

Motivated by comments and questions posed by Douglas West [13], the authors in [7] considered the following slightly more general question.

Question 9. *For $k \geqslant 2$, let H be a hypergraph on n vertices with m edges and with every edge of size at least k. Is it true that $\tau(H) \leqslant \frac{n}{k} + \frac{m}{6}$ holds for all k?*

It is shown in [7] that Question 9 holds for $k = 2$ and a characterization of the extremal hypergraphs is given. Chvátal and McDiarmid [2] proved that Question 9 holds for $k = 3$ and the extremal hypergraphs are characterized in [7]. Question 9 is not always true when $k \geqslant 4$ as shown in [7]. However the family of counterexamples presented in [7] all satisfy $\Delta(H) \geqslant 4$. The authors in [7] pose the following conjecture.

Conjecture 10. ([7]) *For all $k \geqslant 2$, if H is a k-uniform hypergraph on n vertices with m edges satisfying $\Delta(H) \leqslant 3$, then $\tau(H) \leqslant \frac{n}{k} + \frac{m}{6}$.*

As remarked earlier, Conjecture 10 always holds when $k \in \{2, 3\}$ (with no restriction on the maximum degree). In this paper we prove that it holds for $k = 4$ and in [10] it is proved for the case when $k = 6$. In [7] it is furthermore shown that Conjecture 10 is true when $\Delta(H) \leqslant 2$. However, Conjecture 10 appears to be a challenging conjecture for general k and for $\Delta(H) = 3$.

In this paper, we prove that Conjecture 10 is true for 4-uniform hypergraphs as shown in our main result, Theorem 4. As a consequence of this result we show that if H is a 4-uniform hypergraph with n vertices and n edges, then $\tau(H) \leqslant \frac{3}{7}n$, which was the main result of the Thomassé-Yeo paper [11].

The new techniques introduced in this paper are furthermore used in [10] where a proof of the long standing conjecture due to Tuza and Vestergaard [12] that if H is a 3-regular 6-uniform hypergraph, then $\tau(H) \leqslant \frac{1}{4}n$ is given.

4 Preliminary Lemma

We need the following lemma which proves a number of properties of the hypergraphs that belong to the family $\mathcal{B}$.

Lemma 11. *The following properties holds for all $B \in \mathcal{B}$.*

(i): *If B was created from B' in Step (B) or (C) in Definition 1, then $\tau(B) = \tau(B') + 1$.*

(ii): *If B was created from B_1 and B_2 in Step (D) in Definition 1, then $\tau(B) = \tau(B_1) + \tau(B_2)$.*

(iii): $\tau(B) = (6n(B) + 4e_4(B) + 6e_3(B) + 10e_2(B) + 2)/24$.

(iv): *All* (A)*-pairs are vertex disjoint (recall the definition of* (A)*-pairs, below the definition of Steps (A)-(D)).*

(v): *For all* $e \in E(B)$ *we have* $\tau(B - e) = \tau(B) - 1$.

(vi): *For all* $s \in V(B)$ *there exists a* $\tau(B)$*-set containing* s.

(vii): *For all* $s, t \in V(B)$ *there exists a* $\tau(B)$*-set containing both* s *and* t *if and only if* $\{s, t\}$ *is not an* (A)*-pair.*

(viii): *Let* $\{s_1, t_1\}$, $\{s_2, t_2\}$ *and* $\{s_3, t_3\}$ *be three subsets of* $V(B)$*. Then there exists a* $\tau(B)$*-set in* B *intersecting all of these three sets.*

(ix): *There is no* 4*-edge in* B *intersecting three or more* 2*-edges.*

(x): *If* $B \neq H_2$*, then* $\delta(B) \geqslant 2$.

(xi): *If* $d_B(x) = 2$*, then* x *is contained in a* 3*-edge or a* 2*-edge in* B.

(xii): *If* $B \neq H_2$ *and* $e_2(B) > 0$*, then* B *contains either two overlapping* 3*-edges or two* 4*-edges,* e_1 *and* e_2*, with* $|V(e_1) \cap V(e_2)| = 3$.

(xiii): *If* $B \neq H_2$ *and* B *does not contain two* 4*-edges intersecting in three vertices, then every* 2*-edge in* B *intersects two overlapping* 3*-edges.*

Proof. **(i):** Suppose that B was created from B' in Step (B) in Definition 1. Name the vertices as in Definition 1 and let S be a $\tau(B)$-set. Since the set S intersects the 2-edge $\{x, y\}$, we note that $|S \cap \{x, y\}| \geqslant 1$. If $|S \cap \{x, y\}| = 2$, then $(S \cup \{u\}) \setminus \{x\}$ is a $\tau(B)$-set. Hence we may choose the set S so that $|S \cap \{x, y\}| = 1$. This implies that $|S \cap \{u, v\}| \geqslant 1$ and that $S \setminus \{x, y\}$ is a transversal in B' of size $|S| - 1$, and so $\tau(B') \leqslant \tau(B) - 1$. Since every transversal in B' can be extended to a transversal in B by adding to it the vertex x, we have that $\tau(B) \leqslant \tau(B') + 1$. Consequently, $\tau(B) = \tau(B') + 1$, as desired. If B was created from B' in Step (C) in Definition 1, then analogously to when B was created in Step (B), we have that $\tau(B) = \tau(B') + 1$.

(ii): Suppose that B was created from B_1 and B_2 in Step (D). Name the vertices as in Definition 1 and let S be a $\tau(B)$-set. Suppose $x \in S$. Since $S \cap \{u_1, v_1, u_2, v_2\} \neq \varnothing$, we may assume, renaming vertices if necessary, that $u_1 \in S$. Then, $(S \cup \{u_2\}) \setminus \{x\}$ is a $\tau(B)$-set. Hence we may choose the set S so that $x \notin S$. In this case, $S \cap V(B_1)$ is a transversal in B_1 and $S \cap V(B_2)$ is a transversal in B_2, and so $\tau(B_1) + \tau(B_2) \leqslant |S \cap V(B_1)| + |S \cap V(B_2)| = |S| = \tau(B)$. Furthermore, if S_i is a transversal of B_i, for $i = 1, 2$, then $S_1 \cup S_2$ is a transversal of B, and so $\tau(B) \leqslant \tau(B_1) + \tau(B_2)$. Consequently, $\tau(B) = \tau(B_1) + \tau(B_2)$.

(iii): We will show Part (iii) by induction on the order, $n(B)$, of the hypergraph B. If $n(G) = 2$, then $B = H_2$ was created in step (A) in Definition 1. In this case,

$\tau(B) = 1 = (12+10+2)/24 = (6n(B)+4e_4(B)+6e_3(B)+10e_2(B)+2)/24$ and Part (iii) holds in this case. This establishes the base case. Let $k \geqslant 3$ and assume that the formula holds for all $B' \in \mathcal{B}$ with $n(B') < k$ and let $B \in \mathcal{B}$ have order $n(B) = k$.

Suppose that B was created from B' in Step (B) in Definition 1. By Part (i), $\tau(B) = \tau(B') + 1$. Applying the inductive hypothesis to B', we therefore have that

$$\begin{aligned}
\tau(B) &= \tau(B') + 1 \\
&= \tfrac{1}{24}(6n(B') + 4e_4(B') + 6e_3(B') + 10e_2(B') + 2) + 1 \\
&= \tfrac{1}{24}(6(n(B) - 2) + 4e_4(B) + 6(e_3(B) - 2) + 10e_2(B) + 2) + 1 \\
&= \tfrac{1}{24}(6n(B) + 4e_4(B) + 6e_3(B) + 10e_2(B) + 2),
\end{aligned}$$

and so Part (iii) holds in this case. Suppose next that B was created from B' in Step (C) in Definition 1. By Part (i), $\tau(B) = \tau(B') + 1$. Applying the inductive hypothesis to B', we therefore have that

$$\begin{aligned}
\tau(B) &= \tau(B') + 1 \\
&= \tfrac{1}{24}(6n(B') + 4e_4(B') + 6e_3(B') + 10e_2(B') + 2) + 1 \\
&= \tfrac{1}{24}(6(n(B) - 2) + 4(e_4(B) - 2) + 6(e_3(B) + 1) + 10(e_2(B) - 1) + 2) + 1 \\
&= \tfrac{1}{24}(6n(B) + 4e_4(B) + 6e_3(B) + 10e_2(B) + 2),
\end{aligned}$$

and so Part (iii) holds in this case. Suppose finally that B was created from B_1 and B_2 in Step (D). By Part (ii), $\tau(B) = \tau(B_1) + \tau(B_2)$. Applying the inductive hypothesis to B_1 and B_2, we therefore have that

$$\begin{aligned}
\tau(B) &= \tau(B_1) + \tau(B_2) \\
&= \tfrac{1}{24}(6(n(B) - 1) + 4(e_4(B) - 1) + 6(e_3(B) - 2) + 10(e_2(B) + 2) + 2 + 2) \\
&= \tfrac{1}{24}(6n(B) + 4e_4(B) + 6e_3(B) + 10e_2(B) + 2),
\end{aligned}$$

and so Part (iii) holds in this case. This completes the proof of Part (iii).

(iv): Part (iv) follows easily by induction as no operation can make (A)-pairs intersect.

(v): We will prove Part (v) by induction on the order, $n(B)$, of the hypergraph B. Let $e \in E(B)$ be an arbitrary edge in B. If $n(G) = 2$, then $B = H_2$ was created in step (A) in Definition 1. In this case, if e denotes the edge of B, then $\tau(B - e) = 0 = \tau(B) - 1$ and Part (v) holds. This establishes the base case. Let $k \geqslant 3$ and assume that the result holds for all $B' \in \mathcal{B}$ with $n(B') < k$ and let $B \in \mathcal{B}$ have order $n(B) = k$. Let $e \in E(B)$ be an arbitrary edge in B.

Suppose that B was created from B' in Step (B) in Definition 1 and name the vertices as in Definition 1. By Part (i), $\tau(B') = \tau(B) - 1$. Suppose that $e = \{u, v, x\}$ or $e = \{u, v, y\}$. Renaming vertices, if necessary, we may assume without loss of generality that

$e = \{u, v, x\}$. By induction there exists a $\tau(B' - \{u, v\})$-set, S', with $|S'| = \tau(B') - 1$. Since $S' \cup \{y\}$ is a transversal of $B - e$, we note that $\tau(B - e) \leqslant |S'| + 1 = \tau(B') = \tau(B) - 1$, and so $\tau(B - e) \leqslant \tau(B) - 1$. Since deleting an edge from a hypergraph can decrease the transversal number by at most one, we have that $\tau(B - e) \geqslant \tau(B) - 1$. Consequently, $\tau(B - e) = \tau(B) - 1$, as desired. Suppose next that $e = \{x, y\}$. In this case any transversal in B' is a transversal in $B - e$, implying that $\tau(B - e) \leqslant \tau(B') = \tau(B) - 1$. As observed earlier, $\tau(B - e) \geqslant \tau(B) - 1$. Consequently, $\tau(B - e) = \tau(B) - 1$, as desired. Suppose finally that $e \in E(B')$. By induction, $\tau(B' - e) = \tau(B') - 1$. Every $\tau(B' - e)$-set can be extended to a transversal of $B - e$ by adding to it the vertex x, implying that $\tau(B) - 1 \leqslant \tau(B - e) \leqslant \tau(B' - e) + 1 = \tau(B') = \tau(B) - 1$. Consequently, $\tau(B - e) = \tau(B) - 1$, as desired.

If B was created from B' in Step (C) in Definition 1, then the proof that Part (v) holds is analogous to when B was created in Step (B).

Suppose finally that B was created from B_1 and B_2 in Step (D) and name the vertices as in Definition 1. By Part (ii), $\tau(B) = \tau(B_1) + \tau(B_2)$. Suppose first that $e = \{u_i, v_i, x\}$ for some $i \in \{1, 2\}$. By induction there exists a $\tau(B_i - \{u_i, v_i\})$-set, S_i, with $|S_i| = \tau(B_i) - 1$. Let S_{3-i} be any $\tau(B_{3-i})$-set in B_{3-i} and note that $S_1 \cup S_2$ is a transversal in $B - e$, and so $\tau(B) - 1 \leqslant \tau(B - e) \leqslant |S_1| + |S_2| = \tau(B_1) + \tau(B_2) - 1 = \tau(B) - 1$. Consequently, $\tau(B - e) = \tau(B) - 1$, as desired. Suppose next that $e = \{u_1, v_1, u_2, v_2\}$. By induction there exists a $\tau(B_i - \{u_i, v_i\})$-set, T_i, with $|T_i| = \tau(B_i) - 1$. Then, $T_1 \cup T_2 \cup \{x\}$ is a transversal in $B - e$, and so $\tau(B) - 1 \leqslant \tau(B - e) \leqslant |T_1| + |T_2| + 1 = (\tau(B_1) - 1) + (\tau(B_2) - 1) + 1 = \tau(B_1) + \tau(B_2) - 1 = \tau(B) - 1$. Consequently, $\tau(B - e) = \tau(B) - 1$, as desired. Suppose finally that $e \in E(B_i)$ for some $i \in \{1, 2\}$. By induction there exists a $\tau(B_i - e)$-set, D_i, with $|D_i| = \tau(B_i) - 1$. Let D_{3-i} be any $\tau(B_{3-i})$-set in B_{3-i} and note that $D_1 \cup D_2$ is a transversal in $B - e$, and so $\tau(B) - 1 \leqslant \tau(B - e) \leqslant |D_1| + |D_2| = \tau(B_1) + \tau(B_2) - 1 = \tau(B) - 1$. Consequently, $\tau(B - e) = \tau(B) - 1$, as desired. This completes the proof of Part (v).

(vi): We will prove Part (vi) by induction on the order, $n(B)$, of the hypergraph B. Let $s \in V(B)$ be an arbitrary vertex in B. If $n(G) = 2$, then $B = H_2$ was created in step (A) in Definition 1. In this case, there clearly exists a $\tau(B)$-set containing s and Part (vi) holds. This establishes the base case. Let $k \geqslant 3$ and assume that the result holds for all $B' \in \mathcal{B}$ with $n(B') < k$ and let $B \in \mathcal{B}$ have order $n(B) = k$. Let $s \in V(B)$ be an arbitrary vertex in B.

Suppose that B was created from B' in Step (B) in Definition 1 and name the vertices as in Definition 1. By Part (i), $\tau(B') = \tau(B) - 1$. On the one hand, if $s \in \{x, y\}$, then adding the vertex s to any $\tau(B')$-set produces a transversal in B of size $\tau(B') + 1 = \tau(B)$ containing s. On the other hand, if $s \notin \{x, y\}$, then by induction let S be any $\tau(B')$-set containing s and note that $S \cup \{x\}$ is a transversal of size $\tau(B') + 1 = \tau(B)$ in B containing s. In both cases, there exists a $\tau(B)$-set containing s.

If B was created from B' in Step (C) in Definition 1, then the proof that Part (vi) holds is analogous to when B was created in Step (B).

Suppose finally that B was created from B_1 and B_2 in Step (D) and name the vertices as in Definition 1. By Part (ii), $\tau(B) = \tau(B_1) + \tau(B_2)$. Suppose first that $s = x$. In this case, let S_1 be any $\tau(B_1)$-set and let S_2 be any $\tau(B_2 - \{u_2, v_2\})$-set. By Part (v),

$|S_2| = \tau(B_2) - 1$. Thus the set $S_1 \cup S_2 \cup \{x\}$ is a transversal in B containing s of size $\tau(B_1) + (\tau(B_2) - 1) + 1 = \tau(B)$, as desired. Suppose next that $s \neq x$. Renaming B_1 and B_2, if necessary, we may assume that $s \in V(B_1)$. Applying the inductive hypothesis to B_1, there exists a $\tau(B_1)$-set, S_1, containing s. Let S_2 be a $\tau(B_2)$-set. Then, $S_1 \cup S_2$ is a transversal in B containing s of size $\tau(B_1) + \tau(B_2) = \tau(B)$, which completes the proof of Part (vi).

(vii): We will prove Part (vii) by induction on the order, $n(B)$, of the hypergraph B. Let $s, t \in V(B)$ be distinct arbitrary vertices. If $n(G) = 2$, then $B = H_2$ was created in step (A) in Definition 1. In this case, $\{s, t\}$ is an (A)-pair and there is no $\tau(B)$-set containing both s and t. This establishes the base case. Let $k \geqslant 3$ and assume that the result holds for all $B' \in \mathcal{B}$ with $n(B') < k$ and let $B \in \mathcal{B}$ have order $n(B) = k$. Let $s, t \in V(B)$ be distinct arbitrary vertices.

Suppose that B was created from B' in Step (B) in Definition 1 and name the vertices as in Definition 1. By Part (i), $\tau(B') = \tau(B) - 1$. Suppose first that $\{s, t\} = \{x, y\}$. Let S' be any $\tau(B' - \{u, v\})$-set. By Part (v), $|S'| = \tau(B') - 1$. The set $S' \cup \{s, t\}$ is a transversal in B of size $(\tau(B') - 1) + 2 = \tau(B)$ containing s and t, as desired. Suppose next that $|\{s, t\} \cap \{x, y\}| = 1$. By Part (vi) there exists a $\tau(B')$-set, S'', containing the vertex in the set $\{s, t\} \setminus \{x, y\}$. Adding the vertex in $\{s, t\} \cap \{x, y\}$ to S'' produces a transversal of size $\tau(B') + 1 = \tau(B)$ in B containing s and t, as desired. Finally consider the case when $\{s, t\} \cap \{x, y\} = \varnothing$. If there exists a $\tau(B')$-set containing both s and t, then add x to such a set in order to obtain a $\tau(B)$-set containing s and t. If there is no $\tau(B')$-set containing both s and t, then, by induction, $\{s, t\}$ is an (A)-pair in B' and therefore also an (A)-pair in B.

We will now show that if $\{s, t\}$ is an (A)-pair in B' (and therefore in B) there is no $\tau(B)$-set containing s and t. For the sake of contradiction, assume that S is a $\tau(B)$-set containing s and t. If $S \cap V(B')$ is a transversal in B', then since there is no $\tau(B')$-set containing both s and t and $\{s, t\} \subseteq S \cap V(B')$, we have that $\tau(B') < |S \cap V(B')|$. However since $|S \cap \{x, y\}| \geqslant 1$, this implies that $|S| \geqslant |S \cap V(B')| + 1 > \tau(B') + 1 = \tau(B)$, a contradiction. Hence, the set $S \cap V(B')$ is not a transversal in B'. The only edge of B' that does not intersect S is the edge $\{u, v\}$, implying that $\{u, v\} \cap S = \varnothing$ and $\{x, y\} \subseteq S$. In this case, $|S \cap V(B')| = |S| - 2 = \tau(B) - 2 = \tau(B') - 1$. Hence adding the vertex v to the set $S \cap V(B')$ produces a transversal in B' of size $\tau(B')$ containing both s and t, a contradiction. Therefore if $\{s, t\}$ is an (A)-pair in B', then there is no $\tau(B)$-set containing s and t.

If B was created from B' in Step (C) in Definition 1, then the proof that Part (vii) holds is analogous to when B was created in Step (B).

Suppose finally that B was created from B_1 and B_2 in Step (D) and name the vertices as in Definition 1. By Part (ii), $\tau(B) = \tau(B_1) + \tau(B_2)$. Suppose $x \in \{s, t\}$. Without loss of generality we assume that $x = s$ and $t \in V(B_1)$. By Part (vi) there exists a $\tau(B_1)$-set, S_1, containing the vertex t. Let S_2 be a $\tau(B_2 - \{u_2, v_2\})$-set. By part (v), $|S_2| = \tau(B_2) - 1$. Now the set $S_1 \cup S_2 \cup \{x\}$ is a transversal in B containing s and t of size $|S_1| + |S_2| + 1 = \tau(B_1) + (\tau(B_2) - 1) + 1 = \tau(B)$. Hence we may assume that $x \notin \{s, t\}$, for otherwise the desired result follows. Suppose $|\{s, t\} \cap V(B_1)| = 1$. Renaming vertices if necessary, we may assume that $s \in V(B_1)$ and $t \in V(B_2)$. By Part (vi) there exists a $\tau(B_1)$-set, S_1, containing the vertex s and a $\tau(B_2)$-set, S_2, containing the vertex t. In this case, the set $S_1 \cup S_2$ is a transversal in B containing s and t of size $|S_1|+|S_2| = \tau(B_1)+\tau(B_2) = \tau(B)$. Hence without loss of generality we may assume that $\{s,t\} \subseteq V(B_1)$.

If there exists a $\tau(B_1)$-set containing both s and t, then such a set can be extended to a $\tau(B)$-set containing s and t by adding to it a $\tau(B_2)$-set. Hence we may assume that there is no $\tau(B_1)$-set containing both s and t, for otherwise we are done. By induction, the set $\{s,t\}$ is an (A)-pair in B_1 and therefore also an (A)-pair in B. We will now show that in this case there is no $\tau(B)$-set containing s and t, which would complete the proof of Part (vii). For the sake of contradiction, assume that S is a $\tau(B)$-set containing s and t.

If $S \cap V(B_1)$ is a transversal in B_1, then since there is no $\tau(B_1)$-set containing both s and t and $\{s,t\} \subseteq S \cap V(B_1)$, we have that $\tau(B_1) < |S \cap V(B_1)|$. However $|S \cap (V(B_2) \cup \{x\})| \geqslant \tau(B_2)$, implying that $|S| = |S \cap V(B_1)| + |S \cap (V(B_2) \cup \{x\})| > \tau(B_1) + \tau(B_2) = \tau(B)$, a contradiction. Hence, the set $S \cap V(B_1)$ is not a transversal in B'.

The only edge in B_1 that is not intersected by the set S is the edge $\{u_1, v_1\}$, implying that $S \cap \{u_1, v_1\} = \varnothing$. Since $|S \cap \{x, u_1, v_1\}| \geqslant 1$, this implies that $x \in S$. Further since $|S \cap \{u_1, u_2, v_1, v_2\}| \geqslant 1$, this in turn implies that $S \cap \{u_2, v_2\} \neq \varnothing$ and that the set $S \cap V(B_2)$ is a transversal in B_2. Therefore, $|S \cap V(B_2)| \geqslant \tau(B_2)$. Since the set $S \cap V(B_1)$ is a transversal in $B_1 - \{u_1, v_1\}$, by Part (v) we have that $|S \cap V(B_1)| \geqslant \tau(B_1 - \{u_1, v_1\}) = \tau(B_1) - 1$. Hence, $\tau(B_1) + \tau(B_2) = |S| = |S \cap V(B_1)| + |\{x\}| + |S \cap V(B_2)| \geqslant (\tau(B_1) - 1) + 1 + \tau(B_2) = \tau(B_1) + \tau(B_2)$. Thus we must have equality throughout this inequality chain. In particular, we have $|S \cap V(B_1)| = \tau(B_1) - 1$. But then the set $(S \cap V(B_1)) \cup \{u_1\}$ is a transversal in B_1 of size $\tau(B_1)$ containing both s and t, a contradiction. Therefore if $\{s,t\}$ is an (A)-pair in B_1, then there is no $\tau(B)$-set containing s and t, which completes the proof of Part (vii).

(viii): We will prove Part (viii) by induction on the order, $n(B)$, of the hypergraph B. Let $Y_1 = \{s_1, t_1\}$, $Y_2 = \{s_2, t_2\}$ and $Y_3 = \{s_3, t_3\}$. If $n(G) = 2$, then $B = H_2$ was created in step (A) in Definition 1. In this case, $Y_1 = Y_2 = Y_3$ and the result holds trivially. This establishes the base case. Let $k \geqslant 3$ and assume that the result holds for all $B' \in \mathcal{B}$ with $n(B') < k$ and let $B \in \mathcal{B}$ have order $n(B) = k$.

Assume that Y_1, Y_2 and Y_3 are not vertex disjoint. Renaming vertices, we may assume that $s_1 = s_2$. If $s_1 \in Y_3$, then we are done by part (vi) since there exists a $\tau(B)$-set containing s_1. Hence we may assume that $s_1 \notin Y_3$. However by Part (iv) either $\{s_1, s_3\}$ or $\{s_1, t_3\}$ is not an (A)-pair. Renaming vertices in Y_3 if necessary, we may assume that $\{s_1, s_3\}$ is not an (A)-pair. We are now done by Part (vii) since there exists a $\tau(B)$-set containing s_1 and s_3. Hence we may assume that Y_1, Y_2 and Y_3 are vertex disjoint, for otherwise the desired result follows. Let $X = \{s_1, t_1, s_2, t_2, s_3, t_3\}$, and so $|V(B)| \geqslant |X| = 6$.

Suppose that B was created from B' in Step (B) in Definition 1 and name the vertices as in Definition 1. By Part (i), $\tau(B') = \tau(B) - 1$. Suppose $\{x, y\} \cap X = \varnothing$. Applying the inductive hypothesis to B', there exists a $\tau(B')$-set, S', intersecting Y_1, Y_2 and Y_3. But then the set $S' \cup \{x\}$ is a $\tau(B)$-set intersecting Y_1, Y_2 and Y_3. Hence we may assume, renaming vertices if necessary, that $s_1 = x$. Since Y_2 and Y_3 are vertex disjoint sets, the vertex y belongs to at most one of the sets, implying that there exists a vertex, w_1, in $Y_2 \setminus \{y\}$ and a vertex, w_2, in $Y_3 \setminus \{y\}$ that together do not form an (A)-pair by Part (iv). However, by Part (vii), this implies that there exists a $\tau(B')$-set, S', containing w_1 and w_2. Thus the set $S' \cup \{x\}$ is a $\tau(B)$-set covering Y_1, Y_2 and Y_3.

If B was created from B' in Step (C) in Definition 1, then the proof that Part (viii) holds is analogous to when B was created in Step (B).

Suppose finally that B was created from B_1 and B_2 in Step (D) and name the vertices as in Definition 1. By Part (ii), $\tau(B) = \tau(B_1) + \tau(B_2)$. For $i = 1, 2$, let $X_i = X \cap V(B_i)$. Then, $|X_1| \geqslant 3$ or $|X_2| \geqslant 3$. Renaming B_1 and B_2 if necessary, we may assume without loss of generality that $|X_1| \geqslant 3$.

If $|X_1| = 6$, then by induction there exists a $\tau(B_1)$-set, S_1, covering all three sets, Y_1, Y_2 and Y_3. Let S_2 be a $\tau(B_2)$-set. Then, $S_1 \cup S_2$ is a $\tau(B)$-set covering Y_1, Y_2 and Y_3. Hence we may assume that $3 \leqslant |X_1| \leqslant 5$. Further renaming Y_1, Y_2 and Y_3 if necessary, we may assume by Part (iv) that $\{s_1, s_2\} \subset V(B_1)$ and that $\{s_1, s_2\}$ is not an (A)-pair. Further since $|X_1| \leqslant 5$, we may assume that $|Y_3 \cap V(B_1)| \leqslant 1$. By Part (vii), there exists a $\tau(B_1)$-set, S_1, containing s_1 and s_2. On the one hand if $x \in Y_3$, then let S'_2 be a $\tau(B_2 - \{u_2, v_2\})$-set. By part (v), $|S'_2| = \tau(B_2) - 1$. In this case, the set $S_1 \cup S'_2 \cup \{x\}$ is a transversal in B of size $|S_1| + |S'_2| + 1 = \tau(B_1) + (\tau(B_2) - 1) + 1 = \tau(B)$ covering Y_1, Y_2 and Y_3. On the other hand, if $x \notin Y_3$, then $|Y_3 \cap V(B_2)| \geqslant 1$ and we may assume, renaming s_3 and t_3 if necessary, that $s_3 \in V(B_2)$. By Part (vi), there exists a $\tau(B_2)$-set, S_2, containing s_3. In this case the set $S_1 \cup S_2$ is a $\tau(B)$-set covering Y_1, Y_2 and Y_3, which completes the proof of Part (viii).

(ix): We will prove Part (ix) by induction on the order, $n(B)$, of the hypergraph B. Clearly, Part (ix) is vacuously true if $B = H_2$. This establishes the base case. Let $k \geqslant 3$ and assume that the result holds for all $B' \in \mathcal{B}$ with $n(B') < k$ and let $B \in \mathcal{B}$ have order $n(B) = k$. We first note that no 2-edges intersect in any hypergraph in $\mathcal{B}$, as none of the steps (A)-(D) in Definition 1 cause 2-edges to intersect. In particular, we note that in Step (B) the 2-edge $\{u, v\}$ in B' does not intersect any other 2-edge in B'. We now observe that no 3-edge in any $B \in \mathcal{B}$ can intersect two 2-edges in B, as again none of the steps (A)-(D) in Definition 1 can cause this to happen. In particular, we observe that in Step (C) the 3-edge $\{u, v, w\}$ in B' intersects at most one other 2-edge in B'. Finally we observe that no 4-edge in $B \in \mathcal{B}$ can intersect three 2-edges in B, as again none of the steps (A)-(D) in Definition 1 can cause this to happen. Therefore, Part (ix) follows easily by induction.

(x): Part (x) follows easily by induction and the observation that Steps (B)-(D) all increase the degrees of existing vertices being operated on and introduce new vertices of degree two.

(xi): We will prove Part (xi) by induction on the order, $n(B)$, of the hypergraph B. Clearly, Part (xi) is vacuously true if $B = H_2$. This establishes the base case. Let $k \geqslant 3$ and assume that the result holds for all $B' \in \mathcal{B}$ with $n(B') < k$ and let $B \in \mathcal{B}$ have order $n(B) = k$. Let $x \in V(B)$ be chosen such that $d_B(x) = 2$. As observed in the proof of Part (x), Steps (B)-(D) all increase the degrees of existing vertices being operated on and introduce new vertices of degree two. If x is a new vertex of degree two added when constructing B, then by construction the vertex x belongs to a 2-edge or a 3-edge. If x is not a new vertex added when constructing B, then by considering Steps (A)-(D) and Part (x) above it is not difficult to see that Part (xi) holds. This completes the proof of Lemma 11.

(xii): We will prove Part (xii) by induction on the order, $n(B)$, of the hypergraph B. It is not difficult to see that Part (xii) holds if the order is at most four. Let $k \geqslant 5$ and assume that the result holds for all $B' \in \mathcal{B}$ with $n(B') < k$ and let $B \in \mathcal{B}$ have order $n(B) = k$. If B was created using Step (B) or (C), then clearly Part (xii) holds. If B was created using Step (D), then without loss of generality there is a 2-edge in B_1 different from $\{u_1, v_1\}$ (otherwise there is a 2-edge in B_2 different from $\{u_2, v_2\}$) and Part (xii) follows by induction on B_1.

(xiii): As B does not contain two 4-edges intersecting in three vertices we note that Step (C) was never performed in any step of constructing B (as no operation removes 4-edges). As Step (C) was never performed we note that no operation removes 3-edges. As all 2-edges in B are created using Step (B) (any 2-edge created in Step (A) will be removed again by Step (B) or Step (D)) we note that all 2-edges in B intersects two overlapping 3-edges. □

5 Proof of Main Result

In this section, we present a proof of our main result, namely Theorem 3. Recall its statement, where $\mathcal{H}$ denotes the class of hypergraphs where all edges have size at most four and at least two and with maximum degree at most three.

Theorem 3. *If $H \in \mathcal{H}$, then*

$$24\tau(H) \leqslant 6n(H) + 4e_4(H) + 6e_3(H) + 10e_2(H) + 2b(H) + b^1(H).$$

Furthermore if $b^1(H)$ is odd, then the above inequality is strict.

Proof. Given any $H' \in \mathcal{H}$, let

$$\phi(H') = 6n(H') + 4e_4(H') + 6e_3(H') + 10e_2(H') + 2b(H') + b^1(H').$$

We note that if $b^1(H')$ is odd, then $\phi(H')$ is odd. Hence if $24\tau(H) \leqslant \phi(H')$ and $b^1(H')$ is odd, then $24\tau(H) < \phi(H')$.

If $e \in E(H')$, we let $\omega_{H'}(e)$, or simply $\omega(e)$ if H' is clear from the context, denote the contribution of the edge e to the expression $\phi(H')$; that is,

$$\omega(e) = \begin{cases} 4 & \text{if } e \text{ is a 4-edge} \\ 6 & \text{if } e \text{ is a 3-edge} \\ 10 & \text{if } e \text{ is a 2-edge} \end{cases}$$

We refer to $\omega(e)$ as the *weight* of the edge e. Suppose to the contrary that the theorem is false. Among all counterexamples, let H be chosen so that $n(H)+m(H)$ is minimum. In particular, $24\tau(H) > \phi(H)$. We will often use the following fact.

Fact 1: Let $H' \in \mathcal{H}$ be a hypergraph with $n(H')+m(H') < n(H)+m(H)$. Then the following holds.
(a) $\phi(H) - \phi(H') < 24(\tau(H) - \tau(H'))$.
(b) If $H' = H(X,Y)$, then $\phi(H) - \phi(H') < 24|X|$.

Proof. (a) Let $H' \in \mathcal{H}$ satisfy $n(H')+m(H') < n(H)+m(H)$. If $\phi(H)-\phi(H') \geqslant 24\tau(H) - 24\tau(H')$, then $24\tau(H') \geqslant \phi(H') + (24\tau(H) - \phi(H)) > \phi(H')$, contradicting the minimality of H. Hence, $\phi(H) - \phi(H') < 24\tau(H) - 24\tau(H')$.

(b) Further suppose $H' = H(X,Y)$. If X' is a $\tau(H')$-set, then $X \cup X'$ is a transversal in H, implying that $\phi(H) < 24\tau(H) \leqslant 24|X| + 24|X'| = 24\tau(H') + 24|X| \leqslant \phi(H') + 24|X|$, or, equivalently, $\phi(H) - \phi(H') < 24|X|$. (□)

In what follows we present a series of claims describing some structural properties of H which culminate in the implication of its non-existence.

Claim 12. *No edge of H is contained in another edge of H.*

Proof. Let e and f be two distinct edges of H and suppose to the contrary that $V(e) \subseteq V(f)$. Let $H' = H - f$. By the minimality of H, we have that $24\tau(H') \leqslant \phi(H')$. Since every transversal of H' is a transversal of H, and every transversal of H is a transversal of H, we have that $\tau(H) = \tau(H')$. Hence, $24\tau(H) \leqslant 24\tau(H') \leqslant \phi(H') = \phi(H) - \omega(f) \leqslant \phi(H) - 4 < \phi(H)$, a contradiction. (□)

Claim 13. *The following hold in the hypergraph H.*
(a) *H is connected.*
(b) *$b(H) = 0$.*
(c) *$b^1(H) = 0$.*

Proof. (a) If H is disconnected, then by the minimality of H we have that the theorem holds for all components of H and therefore also for H, a contradiction.

(b) If $b(H) > 0$, then by Part (a), $H \in \mathcal{B}$ and by Lemma 11(iii) we note that H is not a counter-example to the theorem, a contradiction.

(c) Suppose to the contrary that $b^1(H) > 0$. Let $B \in \mathcal{B}$ be a subhypergraph in H and let $e \in E(H)$ be the (unique) edge of $E(H) \setminus E(B)$ intersecting B in H. By Lemma 11(vi) there exists a transversal S of B containing a vertex, v, in e. Let $H' = H(S, V(B) \setminus S)$. If a vertex, v', in $V(e) \setminus \{v\}$ belongs to some subhypergraph B' which contributes one to $b(H')$, then necessarily $B' \in \mathcal{B}$ is a component of H' but not a component of H and therefore contributes one to $b^1(H)$ and zero to $b(H)$. In this case, B' contributes one to $2b(H) + b^1(H)$ and two to $2b(H') + b^1(H')$. If a vertex, v', in $V(e) \setminus \{v\}$ belongs to some subhypergraph B' which contributes one to $b^1(H')$, then B' contributes zero to each of $b(H)$, $b^1(H)$ and $b(H')$, and contributes one to $b^2(H)$. In this case, B' contributes zero to $2b(H) + b^1(H)$ and one to $2b(H') + b^1(H')$. In both cases, the vertex v' belongs to a subhypergraph in H' that increases $2b(H')+b^1(H')$ by one. Since $|V(e)| \leqslant 4$, we note that $|V(e) \setminus \{v\}| \leqslant 3$. Thus since each vertex in $V(e) \setminus \{v\}$ belongs to a subhypergraph in H' that increases $2b(H')+b^1(H')$ by at most one, and since the deletion of the subhypergraph B from H decreases $2b(H') + b^1(H')$ by one, we have that $2b(H') + b^1(H')$ is at most two larger than $2b(H) + b^1(H)$; that is,

$$(2b(H) + b^1(H)) - (2b(H') + b^1(H')) \geqslant -2.$$

Further since $\omega_H(e) \geqslant 4$, and applying Lemma 11(iii) to $B \in \mathcal{B}$, we have that

$$\begin{array}{rcl} \phi(H) - \phi(H') & = & (6n(B) + 4e_4(B) + 6e_3(B) + 10e_2(B)) + \omega_H(e) \\ & & +(2b(H) + b^1(H)) - (2b(H') + b^1(H') \\ & \geqslant & (24|S| - 2) + 4 - 2 \;\; = \;\; 24|S|, \end{array}$$

contradicting Fact 1. Therefore, $b^1(H) = 0$. (□)

Claim 14. $b^2(H) = 0$.

Proof. Suppose to the contrary that $b^2(H) > 0$. Let $B \in \mathcal{B}$ be any subhypergraph in H contributing to $b^2(H)$ and let $f_1, f_2 \in E(H)$ be the two edges of $E(H) \setminus E(B)$ intersecting B in H. We now show a number of subclaims.

Subclaim 3(a) *$|V(f_i) \cap V(B)| = 1$ for $i = 1, 2$. Further if $V(f_i) \cap V(B) = \{s_i\}$, then $s_1 \neq s_2$ and $\{s_1, s_2\}$ is an (A)-pair in B.*

Proof of Subclaim 3(a). Suppose to the contrary that $|V(f_i) \cap V(B)| \geqslant 2$ for some $i = 1, 2$ or that $V(f_i) \cap V(B) = \{s_i\}$ but $\{s_1, s_2\}$ is not an (A)-pair in B. We now choose a $\tau(H)$-set, S, as follows. If there exists a vertex $v \in V(f_1) \cap V(f_2)$, then by Lemma 11(vi), let S be chosen to contain v. If f_i intersects B in at least two vertices for some $i \in \{1, 2\}$, then by Lemma 11(iv) we can find vertices $s_j \in V(f_j)$ such that $\{s_1, s_2\}$ is not an (A)-pair in B. By Lemma 11(vii), let S be chosen to contain s_1 and s_2. Finally if $V(f_i) \cap V(B) = \{s_i\}$ where $s_1 \neq s_2$ but $\{s_1, s_2\}$ is not an (A)-pair in B, then by Lemma 11(vii) let S be chosen to contain s_1 and s_2. In all three cases, we have that the $\tau(B)$-set, S, covers f_1 and f_2. Let $H' = H(S, V(B) \setminus S)$. A similar argument as in the proof of Claim 13(c) shows that each vertex in $(V(f_1) \cap V(f_2)) \setminus V(B)$ belongs to a subhypergraph in H' that increases $2b(H') + b^1(H')$ by at most two, while for $i \in \{1, 2\}$ each vertex in $V(f_i) \setminus V(f_{3-i})$ that is not in $V(B)$ belongs to a subhypergraph in H' that increases $2b(H')+b^1(H')$ by at most one. Hence since $2b(H)+b^1(H) = 0$, $|V(f_1) \setminus S| \leqslant 3$ and $|V(f_2) \setminus S| \leqslant 3$, we have that $2b(H') + b^1(H') \leqslant 6$, and so

$$(2b(H) + b^1(H)) - (2b(H') + b^1(H')) \geqslant -6.$$

Further since $\omega_H(f_i) \geqslant 4$ for $i \in \{1, 2\}$, applying Lemma 11(iii) to $B \in \mathcal{B}$, we have that

$$\begin{aligned}\phi(H)-\phi(H') &= (6n(B)+4e_4(B)+6e_3(B)+10e_2(B))+\omega_H(f_1)+\omega_H(f_2)\\ &\quad +(2b(H)+b^1(H))-(2b(H')+b^1(H')\\ &\geqslant (24|S|-2)+4+4-6 \;=\; 24|S|,\end{aligned}$$

contradicting Fact 1 and proving Subclaim 3(a). (□)

Subclaim 3(b) *$B = H_2$.*

Proof of Subclaim 3(b). By Subclaim 3(a), we may assume relabeling vertices if necessary that $V(f_1)\cap V(B)=\{s_1\}$ and $V(f_2)\cap V(B)=\{s_2\}$ and that $\{s_1,s_2\}$ is an (A)-pair in B. Suppose to the contrary that $B\neq H_2$. Let H' be obtained from H by removing all edges in B and all vertices $V(B)\setminus\{s_1,s_2\}$ and adding the 2-edge $\{s_1,s_2\}$. We show that $\tau(H)\leqslant\tau(H')+\tau(B)-1$. Let S' be a $\tau(H')$-set such that $|S'\cap\{s_1,s_2\}|$ is a minimum. Since $\{s_1,s_2\}$ is an edge in H', we have that $|S'\cap\{s_1,s_2\}|\geqslant 1$. If $|S'\cap\{s_1,s_2\}|=2$, then by removing s_2 from S' and replacing it with an arbitrary vertex in $V(f_2)\setminus\{s_2\}$ we get a contradiction to the minimality of $|S'\cap\{s_1,s_2\}|$. Therefore, $|S'\cap\{s_1,s_2\}|=1$. Renaming vertices if necessary, we may assume that $s_1\in S'$. By Lemma 11(vi) there exists a transversal, S_B, of B containing the vertex s_1. Thus, $S'\cup S_B$ is a transversal in H and $S'\cap S_B=\{s_1\}$, and so $\tau(H)\leqslant|S'|+|S_B|-1=\tau(H')+\tau(B)-1$, as desired. Equivalently, $\tau(B)-1\geqslant\tau(H)-\tau(H')$. By Lemma 11(iii), we therefore have that

$$\begin{aligned}\phi(H)-\phi(H') &= \phi(B)-\phi(H_2) \;=\; 24\tau(B)-24\tau(H_2) \;=\; 24(\tau(B)-\tau(H_2))\\ &= 24(\tau(B)-1) \;\geqslant\; 24(\tau(H)-\tau(H')),\end{aligned}$$

where H_2 is defined in Definition 1(A), contradicting Fact 1 and proving Subclaim 3(b). (□)

Subclaim 3(c) *There is no edge $e\in E(H)$ with $V(e)\subseteq(V(f_1)\cup V(f_2))\setminus\{s_1,s_2\}$.*

Proof of Subclaim 3(c). Suppose to the contrary that there is an edge $e\in E(H)$ such that $V(e)\subseteq(V(f_1)\cup V(f_2))\setminus\{s_1,s_2\}$. Let H' be obtained from H by deleting the vertices s_1 and s_2 and the edges $f_1,f_2,\{s_1,s_2\}$; that is, $H'=H(\{s_1,s_2\},\varnothing)$. Let S' be a $\tau(H')$-set. Due to the existence of the edge e we may assume without loss of generality that f_1 contains a vertex from S'. But then $S'\cup\{s_2\}$ is a transversal of H, implying that $\tau(H)\leqslant|S'|+1=\tau(H')+1$. Each vertex in $(V(f_1)\cup V(f_2))\setminus\{s_1,s_2\}$ increases $2b(H')+b^1(H')$ by at most one. Thus since $2b(H)+b^1(H)=0$ and $|(V(f_1)\cup V(f_2))\setminus\{s_1,s_2\}|\leqslant 6$, we have that $2b(H')+b^1(H')\leqslant 6$. Further, $\omega(f_1)\geqslant 4$, $\omega(f_2)\geqslant 4$ and $\omega(\{s_1,s_2\})=10$. Therefore since the vertices s_1 and s_2 and the edges $f_1,f_2,\{s_1,s_2\}$ are removed from H when constructing H', we have that

$$\begin{aligned}\phi(H)-\phi(H') &= 6|\{s_1,s_2\}|+\omega(f_1)+\omega(f_2)+\omega(\{s_1,s_2\})-(2b(H')+b^1(H'))\\ &= 12+4+4+10-6 \;=\; 24 \;\geqslant\; 24(\tau(H)-\tau(H')),\end{aligned}$$

contradicting Fact 1 and proving Subclaim 3(c). (□)

Subclaim 3(d) $b(H - f_1 - f_2) = 1$.

Proof of Subclaim 3(d). Since B is a component of $H - f_1 - f_2$, we have that $b(H - f_1 - f_2) \geqslant 1$. We show that $b(H - f_1 - f_2) = 1$. Suppose to the contrary that there exists a component, $R \in \mathcal{B}$, in $H - f_1 - f_2$ which is different from B. Since $b(H) = b^1(H) = 0$, the subhypergraph R contributes to $b^2(H)$, which by Subclaim 3(b) implies that $R = H_2$. By Subclaim 3(a) we note that the 2-edge in R is a subset of $(V(f_1) \cup V(f_2)) \setminus \{s_1, s_2\}$, a contradiction to Subclaim 3(c). ($\Box$)

We now return to the proof of Claim 14. By Subclaim 3(a) and 3(b), we may assume that $B = H_2$, $V(B) = \{s_1, s_2\}$ and $V(f_i) \cap V(B) = \{s_i\}$ for $i = 1, 2$. Let $X = (V(f_1) \cup V(f_2)) \setminus \{s_1, s_2\}$ and assume without loss of generality that $|V(f_1)| \leqslant |V(f_2)|$. Clearly, $1 \leqslant |X| \leqslant 6$. We now consider a number of different cases.

First consider the case when $|X| = 1$. Assume that $X = \{x\}$, which implies that $f_1 = \{s_1, x\}$ and $f_2 = \{s_2, x\}$. Let $H' = H(\{x\}, \varnothing)$. Suppose $d_H(x) = 2$. Then, $H' = B$, $b(H') = 1$ and $b^1(H) = 0$, implying that $\phi(H) - \phi(H') = 6|\{x\}| + \omega(f_1) + \omega(f_2) - (2b(H') + b^1(H')) = 6 + (2 \times 10) - 2 = 24 = 24|X|$, contradicting Fact 1. Hence, $d_H(x) \geqslant 3$. Consequently since $\Delta(H) = 3$, we have that $d_H(x) = 3$. Let e be the edge of H different from f_1 and f_2 containing x and note that $2b(H - e) + b^1(H - e) \leqslant 3$, which implies that $2b(H') + b^1(H') \leqslant 5$. Therefore, $\phi(H) - \phi(H') = 6|\{x\}| + \omega(e) + \omega(f_1) + \omega(f_2) - (2b(H') + b^1(H')) \geqslant 6 + 4 + (2 \times 10) - 5 > 24 = 24|X|$, contradicting Fact 1. Hence, $|X| \geqslant 2$.

Suppose $2 \leqslant |X| \leqslant 4$. In this case we let H' be obtained from H by deleting the vertices s_1 and s_2 and the edges $f_1, f_2, \{s_1, s_2\}$ and adding the new edge $f = X$. By Subclaim 3(d), $b(H - f_1 - f_2) = 1$ and therefore B is the only component of $H - f_1 - f_2$ in $\mathcal{B}$. This implies that if $b(H') > 0$ or $b^1(H') > 0$, then the new edge f belongs to some subhypergraph R which contributes to $b(H')$ or $b^1(H')$, and this R is the only subhypergraph that contributes to $2b(H') + b^1(H')$. Therefore, $2b(H') + b^1(H') \leqslant 2$. We now show that $\tau(H) \leqslant \tau(H') + 1$. Assume that S' is a $\tau(H')$-set and note that some vertex in X belongs to S'. Without loss of generality we may assume that there is a vertex in $S' \cap X$ belonging to f_1. This implies that $S' \cup \{s_2\}$ is a transversal of H, and so $\tau(H) \leqslant |S'| + 1 = \tau(H') + 1$. We now consider the following possibilities.

Suppose that $|X| = 2$. Suppose that $|V(f_1)| = 2$. As observed earlier, $2b(H') + b^1(H') \leqslant 2$. Since $|V(f_2)| \leqslant |X| + 1 = 3$, we have that $\omega(f_2) \geqslant 6$. Thus,

$$\begin{aligned}\phi(H) - \phi(H') =& 6|\{s_1, s_2\}| + \omega(f_1) + \omega(f_2) + \omega(\{s_1, s_2\}) - \omega(f) - (2b(H') + b^1(H')) \\ \geqslant& (6 \times 2) + 10 + 6 + 10 - 10 - 2 \ > \ 24 \ \geqslant \ 24(\tau(H) - \tau(H')),\end{aligned}$$

contradicting Fact 1. Hence, $|V(f_1)| = 3$. Thus, $3 = |V(f_1)| \leqslant |V(f_2)| \leqslant |X| + 1 = 3$, implying that $|V(f_2)| = 3$. Assume that $X = \{x, y\}$, which implies that $f_1 = \{s_1, x, y\}$ and $f_2 = \{s_1, x, y\}$. If $b(H') = b^1(H') = 0$, then $\phi(H) - \phi(H') \geqslant (2 \times 6) + (2 \times 6) + 10 - 10 = 24 = 24(\tau(H) - \tau(H'))$, contradicting Fact 1. Hence, $2b(H') + b(H') > 0$. This implies that the new edge f belongs to some subhypergraph R which contributes to $b(H')$ or $b^1(H')$, and this R is the only subhypergraph that contributes to $2b(H') + b^1(H')$. Since $\{x, y\}$ is a 2-edge in R, using Step (B) in Definition 1 we can extend R to a subhypergraph

$R' \in \mathcal{B}$, by adding the vertices $\{s_1, s_2\}$ and the edges f_1, f_2 and $\{s_1, s_2\}$ and deleting the edge $\{x, y\}$. However this implies that R' is a subhypergraph in H contributing to $b(H)$ or $b^1(H)$, a contradiction. Hence, $|X| \geqslant 3$.

Suppose that $|X| = 3$. Then, $\omega(f) = 6$. Suppose that $|V(f_1)| \leqslant 3$. Then, $\omega(f_1) \geqslant 6$, while $\omega(f_2) \geqslant 4$. As observed earlier, $2b(H') + b^1(H') \leqslant 2$. Thus, $\phi(H) - \phi(H') \geqslant (2\times6)+6+4+10-6-2 = 24 = 24(\tau(H)-\tau(H'))$, contradicting Fact 1. Hence, $|V(f_1)| \geqslant 4$, implying that $|V(f_1)| = |V(f_2)| = 4$. If $b(H') = b^1(H') = 0$, then $\phi(H)-\phi(H') \geqslant (2\times6)+(2\times4)+10-6 = 24 = 24(\tau(H)-\tau(H'))$, contradicting Fact 1. Hence, $2b(H')+b(H') > 0$. This implies that the new edge f belongs to some subhypergraph R which contributes to $b(H')$ or $b^1(H')$, and this R is the only subhypergraph that contributes to $2b(H')+b^1(H')$. Since f is a 3-edge in R, using Step (C) in Definition 1 we can extend R to a subhypergraph $R' \in \mathcal{B}$, by adding the vertices $\{s_1, s_2\}$ and the edges f_1, f_2 and $\{s_1, s_2\}$ and deleting the edge f, a contradiction.

Hence, $|X| = 4$, and so $\omega(f) = 4$. As observed earlier, $2b(H') + b^1(H') \leqslant 2$. Thus, $\phi(H) - \phi(H') \geqslant (2 \times 6) + (2 \times 4) + 10 - 4 - 2 = 24 = 24(\tau(H) - \tau(H'))$, contradicting Fact 1. This completes the case when $2 \leqslant |X| \leqslant 4$.

It remains for us to consider the case when $5 \leqslant |X| \leqslant 6$. In this case we note that $|V(f_1) \cap V(f_2)| \leqslant 1$. Further, $|V(f_1)| \geqslant 3$, and so neither f_1 nor f_2 is a 2-edge. Let X' be the set of vertices from X which belong to some 2-edge in H. We note that by Subclaim 3(c), every 2-edge in H contains at most one vertex of X.

Suppose that $|X'| \leqslant 3$. Let $f \subseteq X$ be chosen such that $|V(f)| = 4$, $X' \subseteq V(f)$ and if any vertex belongs to $V(e_1) \cap V(e_2)$, then it also belongs to f. In particular, we note that $\omega(f) = 4$. Let H' be obtained from H by deleting the vertices s_1 and s_2 and the edges $f_1, f_2, \{s_1, s_2\}$ and adding the new edge f. Analogously to the case when $2 \leqslant |X| \leqslant 4$, we have that $\tau(H) \leqslant \tau(H')+1$. By Subclaim 3(d), $b(H - f_1 - f_2) = 1$ and therefore B is the only component of $H - f_1 - f_2$ in $\mathcal{B}$. This implies that if $2b(H') + b^1(H') \geqslant 3$, then there must exists a subhypergraph $R \in \mathcal{B}$ which does not contain the edge f but contributes to $2b(H') + b^1(H')$. But then R contributed to $b^2(H)$, which by Subclaim 3(b) implies that $R = H_2$, a contradiction to the definition of X'. Therefore, $2b(H') + b^1(H') \leqslant 2$. Hence, $\phi(H) - \phi(H') \geqslant (2 \times 6) + (2 \times 4) + 10 - 4 - 2 = 24 = 24(\tau(H) - \tau(H'))$, contradicting Fact 1. Hence, $|X'| \geqslant 4$.

Let $f \subseteq X'$ be chosen such that $|V(f)| = 4$. Let H'' be obtained from H by deleting the vertices s_1 and s_2 and the edges $f_1, f_2, \{s_1, s_2\}$ and adding the new edge f. Analogously to the case when $2 \leqslant |X| \leqslant 4$, we have that $\tau(H) \leqslant \tau(H'') + 1$. By Subclaim 3(d), $b(H - f_1 - f_2) = 1$ and therefore B is the only component of $H - f_1 - f_2$ in $\mathcal{B}$. For the sake of contradiction suppose that there exists a subhypergraph $R \in \mathcal{B}$ which contains the edge f and contributes to $2b(H'') + b^1(H'')$. By Lemma 11(ix) and Subclaim 3(c) we note that at most two of the four 2-edges intersecting f can belong to R. As observed earlier, neither f_1 nor f_2 is a 2-edge. But this implies that the subhypergraph $R \in \mathcal{B}$ is intersected by at least two 2-edges in H'' that do not belong to R, contradicting the fact that R contributes to $2b(H'') + b^1(H'')$. Therefore, $2b(H'') + b^1(H'') \leqslant |X \setminus V(f)| \leqslant 2$. Hence, $\phi(H) - \phi(H'') \geqslant (2 \times 6) + (2 \times 4) + 10 - 4 - 2 = 24 = 24(\tau(H) - \tau(H''))$, contradicting Fact 1. This completes the proof of Claim 14. ($\Box$)

Claim 15. *No 2-edges in H intersect.*

Proof. Suppose to the contrary that there are two 2-edges, e and e', that intersect in H and let x be the vertex common to both edges. Let $H' = H(\{x\}, \varnothing)$ and let $X = \{x\}$. If $d(x) = 2$, then Claim 13 and 14 imply that $b(H') = 0$ and $b^1(H') \leqslant 1$, and so $2b(H') + b^1(H') \leqslant 1$. This implies that, $\phi(H) - \phi(H') \geqslant 6 + (2 \times 10) - (2b(H') + b^1(H')) > 24 = 24|X|$, contradicting Fact 1. Therefore, $d(x) = 3$, which by Claim 13 and 14 implies that $2b(H') + b^1(H') \leqslant 2$ and therefore that $\phi(H) - \phi(H') \geqslant 6 + (2 \times 10) + 4 - (2b(H') + b^1(H')) > 24 = 24|X|$, contradicting Fact 1. (□)

Claim 16. *If $e = \{x, y\}$ is a 2-edge in H and $d_H(x) = 3$, then x is contained in two distinct 4-edges.*

Proof. Assume that $e = \{x, y\}$ is a 2-edge in H and $d_H(x) = 3$. Let e, e' and e'' be the three distinct edges in H containing x. By Claim 15, neither e' nor e'' is a 2-edge. Suppose to the contrary that e' is a 3-edge. Let $H' = H(\{x\}, \varnothing)$. Then, $\tau(H) \leqslant \tau(H') + 1$.

Suppose that e'' is a 4-edge. If $b(H') > 0$, then by Claim 13 and 14 we note that any component $R \in \mathcal{B}$ in H must intersect e, e' and e'' and therefore contain y. This implies that $b(H') \leqslant 1$. Since $|V(e) \setminus \{x\}| + |V(e') \setminus \{x\}| + |V(e'') \setminus \{x\}| = 6$, we note that by Claim 13 and 14 either $b(H') = 1$ and $b^1(H') \leqslant 1$ or $b(H') = 0$ and $b^1(H') \leqslant 3$. Thus, $2b(H') + b^1(H') \leqslant 3$. Furthermore if $2b(H') + b^1(H') = 3$, then $b^1(H')$ is odd. By the minimality of H we have $24\tau(H') \leqslant \phi(H')$ when $2b(H') + b^1(H') \leqslant 2$ and $24\tau(H') \leqslant \phi(H') - 1$ when $2b(H') + b^1(H') = 3$. On the one hand if $2b(H') + b^1(H') = 3$, then

$$\begin{aligned} 24\tau(H) &\leqslant 24(\tau(H') + 1) \leqslant (\phi(H') - 1) + 24 \\ &= [\phi(H) - 6|\{x\}| - \omega(e) - \omega(e') - \omega(e'') + 2b(H') + b^1(H') - 1] + 24 \\ &= [\phi(H) - 6 - 10 - 6 - 4 + 3 - 1] + 24 = \phi(H), \end{aligned}$$

a contradiction. On the other hand if $2b(H') + b^1(H') \leqslant 2$, then $24\tau(H) \leqslant 24(\tau(H') + 1) \leqslant \phi(H') + 24 = [\phi(H) - 6 - 10 - 6 - 4 + 2] + 24 = \phi(H)$, once again a contradiction. Hence, e'' is not a 4-edge, implying that e'' is a 3-edge. Since $|V(e) \setminus \{x\}| + |V(e') \setminus \{x\}| + |V(e'') \setminus \{x\}| = 5$, we note that by Claim 13 and 14 $2b(H') + b^1(H') \leqslant 3$. Therefore,

$$\begin{aligned} \phi(H) - \phi(H') &\geqslant 6|\{x\}| + \omega(e) + \omega(e') + \omega(e'') - (2b(H') + b^1(H')) \\ &\geqslant 6 + 10 + (2 \times 6) - 3 > 24 = 24|\{x\}|, \end{aligned}$$

a contradiction. This completes the proof of Claim 16. (□)

Claim 17. *If $R \in \mathcal{B}$ is a subhypergraph in H and e is a 2-edge in $E(H) \setminus E(R)$, then $V(e) \cap V(R) = \varnothing$.*

Proof. Assume that $R \in \mathcal{B}$ is a subhypergraph in H and e is a 2-edge in $E(H) \setminus E(R)$. Suppose to the contrary that $V(e) \cap V(R) \neq \varnothing$ and let $x \in V(e) \cap V(R)$. If $d_R(x) = 1$, then by Lemma 11(x) we have that $R = H_2$ and so x belongs to a 2-edge in R, a contradiction to Claim 15. Hence, $d_R(x) \geqslant 2$. However since $\Delta(H) \leqslant 3$ and the edge $e \notin E(R)$ contains the vertex x, we have that $d_R(x) \leqslant 2$. Consequently, $d_R(x) = 2$. By Lemma 11(xi), x is therefore contained in a 3-edge or a 2-edge in R, a contradiction to Claim 16. (□)

Claim 18. *If $B \in \mathcal{B}$ contributes to $b^3(H)$, then $B = H_2$.*

Proof. Assume that $b^3(H) > 0$ and that $B \in \mathcal{B}$ is a subhypergraph in H that contributes to $b^3(H)$. Suppose to the contrary that $B \neq H_2$. Let $f_1, f_2, f_3 \in E(H) \setminus E(B)$ be the three edges in H that intersect B.

Suppose that $|V(f_i) \cap V(B)| \geqslant 2$ for all $i = 1, 2, 3$. Then by Lemma 11(viii) there exists a $\tau(B)$-set, S, intersecting f_1, f_2 and f_3. Let $H' = H(S, V(B) \setminus S)$. By Claim 13 and 14 we note that any component $R \in \mathcal{B}$ in H' must intersect all of f_1, f_2 and f_3, while any subhypergraph in H' that contributes to $b^1(H')$ must intersect at least two of f_1, f_2 and f_3. Since $|(V(f_1) \cup V(f_2) \cup V(f_3)) \setminus V(B)| \leqslant 6$, this implies that $2b(H') + b^1(H') \leqslant 4$. Therefore by Lemma 11(iii), we have that

$$\begin{aligned} \phi(H) - \phi(H') &\geqslant 6n(B) + 4e_4(B) + 6e_3(B) + 10e_2(B) \\ &\qquad + \omega(f_1) + \omega(f_2) + \omega(f_3) - (2b(H') + b^1(H')) \\ &\geqslant (24|S| - 2) + 12 - 4 \;>\; 24|S|, \end{aligned}$$

contradicting Fact 1. Hence we may assume without loss of generality that $|V(f_1) \cap B| = 1$.

If there is no $\tau(B)$-set intersecting both f_2 and f_3, then by Lemma 11(vii) we must have $V(f_2) \cap B = \{b_2\}$ and $V(f_3) \cap B = \{b_3\}$ and $\{b_2, b_3\}$ is an (A)-pair in B. However in this case by Lemma 11(iv) there exists a $\tau(B)$-set intersecting both f_1 and f_2. Hence in both cases there exists a $\tau(B)$-set intersecting two of f_1, f_2, f_3 such that the edge not covered intersects B in exactly one vertex. Without loss of generality we may assume that $V(f_1) \cap B = \{b_1\}$ and that S_B is a $\tau(B)$-set intersecting both f_2 and f_3.

Let $H_1^* = H(V(B), \varnothing)$. If $b^1(H_1^*) > 0$, then let $B_1 \in \mathcal{B}$ be a subhypergraph in H_1^* and let $e_1 \in E(H_1^*)$ be the only edge intersecting B_1 in H_1^*. In this case let $H_2^* = H_1^*(V(B_1), \varnothing)$. If $b^1(H_2^*) > 0$, then let $B_2 \in \mathcal{B}$ be a subhypergraph in H_2^* and let $e_2 \in E(H_2^*)$ be the only edge intersecting B_2 in H_2^*. In this case let $H_3^* = H_2^*(V(B_2), \varnothing)$. Continue the above process until $b^1(H_\ell^*) = 0$, for some $\ell \geqslant 1$. This defines $H_1^*, H_2^*, \ldots, H_\ell^*$ and $B_1, B_2, \ldots, B_{\ell-1}$ and $e_1, e_2, \ldots, e_{\ell-1}$.

We first consider the case when $b(H_\ell^*) = 0$. Recall that S_B is a $\tau(B)$-set intersecting both f_2 and f_3. Let $S' = S_B$. We now construct a hypergraph H' where initially we let $H' = H(S_B, V(B) \setminus (S_B \cup \{b_1\}))$. If $b^1(H') > 0$, let $R \in \mathcal{B}$ be a subgraph in H' intersected by exactly one edge $e \in E(H') \setminus E(R)$ and do the following. Let S_R be a $\tau(R)$-set intersecting e (which exists by Lemma 11(vi)) and add S_R to S' and let H' be $H'(S_R, V(R) \setminus S_R)$. We continue this process until $b^1(H') = 0$. When the above process stops assume that $b^1(H') > 0$ was true r times. Let S' consist of the set S_B and the r $\tau(R)$-sets S_R resulting from constructing H'.

We show first that $b(H') = 0$. Suppose to the contrary that $b(H') > 0$ and let $R^* \in \mathcal{B}$ be a component in H'. This implies that R must contain the edge f_1, for if this were not the case, then such a component would also be a component in H_ℓ^*, but $b^1(H_\ell^*) = b(H_\ell^*) = 0$. However, f_1 is not a 2-edge by Claim 17, but it does contain a vertex of degree one in H'

(namely b_1). However this is a contradiction to Lemma 11(x). Therefore, $b(H') = 0$ and $2b(H') + b^1(H') = 0$.

Let V' denote all vertices removed from H to obtain H' and let E' be all edges removed. We note that $H' = H(S', V' \setminus (S' \cup \{b_1\}))$. Furthermore the vertex b_1 was not removed from H when we initialized H' for the first time. By applying Lemma 11(iii) $r+1$ times, we note that $24|S'| = 6(|V'|+1) + 4e_4(E') + 6e_3(E') + 10e_2(E') + 2(r+1)$. Note that apart from the vertices and edges in subhypergraphs from $\mathcal{B}$ that were deleted when constructing H' from H, a further $r+2$ edges have been removed, namely the two edges f_2 and f_3 and the r edges from subhypergraphs contributing to $b^1(H')$ when constructing H'. Therefore since we have removed in total $r+1$ subhypergraphs in H belonging to $\mathcal{B}$, we have that

$$\begin{aligned}\phi(H) - \phi(H') &\geqslant 6|V'| + 4e_4(E') + 6e_3(E') + 10e_2(E') + 4(r+2) \\ &= 6(|V'|+1) + 4e_4(E') + 6e_3(E') + 10e_2(E') + 2(r+1) + 2r \\ &\geqslant 24|S'| + 2r \;\geqslant\; 24|S'|,\end{aligned}$$

contradicting Fact 1. Hence, $b(H_\ell^*) > 0$.

Since $B \neq H_2$, we have by Lemma 11(x) that $\delta(B) \geqslant 2$. Since $\Delta(H) = 3$, each vertex in $V(B)$ is intersected by at most one of the three edges f_1, f_2 and f_3, implying that $V(f_1) \cap V(B)$, $V(f_2) \cap V(B)$ and $V(f_3) \cap V(B)$ are distinct sets. By Lemma 11(iv) and 11(vii), we may assume that there exists a $\tau(B)$-set intersecting both f_1 and f_2 and a $\tau(B)$-set intersecting both f_2 and f_3 (by renaming f_1, f_2 and f_3 if necessary).

Let $R \in \mathcal{B}$ be a component in H_ℓ^*. Recall by Claim 13 and 14 that we have $b(H) = b^1(H) = b^2(H) = 0$. This implies that there is an edge in $\{f_1, f_3, e_1, e_2, \ldots, e_{\ell-1}\}$ that intersects R. Assume it is e_{j_1}. However now there is an edge in $\{f_1, f_3, e_1, e_2, \ldots, e_{j_1-1}\}$ that intersects B_{j_1}. Assume it is e_{j_2}. However now there is an edge in $\{f_1, f_3, e_1, e_2, \ldots, e_{j_2-1}\}$ that intersects B_{j_2}. Assume it is e_{j_3}. Continuing the above process we note that $j_1 > j_2 > j_3 > \cdots > j_s$ and the edge that intersects B_{j_s} is without loss of generality f_1 (otherwise it is f_3). By Lemma 11(vi) we can find a minimum transversal in B_{j_i} that covers the edge $e_{j_{i+1}}$ for each $i \in \{1, 2, \ldots\}$. Furthermore we can find a $\tau(B_{j_s})$-set that covers f_1 and a $\tau(R)$-set covering e_{j_1}. Taking the union of all of these transversals we obtain a minimum transversal in each of $R, B_{j_1}, B_{j_2}, \ldots, B_{j_s}$ that together cover all the edges $e_{j_1}, e_{j_2}, \ldots, e_{j_s}, f_1$. Similarly by Lemma 11(vi) we can readily find a minimum transversal in each hypergraph in $\{B_1, B_2, \ldots, B_{\ell-1}\} \setminus \{B_{j_1}, B_{j_2}, \ldots, B_{j_s}\}$ that cover all edges in $\{e_1, e_2, \ldots, e_{\ell-1}\} \setminus \{e_{j_1}, e_{j_2}, \ldots, e_{j_s}\}$. Let S_B be a $\tau(B)$-set covering f_2 and f_3 (if f_3 would have intersected B_{j_s} instead of f_1, then we would have let S_B cover f_1 and f_2). Let S^* denote the union of all of these transversals together with S_B. Then, S^* covers every edge in $E^* \cup E^{**}$, where $E^* = \{f_1, f_2, f_3, e_1, e_2, \ldots, e_{\ell-1}\}$ and $E^{**} = E(R \cup B \cup B_1 \cup B_2 \cup \cdots \cup B_{\ell-1})$.

Let H' be obtained from H be removing S^* and all edges incident with S^* and all resulting isolated vertices. Since $b(H) = b^1(H) = b^2(H) = 0$, we note that every component in H_ℓ^* which belong to $\mathcal{B}$ is incident with at least three edges from E^*. Further every edge in E^* intersects at most three such components, implying that $b(H_\ell^*) \leqslant |E^*| = \ell + 2$. Recall that $b^1(H_\ell^*) = 0$. Since H' is obtained from H_ℓ^* by removing vertices from the component R, we have that $b(H') \leqslant \ell + 1$ and $b^1(H') = 0$, and so $2b(H') + b^1(H') \leqslant 2(\ell + 1)$.

Let $V^* = V(R \cup B \cup B_1 \cup B_2 \cup \cdots \cup B_{\ell-1})$ and note that $H' = H(S^*, V^* \setminus S^*)$. Applying Lemma 11(iii) to the $\ell + 1$ hypergraphs $R, B, B_1, B_2, \ldots, B_{\ell-1}$, we therefore have that

$$\begin{aligned}\phi(H) - \phi(H') &\geqslant 6|V^*| + 4e_4(E^{**}) + 6e_3(E^{**}) + 10e_2(E^{**}) + 4|E^*| - 2(\ell+1)\\ &= (24|S^*| - 2(\ell+1)) + 4(\ell+2) - 2(\ell+1)\\ &= 24|S^*| + 4 \;\geqslant\; 24|S^*|,\end{aligned}$$

contradicting Fact 1. This completes the proof of Claim 18. $(\square)$

Claim 19. $e_2(H) = 0$, *which by Claim 18 also implies that* $b^3(H) = 0$.

Proof. Suppose to the contrary that $e = \{x, y\}$ is a 2-edge in H. Recall by Claim 13 and 14 that $b(H) = b^1(H) = b^2(H) = 0$. Hence since $H_2 \in \mathcal{B}$, we have that $d_H(x) = 3$ or $d_H(y) = 3$ (or both). Renaming vertices if necessary, we may assume that $d_H(x) = 3$. Let e, e_1 and e_2 be the edges in H containing x. By Claim 16, the edges e' and e'' are both 4-edges. Let $e' = \{x, u_1, v_1, w_1\}$ and $e'' = \{x, u_2, v_2, w_2\}$. Let $H' = H(\{x\}, \varnothing)$ and let $X = \{x\}$.

If $b(H') > 0$, then since $b(H) = b^1(H) = b^2(H) = 0$ the component contributing to $b(H')$ must intersect e, e' and e'' and therefore contains the vertex y, contradicting Claim 17. Therefore, $b(H') = 0$. If $b^1(H') = 0$, then

$$\begin{aligned}\phi(H) - \phi(H') &\geqslant 6|X| + \omega(e) + \omega(e') + \omega(e'') - (2b(H') + b^1(H'))\\ &= 6 + 10 + (2 \times 4) - 0 \;=\; 24 \;=\; 24|X|,\end{aligned}$$

contradicting Fact 1. Hence, $b^1(H') \geqslant 1$. If $b^1(H') = 1$, then $24\tau(H') < \phi(H')$ since $b^1(H')$ is odd, and so

$$\begin{aligned}24\tau(H) &\leqslant 24(\tau(H') + 1) \;\leqslant\; (\phi(H') - 1) + 24\\ &= [\phi(H) - 6|X| - \omega(e) - \omega(e') - \omega(e'') + 2b(H') + b^1(H') - 1] + 24\\ &= [\phi(H) - 6 - 10 - (2 \times 4) + 1 - 1] + 24 \;=\; \phi(H),\end{aligned}$$

a contradiction. Hence, $b^1(H') \geqslant 2$. Let $R \in \mathcal{B}$ contribute to $b^1(H')$. By Claim 17, the vertex $y \notin V(R)$ and therefore R contributes to $b^3(H)$ and is intersected by both e' and e''. By Claim 18, we have that $R = H_2$. Let $e_2 = \{z, w\}$ denote the edge in R, and so $y \notin \{z, w\}$.

Suppose that the edges e' and e'' intersect the edge e_2 in the same vertex, say $z \in V(e_2) \cap V(e') \cap V(e'')$. Now let H^* be obtained from H by deleting the vertices x and z and edges e, e', e'' and e_2 and adding a 2-edge $\{y, w\}$. Let S^* be a $\tau(H^*)$-set. In order to cover the 2-edge $\{y, w\}$, we note that $|S^* \cap \{y, w\}| \geqslant 1$. If $y \in S^*$, then $S^* \cup \{z\}$ is a transversal in H. If $w \in S^*$, then $S^* \cup \{x\}$ is a transversal in H. In both cases, there exists a transversal in H of size $|S^*| + 1$, implying that $\tau(H) \leqslant \tau(H^*) + 1$. Furthermore since $|V(e) \setminus \{x\}| + |V(e') \setminus \{x, z\}| + |V(e'') \setminus \{x, z\}| + |V(e_2) \setminus \{z\}| = 6$ and since we added the edge $\{y, w\}$, we note that $2b(H^*) + b^1(H^*) \leqslant 6$ (in fact one can show that it is at most 3). Therefore,

$$\begin{aligned}\phi(H)-\phi(H^*) &\geqslant 6|\{x,z\}|+\omega(e')+\omega(e'')+\omega(e)+\omega(e_2)\\ &\qquad -\omega(\{y,w\})-(2b(H^*)+b^1(H^*))\\ &= (2\times 6)+(2\times 4)+(2\times 10)-10-6\\ &= 24 \;\geqslant\; 24(\tau(H)-\tau(H^*),\end{aligned}$$

contradicting Fact 1. Hence, e' and e'' do not intersect R in the same vertex. Renaming vertices in e' and e'', if necessary, we may assume that $e_2 = \{u_1, u_2\}$, where we recall that $e' = \{x, u_1, v_1, w_1\}$ and $e'' = \{x, u_2, v_2, w_2\}$. Since $b^1(H') \geqslant 2$ there is also another subhypergraph $R' \in \mathcal{B}$ which contributes to $b^1(H')$. Analogously to the above arguments for R, we have that R' contributes to $b^3(H)$, R' is isomorphic to H_2 and we may assume that the edge, e_3, in R' is $\{v_1, v_2\}$. Since R contributes to $b^3(H)$, there is an edge f in $E(H)\setminus\{e_2\}$ that intersects R distinct from e' and e''. By Claim 12, the edge f contains exactly one of u_1 and u_2. Therefore exactly one vertex in $\{u_1, u_2\}$ has degree 2 in H and the other vertex has degree 3 in H. Analogously, there is an edge f' in $E(H)\setminus\{e_3\}$ that intersects R' distinct from e' and e''. Further, exactly one vertex in $\{v_1, v_2\}$ has degree 2 in H and the other vertex has degree 3 in H. Without loss of generality we may assume that $d_H(u_1) = 3$ (and so, $d_H(u_2) = 2$). By Claim 17, we note that f and f' are 4-edges.

Suppose that $d_H(v_2) = 3$. In this case, we let $H'' = H(Y, Y')$, where $Y = \{u_1, v_2\}$ and $Y' = \{u_2, v_1\}$. It is not difficult to see that $2b(H'') + b^1(H'') \leqslant 6$. Therefore the following holds (even if $f = f'$).

$$\begin{aligned}\phi(H)-\phi(H'') &\geqslant 6|Y|+6|Y'|+\omega(e')+\omega(e'')+\omega(e_2)+\omega(e_3)\\ &\qquad +\omega(f)-(2b(H'')+b^1(H''))\\ &= (4\times 6)+(4\times 3)+(2\times 10)-6\\ &> 48 \;=\; 24|Y|,\end{aligned}$$

contradicting Fact 1. Therefore, $d_H(v_1) = 3$. Let H^x be obtained from H by deleting the vertices u_2 and v_2 and edges e', e'', e_2 and e_3 and adding the 2-edge $e^x = \{u_1, v_1\}$. Let S^x be a $\tau(H^x)$-set. In order to cover the 2-edge e^x, we note that $|S^x \cap \{u_1, v_1\}| \geqslant 1$. If $u_1 \in S^x$, then $S^x \cup \{v_2\}$ is a transversal in H. If $v_1 \in S^x$, then $S^x \cup \{u_2\}$ is a transversal in H. In both cases, there exists a transversal in H^x of size $|S^x| + 1$, implying that $\tau(H) \leqslant \tau(H^x) + 1$.

If H^x contains a component, R^x, that belongs to $\mathcal{B}$, then since $b(H) = b^1(H) = b^2(H) = 0$ the component R^x must intersect at least three of the edges e', e'', e_2 and e_3 and therefore contains both vertices u_1 and v_1 (recall that $\{u_1, v_1\}$ is an edge in H^x). Hence, $b(H^x) \leqslant 1$. Suppose $b^1(H^x) \geqslant 1$. In this case, let $R^x \in \mathcal{B}$ contribute to $b^1(H^x)$. Since $b(H) = b^1(H) = b^2(H) = 0$, the subhypergraph R^x must intersect at least two of the edges e', e'', e_2 and e_3. In particular, if $w_1 \in V(R^x)$, then R^x must contains at least one of the vertices u_1, x and w_2. An analogous argument holds if $w_2 \in V(R^x)$. Further since $\{u_1, v_1\}$ is an edge of H^x, this implies that $b^1(H^x) \leqslant 3$. Moreover, if $b^1(H^x) = 3$, then $b(H^x) = 0$. Thus if $b(H^x) = 1$, then $b^1(H^x) \leqslant 2$. Hence, $2b(H^x) + b^1(H^x) \leqslant 4$. Therefore,

$$\begin{aligned}\phi(H) - \phi(H^x) &\geqslant 6|\{u_2, v_2\}| + \omega(e') + \omega(e'') + \omega(e_2) + \omega(e_3) \\ &\quad -\omega(\{u_1, v_1\}) - (2b(H^x) + b^1(H^x)) \\ &= (2 \times 6) + (2 \times 4) + (2 \times 10) - 10 - 4 \\ &> 24 \geqslant 24(\tau(H) - \tau(H^x),\end{aligned}$$

contradicting Fact 1. This completes the proof of Claim 19. (□)

Claim 20. *There are no* 3*-edges* $e_1, e_2 \in E(H)$ *with* $|V(e_1) \cap V(e_2)| = 2$.

Proof. Suppose to the contrary that $e_1, e_2 \in E(H)$ are 3-edges and $|V(e_1) \cap V(e_2)| = 2$. Let H' be obtained from H by removing e_1 and e_2 and adding the edge $f = V(e_1) \cap V(e_2)$. Every transversal in H' is also a transversal in H, and so $\tau(H) \leqslant \tau(H')$. By Claims 13, 14 and 19 we have that $b(H) = b^1(H) = b^2(H) = b^3(H) = 0$. This implies that $b(H - e_1 - e_2) = b^1(H - e_1 - e_2) = 0$, which in turn implies that $2b(H') + b^1(H') \leqslant 2$. Therefore,

$$\begin{aligned}\phi(H) - \phi(H') &\geqslant \omega(e_1) + \omega(e_2) - \omega(f) - (2b(H') + b^1(H')) \\ &\geqslant 6 + 6 - 10 - 2 = 0 \geqslant 24(\tau(H) - \tau(H')),\end{aligned}$$

contradicting Fact 1. (□)

Claim 21. *There are no* 4*-edges* $e_1, e_2 \in E(H)$ *with* $|V(e_1) \cap V(e_2)| = 3$.

Proof. This is proved analogously to Claim 20. Suppose to the contrary that $e_1, e_2 \in E(H)$ are 4-edges and $|V(e_1) \cap V(e_2)| = 3$. Let H' be obtained from H by removing e_1 and e_2 and adding the edge $f = V(e_1) \cap V(e_2)$. Then, $\tau(H) \leqslant \tau(H')$ and $2b(H') + b^1(H') \leqslant 2$. Therefore,

$$\begin{aligned}\phi(H) - \phi(H') &\geqslant \omega(e_1) + \omega(e_2) - \omega(f) - (2b(H') + b^1(H')) \\ &\geqslant 4 + 4 - 6 - 2 = 0 \geqslant 24(\tau(H) - \tau(H')),\end{aligned}$$

contradicting Fact 1. (□)

Claim 22. *There is no* 3*-edge* e_1 *and* 4*-edge* e_2 *in* H *with* $|V(e_1) \cap V(e_2)| = 2$.

Proof. Suppose to the contrary that $e_1 = \{u, v, x\}$ is a 3-edges and $e_2 = \{u, v, s, t\}$ is a 4-edge with $V(e_1) \cap V(e_2) = \{u, v\}$. Suppose that $d_H(u) = 3$ and let e_u be the third edge that contains u. If $d_{H(\{u\},\varnothing)}(v) = 0$, then let $H' = H(\{u\}, \{v\})$. In this case, we note that since $b(H) = b^1(H) = b^2(H) = b^3(H) = 0$, we have $2b(H') + b^1(H') \leqslant 1$. Therefore,

$$\begin{aligned}\phi(H) - \phi(H') &\geqslant 2|\{u, v\}| + \omega(e_1) + \omega(e_2) + \omega(e_u) - (2b(H') + b^1(H')) \\ &\geqslant 12 + 6 + 4 + 4 - 1 > 24 = 24|\{u\}|,\end{aligned}$$

contradicting Fact 1. Hence, $d_{H(\{u\},\varnothing)}(v) > 0$. In this case let H' be obtained from H by removing e_1 and e_2 and adding the edge $f = \{u, v\}$. Since $b(H - e_1 - e_2) = b^1(H - e_1 - e_2) =$

0, we note that if $2b(H') + b^1(H') > 0$, then the edge f must belong to a subhypergraph $R \in \mathcal{B}$ which contributes to $b(H')$ or $b^1(H')$. Since $d_H(u) = 3$ and $d_{H(\{u\},\varnothing)}(v) > 0$, we note that $R \neq H_2$. By Lemma11(xii) we note that R contains two 3-edges overlapping in two vertices or two 4-edges overlapping in three vertices, a contradiction against Claim 20 and 21. Therefore $2b(H') + b^1(H') = 0$ and $\phi(H) - \phi(H') = 6+4-10 = 0$, a contradiction to Fact 1. Therefore, $d_H(u) = 2$. Analogously, $d_H(v) = 2$.

Let $H^* = H(\varnothing, \{u\})$. Hence, H^* is obtained from H by deleting the vertex u and the two edges e_1 and e_2 and adding the 2-edge $e'_1 = \{x, v\}$ and the 3-edge $e'_2 = \{v, s, t\}$. Since every transversal in H^* is a transversal in H, we have that $\tau(H) \leqslant \tau(H^*)$. If $b(H^*) = b^1(H^*) = 0$, then we have that

$$\begin{array}{rcl} \phi(H) - \phi(H^*) & \geqslant & 6|\{u\}| + \omega(e_1) + \omega(e_2) - \omega(e'_1) - \omega(e'_2) - (2b(H^*) + b^1(H^*)) \\ & \geqslant & 6 + 6 + 4 - 10 - 6 \;=\; 0 \;\geqslant\; 24(\phi(H) - \phi(H^*)), \end{array}$$

contradicting Fact 1. Hence, $2b(H^*) + b^1(H^*) > 0$. Let $R \in \mathcal{B}$ be a subhypergraph in H^* contributing to $b(H^*)$ or $b^1(H^*)$. Since $b(H - e_1 - e_2) = b^1(H - e_1 - e_2) = 0$, the edge e'_1 or e'_2 must belong to R, implying that $v \in V(R)$. However we note that $d_{H^*}(v) = 2$ and that v is incident to the 2-edge $e'_1 = \{x, v\}$ and the 3-edge $e'_2 = \{v, s, t\}$.

Suppose $d_R(v) = 1$. Then by Lemma 11(x) we have that $R = H_2$. But since the edge e'_2 intersects R, we have that R contributes to $b^1(H^*)$ and that e'_2 is the only edge intersecting R. This in turn implies that $d_H(x) = 1$. But then letting $H^x = H(\{u, v, x\}, \varnothing)$, we have that every transversal in H^x can be extended to a transversal in H by adding to it the vertex u, and so $\tau(H) \leqslant \tau(H^x)+1$. Further, $b(H^x) = b^1(H^x) = 0$, and so $\phi(H)-\phi(H^x) = 6|\{u, v, x\}| + \omega(e_1) + \omega(e_2) - (2b(H^x) + b^1(H^x)) = 18 + 6 + 4 > 24 \leqslant 24(\tau(H) - \tau(H^x))$, contradicting Fact 1. Hence, $d_R(v) = 2$.

Since $d_R(v) = 2$, both edges e'_1 and e'_2 belong to R. By Lemma11(xii) we note that R contains two 3-edges overlapping in two vertices or two 4-edges overlapping in three vertices. By Claim 20 and 21 we note that R contains two 3-edges overlapping in two vertices and $e'_2 = \{v, s, t\}$ is one of these 3-edges. By Lemma11(xiii) and Claim 21 we note that $\{x, s, t\}$ is an edge in R and therefore also in H^* and H. Considering the edges $\{x, s, t\}$ and $\{u, v, s, t\}$ instead of e_1 and e_2, we have that $d_H(s) = d_H(t) = 2$ (analogously to the arguments showing that $d_H(u) = 2$ and $d_H(v) = 2$).

Let F be the hypergraph with $V(F) = \{u, v, x, s, t\}$ and with $E(F) = \{e_1, e_2, e_3\}$. We note that F is obtained by using Step (D) in Definition 1 on two disjoint copies of H_2, and so $F \in \mathcal{B}$. On the one hand, if $d_H(x) = 2$, then $H = F$ since recall that, by Claim 13, H is connected. But this implies that $b(H) = 1$. On the other hand, if $d_H(x) = 3$, then F is a component of $H - e'$, where e' denote the edge of H containing x different from e_1 and e_2. But this implies that the subhypergraph $F \in \mathcal{B}$ contributes to $b^1(H)$, and so $b^1(H) \geqslant 1$. In both cases, we contradict Claim 13. This completes the proof of Claim 22. (□)

Claim 23. *No $B \in \mathcal{B}$ is a subhypergraph of H.*

Proof. Suppose to the contrary that $R \in \mathcal{B}$ is a subhypergraph of H. By Claim 19, we have that $e_2(H) = 0$, implying that in order to create R in Definition 1 we must have used

Step (D) last. However this implies that a 3-edge and a 4-edge overlap in two vertices, a contradiction to Claim 22. $_{(\Box)}$

Claim 24. *There are no overlapping edges in H.*

Proof. Suppose to the contrary that $e_1, e_2 \in E(H)$ have $|V(e_1) \cap V(e_2)| \geqslant 2$. By Claims 19, 20, 21 and 22 we note that e_1 and e_2 are both 4-edges and $|V(e_1) \cap V(e_2)| = 2$. Let $e_1 = \{u, v, x_1, y_1\}$ and $e_2 = \{u, v, x_2, y_2\}$. Suppose that $d_H(u) = 2$. Let $H' = H(\varnothing, \{u\})$. Hence, H' is obtained from H by deleting the vertex u and the two edges e_1 and e_2 and adding the edges $e'_1 = \{v, x_1, y_1\}$ and $e'_2 = \{v, x_2, y_2\}$. Since every transversal in H' is a transversal in H, we have that $\tau(H) \leqslant \tau(H')$. Every $R \in \mathcal{B}$ contributing to $2b(H') + b^1(H')$ must contain the vertex v, implying that $2b(H') + b^1(H') \leqslant 2$. Therefore,

$$\begin{aligned}\phi(H) - \phi(H') &\geqslant 6|\{u\}| + \omega(e_1) + \omega(e_2) - \omega(e'_1) - \omega(e'_2) - (2b(H') + b^1(H')) \\ &\geqslant 6 + 4 + 4 - 6 - 6 - 2 \;=\; 0 \;\geqslant\; 24(\phi(H) - \phi(H')),\end{aligned}$$

contradicting Fact 1. Therefore, $d_H(u) = 3$. Analogously, $d_H(v) = 3$. Let f_u be the edge in $E(H) \setminus \{e_1, e_2\}$ containing u and let f_v be the edge in $E(H) \setminus \{e_1, e_2\}$ containing v. Without loss of generality, we may assume that $|V(f_u)| \geqslant |V(f_v)|$. Suppose that $f_u = f_v$. In this case, let $H' = H(\{u\}, \{v\})$. By Claim 23, no $B \in \mathcal{B}$ is a subhypergraph of H, and so $b(H') = b^1(H') = 0$. Therefore,

$$\begin{aligned}\phi(H) - \phi(H') &\geqslant 6|\{u, v\}| + \omega(e_1) + \omega(e_2) + \omega(f_u) - (2b(H') + b^1(H')) \\ &\geqslant 12 + 4 + 4 + 4 \;=\; 24 \;=\; 24|\{u\}|,\end{aligned}$$

contradicting Fact 1. Hence, $f_u \neq f_v$, implying that $v \notin V(f_u)$ and $u \notin V(F_v)$. By Claims 20, 21 and 22, there is a vertex $w \in V(f_v) \setminus (V(f_u) \cup \{v\})$. Let $f^* = (V(f_u) \setminus \{u\}) \cup \{w\}$. Then, $|V(f^*)| = |V(f_u)| \geqslant |V(f_v)|$. Let H^* be obtained from H be deleting the edges e_1, e_2, f_u, f_v and the vertices u and v, but adding the edge f^*. Let S^* be a $\tau(H^*)$-set and note that $|S^* \cap V(f^*)| \geqslant 1$. If $w \in S^*$, then let $S = S^* \cup \{u\}$, while if $w \notin S^*$, let $S = S^* \cup \{v\}$. In both cases, S is a transversal in H and $|S| = |S^*| + 1 = \tau(H^*) + 1$, implying that $\tau(H) \leqslant \tau(H^*) + 1$. Recalling that $\omega(f^*) = \omega(f_u)$, we have

$$\begin{aligned}\phi(H) - \phi(H^*) &\geqslant 6|\{u, v\}| + \omega(e_1) + \omega(e_2) + \omega(f_u) + \omega(f_v) \\ &\qquad -\omega(f^*) - 2b(H^*) - b^1(H^*) \\ &\geqslant 12 + 4 + 4 + \omega(f_v) - 2b(H^*) - b^1(H^*) \\ &= 20 + w(f_v) - 2b(H^*) - b^1(H^*).\end{aligned}$$

By Claim 23, no $B \in \mathcal{B}$ is a subhypergraph of H. Hence any subgraph $R \in \mathcal{B}$ contributing to $2b(H^*) + b^1(H^*)$ must contain the edge f^*, implying that $2b(H^*) + b^1(H^*) \leqslant 2$. If f_v is a 3-edge, then $w(f_v) = 6$ and $w(f_v) - 2b(H^*) - b^1(H^*) \geqslant 4$. But then $\phi(H) - \phi(H^*) \geqslant 20 + w(f_v) - 2b(H^*) - b^1(H^*) \geqslant 24 \geqslant 24(\tau(H) - \tau(H^*))$, contradicting Fact 1. Hence, f_v is a 4-edge, implying that f^* and f_u are 4-edges. In particular, $\omega(f_v) = 4$. Furthermore if $2b(H^*) + b^1(H^*) = 0$, then $w(f_v) - 2b(H^*) - b^1(H^*) \geqslant 4$ and therefore that

$\phi(H) - \phi(H^*) \geqslant 24(\tau(H) - \tau(H^*))$, contradicting Fact 1. Hence, $2b(H^*) + b^1(H^*) \geqslant 1$. Let $R \in \mathcal{B}$ be a subhypergraph in H^* contributing to $2b(H^*) + b^1(H^*)$.

By Claim 19, we have $e_2(H) = 0$. Since no 2-edges are added when constructing H^*, we therefore have that $e_2(H^*) = 0$. This implies that the last step performed in the creation of R in Definition 1 is Step (D). This in turn implies that in H^* there is a 4-edge intersected by two 3-edges. Moreover, such a 4-edge intersects each of these 3-edges in two vertices. By Claim 22, this 4-edge must therefore be the new edge f^* added when constructing H^*. By Step (D) in Definition 1 we furthermore note that the two 3-edges that intersect the 4-edge f^* intersect it in disjoint sets. Hence there is a 3-edge, not containing the vertex $w \in V(F^*)$, that intersects f^* in two vertices. But this implies it intersected f_u in two vertices, a contradiction to Claim 22. This completes the proof of Claim 24. (□)

Claim 25. $\delta(H) \geqslant 2$.

Proof. Suppose to the contrary that a vertex $x \in V(H)$ has $d_H(x) = 1$. Let e be the edge containing x and let $e' = V(e)\setminus\{x\}$. Let $H' = H(\varnothing, \{x\})$ and note that $\tau(H) = \tau(H')$. By Claim 23, no $B \in \mathcal{B}$ is a subhypergraph of H, implying that $2b(H') + b^1(H') \leqslant 2$. If e is a 3-edge, then $\phi(H) - \phi(H^*) \geqslant 6|\{x\}| + \omega(e) - \omega(e') - 2b(H') - b^1(H') \geqslant 6 + 6 - 10 - 2 \geqslant 0 \geqslant 24(\tau(H) - \tau(H'))$, contradicting Fact 1. Hence, e is a 4-edge. But then $\phi(H) - \phi(H^*) \geqslant 6|\{x\}| + \omega(e) - \omega(e') - 2b(H') - b^1(H') \geqslant 6 + 4 - 6 - 2 > 0 \geqslant 24(\tau(H) - \tau(H'))$, once again contradicting Fact 1. (□)

Claim 26. *Every vertex of degree* 2 *in* H *is incident with two* 3*-edges.*

Proof. Assume that $d_H(x) = 2$. By Claim 19, we have $e_2(H) = 0$. Suppose to the contrary that x is incident with at least one 4-edge, e. Let f denote the remaining edge that contains x. Let $H' = H(\varnothing, \{x\})$ and note that $\tau(H) \leqslant \tau(H')$. We first show that $2b(H') + b^1(H') = 0$. If this is not the case, let $R \in \mathcal{B}$ be a subhypergraph of H' that contributes to $b(H')$ or $b^1(H')$. Since all hypergraphs in $\mathcal{B} \setminus H_2$ have overlapping edges while there are no overlapping edges in H', by Claim 24, we must have that $R = H_2$. By Claim 23, no $B \in \mathcal{B}$ is a subhypergraph of H, implying that f is a 3-edge and R necessarily contains the 2-edge $V(f) \setminus \{x\}$. However since $\delta(H) \geqslant 2$ by Claim 2, both vertices in R are incident with at least one edge in $E(H) \setminus \{e'\}$. Further since there are no overlapping edges in H, these edges are distinct. But this implies that there are least two edges in $E(H') \setminus E(R)$ intersecting $V(R)$, and so R does not contribute to $b(H')$ or $b^1(H')$, a contradiction. Therefore, $2b(H') + b^1(H') = 0$. Letting $e' = V(e) \setminus \{x\}$ and $f' = V(f) \setminus \{x\}$, we note that $\omega(e') - \omega(e) = 2$ and $\omega(f') - \omega(f) \leqslant 4$. Therefore,

$$\begin{aligned}
\phi(H) - \phi(H') &\geqslant 6|\{x\}| + \omega(e) + \omega(f) - \omega(e') - \omega(f') - (2b(H') + b^1(H')) \\
&\geqslant 6 - (\omega(e') - \omega(e)) - (\omega(f') - \omega(f)) - 0 \\
&\geqslant 6 - 2 - 4 \;=\; 0 \;\geqslant\; 24(\tau(H) - \tau(H'),
\end{aligned}$$

contradicting Fact 1. This completes the proof of Claim 26. (□)

Claim 27. *Every vertex of degree* 3 *in* H *is incident with a* 3*-edge and a* 4*-edge.*

Proof. Assume that $d_H(x) = 3$ and suppose to the contrary that x is contained in only 3-edges or only 4-edges. Suppose first that x is contained in only 3-edges and let $H' = H(\{x\}, \emptyset)$. By Claim 23, we have that $b(H') = b^1(H') = 0$. Therefore since each of the three edges incident with x has weight 6, we have that $\phi(H) - \phi(H') = 6 + (3 \times 6) = 24 = 24|\{x\}|$, contradicting Fact 1. Hence, x is contained in only 4-edges.

We now let $H^* = H(\emptyset, \{x\})$ and note that $\tau(H) \leqslant \tau(H')$. Since there are no 2-edges in H^* and no overlapping edges in H^* by Claim 24, we note that $b(H^*) = b^1(H^*) = 0$. Therefore since each of the three deleted edges has weight 4 and each of the three added edges has weight 6, we have that $\phi(H) - \phi(H') = 6 - (3 \times 2) = 0 \geqslant 24(\tau(H) - \tau(H')$, contradicting Fact 1. This completes the proof of Claim 27. (□)

Claim 28. *Every* 3*-edge in* H *contains a vertex of degree* 3.

Proof. Assume that $e = \{u_1, u_2, u_3\} \in E(H)$ and suppose to the contrary that $d_H(u_1) = d_H(u_2) = d_H(u_3) = 2$. For $i = 1, 2, 3$, let e_i be the edge in $E(H) \setminus \{e\}$ containing u_i. By Claim 24, e_1, e_2 and e_3 are distinct edges and by Claim 26 they are all 3-edges.

Suppose first that $|V(e_i) \cup V(e_j)| \leqslant 5$ for all $1 \leqslant i < j \leqslant 3$. In this case, by Claim 24, every pair of edges in $\{e_1, e_2, e_3\}$ intersect in exactly one vertex. So let $V(e_i) \cap V(e_j) = \{v_{i,j}\}$ for $1 \leqslant i < j \leqslant 3$. If $v_{1,2}$, $v_{1,3}$ and $v_{2,3}$ are not distinct vertices, then we must have $v_{1,2} = v_{1,3} = v_{2,3}$, which contradicts Claim 27. Hence, $v_{1,2}$, $v_{1,3}$ and $v_{2,3}$ are distinct vertices. Hence, $e_1 = \{u_1, v_{12}, v_{13}\}$, $e_2 = \{u_2, v_{12}, v_{23}\}$, and $e_3 = \{u_3, v_{13}, v_{23}\}$. Let H' be obtained from H by deleting the edges e, e_1, e_2, e_2 and vertices u_1, u_2, u_3 and adding the edge $f = \{v_{1,2}, v_{1,3}, v_{2,3}\}$. We will first show that $\tau(H) \leqslant \tau(H') + 1$. Let S' be a $\tau(H')$-set. Since S' intersects the edge f, we may assume, renaming vertices if necessary, that $v_{1,2} \in S'$. But then $S' \cup \{u_3\}$ is a transversal of H, and so $\tau(H) \leqslant |S'| + 1 = \tau(H') + 1$, as desired. Clearly $2b(H') + b^1(H') \leqslant 2$ as any subgraph contributing to $2b(H') + b^1(H')$ must contain the added edge f since by Claim 23 no subhypergraph of H belongs to $\mathcal{B}$. Therefore,

$$
\begin{aligned}
\phi(H) - \phi(H') &\geqslant 6|\{u_1, u_2, u_3\}| + \omega(e) + \omega(e_1) + \omega(e_2) + \omega(e_3) \\
&\qquad -\omega(f) - (2b(H') + b^1(H')) \\
&\geqslant (3 \times 6) + (4 \times 6) - 6 - 2 \\
&> 24 \geqslant 24(\tau(H) - \tau(H'),
\end{aligned}
$$

contradicting Fact 1. We may therefore assume, renaming vertices if necessary, that $|V(e_1) \cup V(e_2)| = 6$; that is, the 3-edges e_1 and e_2 do not intersect. Let $f_{1,2} = (V(e_1) \cup V(e_2)) \setminus \{u_1, u_2\}$ and let $f_3 = V(e_3) \setminus \{u_3\}$. Let H^* be obtained from H by deleting the edges e, e_1, e_2, e_3 and vertices u_1, u_2, u_3 and adding the edges $f_{1,2}$ and f_3. We will first show that $\tau(H) \leqslant \tau(H^*) + 1$. Let S^* be a $\tau(H^*)$-set. Since S^* intersects the edge f_{12}, we may assume, renaming vertices if necessary, that $S^* \cap V(e_1) \neq \emptyset$. But then $S^* \cup \{u_2\}$ is a transversal of H, and so $\tau(H) \leqslant |S^*| + 1 = \tau(H^*) + 1$, as desired. Clearly, $2b(H^*) + b^1(H^*) \leqslant 4$ as any subhypergraph contributing to $2b(H^*) + b^1(H^*)$ must contain the edge $f_{1,2}$ or the edge f_3. Therefore, since $f_{1,2}$ is a 4-edge and f_3 a 2-edge, we have that

$$\begin{aligned}\phi(H)-\phi(H^*) &\geqslant 6|\{u_1,u_2,u_3\}|+\omega(e)+\omega(e_1)+\omega(e_2)+\omega(e_3)\\ &\quad -\omega(f_{1,2})-\omega(f_3)-(2b(H^*)+b^1(H^*))\\ &\geqslant (3\times 6)+(4\times 6)-4-10-4\\ &= 24 \ \geqslant\ 24(\tau(H)-\tau(H^*),\end{aligned}$$

contradicting Fact 1. This completes the proof of Claim 28. (□)

Claim 29. *Every 3-edge in H contains at least two vertices of degree 3.*

Proof. Assume that $e=\{u_1,u_2,u_3\}\in E(H)$ and suppose to the contrary that $d_H(u_2)=d_H(u_3)=2$. By Claim 28 we have $d_H(u_1)=3$. Let e_1' and e_2' be the two edges in $E(H)\setminus\{e\}$ containing u_1. For $i=2,3$, let e_i be the edge in $E(H)\setminus\{e\}$ containing u_i and let $f_i=V(e_i)\setminus\{u_i\}$. By Claim 26, the edges e_2 and e_3 are both 3-edges, and so f_2 and f_3 are both 2-edges. Let $H'=H(\{u_1\},\{u_2,u_3\})$. Let $V(f_2)=\{v_2,w_2\}$.

We will first show that $b(H')=0$. If this is not the case, then let $R\in\mathcal{B}$ be a component in H'. By Claim 24, we have that $R=H_2$, which by Claim 19, implies that f_2 or f_3 is the edge in R. Renaming vertices if necessary, we may assume that $E(R)=\{f_2\}$, and so $V(R)=\{v_2,w_2\}$. Since there is no edge in $E(H')\setminus\{f_2\}$ that intersects $V(R)$, we note that the edges f_2 and f_3 do not intersect. By Claim 26, each vertex in $V(R)$ is either incident to three edges in H or two 3-edges in H. Suppose both v_2 and w_2 are incident to two 3-edges in H. This implies that there are two distinct 3-edges that contain (exactly) one of v_2 and w_2 and these two 3-edges are different from the edge e_2 (and from the edge e). Since the vertex u_1, which has degree 3 in H, cannot be incident to three 3-edges by Claim 27, at least one of these 3-edges that contain v_2 or w_2 is different from both e_1' and e_2'. This 3-edge belongs to $E(H')\setminus\{f_2\}$ and intersects $V(R)$, a contradiction to the fact that R is a component in H'. Hence at least one of v_2 and w_2 is incident to three edges in H and the other to at least two edges in H. But once again this implies that there exists an edge that contain v_2 or w_2 and is different from the deleted edges e,e_1',e_1'',e_2,e_3 and the edge f_2, a contradiction again to the fact that R is a component in H'. Therefore, $b(H')=0$. If $b^1(H')=0$, then

$$\begin{aligned}\phi(H)-\phi(H') &\geqslant 6|\{u_1,u_2,u_3\}|+\omega(e)+\omega(e_1')+\omega(e_1'')+\omega(e_2)+\omega(e_3)\\ &\quad -\omega(f_2)-\omega(f_3)-(2b(H')+b^1(H'))\\ &\geqslant (3\times 6)+(3\times 6)+(2\times 4)-(2\times 10)-0\\ &= 24 \ =\ 24|\{u_1\}|,\end{aligned}$$

contradicting Fact 1. Hence, $b^1(H')\geqslant 1$. Let $R\in\mathcal{B}$ be a subhypergraph in H' contributing to $b^1(H')$. By Claim 24, there are no overlapping edges in H and therefore in H', implying that $R=H_2$. This in turn implies by Claim 19 that f_2 or f_3 is the edge in R. Renaming vertices if necessary, we may assume that $E(R)=\{f_2\}$. Let e' be the edge in $E(H')\setminus\{f_2\}$ that intersects R. Since there are no overlapping edges in H, we note that $|V(f_2)\cap V(e')|=1$. Renaming vertices in f_2 if necessary, we may assume that $V(f_2)\cap V(e')=\{v_2\}$.

We now consider the hypergraph $H^* = H'(\{v_2\}, \{w_2\})$ obtained from H' by deleting the vertices v_2 and w_2 and deleting the edge e'. Note that $H^* = H(\{\{u_1, v_2\}, \{u_2, u_3, w_2\})$. By Claims 19, 23 and 24 the only possibly subhypergraph in H^* in $\mathcal{B}$ is the hypergraph isomorphic to H_2 that consists of the 2-edge f_3, implying that $2b(H^*) + b^1(H^*) \leqslant 2$. Therefore,

$$\begin{aligned} \phi(H) - \phi(H^*) &\geqslant 6|\{u_1, u_2, u_3, v_2, w_2\}| + \omega(e) + \omega(e_1') + \omega(e_1'') + \omega(e_2) + \omega(e_3) \\ &\quad +\omega(e') - \omega(f_3) - (2b(H^*) + b^1(H^*)) \\ &\geqslant (5 \times 6) + (3 \times 6) + (3 \times 4) - 10 - 2 \\ &= 48 \geqslant 24|\{u_1, v_2\}|, \end{aligned}$$

contradicting Fact 1. This completes the proof of Claim 29. (□)

Claim 30. *No vertex is contained in two* 3*-edges and one* 4*-edge, such that one of the* 3*-edges contains a degree-*2 *vertex.*

Proof. Assume that $e_1 = \{x, u_1, v_1\}$, $e_2 = \{x, u_2, v_2\}$ and $e_3 = \{x, u_3, v_3, w_3\}$ are edges in H and suppose to the contrary that $d_H(u_1) = 2$. By Claim 26, u_1 is incident with two 3-edges, say e_1 and $f_1 = \{u_1, x_1, y_1\}$. Let $H' = H(\{x\}, \{u_1\})$. If $2b(H') + b^1(H') > 0$, then let $R \in \mathcal{B}$ be a subhypergraph in H' contributing to $2b(H') + b^1(H')$. By Claim 24, there are no overlapping edges in H and therefore in H', implying that $R = H_2$. This in turn implies by Claim 19 that the edge in $E(R)$ is $g = \{x_1, y_1\}$. By supposition, $d_H(u_1) = 2$. Hence by Claim 29 the two vertices, namely x_1 and y_1, in the 3-edge f_1 both have degree 3 in H. Since there are no overlapping edges in H, there are therefore four distinct edges in H excluding the edge f_1 that intersect $V(R)$. Further we note that the vertex u_1 is the only vertex common to both edges e_1 and f_1, implying that the edge e_1 does not intersect $V(R)$. Hence removing the three edges e_1, e_2 and e_3 from H can remove at most two edges intersecting $V(R)$, implying that at least two edges in H that intersect $V(R)$ remain in H'. But then R does not contribute to $2b(H') + b^1(H')$, a contradiction. Therefore, $2b(H') + b^1(H') = 0$. This implies that

$$\begin{aligned} \phi(H) - \phi(H') &\geqslant 6|\{x, u_1\}| + \omega(e_1) + \omega(e_2) + \omega(e_3) + \omega(f_1) \\ &\quad -\omega(g) - (2b(H') + b^1(H')) \\ &\geqslant (2 \times 6) + (3 \times 6) + 4 - 10 - 0 \\ &= 24 = 24|\{x\}|, \end{aligned}$$

contradicting Fact 1. (□)

Claim 31. *H is* 3*-regular.*

Proof. Suppose to the contrary that $\delta(H) = 2$. Let x be a vertex of degree 2 in H. By Claim 26, x is incident with two 3-edges in H, say $e_1 = \{x, u_1, v_1\}$ and $e_2 = \{x, u_2, v_2\}$. By Claims 27, 29 and 30 each vertex in $\{u_1, v_1, u_2, v_2\}$ is contained in one 3-edge and two 4-edges. Let f_1 and f_2 be the two 4-edges containing u_1 and let h_1 and h_2 be the two

4-edges containing v_1. For $i \in \{1,2\}$, let $h'_i = V(h_i) \setminus \{v_1\}$. Let $e'_2 = \{u_2, v_2\}$. We note that h'_1 and h'_2 are both 3-edges. We now consider the hypergraph $H' = H(\{u_1\}, \{x, v_1\})$.

We will first show that $2b(H') + b^1(H') = 0$. If this is not the case, then let $R \in \mathcal{B}$ be a subhypergraph in H' contributing to $2b(H') + b^1(H')$. By Claim 24, there are no overlapping edges in H and therefore in H', implying that $R = H_2$. This in turn implies by Claim 19 that $V(R) = \{u_2, v_2\}$ as e'_2 is the only 2-edge in H'. By Claim 29, both vertices u_2 and v_2 have degree 3 in H. Since removing all edges containing u_1 can remove at most two edges intersecting $V(R)$ in H, at least two edges in H that intersect $V(R)$ remain in H'. But then R does not contribute to $2b(H') + b^1(H')$, a contradiction. Therefore, $2b(H') + b^1(H') = 0$. This implies that

$$\begin{array}{rcl} \phi(H) - \phi(H') & \geqslant & 6|\{x, u_1, v_1\}| + \omega(e_1) + \omega(e_2) + \omega(f_1) + \omega(f_2) + \omega(h_1) + \omega(h_2) \\ & & -\omega(h'_1) - \omega(h'_1) - \omega(e'_2) - (2b(H') + b^1(H')) \\ & \geqslant & (3 \times 6) + (2 \times 6) + (4 \times 4) - (2 \times 6) - 10 - 0 \\ & = & 24 \;\; = \;\; 24|\{u_1\}|, \end{array}$$

contradicting Fact 1. (□)

Claim 32. *All vertices are contained in two* 3*-edges and one* 4*-edge.*

Proof. Suppose to the contrary that there is a vertex x in H that is not adjacent with two 3-edges and one 4-edge. By Claim 32, $d_H(x) = 3$. By Claim 27, the vertex x is incident with a 3-edge and a 4-edge. By our supposition, the remaining edge incident with x is a 4-edge. Let $e_1 = \{x, u_1, v_1\}$, $e_2 = \{x, u_2, v_2, w_2\}$ and $e_3 = \{x, u_3, v_3, w_3\}$ be the three edges incident with x. For $i \in \{1, 2, \}$, let $e'_i = V(e_i) \setminus \{x\}$.

By Claim 27 and 32, we have that $d_H(u_1) = 3$ and u_1 is incident with either two 3-edges and one 4-edge or with one 3-edge and two 4-edges. Suppose that u_1 is incident with two 3-edges, say e_1 and f_1. In this case, let f_2 be the 4-edge that contains u_1. Let $H' = H(\{u_1\}, \{x\})$. Since $e_2(H') = 0$ and there are no overlapping edges in H', we note that $b(H') = b^1(H') = 0$. Therefore,

$$\begin{array}{rcl} \phi(H) - \phi(H') & \geqslant & 6|\{x, u_1\}| + \omega(e_1) + \omega(e_2) + \omega(e_3) + \omega(f_1) + \omega(f_2) \\ & & -\omega(e'_2) - \omega(e'_3) - (2b(H') + b^1(H')) \\ & \geqslant & (2 \times 6) + (2 \times 6) + (3 \times 4) - (2 \times 6) - 0 \\ & = & 24 \;\; = \;\; 24|\{u_1\}|, \end{array}$$

contradicting Fact 1. Hence, u_1 is incident with one 3-edge and two 4-edges. Analogously, v_1 is incident with one 3-edge and two 4-edges. Let h_1 and h_2 be the two 4-edges containing u_1 and let g_1 and g_2 be the two 4-edges containing v_1. For $i \in \{1, 2\}$, let $h'_i = V(h_i) \setminus \{u_1\}$ and let $g'_i = V(g_i) \setminus \{v_1\}$. We now consider the hypergraph $H^* = H(\{x\}, \{u_1, v_1\})$ and note that $e_2(H^*) = 0$. Further since there are no overlapping edges in H^*, we note that $b(H^*) = b^1(H^*) = 0$. Therefore,

$$\begin{array}{rcl} \phi(H) - \phi(H^*) & \geqslant & 6|\{x, u_1, v_1\}| + \omega(e_1) + \omega(e_2) + \omega(e_3) + \omega(h_1) + \omega(h_2) + \omega(g_1) \\ & & +\omega(g_2) - \omega(h'_1) - \omega(h'_2) - \omega(g'_1) - \omega(g'_1) - (2b(H^*) + b^1(H^*)) \\ & \geqslant & (3 \times 6) + 6 + (6 \times 4) - (4 \times 6) - 0 \\ & = & 24 \;\; = \;\; 24|\{x\}|, \end{array}$$

contradicting Fact 1. $(\Box)$

We now return to the proof of Theorem 3 to obtain a final contradiction implying the non-existence of our counterexample, H, to the theorem. Let $e = \{u_1, u_2, u_3\}$ be an arbitrary 3-edge in H. By Claim 32, each vertex of H is contained in two 3-edges and one 4-edge. For $i \in \{1,2,3\}$, let e_i be the 3-edge and f_i the 4-edge in $E(H) \setminus \{e\}$ that contains the vertex u_i. By Claim 24, the edges e_1, e_2 and e_3 are all distinct. For $i \in \{1,2,3\}$, let $f'_i = V(f_i) \setminus \{u_i\}$ and note that f'_i is a 3-edge.

Suppose that $V(e_i) \cap V(e_j) \neq \emptyset$ for all $1 \leqslant i < j \leqslant 3$. Let $V(e_i) \cap V(e_j) = \{v_{i,j}\}$ for $1 \leqslant i < j \leqslant 3$. If $v_{1,2}$, $v_{1,3}$ and $v_{2,3}$ are not distinct vertices, then we must have $v_{1,2} = v_{1,3} = v_{2,3}$, which implies that a vertex is incident with three 3-edges, contradicting Claim 32. Hence, $v_{1,2}$, $v_{1,3}$ and $v_{2,3}$ are distinct vertices. Thus, $e_1 = \{u_1, v_{12}, v_{13}\}$, $e_2 = \{u_2, v_{12}, v_{23}\}$, and $e_3 = \{u_3, v_{13}, v_{23}\}$. Let $h = \{v_{1,2}, v_{1,3}, v_{2,3}\}$. Let H' be obtained by deleting the edges $e, e_1, e_2, e_3, f_1, f_2, f_3$ and vertices u_1, u_2, u_3 and adding the 3-edges f'_1, f'_2, f'_3 and h. By Claim 23 and by construction, we note that if R is a subhypergraph contributing to $2b(H') + b^1(H')$, then R must contain the added 3-edge h, implying that $2b(H') + b^(H') \leqslant 2$. Suppose that S' is a $\tau(H')$-set. Since $|S' \cap V(h)| \geqslant 1$, we may assume renaming vertices if necessary that $v_{1,2} \in S'$. But then $S' \cup \{u_3\}$ is a transversal of H, and so $\tau(H) \leqslant |S'| + 1 = \tau(H') + 1$. Therefore,

$$\begin{array}{rcl}
\phi(H) - \phi(H') & \geqslant & 6|\{u_1, u_2, u_3\}| + \omega(e) + \omega(e_1) + \omega(e_2) + \omega(e_3) + \omega(f_1) + \omega(f_2) \\
& & \quad +\omega(f_3) - \omega(f'_1) - \omega(f'_2) - \omega(f'_3) - \omega(h) - (2b(H') + b^1(H')) \\
& \geqslant & (3 \times 6) + (4 \times 6) + (3 \times 4) - (4 \times 6) - 2 \\
& > & 24 \ \geqslant \ 24(\tau(H) - \tau(H')),
\end{array}$$

contradicting Fact 1. Hence, $V(e_i) \cap V(e_j) = \emptyset$ for some i and j where $1 \leqslant i < j \leqslant 3$. Renaming vertices if necessary, we may assume that $V(e_1) \cap V(e_2) = \emptyset$. For $i \in \{1,2,3\}$, let $e_i = \{u_i, x_i, y_i\}$. Since H has no overlapping edges by Claim 24, we know that $|V(f_2) \cap V(e_1)| \leqslant 1$. Renaming the vertices x_1 and y_1 if necessary, we may assume that $x_1 \notin V(f_2)$. This implies that there is no common edge containing both u_2 and x_1. We now consider the hypergraphs $H^* = H(\{x_1, u_2\}, \{u_1\})$. Then, $e_2(H^*) = 0$ and H^* has no overlapping edges, implying that $b(H^*) = b^1(H^*) = 0$. By Claim 32, the vertex x is contained in two 3-edges, say e_1 and e_x, and in one 4-edge, say f_x. We now have that

$$\begin{array}{rcl}
\phi(H) - \phi(H^*) & \geqslant & 6|\{u_1, u_2, x_1\}| + \omega(e) + \omega(e_1) + \omega(e_2) + \omega(f_1) + \omega(f_2) \\
& & \quad +\omega(e_x) + \omega(f_x) - \omega(f'_1) - (2b(H^*) + b^1(H^*)) \\
& \geqslant & (3 \times 6) + (4 \times 6) + (3 \times 4) - 6 - 0 \\
& = & 48 \ = \ 24|\{x_1, u_2\}|,
\end{array}$$

contradicting Fact 1. This completes the proof of Theorem 3. $\Box$

6 Proof of Theorem 6

Chvátal and McDiarmid proved the following bound in [2].

Theorem 33. ([2]) *If H is a 4-uniform hypergraph, then $6\tau(H) \leqslant n(H) + 2m(H)$.*

We are now in a position to prove Theorem 6. Recall the statement of the theorem.

Theorem 6. *If H is a 4-uniform hypergraph with n vertices and n edges, then $\tau(H) \leqslant \frac{3}{7}n$.*

Proof. Let H be a 4-uniform hypergraph with n vertices and n edges. Let x_1 be a vertex of maximum degree in H. Let x_2 be a vertex of maximum degree in $H - \{x_1\}$. Let x_3 be a vertex of maximum degree in $H - \{x_1, x_2\}$. Continue this process as long as the maximum degree in the resulting hypergraph is at least four and let $X = \{x_1, x_2, \ldots, x_\ell\}$ be the resulting set of chosen vertices. Let $H' = H - X$. By construction, $\Delta(H') \leqslant 3$, $n(H') = n(H) - |X|$ and $m(H') \leqslant m(H) - 4|X|$. Every $\tau(H')$-set can be extended to a transversal in H by adding to it the set X, implying that $\tau(H) \leqslant \tau(H') + |X|$. By Theorem 4, we have $\tau(H') \leqslant n(H')/4 + m(H')/6$ and by Theorem 33, we have $6\tau(H') \leqslant n(H') + 2m(H')$. The above observations imply that

$$
\begin{array}{rcl}
7\tau(H) & \leqslant & 7(\tau(H') + |X|) \\
& = & 6\tau(H') + \tau(H') + 7|X| \\
& \leqslant & 6\left(\frac{n(H')}{4} + \frac{m(H')}{6}\right) + \left(\frac{n(H')+2m(H')}{6}\right) + 7|X| \\
& = & \frac{20n(H')}{12} + \frac{16m(H')}{12} + 7|X| \\
& \leqslant & \frac{20(n(H)-|X|)}{12} + \frac{16(m(H)-4|X|)}{12} + 7|X| \\
& = & 3n + \left(7 - \frac{20+64}{12}\right)|X| \;\; = \;\; 3n.
\end{array}
$$

This completes the proof of Theorem 6. □

In order to present a proof of Theorem 7, we shall need a characterization of the hypergraphs that achieve equality in Theorem 33. For this purpose, let H_4 be the hypergraph on four vertices with only one hyperedge containing all four of these vertices. Let H_6 be the hypergraph with vertex set $\{a_1, a_2, b_1, b_2, c_1, c_2\}$ and edge set $E(H_6) = \{\{a_1, a_2, b_1, b_2\}, \{a_1, a_2, c_1, c_2\}, \{b_1, b_2, c_1, c_2\}\}$. The following result is given in [3].

Theorem 34. ([3]) *Let H be a 4-uniform hypergraph. If $6\tau(H) = n(H) + 2m(H)$, then every component of H is isomorphic to H_4 or H_6.*

We shall also need the following result in [4], where we recall that for a graph G, the *open neighborhood hypergraph*, abbreviated ONH, of G is the hypergraph H_G with vertex set $V(H_G) = V(G)$ and with edge set $E(H_G) = \{N_G(x) \mid x \in V\}$ consisting of the open neighborhoods of vertices in G. The transversal number of the ONH of a graph is precisely the total domination number of the graph; that is, for a graph G, we have $\gamma_t(G) = \tau(H_G)$.

Theorem 35. ([4]) *The ONH of a connected bipartite graph consists of two components, while the ONH of a connected graph that is not bipartite is connected.*

We are now in a position to prove Theorem 7. Recall the statement of the theorem.

Theorem 7. *If G is a connected graph of order n with $\delta(G) \geqslant 4$, then $\gamma_t(G) \leqslant 3n/7$, with equality if and only if G is the bipartite complement of the Heawood Graph.*

Proof. Let G be a connected graph of order n with $\delta(G) \geqslant 4$ and let H_G be the ONH of G. If G is not 4-regular, then let x be an arbitrary vertex in G with $d_G(x) \geqslant 5$. Now let H be obtained by shrinking all edges of size greater than four to size four in such a way that we never remove x from any edge. We note that the resulting hypergraph H is 4-uniform with $n(H) = m(H) = n(G) = n$, but H is not 4-regular. Alternatively if G is 4-regular, then let $H = H_G$ in which case again $n(H) = m(H) = n(G) = n$, but in this case H is 4-regular.

Let x_1 be a vertex of maximum degree in H. Let x_2 be a vertex of maximum degree in $H - \{x_1\}$. Let x_3 be a vertex of maximum degree in $H - \{x_1, x_2\}$. Continue this process as long as the maximum degree in the resulting hypergraph is at least four and let $X = \{x_1, x_2, \ldots, x_\ell\}$ be the resulting set of chosen vertices. Let $H' = H - X$ and note that the following holds.

(a): $\Delta(H') \leqslant 3$.

(b): $n(H') \leqslant n(H) - |X|$ and $m(H') \leqslant m(H) - 4|X|$. Furthermore if H is not 4-regular, then since x_1 removes at least five edges from H we have that $m(H') < m(H) - 4|X|$.

If $|X| < n/7$, then by Theorem 4 we have

$$\begin{array}{rcl} \tau(H) & \leqslant & \tau(H') + |X| \\ & \leqslant & \frac{n(H')}{4} + \frac{m(H')}{6} + |X| \\ & \leqslant & \frac{n(H)-|X|}{4} + \frac{m(H)-4|X|}{6} + |X| \\ & = & \left(\frac{1}{4} + \frac{1}{6}\right) n + \left(1 - \frac{1}{4} - \frac{4}{6}\right) |X| \\ & = & \frac{5n}{12} + \frac{|X|}{12} \\ & < & \left(\frac{5}{12} + \frac{1}{7 \times 12}\right) n \\ & = & 3n, \end{array}$$

and the desired result follows from the observation that $\gamma_t(G) = \tau(H_G) \leqslant \tau(H)$. Hence in what follows we may assume that $|X| \geqslant n/7$. By Theorem 33, we now have that

$$\begin{array}{rcl} \tau(H) & \leqslant & \tau(H') + |X| \\ & \leqslant & \left(\frac{n(H')+2m(H')}{6}\right) + |X| \\ & \leqslant & \left(\frac{n(H)-|X|+2(m(H)-4|X|)}{6}\right) + |X| \\ & = & \frac{n}{2} - \frac{9|X|}{6} + |X| \\ & = & \frac{n-|X|}{2} \\ & \leqslant & \frac{n-n/7}{2} \\ & = & 3n. \end{array}$$

Hence, $\gamma_t(G) = \tau(H_G) \leqslant \tau(H) \leqslant 3n/7$, proving the desired upper bound. Suppose that $\gamma_t(G) = 3n/7$. Then we must have equality throughout the above inequality chains. In particular, this implies that the following holds.

(c): $6\tau(H') = n(H') + 2m(H')$.

(d): $n(H') = n(H) - |X|$.

(e): $m(H') = m(H) - 4|X|$.

(f): $|X| = n/7$.

(g): H is 4-regular (by (b) and (e)).

Since (c) holds, Theorem 34 implies that every component of H' is isomorphic to H_4 or H_6. By (d), (e) and (f), and noting that $n(H) = m(H) = n$, we have that $n(H') = 6n/7$ and $m(H') = 3n/7$. Hence if $\overline{d}$ denotes the average degree in H', we have that

$$4m(H') = \sum_{v \in V(H')} d_{H'}(v) = n(H') \cdot \overline{d},$$

and so $\overline{d} = 4m(H')/n(H') = 2$. We show that every component of H' is an H_6-component. Suppose to the contrary that there is an H_4-component in H'. Each vertex in such a component has degree 1 in H'. Since the average degree in H' is 2, this implies that there must also be a vertex of degree 3 in H'. However such a vertex does not belong to an H_4- or an H_6-component, a contradiction. Therefore the following holds.

(h): Every component of H' is an H_6-component.

Suppose that $N_H(u_1) \cap N_H(u_2) \neq \varnothing$ for some $u_1, u_2 \in V(H)$. We show that u_1 and u_2 are contained in a common edge of H. Suppose to the contrary that no edge in H contains both u_1 and u_2 and let $w \in N(u_1) \cap N(u_2)$ be arbitrary. Let $f_1, f_2 \in E(H)$ be chosen so that $\{u_1, w\} \subset V(f_1)$ and $\{u_2, w\} \subset V(f_2)$. By the 4-regularity of H, we can choose the set X by starting with $x_1 = u_1$ and $x_2 = u_2$. We note that with this choice of the set X, the vertex $w \in V(H')$. By (h), the vertex w belongs to some H_6-component in H', implying that there is a vertex $w' \in V(H')$ such that w and w' both belong to two overlapping edges, say e_1 and e_2, in $E(H')$. However, if we had created X starting with $x_1 = w'$, then w would belong to an H_6-component, R, of H'. Since $d_H(w) = 4$ and $d_{H'}(w) = 2$, and since $e_1, e_2 \notin E(H')$, we have that $f_1, f_2 \in E(H')$ and $f_1, f_2 \in E(R)$. In particular, u_1 and u_2 are contained in a common edge of R and therefore of H, a contradiction. Therefore, the following holds.

(i): If $N_H(u_1) \cap N_H(u_2) \neq \varnothing$ for some $u_1, u_2 \in V(H)$, then there exists an edge $e \in E(H)$, such that $u_1, u_2 \in V(e)$.

Let $u_1 \in V(H)$ be arbitrary. If some edge e contains vertices from $N[u_1]$ and from $V(H) \setminus N[u_1]$, then let $u_2 \in V(e) \cap (V(H) \setminus N[u_1])$ and let $w \in V(e) \cap N[u_1]$ be arbitrary. Since u_1 and u_2 are not adjacent, $u_1 \neq w$ and no edge contains both u_1 and u_2. However this is a contradiction by (i). Therefore, the following holds.

(j): If $x \in V(H)$, then $N[x]$ is the vertex set of some component in H.

Let $x \in V(H)$ be arbitrary and let R_x be the component of H containing x. By (j), $V(R_x) = N[x]$. By the 4-regularity of H, we can choose the set X by starting with $x_1 = x$. Thus by (h), $R_x - \{x\}$ only contains components isomorphic to H_6. However since H is a 4-regular 4-uniform hypergraph, and since H_6 is 2-regular, we note that $R_x - \{x\}$ must contain only one component, which is isomorphic to H_6. Therefore, $|R_x| = 7$ and $R_x - \{x\} = H_6$. This is true for every vertex x of H, implying that R_x must be isomorphic to the complement of the Fano plane. Hence, the following holds.

(k): Every component of H is isomorphic to the complement of the Fano plane, which we will denote by $\overline{F_7}$.

By (k), every component of H is isomorphic to $\overline{F_7}$ (the complement of the Fano plane). If $H \neq H_G$, then by construction H is not 4-regular, a contradiction to (g). Hence, $H = H_G$. Since $\overline{F_7}$ is not the ONH of any graph, applying the result of Theorem 35 we have that H consists of precisely two components since G is by assumption connected. Let G' be constructed such that $V(G') = V(\overline{F_7}) \cup E(\overline{F_7})$ and let xy be an edge in G' if and only if x belongs to y in $\overline{F_7}$ (x is a vertex and y is an edge in $\overline{F_7}$). Now it is not difficult to see that G' is the incidence bipartite graph of the complement of the Fano plane and that the ONH of G' is H. Therefore $G' = G$. □

7 Closing Comment

Let H be a 4-uniform hypergraph of order $n = n(H)$ and size $m = m(H)$. In this paper we have shown that if $\Delta(H) \leqslant 3$, then $\tau(H) \leqslant n/4 + m/6$. It is known that $\tau(H) \leqslant n/4 + m/6$ is not always true when $\Delta(H) \geqslant 4$. We close with the following conjectures. Recall that a hypergraph is *linear* if every two edges intersect in at most one vertex.

Conjecture 36. *If H is a 4-uniform linear hypergraph, then* $\tau(H) \leqslant \frac{n}{4} + \frac{m}{6}$.

Conjecture 37. *If H is a 4-uniform linear hypergraph, then* $\tau(H) \leqslant \frac{n+m}{5}$.

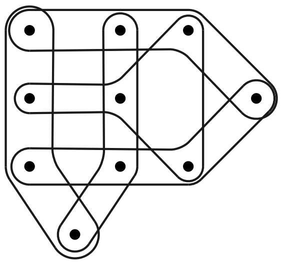


Figure 5: The hypergraph H_{10}.

We remark that Conjecture 37 implies Conjecture 36. If there is a vertex of degree at least 5, then we may remove it and use induction in order to prove Conjecture 36 and if there is no such vertex we note that Conjecture 36 follows from Conjecture 37 as in this case $n/5 + m/5 < n/4 + m/6$. Conjecture 37, if true, would be best possible due to the 4-uniform hypergraph H_{10}, illustrated in Figure 5, of order $n = 10$, size $m = 5$, and $\tau = 3$.

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