Cycles in the graph of overlapping permutations avoiding barred patterns

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Abstract

As a variation of De Bruijn graphs on strings of symbols, the graph of overlapping permutations has a directed edge $\pi(1)\pi(2)\ldots\pi(n+1)$ from the standardization of $\pi(1)\pi(2)\ldots\pi(n)$ to the standardization of $\pi(2)\pi(3)\ldots\pi(n+1)$. In this paper, we consider the enumeration of d-cycles in the subgraph of overlapping $(231, 4\bar{1}32)$-avoiding permutations. To this end, we introduce the notions of marked Motzkin paths and marked Riordan paths, where a marked Motzkin (resp. Riordan) path is a Motzkin (resp. Riordan) path in which exactly one step before the leftmost return point is marked. We show that the number of closed walks of length d in the subgraph of overlapping $(231, 4\bar{1}32)$-avoiding permutations are closely related to the number of marked Motzkin paths and that of marked Riordan paths. By establishing bijections, we get the enumerations of marked Motzkin paths and marked Riordan paths. As a corollary, we provide bijective proofs of two identities involving Catalan numbers in answer to the problem posed by Ehrenborg, Kitaev and Steingrímsson. Moreover, we get the enumerations of $(231, 4\bar{1}32)$-avoiding affine permutations and $(312, 32\bar{4}1)$-avoiding affine permutations.

Keywords: graph of overlapping permutations; cycles; Motzkin paths; Riordan paths.

1 Introduction

This work is motivated by the problem of determining the number of d-cycles in the graph of overlapping pattern avoiding permutations, which was initiated by Ehrenborg et al. [15].

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Denote by $\mathfrak{S}_n$ the set of all permutations on $[n] = \{1, 2, \ldots, n\}$. For any permutation π, we write $|\pi|$ to denote the number of entries of π and we write $\pi(i)$ to denote the ith entry of π. Given a permutation $\pi = \pi(1)\pi(2)\ldots\pi(n) \in \mathfrak{S}_n$ and a permutation $\tau = \tau(1)\tau(2)\ldots\tau(k) \in \mathfrak{S}_k$, we say that π contains the *pattern* τ if there exists a subsequence $\pi(i_1)\pi(i_2)\ldots\pi(i_k)$ of π that is order-isomorphic to τ. Otherwise, π is said to *avoid* the pattern τ or be τ*-avoiding*. We write $\mathfrak{S}_n(\tau)$ the set of all τ-avoiding permutations in $\mathfrak{S}_n$.

The classic problem of enumerating permutations avoiding a given pattern has received a great deal of attention and has led to interesting variations. For a thorough summary of the current status of research, see Bóna's book [6, 7] and Kitaev's book [20].

A *barred* permutation $\overline{\tau}$ of $[k]$ is a permutation of $\mathfrak{S}_k$ having a bar over one of its elements. Let τ be the permutation of $[k]$ by unbarring $\overline{\tau}$, and $\widehat{\tau}$ be the permutation obtained from $\overline{\tau}$ by removing the barred elements. For any permutation π, if every subsequence which is order-isomorphic to $\widehat{\tau}$ can be extended to a subsequence which is order-isomorphic to τ, then we say that π avoids the pattern $\overline{\tau}$. For example, if $\pi = 37258416$ and $\overline{\tau} = 235\bar{1}4$, then we have $\tau = 23514$ and $\widehat{\tau} = 1243$. All subsequences of the pattern 1243 are 3586 and 2586, which are subsequence of 35816 and 25816. So we have $\pi \in \mathfrak{S}_8(235\bar{1}4)$. The classes $\mathfrak{S}_n(321, 3\bar{1}42)$ and $\mathfrak{S}_n(231, 4\bar{1}32)$ are enumerated by the n-th Motzkin number, see [2, 8, 17, 19]. In [9], Chen et al. established a correspondence between Riordan paths and $(321, 3\bar{1}42)$-avoiding derangements.

A *De Bruijn graph* is a directed graph on vertex set $\{0, 1, \ldots, q-1\}^n$, the set of all strings of length n over an alphabet of size q, in which there is a directed edge from the string x to the string y if and only if the last $n-1$ coordinates of x agree with the first $n-1$ coordinates of y. It is well known that the number of directed cycles of length d, for $d \leqslant n$, is given by

$$\frac{1}{d}\sum_{e|d} \mu(d/e) q^e, \tag{1.1}$$

where the sum is over all divisors e of d, and where μ denotes the number theoretic *Möbius function*, see ([18], p.126) for instance. Recall that $\mu(n)$ is $(-1)^k$ if n is a product of k distinct primes and is zero otherwise.

For a permutation $\pi = \pi(1)\pi(2)\ldots\pi(n)$ consisting of distinct real numbers, the *standardization* of π is the unique permutation $\tau \in \mathfrak{S}_n$ which is order-isomorphic to π. For example, the standardization of $4(-1)53$ is 3142.

As a variation of De Bruijn graphs, the graph of *overlapping permutations*, denoted by $G(n)$, has a directed edge $\pi(1)\pi(2)\ldots\pi(n+1)$ from the standardization of $\pi(1)\pi(2)\ldots\pi(n)$ to the standardization of $\pi(2)\pi(3)\ldots\pi(n+1)$, which are also called the *graph of overlapping patterns* in [1] (see also ([20], Section 5.6)). The graph $G(n)$ appeared in [10] in connection with *universal cycles on permutations*, and was used as a tool in determining the asymptotic behaviour of consecutive pattern avoidance in [14].

The graph of *overlapping τ-avoiding permutations*, denoted by $G(n, \tau)$, is the subgraph of $G(n)$ having the vertex set $\mathfrak{S}_n(\tau)$ and the edge set $\mathfrak{S}_{n+1}(\tau)$. For example, the graph $G(2, \{231, 4\bar{1}32\})$ is illustrated in Figure 1. Recently, Ehrenborg et al. [15] derived that, for $d \leqslant n$, the number of cycles of length d in the graph $G(n, 312)$ is given by

$$\frac{1}{d}\sum_{e|d}\mu(d/e)\binom{2e}{e}, \tag{1.2}$$

which can be viewed as an analogous result for De Bruijn graphs. In their paper [15], they also posed the problem of evaluating the number of d-cycles in the graph of overlapping permutations avoiding any set of patterns of length 3 or more. In this paper, we derive that, for $n \geqslant 2$ and $d \leqslant n$, the number of cycles of length d in the graph $G(n, \{231, 4\bar{1}32\})$ is given by

$$\frac{1}{d}\sum_{e|d}\sum_{i=0}^{\lfloor \frac{e}{2} \rfloor}\mu(d/e)\frac{e}{e-i}\binom{e}{2i}\binom{2i}{i}. \tag{1.3}$$

Due to straightforward symmetries, one can easily verify that all the graphs $G(n, \{312, 32\bar{4}1\})$, $G(n, \{132, 23\bar{1}4\})$ and $G(n, \{213, 1\bar{4}23\})$ are isomorphic to the graph $G(n, \{231, 4\bar{1}32\})$.

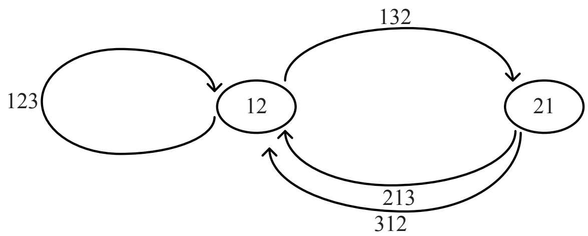

Figure 1: The graph $G(2, \{231, 4\bar{1}32\})$.

In order to get the enumeration of d-cycles in the graph $G(n, \{231, 4\bar{1}32\})$, we introduce the notions of *marked* Motzkin paths, *marked* Riordan paths and free Motzkin paths.

A *Motzkin* path of order n is a lattice path in $Z \times Z$ from $(0, 0)$ to $(n, 0)$ using up steps $U = (1, 1)$, down steps $D = (1, -1)$ and horizontal steps $H = (1, 0)$, and never lying below the x-axis [13]. Denote by $\mathcal{M}_n$ the set of all Motzkin paths of order n. It is well known that Motzkin paths of order n are counted by the n-th Motzkin number $m_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\frac{1}{i+1}\binom{n}{2i}\binom{2i}{i}$. A *free* Motzkin path is just a Motzkin path but without the restrictions that it has to end with a point on the x-axis and that it cannot go below the x-axis. Let $\mathcal{FM}_n$ denote the set of free Motzkin paths from $(0, 0)$ to $(n, 0)$. By simple arguments we have that $|\mathcal{FM}_n| = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2i}\binom{2i}{i}$.

A *marked* Motzkin path is a Motzkin path in which exactly one step before the leftmost return point is distinguished. Recall that for a lattice path, the points on the x-axis except for the initial point are called return points. In this sense, the ending point is always a return point. Denote by $\mathcal{M}_n^*$ the set of all marked Motzkin paths of order n. Let $\alpha_n = m_{n-2} + \delta_{i,1}$ where $m_{-1} = 0$ and $\delta_{n,1} = 1$ if $n = 1$ and 0 otherwise. By simple computation, we derive that the number of marked Motzkin paths of order n with k return points is given by

$$\sum_{P} n_1\alpha_{n_1}\alpha_{n_2}\dots\alpha_{n_k}$$

where the sum is over all compositions $P = (n_1, n_2, \dots, n_k)$ of n into k parts. Recall that a *composition* of a non-negative integer n into k parts is a list of k positive integers $(n_1, n_2, \dots, n_k)$ such that their sum is n.

By establishing a bijection Φ between the set $\mathcal{M}_n^*$ and the set $\mathcal{FM}_n$, we derive that

$$\sum_{k\geqslant 1}\sum_{P} n_1\alpha_{n_1}\alpha_{n_2}\dots\alpha_{n_k} = \sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2i}\binom{2i}{i}, \tag{1.4}$$

where the second sum is over all compositions $P = (n_1, n_2, \dots, n_k)$ of n into k parts.

A *Riordan* path is a *Motzkin* path without horizontal steps on the x-axis. Denote by $\mathcal{R}_n$ the set of Riordan paths from $(0, 0)$ to $(n, 0)$. The Riordan number r_n counts the number of Riordan paths from $(0, 0)$ to $(n, 0)$, see [3] and sequence A005043 in [21] for other combinatorial interpretations of r_n. A *marked* Riordan path is a Riordan path in which exactly one step before the leftmost return point is distinguished. Denote by $\mathcal{R}_n^*$ the set of all marked Riordan paths from $(0, 0)$ to $(n, 0)$. Furthermore, denote by $\mathcal{FM}(n, k)$ the set of all free Motzkin paths from $(0, 0)$ to (n, k).

Let $\mathcal{R}_n^*(U)$ denote the subset of $\mathcal{R}_n^*$ in which the marked step of each path is an up step. Analogously, let $\mathcal{R}_n^*(H)$ (resp. $\mathcal{R}_n^*(D)$) denote the subset of $\mathcal{R}_n^*$ in which the marked step of each path is a horizontal (resp. down) step.

By establishing a bijection Γ between the set $\mathcal{R}_n^*(U)\cup\mathcal{R}_n^*(H)$ and the set $\mathcal{FM}(n-1, 1)$, and a bijection Υ between the subset of $\mathcal{R}_n^*(U)$ in which each path has k return points and the set $\mathcal{FM}(n-1-k, k-1)$, we derive that

$$\sum_{k\geqslant 1}\sum_{P} n_1 m_{n_1-2}m_{n_2-2}\dots m_{n_k-2} = \sum_{i=1}^{\lfloor \frac{n}{2}\rfloor}\binom{n-i-1}{i-1}\binom{n}{i}, \tag{1.5}$$

where the second sum is over all compositions $P = (n_1, n_2, \dots, n_k)$ of n into k parts.

Note that each connected subgraph of a path is also a path. Hence a composition $(n_1, n_2, \dots, n_k)$ of n can be thought of as a subgraph of the path on n vertices, where n_i is the size of the ith connected component. The number of connected components of the subgraph is the number of parts of the composition. Analogously, a *cyclic composition* of n is defined to be a subgraph of the labeled cycle on n vertices where each component is a path.

Given a cyclic composition P of n into k parts, we first label the path containing the vertex 1 by B_1. Then label the connected components of P clockwise by $B_2, B_3, \dots, B_k$. The *type* of P, denoted by *type(P)*, is defined to be $(n_1, n_2, \dots, n_k)$, where n_i is the size of B_i. The compositions and the cyclic compositions of n are closely related by the following trivial observation.

Observation 1.1. *There are n_1 cyclic compositions P of n into k parts with type$(P) = (n_1, n_2, \dots, n_k)$.*

Let $c_n = \frac{1}{n+1}\binom{2n}{n}$ and $\beta_n = c_{n-1} + \delta_{n,1}$. Using generating functions, Ehrenborg et al. [15] proved that

$$\sum_{k\geqslant 1}\sum_{P}\beta_{n_1}\beta_{n_2}\ldots\beta_{n_k} = \binom{2n}{n} \tag{1.6}$$

$$\sum_{P} c_{n_1-1}c_{n_2-1}\ldots c_{n_k-1} = \binom{2n-k-1}{n-1}, \tag{1.7}$$

where the sum is over all cyclic compositions P of type $(n_1, n_2, \ldots, n_k)$. They also asked for bijective proofs of (1.6) and (1.7).

By Observation 1.1, Formulae (1.6) and (1.7) can be rewritten as

$$\sum_{k\geqslant 1}\sum_{P} n_1\beta_{n_1}\beta_{n_2}\ldots\beta_{n_k} = \binom{2n}{n} \tag{1.8}$$

$$\sum_{P} n_1 c_{n_1-1}c_{n_2-1}\ldots c_{n_k-1} = \binom{2n-k-1}{n-1}, \tag{1.9}$$

where the sum is over all compositions $P = (n_1, n_2, \ldots, n_k)$ of n into k parts.

Relying on the bijections Φ and Υ, we provide bijective proofs of Formulae (1.8) and (1.9) in answer to the problem posed by Ehrenborg et al. [15].

Let $\widetilde{\mathfrak{S}}_n$ denote the set of all bijections $\pi : \mathbb{Z} \to \mathbb{Z}$ such that

$$\pi(i+n) = \pi(i) + n, \tag{1.10}$$

$$\sum_{i=1}^{n} \pi(i) = \binom{n+1}{2}. \tag{1.11}$$

$\widetilde{\mathfrak{S}}_n$ is called the *affine symmetric group*, and the elements of $\widetilde{\mathfrak{S}}_n$ are called *affine permutations*. The combinatorial description of affine permutations is due to Lusztig and the first combinatorial study of them was conducted in [10, 16].

As an interesting variation of pattern avoidance on ordinary permutations, Crites [11] studied the generating functions for affine permutations avoiding a given pattern. In his paper [11], he derived that the number of 231-avoiding affine permutations in $\widetilde{\mathfrak{S}}_n$ is given by $\binom{2n-1}{n}$. We denote by $\widetilde{\mathfrak{S}}_n(\tau)$ the set of all τ-avoiding permutations in $\widetilde{\mathfrak{S}}_n$.

Based on Formula (1.4), we derive that

$$|\widetilde{\mathfrak{S}}_n(231, 4\bar{1}32)| = |\widetilde{\mathfrak{S}}_n(312, 32\bar{4}1)| = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i}\binom{2i}{i}.$$

2 The bijective proofs of (1.4) and (1.5)

Throughout this section we identify a path with a word by encoding each up step by the letter U, each down step by D and each horizontal step by H. If $P = p_1p_2\ldots p_n$ is a path, then the *reverse* of the path, denoted by $\overline{P}$, is defined by $p_np_{n-1}\ldots p_1$. For example, the reverse of the path $P = HUDDUDH$ is given by $HDUDDUH$.

Theorem 2.1. *There is a bijection* Φ *between the set of* $\mathcal{M}_n^*$ *and the set* $\mathcal{FM}_n$*.*

Proof. We first define a map Φ from the set $\mathcal{M}_n^*$ to the set $\mathcal{FM}_n$. Given a marked Motzkin path $P \in \mathcal{M}_n^*$, we define $\Phi(P)$ as follows. Let A be the starting point of the marked step. Denote by P_+ the section of the P which goes from $(0, 0)$ to the point A. Let P_- denote the remaining section of P. Define $\Phi(P) = P_-P_+$.

According to the construction of the map Φ, we preserve the number of up steps, the number of down steps and the number of horizontal steps. Hence, the map Φ is well defined, that is, $\Phi(P) \in \mathcal{FM}_n$.

In order to show that the map Φ is a bijection, we describe a map Φ' from the set $\mathcal{FM}_n$ to the set $\mathcal{M}_n^*$. Given a free Motzkin path $L \in \mathcal{FM}_n$, we define $\Phi'(L)$ as follows. Let B be the lowest point of the path L. If there are more than one such lowest point, we choose B to be the rightmost one. Denote by L_- the section of L which goes from $(0, 0)$ to the point B. Let L_+ denote the remaining section of L. Denote by L_-^* the path obtained from L_- by marking its first step. Define $\Phi'(L) = L_+L_-^*$.

Since the map Φ' preserves the number of up steps, the number of down steps, and the number of horizontal steps, the resulting path is from $(0, 0)$ to $(n, 0)$. Since B is the (rightmost) lowest point of L, the path L_+ has only one lowest point in L. This implies that L_+ has only one lowest point and its initial point is the lowest point of L_+. Clearly, the ending point of L_- is the lowest point in both L and $\Phi'(L)$. Hence, the resulting path $\Phi'(L)$ is a Motzkin path of order n. Moreover, the marked step is to the left of the first return point of $\Phi'(L)$. This implies that the map Φ' is well defined.

It is easily seen that the starting point of P_+ is the (rightmost) lowest point of $\Phi(P)$. This ensures that the maps Φ and Φ' are inverse of each other. Hence, the map Φ is a bijection, which completes the proof. ∎

Figure 2 shows, as an example, a marked Motzkin path $P \in \mathcal{M}_8^*$ in which a down step is marked, and Figure 3 shows a free Motzkin path $\Phi(P) \in \mathcal{FM}_8$, whose (rightmost) lowest point is B.

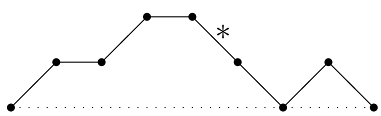

Figure 2: A marked Motzkin path $P \in \mathcal{M}_8^*$.

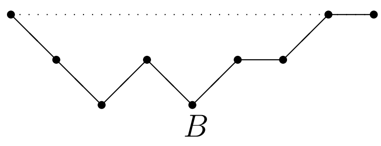

Figure 3: The corresponding free Motzkin $\Phi(P) \in \mathcal{FM}_8$.

A *Dyck* path of order n is a lattice path in $Z\times Z$ from $(0,0)$ to $(2n,0)$ using up steps $U=(1,1)$ and down steps $D=(1,-1)$, and never lying below the x-axis [12]. Denote by $\mathcal{D}_n$ the set of all Dyck paths of order n. It is well known that Dyck paths of order n are counted by the n-th Catalan number $c_n=\frac{1}{n+1}\binom{2n}{n}$. A *free* Dyck path is just a Dyck path but without the restrictions that it has to end with a point on the x-axis and that it cannot go below the x-axis. Let $\mathcal{FD}_n$ denote the set of free Dyck paths from $(0,0)$ to $(2n,0)$. By simple arguments we have that $|\mathcal{FD}_n|=\binom{2n}{n}$. A *marked* Dyck path is a Dyck path in which exactly one up step before the leftmost return point is distinguished, and each peak of height one maybe marked or not. Recall that the *height* of a step is defined to be the y-coordinate of its ending point. The *height* of a peak is the defined to be the height of its up step. Denote by $\mathcal{D}^*_n$ the set of all marked Dyck paths of order n.

From the construction of the bijection Φ, one can easily verify that for any marked Motzkin path L in which the marked step is an up step, its corresponding free Motzkin path $\Phi(L)$ starts with an up step. Hence, we have the following result.

Theorem 2.2. *The map Φ induces a bijection between the set of marked Dyck paths of order n but without any marked peaks and the set of free Dyck paths of order n that starts with an up step.*

By simple arguments, we have that the number of free Dyck paths from $(0,0)$ to $(2n,0)$ that start with an up step is equal to $\frac{1}{2}\binom{2n}{n}=\binom{2n-1}{n}$. Moreover, the number of marked Dyck paths of order n but without any marked peaks is counted by

$$\sum_{k\geqslant 1}\sum_{P} n_1 c_{n_1-1}c_{n_2-1}\dots c_{n_k-1},$$

where the second sum is over all compositions $P=(n_1,n_2,\dots,n_k)$ of n into k parts. By Theorem 2.2, we derive that

$$\sum_{k\geqslant 1}\sum_{P} n_1 c_{n_1-1}c_{n_2-1}\dots c_{n_k-1}=\binom{2n-1}{n}. \tag{2.1}$$

By simple computation, we get that the number of marked Dyck paths of order n with k return points is given by

$$\sum_{P} n_1\beta_{n_1}\beta_{n_2}\dots\beta_{n_k},$$

where the sum is over all compositions $P=(n_1,n_2,\dots,n_k)$ of n into k parts. Thus, the left-hand side of (1.8) counts the number of all marked Dyck paths of order n, while the right-hand side of (1.8) counts the number of all free Dyck paths of order n. In order to prove (1.8), it suffices to establish a bijection between the set $\mathcal{D}^*_n$ and the set $\mathcal{FD}_n$.

Theorem 2.3. *There is a bijection between the set of $\mathcal{D}^*_n$ and the set $\mathcal{FD}_n$.*

Proof. We first define a map Ψ from the set $\mathcal{D}^*_n$ to the set $\mathcal{FD}_n$. Given a marked Dyck path $P\in\mathcal{D}^*_n$, we define $L=\Psi(P)$ by the following rules:

- Let O and N be the initial point and the ending point of P, respectively.
- Let A is the starting point of the up step in the leftmost marked peak. If there is no such marked peak, let $A = N$.
- If P has exactly t marked peaks, then it can be uniquely decomposed as

$$P = P_{OA}(UD)^* P_1 (UD)^* P_2 \dots (UD)^* P_t,$$

  where each P_i is a (possibly empty) Dyck path.
- Define $L = \Psi(P)$ by letting $L = D\overline{P_1}UD\overline{P_2}U \dots D\overline{P_t}U\Phi(P_{OA})$ if there are exactly t marked peaks, and letting $L = \Phi(P_{OA})$, otherwise.

According to the construction of the map Ψ, we preserve the number of up steps and the number of down steps. Hence, the map Ψ is well defined, that is, $\Psi(P) \in \mathcal{FD}_n$.

In order to show that the map Ψ is a bijection, we describe a map Ψ' from the set $\mathcal{FD}_n$ to the set $\mathcal{D}_n^*$. Given a free Dyck path $L \in \mathcal{FD}_n$, we define $\Psi'(L)$ by the following rules:

- Let N be the ending point of L.
- Let A be the starting point of the leftmost up step that is above the x-axis. If there is no such up step, let $A = N$.
- Denote by L_{AN} the section of L that goes from the point A to the point N.
- If L starts with a down step, then L is uniquely decomposed as

$$L = D\overline{L_1}UD\overline{L_2}U \dots D\overline{L_t}UL_{AN},$$

  where each L_i is a (possibly empty) Dyck path. Set

$$\Psi'(L) = \Phi^{-1}(L_{AN})(UD)^* L_1 (UD)^* L_2 \dots (UD)^* L_t.$$

- If L starts with an up step, then set $\Psi'(L) = \Phi^{-1}(L_{AN})$.

By Theorem 2.2, the path $\Phi^{-1}(L_{AN})$ is a marked Dyck path without any marked peaks. Hence, the resulting path $\Psi'(L)$ is a marked Dyck path of order n. This implies that the map Ψ' is well defined. Since Φ is a bijection, one can easily check that the map Ψ and Ψ' are inverses of each other. Thus, the map Ψ is a bijection. This completes the proof. ∎

Now we proceed to give a combinatorial proof of Formula (1.5). A *lifted* Motzkin path of order n is a free Motzkin path from $(0, 0)$ to $(n, 1)$ such that it starts with an up step and all the points are above the x-axis except for the initial point. Denote by $\mathcal{LM}_n$ the set of lifted Motzkin paths of order n. A *marked* lifted Motzkin path is a lifted Motzkin path in which exactly one step is marked. Let $\mathcal{LM}_n^*$ denote the set of marked lifted Motzkin paths of order n.

Theorem 2.4. *There is a bijection between the set $\mathcal{LM}_n^*$ and the set $\mathcal{FM}(n,1)$.*

Proof. Given a marked lifted Motzkin path $P \in \mathcal{LM}_n^*$, we shall construct a path $\Omega(P) \in \mathcal{FM}(n,1)$ as follows. If the first step of P is marked, then let $\Omega(P) = P$. Otherwise, let A be the starting point of the marked step. Denote by P_+ the section of P which goes from $(0,0)$ to the point A, and P_- denote the remaining section of P. Define $\Omega(P) = P_-P_+$. It is easily seen that the map Ω preserves the number of up steps, the number of horizontal steps and the number of down steps. This yields that the resulting path $\Omega(P)$ is a free Motzkin path from $(0,0)$ to $(n,1)$, that is, the map Ω is well defined.

Conversely, given a free Motzkin path $L \in \mathcal{FM}(n,1)$, we wish to recover a path $\Omega'(L) \in \mathcal{LM}_n^*$ as follows. Let B be the lowest point of the path L. If there are more than one such lowest point, we choose B to be the rightmost one. If B is the initial point of L, define $\Omega'(L) = L$. Otherwise, let L_- denote the section of L which goes from $(0,0)$ to the point B, and L_+ denote the remaining section of L. Denote by L_-^* the path obtained from L_- by marking its first step. Define $\Omega'(L) = L_+L_-^*$.

Since the map Ω' preserves the number of up steps, the number of down steps and the number of horizontal steps, the resulting path is from $(0,0)$ to $(n,1)$. One can easily verify that the resulting path $\Omega'(L)$ is a lifted Motzkin path when B is the initial point of L. In order to show that the map Ω' is well defined, it remains to show that $\Omega'(L) \in \mathcal{LM}_n^*$ when B is not the initial point of L. It is easy to check that the point B is the lowest point of L_- both in L and $\Omega'(L)$. Obviously, B is the ending point of $\Omega'(L)$ and the y-coordinate of B is 1. Hence, all the points of L_- are weakly above the line $y = 1$ in $\Omega'(L)$. Moreover, the path L_+ has exactly one lowest point according to the definition of B. Clearly, the initial point of L_+ is such point. This implies that all the remaining points of L_+ are above the x-axis in $\Omega'(L)$. Thus, the resulting path $\Omega'(L)$ is a lifted Motzkin path of order n.

It is easily seen that the ending point of P_- is the (rightmost) lowest point of $\Omega(P)$. Since the section P_- contains at least one step, the initial point of $\Omega(P)$ is not the (rightmost) lowest point of $\Omega(P)$. This ensures that the maps Ω and Ω' are inverse of each other. Hence, the map Ω is a bijection, which completes the proof. ∎

Theorem 2.5. *There is a bijection between the set $\mathcal{R}_n^*(U) \cup \mathcal{R}_n^*(H)$ and the set $\mathcal{FM}(n-1,1)$.*

Proof. First, we describe a map Γ from the set $\mathcal{R}_n^*(U) \cup \mathcal{R}_n^*(H)$ to the set $\mathcal{FM}(n-1,1)$. Given a path $L \in \mathcal{R}_n^*(U) \cup \mathcal{R}_n^*(H)$, we construct a path $\Gamma(L)$ by the following procedure.

- Let O and A be the initial point and the leftmost return point of L, respectively.
- Let L_{OA} denote the section of L which goes from O to A, and let L' denote the remaining section of L.
- Let L'' be the path obtained from L_{OA} by removing its last down step.
- Define $\Gamma(L) = \overline{L'}\Omega(L'')$.

By Theorem 2.4, the path $\Omega(L'') \in \mathcal{FM}(n-1,1)$ which starts with either an up step or a horizontal step. Meanwhile, the path $\overline{L'}$ is a free Motzkin path ending on the x-axis in which each step is below the x-axis. Hence, the resulting path $\Gamma(L) \in \mathcal{FM}(n-1,1)$.

Conversely, given a path $P \in \mathcal{FM}(n-1,1)$, we can recover a path $\Gamma'(P) \in \mathcal{R}_n^*(U) \cup \mathcal{R}_n^*(H)$ as follows.

- Let s be the leftmost step in P which is weakly above the x-axis.
- Let P_+ denote the section of P which goes from $(0,0)$ to the starting point of s, and let P_- denote the remaining section of P.
- Define $\Gamma'(P) = \Omega^{-1}(P_-)D\overline{P_+}$.

By Theorem 2.4, the path $\Omega^{-1}(P_-)$ is a marked lifted Motzkin path in which either an up step or a horizontal step is marked. This implies that the marked step s is to the left of the leftmost return point in the resulting path $\Gamma'(P)$. Since each step of P_+ is below the x-axis, all the steps of $\overline{P_+}$ is above the x-axis. Hence, we deduce that $\Gamma'(P) \in \mathcal{R}_n^*(U) \cup \mathcal{R}_n^*(H)$. Since Ω is a bijection, one can easily check that the maps Γ and Γ' are inverses of each other. Thus, the map Γ is a bijection, which completes the proof. ∎

Figure 4 shows, as an example, a marked Riordan path $L \in \mathcal{R}_8^*(U)$ in which an up step is marked, and Figure 5 shows a free Motzkin path $\Gamma(L) \in \mathcal{FM}(7,1)$.

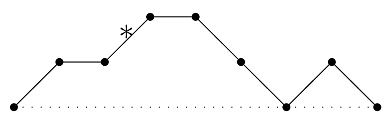

Figure 4: A marked Riordan path $L \in \mathcal{R}_8^*(U)$.

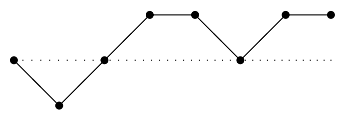

Figure 5: The corresponding free Motzkin $\Gamma(L) \in \mathcal{FM}(7,1)$.

Theorem 2.6. *Fix $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$. There is a bijection between the subset of $\mathcal{R}_n^*(U)$ in which each path has exactly k return points and the set $\mathcal{FM}(n-1-k, k-1)$.*

Proof. First, we describe a map Υ from the subset of $\mathcal{R}_n^*(U)$ in which each path has k return points to the set $\mathcal{FM}(n-1-k, k-1)$. Given a path $L \in \mathcal{R}_n^*(U)$ with exactly k return points, it can be uniquely decomposed as

$$L = L_1 L_2 \dots L_k,$$

where each L_i is a Riordan path having exactly one return point.

Now we proceed to construct a path $\Upsilon(L)$ by the following procedure.

- For all $1 \leqslant i \leqslant k$, denote by L'_i the path obtained from L_i by removing its rightmost step.
- Let L' be the path obtained from $\Omega(L'_1)$ by removing its leftmost step.
- Define $\Upsilon(L) = L'L'_2L'_3 \dots L'_k$.

By Theorem 2.4, it is easy to check that $\Omega(L'_1)$ is a free Motzkin path which starts with an up step and ends on the line $y = 1$. From the construction of the map Υ, we remove altogether k down steps and one up step. Hence, the resulting path $\Upsilon(L)$ is a free Motzkin path from $(0, 0)$ to $(n - 1 - k, k - 1)$.

Conversely, given a path $P \in \mathcal{FM}(n - 1 - k, k - 1)$, we wish to recover a path $\Upsilon'(P) \in \mathcal{R}^*_n(U)$. Clearly, the path P can be uniquely decomposed as

$$P = P_1s_1P_2s_2P_3 \dots s_{k-1}P_k,$$

where the step s_i is last step that leaves the line $y = i - 1$ for all $1 \leqslant i \leqslant k - 1$, P_1 is the section of P from $(0, 0)$ to the starting point of s_1, each P_i is the section of P between the steps s_{i-1} and s_i for all $2 \leqslant i \leqslant k - 1$, and P_k is the remaining section of P.

Obviously, each s_i is an up step. Moreover, P_1 is a free Motzkin path ending on the x-axis and each P_i is a Motzkin path in which each step is weakly above the line $y = i-1$ for all $2 \leqslant i \leqslant k$.

Define $\Upsilon'(P) = \Omega^{-1}(UP_1)Ds_1P_2Ds_2P_3D \dots s_{k-1}P_kD$. By Theorem 2.4, one can easily check that the resulting path $\Upsilon'(P) \in \mathcal{R}^*_n(U)$. Moreover, the maps Υ and Υ' are inverses of each other. Hence, the map Υ is the desired bijection. This completes the proof. ∎

From Theorems 2.5 and 2.6, it follows that

$$\begin{aligned}
|\mathcal{R}^*_n| &= |\mathcal{R}^*_n(U)| + |\mathcal{R}^*_n(H)| + |\mathcal{R}^*_n(D)| \\
&= |\mathcal{R}^*_n(U)| + |\mathcal{R}^*_n(H)| + |\mathcal{R}^*_n(U)| \\
&= |\mathcal{FM}(n-1, 1)| + \textstyle\sum_{k=0}^{n-2} |\mathcal{FM}(n-2-k, k)| \\
&= \textstyle\sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1}{2i+1}\binom{2i+1}{i} + \sum_{k=0}^{n-2} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{2i-k}\binom{2i-k}{i-k} \\
&= \textstyle\sum_{k=-1}^{n-2} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{2i-k}\binom{2i-k}{i-k} \\
&= \textstyle\sum_{k=0}^{n-1} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1-k}{2i+1-k}\binom{2i+1-k}{i+1-k} \\
&\text{(using } \textstyle\binom{n}{k}\binom{k}{i} = \binom{n}{i}\binom{n-i}{k-i}) \\
&= \textstyle\sum_{k=0}^{n-1} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1-k}{i+1-k}\binom{n-2-i}{i} \\
&= \textstyle\sum_{k=0}^{n-1} \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1-k}{n-2-i}\binom{n-2-i}{i} \\
&\text{(using } \textstyle\sum_{k=0}^{n} \binom{k}{\ell} = \binom{n+1}{\ell+1}) \\
&= \textstyle\sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-i}{i}\binom{n}{n-1-i} \\
&= \textstyle\sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-i}{i}\binom{n}{i+1} \\
&= \textstyle\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-i}{i-1}\binom{n}{i}.
\end{aligned}$$

Hence, we have

$$|\mathcal{R}_n^*| = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-i}{i-1} \binom{n}{i}. \tag{2.2}$$

On the other hand, by simple computation, we have

$$|\mathcal{R}_n^*| = \sum_{k \geqslant 1} \sum_{P} n_1 m_{n_1-2} m_{n_2-2} \dots m_{n_k-2}, \tag{2.3}$$

where the second sum is over all compositions $P = (n_1, n_2, \dots, n_k)$ of n into k parts. Combining Formulae (2.2) and (2.3), we are led to Formula (1.5).

Notice that our bijection Υ reduces to a bijection between the subset of $\mathcal{D}_n^*$ in which each path has k return points but without marked peaks and the set of free Dyck paths from $(0,0)$ to $(2n-1-k, k-1)$. By simple computation, one can verify that the former set is counted by the left-hand side of Formula (1.9), while the latter set is counted by the right-hand side of Formula (1.9). This leads to a bijective proof of (1.9).

3 Cycles in the graph of $(231, 4\bar{1}32)$-avoiding permutations

We begin with some definitions and notations. A *cut point* of a permutation $\pi \in \mathfrak{S}_n$ is an index j with $1 \leqslant j \leqslant n-1$ such that for all i and k satisfying $1 \leqslant i \leqslant j < k \leqslant n$ we have $\pi(i) < \pi(k)$. The cut points split a permutation into *components*, each ending at a cut point. A permutation without cut points is said to be *indecomposable*.

A permutation π is said to a *L-marked* (resp. *R-marked*) permutation if exactly one entry of the leftmost (resp. rightmost) component of π is marked. Denote by $\mathfrak{S}_n^L$ and $\mathfrak{S}_n^R$ the set of L-marked permutations and R-marked permutations, respectively.

For two positive integers a and b with $a \leqslant b$, we write $[a, b]$ to denote the sets of all integers that are larger than $a-1$ and smaller than $b+1$. Given two permutations $\tau = \tau(1)\tau(2) \dots \tau(k) \in \mathfrak{S}_k$ and $\sigma = \sigma(1)\sigma(2) \dots \sigma(\ell) \in \mathfrak{S}_\ell$, we write $\tau \oplus \sigma$ to denote the permutation $\tau(1)\tau(2) \dots \tau(k)(\sigma(1)+k)(\sigma(2)+k) \dots (\sigma(\ell)+k)$, and we write $\tau \ominus \sigma$ to denote the permutation $(\tau(1)+\ell)(\tau(2)+\ell) \dots (\tau(k)+\ell)\sigma(1)\sigma(2) \dots \sigma(\ell)$.

A *closed walk* of length d in a graph is a list of d edges $(e_1, e_2, \dots, e_d)$ such that $head(e_i) = tail(e_{i+1})$ for $1 \leqslant i \leqslant d-1$ and $head(e_d) = tail(e_1)$, where for a directed edge e, $head(e)$ is the node the edge points to, while $tail(e)$ is the other node incident to e. Two closed walks $(e_1, e_2, \dots, e_d)$ and $(e_i, e_{i+1}, \dots, e_d, e_1, e_2, \dots, e_{i-1})$ are said to be *equivalent*. Then a *d-cycle* is defined to be an equivalence class of size d. For example, the graph $G(2, \{231, 4\bar{1}32\})$ in Figure 1 has 5 closed walks of length 2, namely,

$$(132, 312), (312, 132), (132, 213), (213, 132), (123, 123).$$

However, the graph $G(2, \{231, 4\bar{1}32\})$ has only two 2-cycles, since the first (resp. third) closed walk is equivalent to the second (resp. fourth) walk, while the fifth walk yields a 1-cycle.

Denote by $\mathcal{W}_{n,d}$ the set of all closed walks of length d in the graph $G(n)$. Denote by $\mathfrak{S}_{n,d}$ the set of permutations $\pi = \pi(1)\pi(2)\dots\pi(n+d) \in \mathfrak{S}_{n+d}$ such that $\pi(1)\pi(2)\dots\pi(n)$ is order-isomorphic to $\pi(d+1)\pi(d+2)\dots\pi(n+d)$. Let $\mathcal{W}_{n,d}(\tau)$ denote the set of all closed walks of length d in the graph $G(n,\tau)$. Denote by $\mathfrak{S}_{n,d}(\tau)$ the subset of $\mathfrak{S}_{n,d}$ in which each permutation avoids the pattern τ.

Lemma 3.1. *Let $n \geqslant 2$ and $n \geqslant d \geqslant 1$. There is a bijection between the set $\mathcal{W}_{n,d}(231, 4\bar{1}32)$ and the set $\mathfrak{S}_{n,d}(231, 4\bar{1}32)$.*

Proof. We first define a map f from the set $\mathcal{W}_{n,d}(231, 4\bar{1}32)$ to the set $\mathfrak{S}_{n,d}(231, 4\bar{1}32)$. Given a closed walk $W = (\sigma_1, \sigma_2, \dots, \sigma_d) \in \mathcal{W}_{n,d}(231, 4\bar{1}32)$, we recursively generate a sequence $(\pi_1, \pi_2, \dots, \pi_d)$ of permutations in which $\pi_i \in \mathfrak{S}_{i+n}$ for all $i = 1, 2, \dots, d$. Let $\pi_1 = \sigma_1$. Suppose that we have obtained π_{i-1}. Now we proceed to construct π_i from π_{i-1} by the following *insertion algorithm*.

- If $\sigma_i(n+1) = n+1$, then let $\pi_i(n+i) = n+i$ and let $\pi_i(k) = \pi_{i-1}(k)$ for all $1 \leqslant k \leqslant n+i-1$.
- Otherwise, suppose that $\sigma_i(a) = \sigma_i(n+1)+1$. Then let $\pi_i(n+i) = \pi_{i-1}(a+i-1)$ and let
$$\pi_i(k) = \begin{cases} \pi_{i-1}(k)+1 & \text{if } \pi_{i-1}(k) \geqslant \pi_{i-1}(a+i-1), \\ \pi_{i-1}(k) & \text{otherwise.} \end{cases}$$

Define $f(W) = \pi_d$.

Claim 1. The subsequence $\pi_i(i)\pi_i(i+1)\dots\pi_i(i+n)$ of π_i is order-isomorphic to σ_i.

We prove Claim 1 by induction on i. Since $\pi_1 = \sigma_1$, the claim obviously holds for $i = 1$. Assume that the claim also holds for $i-1$, that is, the subsequence $\pi_{i-1}(i-1)\pi_{i-1}(i)\dots\pi_{i-1}(i-1+n)$ of π_{i-1} is order-isomorphic to σ_{i-1}. From the construction of π_i, it is easily seen that the subsequence $\pi_i(1)\pi_i(2)\dots\pi_i(n+i-1)$ is order-isomorphic to π_{i-1}. This implies that the subsequence $\pi_i(i)\pi_i(i+1)\dots\pi_i(i-1+n)$ is order-isomorphic to the subsequence $\pi_{i-1}(i)\pi_{i-1}(i+1)\dots\pi_{i-1}(i-1+n)$. Recall that $head(\sigma_{i-1}) = tail(\sigma_i)$. By the induction hypothesis, the subsequence $\pi_i(i)\pi_i(i+1)\dots\pi_i(i-1+n)$ is order-isomorphic to $\sigma_i(1)\sigma_i(2)\dots\sigma_i(n)$. Now we proceed to show that the subsequence $\pi_i(i)\pi_i(i+1)\dots\pi_i(i+n)$ is order-isomorphic to $\sigma_i(1)\sigma_i(2)\dots\sigma_i(n+1)$. We have two cases.

If $\sigma_i(n+1) = n+1$, then we have $\pi_i(i+n) = i+n$. Clearly, we have $\sigma_i(j) < \sigma_i(n+1)$ and $\pi_i(j+i-1) < \pi_i(n+i)$ for all $1 \leqslant j \leqslant n$.

If $\sigma_i(a) = \sigma_i(n+1)+1$, then we have $\pi_i(i+n) = \pi_{i-1}(a+i-1)$. It is easily seen that $\pi_i(a+i-1) = \pi_i(n+i)+1$. Moreover, we have $\pi_i(j+i-1) < \pi_i(n+i)$ if and only if $\sigma_i(j) < \sigma_i(n+1)$ for all $1 \leqslant j \leqslant n$.

In both cases, one can verify that the subsequence $\pi_i(i)\pi_i(i+1)\dots\pi_i(i+n)$ is order-isomorphic to $\sigma_i(1)\sigma_i(2)\dots\sigma_i(n+1)$. Hence, Claim 1 is proved.

Claim 2. The subsequence $\pi_i(j)\pi_i(j+1)\dots\pi_i(n+j)$ is order-isomorphic to σ_j for all $j \leqslant i$.

We prove Claim 2 by induction on i. Since $\pi_1 = \sigma_1$, the claim obviously holds for $i = 1$. Assume that the claim also holds for $i-1$. From the construction of π_i, it is easily seen that the subsequence $\pi_i(1)\pi_i(2)\dots\pi_i(n+i-1)$ is order-isomorphic to π_{i-1}. Hence, the subsequence $\pi_i(j)\pi_i(j+1)\dots\pi_i(n+j)$ is order-isomorphic to σ_j for all $j \leqslant i-1$. From Claim 1, it follows that the subsequence $\pi_i(i)\pi_i(i+1)\dots\pi_i(i+n)$ of π_i is order-isomorphic to σ_i. This completes the proof of Claim 2.

By Claim 2, the subsequence $\pi_d(1)\pi_d(2)\dots\pi_d(n+1)$ is order-isomorphic to σ_1, and the subsequence $\pi_d(d)\pi_d(d+1)\dots\pi_d(n+d)$ is order-isomorphic to σ_d. Since $head(\sigma_d) = tail(\sigma_1)$, the subsequence $\pi_d(1)\pi_d(2)\dots\pi_d(n)$ is order-isomorphic to the subsequence $\pi_d(d+1)\pi_d(d+2)\dots\pi_d(d+n)$ in π_d. This yields that $\pi_d \in \mathfrak{S}_{n,d}$.

In order to show that $\pi_d \in \mathfrak{S}_{n,d}(231, 4\bar{1}32)$, it remains to show that π_d avoids the patterns 231 and $4\bar{1}32$. First, we aim to show that π_d avoids the pattern 231. In fact, we show that $\pi_i \in \mathfrak{S}_{n+i}(231)$ for all $1 \leqslant i \leqslant d$. We prove the assertion by induction on i. Since $\sigma_1 \in \mathfrak{S}_{n+1}(231)$ and $\pi_1 = \sigma_1$, the assertion holds for $i = 1$. Assume that the assertion holds for $m \leqslant i-1$, that is, $\pi_m \in \mathfrak{S}_{n+m}(231)$ for all $m \leqslant i-1$. Now we proceed to show that $\pi_i \in \mathfrak{S}_{n+i}(231)$. Suppose that the subsequence $\pi_i(\ell)\pi_i(j)\pi_i(k)$ is an occurrence of 231 where $\ell < j < k$. Recall that the subsequence $\pi_i(1)\pi_i(2)\dots\pi_i(n+i-1)$ is order-isomorphic to π_{i-1}, and the subsequence $\pi_i(i)\pi_i(i+1)\dots\pi_i(n+i)$ is order isomorphic to σ_i. Thus, from $\sigma_i \in \mathfrak{S}_{n+1}(231)$ and the induction hypothesis that $\pi_{i-1} \in \mathfrak{S}_{n+i-1}(231)$, it follows that $\ell < i$ and $k = n+i$. According to the construction of π_i, we have $\pi_i(i-1+a) = \pi_{i-1}(i-1+a)+1 = \pi_i(i+n)+1$. This yields that $\pi_i(j) > \pi_i(i-1+a)$. Then either $\pi_i(\ell)\pi_i(j)\pi_i(i-1+a)$ or $\pi_i(i-1+a)\pi_i(j)\pi_i(i+n)$ would form an occurrence of 231. In the former case, $\pi_{i-1}(\ell)\pi_{i-1}(j)\pi_{i-1}(i-1+a)$ would form an occurrence of 231 in π_{i-1}. This contradicts the fact that $\pi_{i-1} \in \mathfrak{S}_{n+i-1}(231)$. In the latter case, the subsequence $\sigma_i(a)\sigma_i(j-i-1)\sigma_i(n+1)$ would form an occurrence of 231 in σ_i. This contradicts the fact that $\sigma_i \in \mathfrak{S}_{n+1}(231)$. Hence, we have $\pi_i \in \mathfrak{S}_{n+i}(231)$.

Our next goal is to show that π_d also avoids the pattern $4\bar{1}32$. We claim that $\pi_i \in \mathfrak{S}_{n+i}(4\bar{1}32)$ for all $i = 1, 2, \dots, d$. We prove the claim by induction on i. Obviously, the claim holds for $i = 1$ since $\pi_1 = \sigma_1$ and $\sigma_1 \in \mathfrak{S}_{n+1}(4\bar{1}32)$. Assume that $\pi_m \in \mathfrak{S}_{n+m}(4\bar{1}32)$ for all $m \leqslant i-1$. Now we proceed to show that $\pi_i \in \mathfrak{S}_{n+i}(4\bar{1}32)$. If not, there must exist three indices ℓ, j, k ($\ell < j < k$) such that $\pi_i(\ell)\pi_i(j)\pi_i(k)$ is an occurrence of 321 which cannot be extended to an occurrence of $4\bar{1}32$. More precisely, we have $\pi_i(m) > \pi_i(k)$ for all $\ell < m < j$. Recall that the subsequence $\pi_i(1)\pi_i(2)\dots\pi_i(n+i-1)$ is order-isomorphic to π_{i-1} and the subsequence $\pi_i(i)\pi_i(i+1)\dots\pi_i(n+i)$ is order-isomorphic to σ_i. Since both π_{i-1} and σ_i avoids the pattern $4\bar{1}32$, we have $\ell < i$ and $k = n+i$. Then we have $1 \leqslant \pi_i(n+i) < n+i$. Since $\sigma_i \in \mathfrak{S}_{n+1}(231, 4\bar{1}32)$ and $n \geqslant 2$, it follows that $1 < \sigma_i(n+1) < n+1$.

Suppose that $\sigma_i(b) = \sigma_i(n+1)-1$. Since the subsequence $\pi_i(i)\pi_i(i+1)\dots\pi_i(n+i)$ is order-isomorphic to σ_i, we have $\pi_i(i-1+b) < \pi_i(n+i)$. Recall that $\pi_i(m) > \pi_i(n+i)$ for all $\ell < m < j$. This implies that $j < i-1+b$. Then the subsequence $\pi_i(\ell)\pi_i(j)\pi_i(i-1+b)$ would from an occurrence of 321. Since $\pi_i(1)\pi_i(2)\dots\pi_i(n+i-1)$ avoids the pattern $4\bar{1}32$, there exists an integer m such that $\ell < m < j$ and $\pi_i(\ell)\pi_i(m)\pi_i(j)\pi_i(i-1+b)$ forms an occurrence of 4132. Then we have $\pi_i(m) < \pi_i(i-1+b) < \pi_i(n+i)$. This yields a contradiction with the fact that $\pi_i(m) > \pi_i(n+i)$ for all $\ell < m < j$. Thus, we deduce that the indices ℓ, j, k do not exist. Hence, we have $\pi_i \in \mathfrak{S}_{n+i}(4\bar{1}32)$.

So far, we have concluded the map f is well defined, that is, $f(W) \in \mathfrak{S}_{n,d}(231, 4\bar{1}32)$.

Conversely, given a permutation $\pi \in \mathfrak{S}_{n,d}(231, 4\bar{1}32)$, we can recover a closed walk $f'(\pi) = (\sigma_1, \sigma_2, \ldots, \sigma_d) \in \mathcal{W}_{n,d}(231, 4\bar{1}32)$ by letting σ_i be the permutation of $[n+1]$ which is isomorphic to $\pi(i)\pi(i+1)\ldots\pi(i+n)$ for all $i = 1, 2, \ldots, d$.

In order to show that the map f is a bijection, it suffices to show that the maps f and f' are inverses of each other. First, we aim to show that $f'(f(W)) = W$ for any closed walk $W = (\sigma_1, \sigma_2, \ldots, \sigma_d) \in \mathcal{W}_{n,d}(231, 4\bar{1}32)$. By the construction of the map f, we recursively generate a sequence $(\pi_1, \pi_2, \ldots, \pi_d)$ of permutations in which each π_i is a permutation of $[n+i]$ and $f(W) = \pi_d$. By Claim 2, the subsequence $\pi_d(i)\pi_d(i+1)\ldots\pi_d(i+n)$ is order-isomorphic to σ_i. From the construction of the map f', it is easily seen that $f'(f(W)) = W$.

Next we turn to the proof of the equality $f(f'(\pi)) = \pi$ for any $\pi \in \mathfrak{S}_{n,d}(231)$. Suppose that $f'(\pi) = (\sigma_1, \sigma_2, \ldots, \sigma_d)$. When apply the map f to $f'(\pi)$, we recursively get a sequence $(\pi_1, \pi_2, \ldots, \pi_d)$ of permutations in which each π_i is a permutation of $[n+i]$ and $f(f'(\pi)) = \pi_d$. In order to show that $f(f'(\pi)) = \pi$, it suffices to show that π_i is order-isomorphic to $\pi(1)\pi(2)\ldots\pi(n+i)$ for all $1 \leqslant i \leqslant d$. We prove by induction on i. Since $\pi_1 = \sigma_1$ and σ_1 is order-isomorphic to $\pi(1)\pi(2)\ldots\pi(n+1)$, the assertion holds for $i = 1$. Assume that π_{i-1} is order-isomorphic to $\pi(1)\pi(2)\ldots\pi(n+i-1)$. Now we proceed to show that the assertion also holds for i. We have two cases.

Case 1. $\sigma_i(n+1) = n+1$. From the construction of π_i, we have $\pi_i(n+i) = n+i$ and $\pi_{i-1}(k) = \pi_i(k)$ for all $1 \leqslant k \leqslant n+i-1$. By the induction hypothesis that π_{i-1} is order-isomorphic to $\pi(1)\pi(2)\ldots\pi(n+i-1)$, in order to prove the assertion, it suffices to show that $\pi(k) < \pi(n+i)$ for all $1 \leqslant k \leqslant n+i-1$. Since $\pi(i)\pi(i+1)\ldots\pi(i+n)$ is order-isomorphic to σ_i, we have $\pi(k) < \pi(i+n)$ for all $i \leqslant k \leqslant i+n-1$. It remains to show that $\pi(k) < \pi(i+n)$ for all $1 \leqslant k \leqslant i-1$. Recall that $\pi(1)\pi(2)\ldots\pi(n)$ is order-isomorphic to $\pi(d+1)\pi(d+2)\ldots\pi(d+n)$. This yields that $\pi(d+1)\pi(d+2)\ldots\pi(i+n)$ is order-isomorphic to $\pi(1)\pi(2)\ldots\pi(i+n-d)$. Since $\pi(k) < \pi(i+n)$ for all $d+1 \leqslant k < i+n$, we have $\pi(k) < \pi(i+n-d)$ for all $1 \leqslant k < i+n-d$. Thus, it follows that $\pi(k) < \pi(i+n-d) < \pi(n+i)$ for all $1 \leqslant k \leqslant i-1$.

Case 2. $1 \leqslant \sigma_i(n+1) < n+1$. Suppose that $\sigma_i(a) = \sigma_i(n+1)+1$. According to the construction of π_i, we have $\pi_i(n+i) = \pi_{i-1}(a+i-1)$ and

$$\pi_i(k) = \begin{cases} \pi_{i-1}(k)+1 & \text{if } \pi_{i-1}(k) \geqslant \pi_{i-1}(a+i-1), \\ \pi_{i-1}(k) & \text{otherwise.} \end{cases}$$

Obviously, we have $\pi_i(a+i-1) = \pi_{i-1}(a+i-1)+1$, which implies that $\pi_i(a+i-1) = \pi_i(i+n)+1$. Recall that $\pi_i(1)\pi_i(2)\ldots\pi_i(n+i-1)$ is order-isomorphic to π_{i-1} and π_{i-1} is order-isomorphic to $\pi(1)\pi(2)\ldots\pi(n+i-1)$. In order to prove the assertion, it suffices to show that $\pi(a+i-1) = \pi(i+n)+1$. Since σ_i is order-isomorphic to $\pi(i)\pi(i+1)\ldots\pi(i+n)$, the inequality $\sigma_i(a) > \sigma_i(n+1)$ leads to $\pi(i-1+a) > \pi(n+i)$. Suppose that $\pi(a+i-1) > \pi(i+n)+1$ and $\pi(m) = \pi(i+n)+1$. We claim that $m < i$. If not, suppose that $m \geqslant i$. Since σ_i is order-isomorphic to the subsequence $\pi(i)\pi(i+1)\ldots\pi(n+i)$, we have $m = a+i-1$, which contradicts the assumption $\pi(a+i-1) > \pi(i+n)+1$ and $\pi(m) = \pi(i+n)+1$. Hence we have $m < i$. Then the subsequence $\pi(m)\pi(a+i-1)\pi(n+i)$ would form an occurrence of 231 in π, which yields a contradiction with the fact that π avoids the pattern 231. This completes the proof. ∎

Given a permutation $\pi \in \mathfrak{S}_{n,d}(231, 4\bar{1}32)$, suppose that the entry $\pi(d+a)$ is the maximum among the entries $\pi(d+1), \pi(d+2), \dots, \pi(2d)$. The permutation is said to be of type one if $\pi(k) < \pi(d+a)$ for all $1 \leqslant k \leqslant d$. Otherwise, it is said to be of type two. We partition the set $\mathfrak{S}_{n,d}(231, 4\bar{1}32)$ into two subsets X and Y, where X is the set all permutations of type one and Y is the set of all permutations of type two.

Observation 3.2. *Let $n \geqslant d \geqslant 1$. Given a permutation $\pi \in \mathfrak{S}_{n,d}$, we have $\pi(d+x) > \pi(d+y)$ if and only if $\pi(x) > \pi(y)$, where $1 \leqslant x < y \leqslant n$.*

Lemma 3.3. *Let $n \geqslant d \geqslant 1$. Given a permutation $\pi \in \mathfrak{S}_{n+d}$, suppose that $\pi(d+a)$ is the maximum among the elements $\pi(d+1), \pi(d+2), \dots, \pi(2d)$. Then $\pi \in X$ if and only if π has the following properties.*

(i) *The subsequence $\pi(a)\pi(a+1)\dots\pi(a+d-1)$ is a $(231, 4\bar{1}32)$-avoiding permutation of $[a, d-1+a]$.*

(ii) *The subsequence $\pi(1)\pi(2)\dots\pi(a-1)$ is a permutation of $[a-1]$ which is order-isomorphic to the subsequence $\pi(d+1)\pi(d+2)\dots\pi(d+a-1)$.*

(iii) *$\pi(i+d) = \pi(i) + d$ for all $i \geqslant a$.*

Proof. It is easily seen that if π has properties (i), (ii) and (iii), then $\pi \in X$.

Now suppose that $\pi \in X$, we shall show that π verifies properties (i), (ii) and (iii).

(i) By Observation 3.2, the entry $\pi(a)$ is also the maximum among the entries $\pi(1), \pi(2), \dots, \pi(d)$. This implies that $\pi(i) < \pi(a)$ for all $i \leqslant a-1$. Since π avoids the pattern 231, we have $\pi(i) < \pi(j)$ for all $i < a < j$. By similar reasoning, it is easy to verify that $\pi(i) < \pi(j)$ for all $i < a+d < j$. Hence, the permutation $\pi(a)\pi(a+1)\dots\pi(a+d-1)$ is a $(231, 4\bar{1}32)$-avoiding permutation of $[a, a+d-1]$.

(ii) Recall that the subsequence $\pi(1)\pi(2)\dots\pi(n)$ is order-isomorphic to the subsequence $\pi(d+1)\pi(d+2)\dots\pi(d+n)$. Moreover, we have $\pi(k) < a$ for all $k \leqslant a-1$. This implies that the subsequence $\pi(1)\pi(2)\dots\pi(a-1)$ is a permutation of $[a-1]$ which is order-isomorphic to the subsequence $\pi(d+1)\pi(d+2)\dots\pi(d+a-1)$.

(iii) By Observation 3.2, the subsequence $\pi((k+1)d+a)\pi((k+1)d+a+1)\dots\pi((k+2)d+a-1)$ is order-isomorphic to the subsequence $\pi(kd+a)\pi(kd+a+1)\dots\pi((k+1)d+a-1)$ for all $k \geqslant 0$. In order to verify (iii), it remains to show that $\pi(kd+a)\pi(kd+a+1)\dots\pi((k+1)d+a-1)$ is permutation of $[kd+a, (k+1)d+a-1]$ for all $k \geqslant 0$. We prove by induction on k. By property (i), the assertion holds for $k=0$. Assume that the assertion holds for all $m \leqslant k-1$. Now we proceed to show that the assertion also holds for k. By the induction hypothesis, we have $\pi((k-1)d+x) < \pi(y)$ for all $a \leqslant x \leqslant d+a-1$ and $y > kd+a-1$. By Observation 3.2, we have $\pi(kd+x) < \pi(d+y)$ for all $a \leqslant x \leqslant d+a-1$ and $y \geqslant kd+a-1$. Hence, by the induction hypothesis, the subsequence $\pi(kd+a)\pi(kd+a+1)\dots\pi((k+1)d+a-1)$ is permutation of $[kd+a, (k+1)d+a-1]$. This completes the proof. ∎

Lemma 3.4. *Given a permutation $\pi \in \mathfrak{S}_{n+d}$, suppose that $\pi(d+a)$ is the maximum element among the elements $\pi(d+1), \pi(d+2), \ldots, \pi(2d)$ and there are exactly k entries among the elements $\pi(1), \pi(2), \ldots, \pi(d)$ which are larger than π_{d+a}. Let $\pi(b_1), \pi(b_2), \ldots, \pi(b_k)$ be such entries, where $b_1 < b_2 < \ldots < b_k < d+a$. Then $\pi \in Y$ if and only if π has the following properties.*

(i)$'$ $a = b_1$.

(ii)$'$ $\pi \in \mathfrak{S}_{n,d}$.

(iii)$'$ *For all $1 \leqslant i \leqslant k$, the subsequence $\pi(b_i+1)\pi(b_i+2)\ldots\pi(b_{i+1}-1)$ avoids the patterns 231 and $4\bar{1}32$ with the assumption $b_{k+1} = d + b_1$.*

(iv)$'$ *For all $p \geqslant 0$ and $1 \leqslant i \leqslant k$, we have $\pi(pd + b_i) = n + d + 1 - i - pk$.*

(v)$'$ *For all $p \geqslant 0$ and $1 \leqslant i \leqslant k$, we have $\pi(x) < \pi(y)$ if $x < pd+b_i \leqslant y$ and $x \neq md+b_j$ for all $m \geqslant 0$ and $1 \leqslant j \leqslant k$.*

(vi)$'$ *For all $1 \leqslant i \leqslant k$, we have $b_{i+1} - b_i \geqslant 2$ with the assumption $b_{k+1} = d + b_1$.*

Proof. It is easy to check that if π has properties (ii)$'$-(v)$'$, we have $\pi \in \mathfrak{S}_{n,d}(231)$. It remains to show that π avoids the pattern $4\bar{1}32$. We argue by contradiction. Suppose that there exist three indices x, y, z with $x < y < z$ such that $\pi(x)\pi(y)\pi(z)$ $(x < y < z)$ is a pattern of 321, which cannot be extended to a pattern 4132. By properties (ii)$'$-(v)$'$, we have $x = pd + b_i$ for some nonnegative integer p and some positive integer i. By property (ii)$'$, the subsequence $\pi(b_i)\pi(y-pd)\pi(z-pd)$ also forms a pattern 321 which cannot be extended to a pattern 4132. By property (iii)$'$, we have $z - pd > b_1 + d - 1$. From property (v)$'$, we have $y - pd \geqslant b_{i+1}$ since $\pi(y-pd) > \pi(z-pd)$. By property (v)$'$ and (vi)$'$, there exists an entry $\pi(\ell)$ such that $\pi(\ell) < \pi(z-pd)$ and $b_i < \ell < b_{i+1}$. Hence, the subsequence $\pi(b_i)\pi(y-pd)\pi(z-pd)$ forms a pattern 321 which can be extended to a pattern 4132, which yields a contradiction. So such indices x, y, z do not exist. This implies that π avoids the pattern $4\bar{1}32$.

Our next goal is to show that if $\pi \in Y$, then it verifies Properties (i)$'$-(vi)$'$.

(i)$'$ Recall that the entry $\pi(d+a)$ is the maximum among the entries $\pi(d+1), \pi(d+2), \ldots, \pi(d+a)$. From Observation 3.2, it follows that that the entry $\pi(a)$ is also the maximum among the entries $\pi(1), \pi(2), \ldots, \pi(d)$. Since π avoids the pattern 231, we have $\pi(b_1) > \pi(b_2) > \ldots > \pi(b_k) > \pi(d+a)$. Notice that we have $\pi(j) < \pi(d+a)$ for all $j \leqslant d$ and $j \notin \{b_1, b_2, \ldots, b_k\}$. Thus, the entry $\pi(b_1)$ is the maximum among the entries $\pi(1), \pi(2), \ldots, \pi(d)$, which implies that $a = b_1$. This verifies property (i)$'$.

Properties (ii)$'$ and (iii)$'$ follow immediately from the fact that $\pi \in \mathfrak{S}_{n,d}(231, 4\bar{1}32)$.

(iv)$'$ We proceed by induction on p. First, we aim to show that the assertion holds for $p = 0$. Recall $\pi(j) < \pi(d+a)$ for all $j \leqslant d$ and $j \notin \{b_1, b_2, \ldots, b_k\}$ and $\pi(b_1) > \pi(b_2) > \ldots > \pi(b_k) > \pi(d+a) = \pi(d+b_1)$. This implies that $\pi(b_1) > \pi(b_2) > \ldots > \pi(b_k) > \pi(j)$ for all $j \leqslant d$ and $j \notin \{b_1, b_2, \ldots, b_k\}$. In order to verify $\pi(b_i) = n + d + 1 - i$, it remains to show that $\pi(b_1) > \pi(b_2) > \ldots > \pi(b_k) > \pi(j)$ for all $j > d$. Notice that the entry $\pi(d+b_1)$ is the maximum among the entries $\pi(d+1)\pi(d+2)\ldots\pi(2d)$. This implies that

$\pi(b_k) > \pi(d+j)$ for all $1 \leqslant j \leqslant d$. By Observation 3.2, we have $\pi(pd+b_k) > \pi((p+1)d+j)$ for all $1 \leqslant j \leqslant d$. This yields that $\pi(b_1) > \pi(b_2) > \ldots > \pi(b_k) > \pi(j)$ for all $j > d$. Thus, we have $\pi(b_i) = n + d + 1 - i$ for all $1 \leqslant i \leqslant k$.

Now assume that $\pi(md+b_j) = n+d+1-j-mk$ for all $m \leqslant p-1$ and $1 \leqslant j \leqslant k$. We shall show that $\pi(pd + b_j) = n + 1 - j - pk$ for all $1 \leqslant j \leqslant k$. We argue by contradiction. Suppose that q is the maximum integer such that $q \leqslant k$ and $\pi(pd+b_\ell) = n+d+1-\ell-pk$ for all $\ell < q$. In other words, we have $\pi(pd + b_q) < n + d + 1 - q - pk$. Suppose that $\pi(x) = n+d+1-q-pk$. Notice that we have showed that $\pi(b_1) > \pi(b_2) > \ldots > \pi(b_k) > \pi(y)$ for all $y \notin \{b_1, b_2, \ldots, b_k\}$. By Observation 3.2, we have $\pi(pd + b_1) > \pi(pd + b_2) > \ldots > \pi(pd + b_k) > \pi(pd + y)$ for all $y \notin \{b_1, b_2, \ldots, b_k\}$. So we have $x \leqslant pd$. If $q > 1$, then the subsequence $\pi(x)\pi(pd + b_1)\pi(pd + b_q)$ would form an occurrence of 231 in π. This contradicts the fact that π avoids the pattern 231. If $q = 1$, then the inequality $\pi(x) > \pi(pd + b_1)$ implies that $\pi(x - d) < \pi((p - 1)d + b_1)$ by Observation 3.2. This contradicts the fact that $\pi(y) < \pi((p - 1)d + b_1)$ for all $y < (p - 1)d + b_1$ and $y \neq sd + b_j$ for all $s \geqslant 1$ and $1 \leqslant j \leqslant k$. Thus, such integer q does not exist. This implies that $\pi(pd + b_j) = n + d + 1 - j - pk$ for all $1 \leqslant j \leqslant k$, which verifies property (iv)$'$.

Property (v)$'$ follows immediately from property (iv)$'$ since π avoids the pattern 231.

Now we proceed to show that π has property (vi)$'$. We have two cases.

Case 1. $k > 1$. If $b_{i+1} = b_i+1$ for $1 \leqslant i \leqslant k-1$, then by property (iv)$'$ the subsequence $\pi(b_i)\pi(b_{i+1})\pi(d + b_1)$ would form an occurrence of 321 which cannot be extended to a pattern 4132, which yields a contradiction. This implies that $b_{i+1}-b_i \geqslant 2$ for $1 \leqslant i \leqslant k-1$. If $b_{k+1} = b_k + 1$, then by property (iii)$'$, the subsequence $\pi(b_k)\pi(d + b_1)\pi(d + b_2)$ would form an occurrence of 321 which cannot be extended to a pattern 4132, which yields a contradiction. This yields that $b_{k+1} - b_k \geqslant 2$. Thus, we deduce that when $k > 1$, we have $b_{i+1} - b_i \geqslant 2$ for all $1 \leqslant i \leqslant k$.

Case 2. $k = 1$. We claim that $d > 1$. If not, suppose that $d = 1$. Since $n \geqslant 2$, we have $n+d \geqslant 3$. Recall that $b_1 = 1$ when $d = 1$. By property (iv)$'$, the subsequence $\pi(1)\pi(2)\pi(3)$ would form an occurrence of 321, which cannot be extended to an occurrence of 4132. This yields a contradiction. Hence, the claim is proved, that is, we have $d > 1$. Then we have $b_2 - b_1 = d + b_1 - b_1 = d \geqslant 2$. This completes the proof. ∎

Lemma 3.5. *Let $n \geqslant d \geqslant 1$. There is a bijection between the set X and the set $\mathfrak{S}_d^L(231, 4\bar{1}32)$.*

Proof. Given a permutation $\pi \in X$, we wish to obtain a marked permutation $g(\pi) \in \mathfrak{S}_n^L(231, 4\bar{1}32)$. Suppose that $\pi(d + a)$ is the maximum element among the elements $\pi(d + 1), \pi(d + 2), \ldots, \pi(2d)$. Let π' be the permutation of $[d]$ which is order-isomorphic to $\pi(a)\pi(a + 1) \ldots \pi(a + d - 1)$. Define $g(\pi)$ to be the permutation obtained from π' by marking the entry $\pi'(d - a + 1)$.

Obviously, π' avoids the patterns 231 and $4\bar{1}32$. This implies that the resulting permutation $g(\pi)$ also avoids the patterns 231 and $4\bar{1}32$. Since the subsequence $\pi(1)\pi(2) \ldots \pi(n)$ is order-isomorphic to the subsequence $\pi(d+1)\pi(d+2) \ldots \pi(d+n)$, the element $\pi(a)$ is also the maximum among the elements $\pi(1), \pi(2), \ldots, \pi(d)$. Thus, we have $\pi(i) < \pi(a)$ for all $a+1 < i \leqslant d$. Since π' is order-isomorphic to the subsequence $\pi(a)\pi(a+1) \ldots \pi(a+n-1)$, the leftmost component contains at least $d-a+1$ elements. Hence, the resulting permutation $g(\pi) \in \mathfrak{S}_n^L(231, 4\bar{1}32)$.

Conversely, given a marked permutation $\sigma \in \mathfrak{S}_n^L(231, 4\bar{1}32)$, we wish to recover a permutation $\pi = g'(\sigma) \in X$. Suppose that the b-th entry of its leftmost component is marked. Let $a = d+1-b$. Define π to be the permutation such that $\pi(a)\pi(a+1)\ldots\pi(a+d-1)$ is order-isomorphic to σ and verifies properties (i), (ii) and (iii).

By Lemma 3.3, one can easily verify that $g'(\pi) \in X$ and and the maps g and g' are inverses of each other. Hence, the map g is a bijection. This completes the proof. ∎

Lemma 3.6. *Let $n \geqslant 2$ and $n \geqslant d \geqslant 1$. There is a bijection between the set Y and the set of permutations $\pi \in \mathfrak{S}_d^R(231, 4\bar{1}32)$ in which each component contains at least two entries.*

Proof. Given a permutation $\pi \in Y$, suppose that $\pi(d+a)$ is the maximum among the entries $\pi(d+1), \pi(d+2), \ldots, \pi(2d)$. Suppose that there are exactly k entries among the entries $\pi(1), \pi(2), \ldots, \pi(d)$ which are larger than π_{d+a}. Let $\pi(b_1), \pi(b_2), \ldots, \pi(b_k)$ be such entries, where $b_1 < b_2 < \ldots < b_k < d+a$. By Lemma 3.4, π has Properties (i)$'$-(vi)$'$.

Now we proceed to define a map ξ from the set Y to the set of permutations $\pi \in \mathfrak{S}_d^R(231, 4\bar{1}32)$ in which each component contains at least two entries. Assume that $b_{k+1} = d + b_1$. Let σ_i be the permutation of $[b_{i+1} - b_i - 1]$ which is order-isomorphic to the subsequence $\pi(b_i+1)\pi(b_i+2)\ldots\pi(b_{i+1}-1)$ for all $1 \leqslant i \leqslant k$. Let $\sigma = (1 \ominus \sigma_1) \oplus (1 \ominus \sigma_2) \ldots \oplus (1 \ominus \sigma_k)$. Define $\xi(\pi)$ to be the permutation obtained from σ by marking the a-th entry of the rightmost component of σ.

By properties (iv)$'$ and (v)$'$, we have that the subsequence $\pi(b_i)\pi(b_i+1)\ldots\pi(b_{i+1}-1)$ is a $(231, 4\bar{1}23)$-avoiding permutation which is order-isomorphic to $1 \ominus \sigma_i$ for all $1 \leqslant i \leqslant k$. Hence, the resulting permutation $\sigma \in \mathfrak{S}_d(231, 4\bar{1}32)$. Since $b_k \leqslant d$ and $b_{k+1} = d+a$, we have that $|\sigma_k| = b_{k+1} - b_k - 1 \geqslant d + a - 1 - d = a - 1$. This ensures that the rightmost component of σ contains at least a entries. Thus, we have $\xi(\pi) \in \mathfrak{S}_d^R(231, 4\bar{1}32)$. In order to show that the map ξ is well defined, it remains to show that each component of $\xi(\pi)$ contains at least two entries. By property (vi)$'$, we have $|\sigma_i| = b_{i+1} - b_i - 1 \geqslant 1$ for all $1 \leqslant i \leqslant k$. Hence, we have concluded that the map ξ is well defined.

Conversely, given a permutation $\sigma \in \mathfrak{S}_d^R(231, 4\bar{1}32)$ in which each component contains at least two entries, we shall recover a permutation $\xi'(\sigma) \in Y$. Suppose that σ is uniquely decomposes as

$$\sigma = (1 \ominus \sigma_1) \oplus (1 \ominus \sigma_2) \ldots \oplus (1 \ominus \sigma_k),$$

where each σ_i is a $(231, 4\bar{1}32)$-avoiding permutation. Assume that the a-th entry of the last component of σ is marked. Let $b_1 = a$ and $b_{i+1} = b_i + |\sigma_i|$ for all $1 \leqslant i \leqslant k$. Let $\xi'(\sigma)$ to the permutation π such that the subsequence $\pi(b_i+1)\pi(b_i+2)\ldots\pi(b_{i+1}-1)$ is order-isomorphic to σ_i for all $1 \leqslant i \leqslant k$, and satisfies properties (ii)$'$-(vi)$'$.

Recall that $b_{i+1} - b_i = |\sigma_i| \geqslant 1$. By Lemma 3.4, we have $\pi \in Y$ and the maps ξ and ξ' are inverses of each other. Hence, the map ξ is the desired bijection. This completes the proof. ∎

Next we give an overview of Guibert's bijection [19] between the set $\mathfrak{S}_n(231, 4\bar{1}32)$ and the set $\mathcal{M}_n$. Given a permutation $\pi \in \mathfrak{S}_n(231, 4\bar{1}32)$, we recursively construct a Motzkin path $\chi(\pi)$ as follows.

- Suppose that $\pi = \pi' n$, where $\pi' \in \mathfrak{S}_n(231, 4\bar{1}32)$. Set $\chi(\pi) = \chi(\pi')H$.
- Suppose that $\pi = \pi' n(i+1)\widetilde{\pi''}$ for some $0 \leqslant i \leqslant n-2$, where $\pi' \in \mathfrak{S}_i(231, 4\bar{1}32)$, $\pi'' \in \mathfrak{S}_{n-2-i}(231, 4\bar{1}32)$ and $\widetilde{\pi''}$ is obtained from π'' by adding $i+1$ to every entry. Set $\chi(\pi) = \chi(\pi')U\chi(\pi'')D$.

Notice that the bijection χ induces an one-to-one correspondence between the set of indecomposable $(231, 4\bar{1}32)$-avoiding permutations of $[n]$ and the set of Motzkin paths of order n with exactly one return point.

Lemma 3.7. *For $n \geqslant 1$, the number of indecomposable $(231, 4\bar{1}32)$-avoiding permutations of $[n]$ is given by $m_{n-2} + \delta_{n,1}$.*

By Lemma 3.7, it is easy to verify that the cardinality of the set $\mathfrak{S}_d^L(231, 4\bar{1}32)$ is given by the left-hand side of Formula 1.4, while the cardinality of the subset of $\mathfrak{S}_d^R(231, 4\bar{1}32)$ in which each component contains at least two entries is given by the left-hand side of Formula 1.5. Together with Formulae (1.4) and (1.5), Lemmas 3.5 and 3.6 imply that, for $n \geqslant 2$ and $n \geqslant d \geqslant 1$,

$$\begin{aligned} |\mathfrak{S}_{n,d}(231, 4\bar{1}32)| &= |X| + |Y| \\ &= \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{d}{2i}\binom{2i}{i} + \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} \binom{d-i-1}{i-1}\binom{d}{i} \\ &= \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{d}{2i}\binom{2i}{i} + \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \frac{i}{d-i}\binom{d}{2i}\binom{2i}{i} \\ &= \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \frac{d}{d-i}\binom{d}{2i}\binom{2i}{i}. \end{aligned} \tag{3.1}$$

Combining Lemma 3.1 and Formula (3.1), we get the enumeration of closed walks of length d in the graph $G(n, 231, 4\bar{1}32)$.

Theorem 3.8. *For $n \geqslant 2$ and $n \geqslant d \geqslant 1$, the number of closed walks of length d in the graph $G(n, 231, 4\bar{1}32)$ is given by $\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \frac{d}{d-i}\binom{d}{2i}\binom{2i}{i}$.*

Following the approach given in [15], we get the enumeration of d-cycles in the graph $G(n, 231, 4\bar{1}32)$.

Theorem 3.9. *For $n \geqslant 2$ and $n \geqslant d \geqslant 1$, the number of d-cycles in the graph $G(n, 231, 4\bar{1}32)$ is given by $\frac{1}{d}\sum_{e|d}\sum_{i=0}^{\lfloor \frac{e}{2} \rfloor} \mu(d/e)\frac{e}{e-i}\binom{e}{2i}\binom{2i}{i}$.*

Proof. Let $h(d)$ denote the number of d-cycles. A closed walk of length d can be obtained by choosing a divisor e of d, an e-cycle and a starting point on the cycle. By repeating the e-cycle d/e times, we obtain a closed walk of length d. Hence, by Theorem 3.8, we have

$$\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \frac{d}{d-i}\binom{d}{2i}\binom{2i}{i} = \sum_{e|d} e \cdot h(e).$$

The result follows by classic Möbius inversion. This completes the proof. ∎

4 Affine permutations avoiding barred patterns

Given a permutation $\pi \in \mathfrak{S}_n$, the *reverse-complement* of π, denoted by π^{rc}, is defined to be the permutation $(n+1-\pi(n))(n+1-\pi(n-1))\ldots(n+1-\pi(1))$. Clearly, the permutation π is indecomposable if and only if its reverse-complement π^{rc} is indecomposable. Moreover, a permutation is $(231, 4\bar{1}32)$-avoiding if and only if its reverse-complement π^{rc} is $(312, 32\bar{4}1)$-avoiding. By lemma 3.7, we get the following result.

Lemma 4.1. *For $n \geqslant 1$, the number of indecomposable $(312, 32\bar{4}1)$-avoiding permutations of $[n]$ is given by $m_{n-2} + \delta_{n,1}$.*

By Lemmas 3.7 and 4.1, we have

$$|\mathfrak{S}_n^L(231, 4\bar{1}32)| = |\mathfrak{S}_n^R(312, 32\bar{4}1)| = \sum_{k\geqslant 1}\sum_{P} n_1 \alpha_{n_1}\alpha_{n_2}\ldots\alpha_{n_k}, \tag{4.1}$$

where the second sum is over all compositions $P = (n_1, n_2, \ldots, n_k)$ of n into k parts. Hence, from Formula (1.4), it follows that

$$|\mathfrak{S}_n^L(231, 4\bar{1}32)| = |\mathfrak{S}_n^R(312, 32\bar{4}1)| = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i}\binom{2i}{i}. \tag{4.2}$$

Lemma 4.2. *There is a bijection between the set $\widetilde{\mathfrak{S}}_n(231, 4\bar{1}32)$ and the set $\mathfrak{S}_n^L(231, 4\bar{1}32)$.*

Proof. Given a permutation $\pi \in \widetilde{\mathfrak{S}}_n(231, 4\bar{1}32)$, we wish to obtain a marked permutation $\theta(\pi) \in \mathfrak{S}_n^L(231, 4\bar{1}32)$. Suppose that $\pi(a)$ is the maximum element among the elements $\pi(1), \pi(2), \ldots, \pi(n)$. By Condition (1.10), $\pi(n+a)$ is also the maximum element among the elements $\pi(n+1), \pi(n+2), \ldots, \pi(2n)$. Moreover, we have $\pi(i) < \pi(a)$ for all $i \leqslant a-1$. Since π avoids the pattern 231, we have $\pi(i) < \pi(j)$ for all $i < a < j$. By similar reasoning, it is easy to verify that $\pi(i) < \pi(j)$ for all $i < a+n < j$. Hence, the permutation $\pi(a)\pi(a+1)\ldots\pi(a+n-1)$ is a $(231, 4\bar{1}32)$-avoiding permutation of $[a, a+n-1]$. Let π' be the permutation of $[n]$ which is order-isomorphic to $\pi(a)\pi(a+1)\ldots\pi(a+n-1)$. Define $\theta(\pi)$ to be the permutation obtained from π' by marking the $(n+1-a)$-th entry.

Obviously, π' avoids the patterns 231 and $4\bar{1}32$. This implies that the resulting permutation $\theta(\pi)$ also avoids the patterns 231 and $4\bar{1}32$. Recall that $\pi(i) < \pi(a)$ for all $a+1 < i \leqslant n$ and π' is order-isomorphic to the subsequence $\pi(a)\pi(a+1)\ldots\pi(a+n-1)$. This yields that the leftmost component of π' contains at least $n+1-a$ elements. Hence, the resulting permutation $\theta(\pi) \in \mathfrak{S}_n^L(231, 4\bar{1}32)$.

Conversely, given a marked permutation $\sigma \in \mathfrak{S}_n^L(231, 4\bar{1}32)$, we wish to recover an affine permutation $\pi = \theta'(\sigma) \in \widetilde{\mathfrak{S}}_n(231, 4\bar{1}32)$. Suppose that the $\sigma(b)$ is marked. Define π to be the bijection $\pi : \mathbb{Z} \to \mathbb{Z}$ such that

- $\pi(n-b+1)\pi(n-b+2)\ldots\pi(2n-b)$ is a permutation of $[n-b+1, 2n-b]$ which is order-isomorphic to σ;
- $\pi(i+n) = \pi(i) + n$.

Clearly, the resulting permutation π avoids the patterns 231 and $4\bar{1}32$. In order to show that $\pi \in \widetilde{\mathfrak{S}}_n(231, 4\bar{1}32)$, it suffices to show that $\sum_{i=1}^{n} \pi(i) = \binom{n+1}{2}$. From $\pi(i+n) = \pi_i + n$ and $\sum_{i=n-b+1}^{2n-b} \pi(i) = \binom{n+1}{2} + n(n-b)$, it follows that

$$\sum_{i=1}^{n} \pi(i) = \sum_{i=n-b+1}^{2n-b} \pi(i) - n(n-b) = \binom{n+1}{2}.$$

Hence, the map θ' is well-defined, that is, $\pi = \theta'(\sigma) \in \widetilde{\mathfrak{S}}_n(231, 4\bar{1}32)$.

It is straightforward to check that the construction of the map θ' reverses each step of the construction of the map θ. Thus the maps θ and θ' are inverses of each other. This yields that the map θ is the desired bijection, which completes the proof. ∎

Our next goal is to establish an analogous bijection between the set $\widetilde{\mathfrak{S}}_n(312, 32\bar{4}1)$ and the set $\mathfrak{S}_n^R(312, 32\bar{4}1)$.

Lemma 4.3. *There is a bijection between the set* $\widetilde{\mathfrak{S}}_n(312, 32\bar{4}1)$ *and the set* $\mathfrak{S}_n^R(312, 32\bar{4}1)$.

Proof. Given a permutation $\pi \in \widetilde{\mathfrak{S}}_n(312, 32\bar{4}1)$, we wish to obtain a marked permutation $\tau(\pi) \in \mathfrak{S}_n^R(312, 32\bar{4}1)$. Suppose that $\pi(a)$ is the minimum element among the elements $\pi(1), \pi(2), \ldots, \pi(n)$. By Condition (1.10), $\pi(a+n)$ is also the minimum element among the elements $\pi(n+1), \pi(n+2), \ldots, \pi(2n)$. Moreover, we have $\pi(i) > \pi(a)$ for all $i \geqslant a+1$. Since π avoids the pattern 312, we have $\pi(i) < \pi(j)$ for all $i < a < j$. By similar reasoning, it is easy to verify that $\pi(i) < \pi(j)$ for all $i < a+n < j$. Hence, the permutation $\pi(a+1) \ldots \pi(a+n)$ is a $(312, 32\bar{4}1)$-avoiding permutation of $[a+1, a+n]$. Let π' be the permutation of $[n]$ which is order-isomorphic to the subsequence $\pi(a+1)\pi(a+2) \ldots \pi(a+n)$. Define $\tau(\pi)$ to be the marked permutation obtained from π' by marking the a-th entry.

Obviously, π' avoids the patterns 312 and $32\bar{4}1$. This implies that the resulting permutation $\tau(\pi)$ also avoids the patterns 312 and $32\bar{4}1$. Recall that $\pi(n+i) > \pi(n+a)$ for all $1 \leqslant i \leqslant a-1$ and π' is order-isomorphic to the subsequence $\pi(a+1)\pi(a+2) \ldots \pi(a+n)$. This yields that the rightmost component of π' contains at least a elements. Hence, the resulting permutation $\tau(\pi) \in \mathfrak{S}_n^R(312, 32\bar{4}1)$.

Conversely, given a marked permutation $\sigma \in \mathfrak{S}_n^R(312, 32\bar{4}1)$, we wish to recover an affine permutation $\pi = \tau'(\sigma) \in \widetilde{\mathfrak{S}}_n(312, 32\bar{4}1)$. Suppose that the b-th entry of its rightmost component is marked. Define π to be the bijection $\pi : \mathbb{Z} \to \mathbb{Z}$ such that

- $\pi(b+1)\pi(b+2) \ldots \pi(n+b)$ is a permutation of $[b+1, n+b]$ which is order-isomorphic to σ;
- $\pi(i+n) = \pi(i) + n$.

Clearly, the resulting permutation π avoids the patterns 312 and $32\bar{4}1$. In order to show that $\pi \in \widetilde{\mathfrak{S}}_n(312, 32\bar{4}1)$, it suffices to show that $\sum_{i=1}^{n} \pi(i) = \binom{n+1}{2}$. Since $\pi(i+n) = \pi_i + n$ and $\sum_{i=b+1}^{n+b} \pi(i) = \binom{n+1}{2} + nb$, we have

$$\sum_{i=1}^{n} \pi(i) = \sum_{i=b+1}^{n+b} \pi(i) - nb = \binom{n+1}{2}.$$

Hence, the map τ' is well-defined, that is, that is, $\sigma = \tau'(P) \in \widetilde{\mathfrak{S}}_n(312, 32\bar{4}1)$. It is easy to verify that the maps τ and τ' are inverses of each other. Thus, the map τ is the desired bijection, which completes the proof. ∎

Together with Formula (4.2), Lemmas 4.2 and 4.3 lead to the following result.

Theorem 4.4. *For $n \geqslant 1$, we have* $|\widetilde{\mathfrak{S}}_n(231, 4\bar{1}32)| = |\widetilde{\mathfrak{S}}_n(312, 32\bar{4}1)| = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i}\binom{2i}{i}$.

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