A short conceptual proof of Narayana's path-counting formula

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Abstract

We deduce Narayana's formula for the number of lattice paths that fit in a Young diagram as a direct consequence of the Gessel-Viennot theorem on non-intersecting lattice paths.

Keywords: lattice paths; Young diagram; Narayana's path-counting formula

Let λ and μ be two partitions so that $\mu \subset \lambda$, and consider the skew Young diagram λ/μ (see Figure 1 for an example). We give a conceptual proof for the fact that the number of minimal lattice paths[1] on $\mathbb{Z}^2$ contained in this skew Young diagram from its southwestern to its northeastern corner is

$$\det\left(\binom{\lambda_j - \mu_i + 1}{j - i + 1}_{1 \leqslant i,j \leqslant n}\right) \tag{1}$$

(n being the number of parts in λ/μ), an extension of Narayana's formula [4] due to Kreweras [3] (see [5]). Narayana's formula is the special case $\mu = \emptyset$, which we include below for completeness.

Theorem 1. *The number of minimal lattice paths on $\mathbb{Z}^2$ contained in the Young diagram of any partition λ with n parts is equal to*

$$\det\left(\binom{\lambda_j + 1}{j - i + 1}_{1 \leqslant i,j \leqslant n}\right). \tag{2}$$

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[1] A minimal lattice path between two lattice points on $\mathbb{Z}^2$ is a lattice path with the smallest possible number of steps connecting the two points.

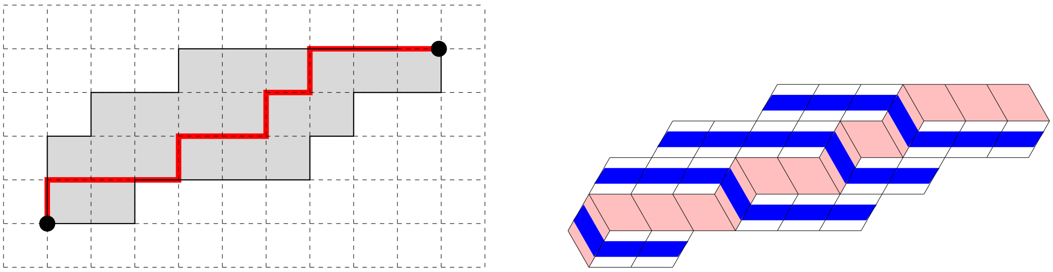

Figure 1: The skew shape $(9, 7, 6, 2)/(3, 1)$ (left); the corresponding region R (right).

Consider the region R on the triangular lattice corresponding to λ/μ indicated by the the outside contour in Figure 2 — it is obtained from the Young diagram of λ/μ by affinely deforming it so that its unit squares become unit rhombi on the triangular lattice, and then translating the southeastern boundary one unit in the $-\pi/3$ polar direction. Recall that lozenge tilings of regions on the triangular lattice are in one-to-one correspondence with families on non-intersecting paths of rhombi (see [1]). The latter can be chosen in three different ways, depending on whether the segments where the paths of lozenges start and end point in the $-\pi/3$, $\pi/3$ or $-\pi$ polar directions. For the region R, the first of these three ways yields a single path of rhombi (lightly shaded in Figure 2), which can be regarded as a lattice path in λ/μ connecting the southwestern and northeastern corners. On the other hand, the second way yields a family of n non-intersecting paths of rhombi, which can be regarded as non-intersecting lattice paths on $\mathbb{Z}^2$. By the Gessel-Viennot theorem [2], their number is the determinant of the $n \times n$ matrix whose (i, j)-entry is the number of minimal lattice paths on Z^2 from the i-th starting point to the j-th ending point. One readily checks that these are precisely the entries of the matrix in (1). This proves formula (1).

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