# Some self-orthogonal codes related to Higman's geometry

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**Abstract**

We examine some self-orthogonal codes constructed from a rank-5 primitive permutation representation of degree 1100 of the sporadic simple group HS of Higman-Sims. We show that $\mathrm{Aut}(C) = \mathrm{HS}{:}2$, where $C$ is a code of dimension 21 associated with Higman's geometry.

**Keywords:** Linear codes; Higman's geometry; Higman-Sims group

## 1 Introduction

The study of binary linear codes invariant under the Higman-Sims group (HS), in particular those constructed from the primitive permutation representations of degrees 100, 176 and 1100 respectively has been carried in [3, 18] and in [15]. Recently, Knapp and Schaeffer [13] using representation theoretic methods provided an elegant account on the binary codes of length 100 related with the Higman-Sims graph. In the paper [16], the

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second author offers an account on non-binary codes from the representations of degree 100 and constructs new 2-designs from the representation of degree 176. It is well known that the Higman-Sims group possesses two inequivalent rank-5 primitive representations of degree 1100, one on the set of edges of $\mathcal{G}$, the Higman-Sims graph with parameters $(100, 22, 0, 6)$, with edge stabilizer isomorphic to $L_3(4){:}2_1$, and the other on the set of conics of G. Higman's geometry (see [4, 9]) with stabilizer of a conic isomorphic to $S_8$. The orbits of the action on the cosets of $L_3(4){:}2_1$ have lengths $1, 42, 105, 280$ and $672$ respectively, while those of the action on the cosets of $S_8$ have lengths $1, 28, 105, 336$ and $630$. In [15], using the orbit of length 672 we constructed the unique and minimal degree faithful irreducible $\mathbb{F}_2$-representation (20-dimensional) of the Higman-Sims group, as a binary $[1100, 20, 480]_2$ code. A review of our paper [On some designs and codes invariant under the Higman-Sims group' (Util. Math., 2011), MR2884789 (2012m:05082)] prompted us to examine the extent of a possible relation between the 20-dimensional code constructed in [15] and Higman's geometry. The careful reader will notice that Higman's geometry originates from an action on the cosets of $S_8$ and not on the cosets of $L_3(4){:}2_1$. Due to this, it would seem that no relation could be borne between the said 20-dimensional code and Higman's geometry. However, on examining the question on the existence of binary codes related with the geometry of G. Higman we were able to show that the 20-dimensional code referred to above is a subcode of codimension 1 in a $[1100, 21, 420]_2$ code constructed in this paper and related to Higman's geometry. To deal with this question we use a method proposed in [7], and construct a self-dual symmetric 1-$(1100, 420, 420)$ design $\mathcal{D}$ taking for point set the conjugacy classes of a maximal subgroup isomorphic to $L_3(4){:}2_1$ and for block set the conjugacy classes of a maximal subgroup isomorphic to $S_8$. The binary row span of the incidence matrix of $\mathcal{D}$ induces a 21-dimensional $[1100, 21, 420]_2$ code whose properties we examine in the sequel.
The paper is organized as follows: in Section 2 we outline our notation and give a brief overview of the HS group in Section 3. In Section 4 we describe the construction method used and in Section 5 and Section 6 we present our results.

## 2 Terminology and notation

Our notation will be standard, and it is as in [1] and $\mathbb{ATLAS}$ [4]. For the structure of groups and their maximal subgroups we follow the $\mathbb{ATLAS}$ notation. The groups $G.H$, $G{:}H$, and $G{}^{\cdot}H$ denote a general extension, a split extension and a non-split extension respectively. For a prime $p$, the symbol $p^m$ denotes an elementary abelian group of that order. The notation $p_+^{1+2n}$ and $p_-^{1+2n}$ are used for extraspecial groups of order $p^{1+2n}$. If $p$ is an odd prime, the subscript is $+$ or $-$ according as the group has exponent $p$ or $p^2$. For $p = 2$ it is $+$ or $-$ according as the central product has an even or odd number of quaternionic factors.

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a $t$-$(v, k, \lambda)$ design, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely $k$ points, and every $t$ distinct points are together incident with precisely $\lambda$ blocks. The **complement** of $\mathcal{D}$ is the structure $\tilde{\mathcal{D}} = (\mathcal{P}, \mathcal{B}, \tilde{\mathcal{I}})$, where $\tilde{\mathcal{I}} = \mathcal{P} \times \mathcal{B} - \mathcal{I}$. The **dual** structure of $\mathcal{D}$

is $\mathcal{D}^t = (\mathcal{B}, \mathcal{P}, \mathcal{I}^t)$, where $(B, p) \in \mathcal{I}^t$ if and only if $(P, B) \in \mathcal{I}$. Thus the transpose of an incidence matrix for $\mathcal{D}$ is an incidence matrix for $\mathcal{D}^t$. We will say that the design is **symmetric** if it has the same number of points and blocks, and **self dual** if it is isomorphic to its dual.

The code $C_F$ of the design $\mathcal{D}$ over the finite field $F$ is the space spanned by the incidence vectors of the blocks over $F$. We take $F$ to be a prime field $F_p$, in which case we write also $C_p$ for $C_F$, and refer to the dimension of $C_p$ as the ***p*-rank** of $\mathcal{D}$.

If the point set of $\mathcal{D}$ is denoted by $\mathcal{P}$ and the block set by $\mathcal{B}$, and if $\mathcal{Q}$ is any subset of $\mathcal{P}$, then we will denote the incidence vector of $\mathcal{Q}$ by $v^{\mathcal{Q}}$. Thus $C_F = \langle v^B \mid B \in \mathcal{B} \rangle$, and is a subspace of $F^{\mathcal{P}}$, the full vector space of functions from $\mathcal{P}$ to $F$. The dual or **orthogonal** code $C_F{}^{\perp}$ of $C_F$ is the orthogonal subspace under the standard inner product. The **hull** of a design's code over some field is the intersection $C_F \cap C_F{}^{\perp}$. If a linear code over a field of order $q$ is of length $n$, dimension $k$, and minimum weight $d$, then we write $[n, k, d]_q$ to represent this information. A **constant word** in the code is a codeword all of whose coordinate entries are the same. The all-one vector will be denoted by $\boldsymbol{\jmath}$, and is the constant vector of weight the length of the code. Two linear codes of the same length and over the same field are **equivalent** if each can be obtained from the other by permuting the coordinate positions and multiplying each coordinate position by a non-zero field element. They are **isomorphic** if they can be obtained from one another by permuting the coordinate positions. An **automorphism** of a code is any permutation of the coordinate positions that maps codewords to codewords. An automorphism thus preserves each weight class of $C$.

## 3 The Higman-Sims group and its automorphism group

As we had in [15], the Higman-Sims simple group can be constructed from the Higman-Sims graph $\mathcal{G}$. Let $\mathcal{G} = (\Omega, \mathcal{E})$ be a graph of valence 22 on the set $\Omega$ of 100 points such that any given vertex has 22 neighbours (points) and the remaining 77 vertices are joined to 6 of these points and may be labelled by the corresponding hexad. Two of the 77 vertices are joined only if the corresponding hexads are disjoint. The hexads form a Steiner system $S(3, 6, 22)$. The Higman-Sims simple group HS is the subgroup of even permutations of $\mathrm{Aut}(\mathcal{G}) \cong$ HS:2, the automorphism group of HS. The point stabilizer of $\mathrm{Aut}(\mathcal{G})$ on $\Omega$ is $\mathrm{Aut}(S(3, 6, 22)) \cong M_{22}$:2 and the order of the Higman-Sims group HS is $44352000 = 2^9{\cdot}3^2{\cdot}5^3{\cdot}7{\cdot}11$. The group HS has two inequivalent representations of degree 1100, one on the set of edges of $\mathcal{G}$ with point stabilizer isomorphic to $L_3(4){:}2_1$ and the other on the set of conics of G. Higman's geometry (see [4]) with point stabilizer isomorphic to $S_8$. The subgroup $S_8$ is also the set stabilizer of a fixed outer automorphism of HS.

**Result 1.** *(Magliveras [14]) The Higman-Sims group* HS *has exactly* 12 *conjugacy classes of maximal subgroups, as follows:*

| | |
|---|---|
| $M_{22}$ | $U_3(5){:}2$ (2 *classes*) |
| $L_3(4){:}2_1$ | $S_8$ |
| $2^4.S_6$ | $4^3{:}L_3(2)$ |
| $M_{11}$ (2 *classes*) | $4{\cdot}2^4{:}S_5$ |
| $2 \times A_6{\cdot}2^2$ | $5{:}4 \times A_5$. |

The primitive representations described in Result 1 are of degrees 100, 176, 176, 1100, 1100, 3850, 4125, 5600, 5600, 5575, 15400 and 36960 respectively. In TABLE 1 below the first column depicts the ordering of the primitive representations of HS and HS:2 respectively, as given by Magma (or the $\mathbb{ATLAS}$ [4]) and as used in our computations; the second gives the maximal subgroups; the third gives the degrees (the number of cosets of the point stabilizer).

TABLE 1: Maximal subgroups of HS and HS:2

| No. | Max. sub. of HS | Deg. | Max. sub. of HS:2 | Deg. |
|---|---|---|---|---|
| 1 | $M_{22}$ | 100 | HS | 2 |
| 2 | $U_3(5):2$ | 176 | $M_{22}:2$ | 100 |
| 3 | $U_3(5):2$ | 176 | $L_3(4){:}2^2$ | 1100 |
| 4 | $L_3(4):2_1$ | 1100 | $S_8 \times 2$ | 1100 |
| 5 | $S_8$ | 1100 | $2^5 \cdot S_6$ | 3850 |
| 6 | $2^4 \cdot S_6$ | 3850 | $4^3{:}(L_3(2) \times 2)$ | 4125 |
| 7 | $4^3 : L_3(2)$ | 4125 | $2_+^{1+6}{:}S_5$ | 5775 |
| 8 | $M_{11}$ | 5600 | $2 \times A_6 \cdot 2^2 \cdot 2$ | 15400 |
| 9 | $M_{11}$ | 5600 | $5^{1+2}{:}(Q_8{:}4)$ | 22176 |
| 10 | $4{\cdot}2^4{:}S_5$ | 5775 | $5{:}4 \times S_5$ | 36960 |
| 11 | $2 \times A_6.2^2$ | 15400 | | |
| 12 | $5 : 4 \times A_5$ | 36960 | | |

# 4 The construction

Crnković and Mikulić in [7] (see also [8]) gave a method that outlines a construction of 1-designs from finite permutation groups, which are not necessarily symmetric, and stabilizers of a point and a block that are not necessarily conjugate. This result is a generalization of an earlier construction of symmetric 1-designs and regular graphs which was described in [10, Proposition 1], used in [12] and later corrected in [11]. For the sake of completeness and readiness of use we state the result below.

**Result 2.** *Let $G$ be a finite permutation group acting primitively on the sets $\Omega_1$ and $\Omega_2$ of size $m$ and $n$, respectively. Let $\alpha \in \Omega_1$ and $\delta \in \Omega_2$ and let $\Delta_2 = \delta G_\alpha$ be the $G_\alpha$-orbit of $\delta \in \Omega_2$ and $\Delta_1 = \alpha G_\delta$ be the $G_\delta$-orbit of $\alpha \in \Omega_1$. If $\Delta_2 \neq \Omega_2$ and $\mathcal{B} = \{\Delta_2 g : g \in G\}$, then $\mathcal{D}(G, \alpha, \delta) = (\Omega_2, \mathcal{B})$ is a 1-$(n, |\Delta_2|, |\Delta_1|)$ design with $m$ blocks, and $G$ acts as an automorphism group, primitive on points and blocks of the design.*

*Remark* 3. Let $M_1$ and $M_2$ to be two maximal subgroups of a finite group $G$. We denote the conjugacy class of $M_i$, $i = 1, 2$, by $ccl_G(M_i)$ and $|ccl_G(M_i)| = [G : N_G(M_i)]$. The elements of $ccl_G(M_i)$, $i = 1, 2$, are denoted by $M_i^{g_1}, M_i^{g_2}, \ldots, M_i^{g_{j_i}}$ and thus we obtain $j_i = [G : N_G(M_i)]$.
Observe that $G$ acts primitively on $ccl_G(M_1)$ and $ccl_G(M_2)$ by conjugation. In this way a primitive $1-$design can be constructed such that:

- the point set of the design is $ccl_G(M_2)$, and the block set is $ccl_G(M_1)$,
- the block $M_1^{g_i}$ is incident with the point $M_2^{h_j}$ if and only if $M_1^{g_i} \cap M_2^{h_j} \cong M_1 \cap M_2$.

In the case when $G$ is a finite simple group and $M_1$ and $M_2$ are two maximal subgroups of $G$, then clearly $N_G(M_i) = M_i$ and hence $j_i = [G : M_i]$ for $i = 1, 2$.

## 5 A $[1100, 21, 420]_2$ code invariant under HS:2.

Recall that our interest is in the construction of designs and codes which bear an association with Higman's geometry. To this end, we consider the description given in Section 3 for the Higman-Sims group and the discussion by G. Higman in [9, Section 1, p.75] (see also [13, Sections 4, and 5] for a model of Higman's geometry) to construct Higman's geometry. Hence, taking for points the edges of the graph $\mathcal{G}$ and for blocks the conics of the geometry we construct a 1-$(1100, 420, 420)$ design $\mathcal{D}$ on which HS acts primitively on points and on blocks. It will be intuitive to notice that if we use Result 2 and Remark 3 we take for point set $\Omega_2$ the conjugacy classes of a maximal subgroup isomorphic to $L_3(4){:}2_1$ and for block set $\Omega_1$ the conjugacy classes of a maximal subgroup isomorphic to $S_8$. Notice that $\Omega_1$, and $\Omega_2$ are primitive HS-sets of degree 1100. In the sequel we examine the properties of a binary $[1100, 21, 420]_2$ self-orthogonal and doubly-even code $C$ determined by the row span of the incidence matrix of $\mathcal{D}$ and explore its possible connections with Higman's geometry.

**Lemma 4.** *Let $G = \mathrm{HS}{:}2$ and let $\mathcal{D} = (\Omega_2, \mathcal{B})$ be a design constructed as in Result 2 taking for point set $\Omega_2$ the conjugacy classes of a maximal subgroup isomorphic to $L_3(4){:}2_1$ and for block set $\Omega_1$ the conjugacy classes of a maximal subgroup isomorphic to $S_8$. Then $\mathcal{D}$ is a self-dual, symmetric* 1-$(1100, 420, 420)$ *design with $G = \mathrm{Aut}(\mathcal{D})$ acting point- and block-primitively.*

*Proof.* From Result 2 it is clear that $G \subseteq \mathrm{Aut}(\mathcal{D})$. Once again, from Result 2, and also from the $\mathbb{ATLAS}$ [4, p.80] we see that $G$ acts primitively on both $\Omega_2$ and $\Omega_1$, where $\Omega_1$ and $\Omega_2$ represent the point and block sets of $\mathcal{D}$ and these are the edges of the Higman-Sims graph $\mathcal{G}$ and the conics of Higman's geometry, respectively, (in terms of Result 2 these are the sets of conjugacy classes of a maximal subgroup isomorphic to $L_3(4){:}2_1$ and of a maximal subgroup isomorphic to $S_8$ respectively) with degree $|\Omega_1| = |\Omega_2| = 1100$. This shows that $\mathcal{D}$ is a point primitive, symmetric 1-design. Moreover, the stabilizers $G_x$ and $G_B$ of a point $x \in \Omega_2$ and of a block $B \in \mathcal{B}$ have five orbits, namely $\Phi_1 = \{x\}$, $\Phi_2$, $\Phi_3$, $\Phi_4$, and $\Phi_5$ with subdegrees:1, 42, 105, 280, and 672; and three orbits namely $\Psi_1$,

$\Psi_2$ and $\Psi_3$ with subdegrees: 120, 420 and 560. It remains to show that $G = \mathrm{Aut}(\mathcal{D})$. Now $G \subseteq \mathrm{Aut}(\mathcal{D}) \subseteq S_{1100}$, so $\mathrm{Aut}(\mathcal{D})$ is a primitive permutation group on $\Omega_2$ of degree 1100. Moreover, $\mathrm{Aut}(\mathcal{D})_x$ must fix $\Delta_2$ setwise, and hence $\mathrm{Aut}(\mathcal{D})_x$ also has orbits of lengths 1, 42, 105, 280, and 672 in $\Omega_2$. The only primitive group of degree 1100, such that $\mathrm{Aut}(\mathcal{D})_x$ has orbit lengths 1, 42, 105, 280, and 672 is HS:2, see [17, Table 9,p.178]. Hence $G = \mathrm{Aut}(\mathcal{D})$. □

Taking the binary row span of the incidence matrix of $\mathcal{D}$ we obtain the 21-dimensional HS:2-invariant $[1100, 21, 420]_2$ code whose properties are discussed in Proposition 5 below.

**Proposition 5.** *Let $C$ be the binary code defined by the incidence matrix of $\mathcal{D}$. Then $C$ is a self-orthogonal doubly-even $[1100, 21, 420]_2$ code. Its dual code $C^\perp$ is a $[1100, 1079, 4]_2$ with words of weight 4. Furthermore, $\jmath \in C^\perp$ and $\mathrm{Aut}(C) \cong$ HS:2.*

*Proof.* The parameters of $C$ were determined through computations with Magma [2]. Since the dimension of the $C$ equals the dimension of $C \cap C^\perp$, we have $C \subseteq C^\perp$ and so $C$ is self orthogonal. Since HS is a normal subgroup of HS:2 it follows that $C$ is HS:2-invariant.

Notice from TABLE 3 that there are exactly 1100 codewords of minimum weight 420 in $C$. Thus the minimum weight codewords are the incidence vectors of the blocks of the design $\mathcal{D}$, and hence spanning vectors of $C$. From this we deduce that $\mathrm{Aut}(\mathcal{D}) \subseteq \mathrm{Aut}(C)$. Now, order considerations shows that $\mathrm{Aut}(C) \cong$ HS:2. Furthermore, since the spanning words of $C$ have weight divisible by four, it follows that $C$ is doubly-even.

The weight distribution of $C$ is listed in TABLE 3. In TABLE 3, $l$ represents the weight of a codeword and $A_l$ denotes the number of codewords of weight $l$.

TABLE 3: The weight distribution of $C$

| $l$ | $A_l$ | $l$ | $A_l$ |
|---|---|---|---|
| 0 | 1 | 544 | 793100 |
| 420 | 1100 | 548 | 500500 |
| 480 | 15400 | 564 | 308000 |
| 484 | 100 | 576 | 231000 |
| 500 | 22176 | 612 | 23100 |
| 512 | 7975 | 672 | 1100 |
| 532 | 193600 | | |

In addition note that the blocks of $\mathcal{D}$ are of even size, so $\jmath$ meets evenly every vector of $C$, and thus $\jmath \in C^\perp$. Finally, using MacWilliams identities and Pless' power moment identities we obtain the minimum weight 4 for $C^\perp$. □

*Remark* 6. The code and designs found above can be described geometrically: the 1100 codewords of weight 420 are the incidence vectors of the blocks of the design $\mathcal{D}$, and these represent the conics of Higman's geometry.

### 5.1 Stabilizer in HS:2 of a word of weight $l$

Let $L = \{420, 480, 484, 500, 512, 672\}$ and $\overline{L} = \{532, 544, 548, 564, 576, 612\}$. For $l \in L \cup \overline{L}$ we define $W_l = \{w_l \in C \mid \mathrm{wt}(w_l) = l\}$. Since $\mathrm{Aut}(C) \cong$ HS:2, in this section we determine the structures of the stabilizers $(\mathrm{HS{:}2})_{w_l}$, for all nonzero weight $l$.

We show in Lemma 7 that for $l \in L$ the stabilizer $(\mathrm{HS{:}2})_{w_l}$ is a maximal subgroup of HS:2, where $(\mathrm{HS{:}2})_{w_{420}} \cong S_8 \times 2$, $(\mathrm{HS{:}2})_{w_{480}} \cong 2 \times A_6 \cdot 2^2 \cdot 2$, $(\mathrm{HS{:}2})_{w_{484}} \cong M_{22}{:}2$, $(\mathrm{HS{:}2})_{w_{500}} \cong 5^{1+2}{:}(Q_8{:}4)$, $(\mathrm{HS{:}2})_{w_{512_1}} \cong 2^5 \cdot S_6$, $(\mathrm{HS{:}2})_{w_{512_2}} \cong 4^3{:}(L_3(2) \times 2)$ and $(\mathrm{HS{:}2})_{w_{672}} \cong L_3(4){:}2^2$. Now for $w_l \in W_l$ we take the support of $w_l$ and orbit it under HS:2 to form the blocks of the 1-$(1100, l, k_l)$ designs $\mathcal{D}_{w_l}$, where $k_l = |(w_l)^{\mathrm{HS:2}}| \times \dfrac{l}{1100}$. We show that for all $l \in L$, HS:2 acts primitively on these designs. Information on these designs is given in TABLE 4 and TABLE 5.

Next in Lemma 8 by considering $w_l$ where $l \in \overline{L}$ we describe the structures of $(\mathrm{HS{:}2})_{w_l}$ and show that these are not maximal in HS:2.

**Lemma 7.** *Let $l \in L$ and $w_l \in W_l$. Then* $(\mathrm{HS{:}2})_{w_l}$ *is a maximal subgroup of* HS:2. *Furthermore* HS:2 *is primitive on $\mathcal{D}_{w_l}$ for each $l$.*

*Proof.* First assume that $l \in \{420, 480, 484, 500, 672\}$. Since HS is transitive on $W_l$, so is HS:2. Hence for $l \in L$, each $W_l$ forms an orbit under the action of HS:2, so that $(\mathrm{HS})_{w_l}$ is subgroup of index 2 in $(\mathrm{HS{:}2})_{w_l}$. Therefore by the orbit stabilizer Theorem and the $\mathbb{ATLAS}$ (or right hand side of TABLE 1) we have $[\mathrm{HS{:}2} : (\mathrm{HS{:}2})_{w_l}] \in \{1100, 15400, 100, 22176, 4125, 3850\}$. Using the list of maximal subgroups of $HS{:}2$ (see right hand side of TABLE 1), we deduce that $(\mathrm{HS{:}2})_{w_{420}} \in \{L_3(4){:}2^2,\ S_8 \times 2,\ S\}$, where $S$ possibly is a subgroup of $M_{22}{:}2$ of index 11. Examining the list of maximal subgroups of $M_{22}{:}2$ in [4] or [5], we can easily see that $M_{22}{:}2$ contains no subgroup of index 11. Also direct calculations of the composition factors of $(\mathrm{HS{:}2})_{w_{420}}$ excludes the first possibility, namely $L_3(4){:}2^2$. Hence $(\mathrm{HS})_{w_{420}} \cong S_8 \times 2$.

Similarly we can deduce that $(\mathrm{HS{:}2})_{w_{480}} \in \{2 \times A_6 \cdot 2^2 \cdot 2,\ H,\ K,\ M, N\}$, where, possibly, $H$ is a subgroup of index 154 in $M_{22}{:}2$, $K$ of index 4 in $2^5 \cdot S_6$, $M$ of index 14 in $S_8 \times 2$ and $N$ of index 14 in $L_3(4){:}2^2$. We deal with the elimination of $H$, $K$, $M$ and $N$ in the following:

(i) From the list of maximal subgroups of $M_{22}{:}2$, there are two possible candidates for $H$, either a subgroup of index 7 in $L_3(4){:}2_2$ or of index 2 in $2^4{:}S_6$. The list of maximal subgroups of $L_3(4)$ shows that it contains no subgroup of index 7. The group $2^4{:}S_6$ is a maximal subgroup of $M_{22}{:}2$ and computations with Magma show that its non-trivial normal subgroups are of type $2^4$, and hence it cannot have a subgroup of index 2.

(ii) We constructed the maximal subgroup $2^5 \cdot S_6$ inside HS:2 and found out that it does not contain a subgroup of index 4.

(iii) Lists of maximal subgroups of $S_8 \times 2$ and $L_3(4){:}2^2$ (see [4]) eliminate the possibilities of $M$ and $N$.

Therefore $(\mathrm{HS}{:}2)_{w_{480}} = 2 \times A_6 \cdot 2^2 \cdot 2$.

Further, we can deduce that $(\mathrm{HS}{:}2)_{w_{484}} \in \{M_{22}{:}2,\ A\}$, where, possibly, $A$ is a subgroup of index 11 in $L_3(4){:}2^2$ or $S_8 \times 2$, or $A$ is a subgroup of index 154 in $2 \times A_6 \cdot 2^2 \cdot 2$. A careful verification of each case rules out all other possibilities except $M_{22}{:}2$. Hence we deduce that $(\mathrm{HS}{:}2)_{w_{484}} \cong M_{22}{:}2$.

Similarly by using the composition factors we deduce that $(\mathrm{HS}{:}2)_{w_{500}} \cong 5^{1+2}{:}(Q_8{:}4)$.

For $l = 672$, we argue similarly as in the case $l = 420$, since $A_{672} = A_{420}$. Thus, we deduce that $(\mathrm{HS}{:}2)_{w_{672}} \in \{L_3(4){:}2^2,\ S_8 \times 2,\ B\}$, where $B$ possibly is a subgroup of $M_{22}{:}2$ of index 11. Since $M_{22}{:}2$ contains no subgroup of index 11 we deduce that $B$ is either a subgroup of $L_3(4){:}2^2$ or a subgroup of $S_8 \times 2$. An examination of the composition factors of $(\mathrm{HS}{:}2)_{w_{672}}$ excludes the second possibility, namely $S_8 \times 2$. Hence $(\mathrm{HS}{:}2)_{w_{672}} = L_3(4){:}2^2$.

It follows by the above case by case analysis that $(\mathrm{HS}{:}2)_{w_{420}}$, $(\mathrm{HS}{:}2)_{w_{480}}$, $(\mathrm{HS}{:}2)_{w_{484}}$, $(\mathrm{HS}{:}2)_{w_{500}}$ and $(\mathrm{HS}{:}2)_{w_{672}}$ are all maximal subgroups of HS:2.

Now, by the transitivity of HS:2 on the code coordinates, the codewords of $W_l$ form a 1-design $\mathcal{D}_{w_l}$ with $A_l$ blocks. This implies that HS:2 is transitive on the blocks of $D_{w_l}$ for each $w_l$ and since $(\mathrm{HS}{:}2)_{w_l}$, for $l \in \{420, 480, 484, 500,\ 672\}$ is a maximal subgroup of HS:2, we deduce that HS:2 acts primitively on $\mathcal{D}_{w_l}$ for $l \in \{420, 480, 484, 500,\ 672\}$. Note that $\mathcal{D}_{w_{420}}$, $\mathcal{D}_{w_{480}}$, $\mathcal{D}_{w_{484}}$, $\mathcal{D}_{w_{500}}$, and $\mathcal{D}_{w_{672}}$ are 1-designs with parameters 1-$(1100, 420, 420)$, 1-$(1100, 480, 6720)$, 1-$(1100, 484, 44)$, 1-$(1100, 500, 10080)$, and 1-$(1100, 672, 672)$ respectively with 1100, 15400, 100, 22176, and 1100 blocks.

Finally for $l = 512$, $W_{512}$ splits into two orbits of lengths 3850 and 4125, namely $W_{(512)_1}$ and $W_{(512)_2}$ respectively. Let $u = w_{(512)_1} \in W_{(512)_1}$ and $v = w_{(512)_2} \in W_{(512)_2}$. Then $(\mathrm{HS}{:}2)_u$ is a subgroup of order 23040, and from the right hand side of TABLE 1 we deduce that $(\mathrm{HS}{:}2)_u \cong 2^5 \cdot S_6$. Similarly $|(\mathrm{HS}{:}2)_v| = 21504$ and $(\mathrm{HS}{:}2)_v$ is a maximal subgroup of HS:2 isomorphic to $4^3{:}(L_3(2) \times 2)$. Notice that $D_u$ is a 1-$(1100, 512, 1792)$ design having 3850 blocks, while $D_v$ is a 1-$(1100, 512, 1920)$ design with 4125 blocks. HS:2 acts primitively on $\mathcal{D}_u$ and $\mathcal{D}_v$. □

**Lemma 8.** *Let $l \in \overline{L}$ and $w_l \in W_l$. Then $(\mathrm{HS}{:}2)_{w_l}$ is a non-maximal subgroup of* HS:2.

*Proof.* We give a description of the cases $l = 532$, $l = 544$ and $l = 548$ since the sets of codewords of these weights split into a number of orbits. The remaining cases, i.e., $l = 564, 576$, and 612 are much simpler to be dealt with using similar arguments. Let $l = 532$. Then $W_{532}$ splits into two orbits of lengths 61600 and 132000, namely $W_{(532)_1}$, and $W_{(532)_2}$ respectively. Let $a = w_{(532)_1} \in W_{(532)_1}$ and $b = w_{(532)_2} \in W_{(532)_2}$. Then $(\mathrm{HS}{:}2)_a$ is a subgroup of order 1440, and thus not maximal in HS:2. Using the composition factors of $(\mathrm{HS}{:}2)_a$ and the information in [6] and [5] we deduce that $(\mathrm{HS})_a \cong A_6{:}2{:}2$. Similarly $|(\mathrm{HS}{:}2)_b| = 672$ and $(\mathrm{HS}{:}2)_b$ is a non-maximal subgroup of HS:2 isomorphic to $L_2(7){:}2{:}2$.

For $l = 544$ we have that $W_{544}$ splits into three orbits of lengths 77000, 346500 and 369600, namely $W_{(544)_1}$, $W_{(544)_2}$ and $W_{(544)_3}$ respectively. Set $x = w_{(544)_1} \in W_{(544)_1}, x' = w_{(544)_2} \in W_{(544)_2}$ and $x'' = w_{(544)_3} \in W_{(544)_3}$. We used Magma and [6], and also the information on maximal subgroups of HS:2, to determine the structure of $(\mathrm{HS}{:}2)_x$ and deduce that $(\mathrm{HS}{:}2)_x = ((\mathrm{HS})_x){:}2 \cong (2^4{:}(S_3 \times S_3)){:}2$. Similarly, since $|(\mathrm{HS}{:}2)_{x'}| = 256$, and so not a maximal subgroup of HS:2. We determined that $(\mathrm{HS}{:}2)_{x'} \cong X{:}2$ where

$X = ((((4 \times 2){:}2){:}2){:}2){:}2$, and $X{:}2 \leqslant P{:}2$ with $P{:}2 \in Syl_2(\mathrm{HS}{:}2)$. We can easily show that

$$P{:}2 \cong ((4.2^4){:}D_8){:}2 \cong 2_+^{1+6}{:}D_8.$$

Clearly, $(\mathrm{HS}{:}2)_{x''} \cong S_5{:}2$.

If $l = 548$, then $W_{548}$ splits into two orbits of lengths 38500 and 462000, namely $W_{(548)_1}$, and $W_{(548)_2}$. Let $s = w_{(548)_1} \in W_{(548)_1}$ and $t = w_{(548)_2} \in W_{(548)_2}$. Then $(\mathrm{HS}{:}2)_s$ is a subgroup of order 2304, and thus not maximal in HS:2. Using the composition factors of $(\mathrm{HS}{:}2)_a$ and the information in [6] we deduce that $(\mathrm{HS})_s \cong 2^4{:}S_3{:}S_3{:}2 \times 2$. Similarly $|(\mathrm{HS}{:}2)_t| = 192$ and $(\mathrm{HS}{:}2)_t$ is a non-maximal subgroup of HS:2 isomorphic to $2^{1+4}{:}S_3$.

Using similar arguments for $l = 564, 576$ and 612 we deduce that $(\mathrm{HS}{:}2)_{w_{564}} \cong (2^3 \times S_3){:}S_3$, and $(\mathrm{HS}{:}2)_{w_{576}} \cong (2^3{\cdot}S_4){:}2$, and $(\mathrm{HS}{:}2)_{w_{612}} \cong ((2^4{:}A_5){:}2){:}2$. □

TABLES 4 and 5 below, list the structures of $(\mathrm{HS}{:}2)_{w_l}$ and $\mathcal{D}_{w_l}$ for all $l$, respectively.

## 5.2 Observations

(i) In TABLE 4 the first column represents the codewords of weight $l$ and the second column represents the stabilizer in HS:2 of a codeword $w_l$ of $W_l$. In the final column we test the maximality of $(\mathrm{HS}{:}2)_{w_l}$ in HS:2.

TABLE 4
Stabilizer in HS:2 of a word $w_l$

| $l$ | $(\mathrm{HS}{:}2)_{w_l}$ | Maximality | $l$ | $(\mathrm{HS}{:}2)_{w_l}$ | Maximality |
|---|---|---|---|---|---|
| 420 | $S_8 \times 2$ | Yes | $(544)_2$ | $2_+^{1+6}{:}D_8$ | No |
| 480 | $2 \times A_6.2^2$ | Yes | $(544)_3$ | $S_5{:}2$ | No |
| 484 | $M_{22}{:}2$ | Yes | $(548)_1$ | $2^4{:}S_3{:}S_3{:}2 \times 2$ | No |
| 500 | $5^{1+2}{:}(Q_8{:}4)$ | Yes | $(548)_2$ | $2^{1+4}{:}S_3$ | No |
| $(512)_1$ | $2^5 \cdot S_6$ | Yes | 564 | $(2^3 \times S_3){:}S_3$ | No |
| $(512)_2$ | $4^3(L_3(2) \times 2)$ | Yes | 576 | $(2^3{\cdot}S_4){:}2$ | No |
| $(532)_1$ | $A_6{:}2{:}2$ | No | 612 | $((2^4{:}A_5){:}2){:}2$ | No |
| $(532)_2$ | $L_2(7){:}2{:}2$ | No | 672 | $L_3(4) : 2^2$ | Yes |
| $(544)_1$ | $(2^4{:}(S_3 \times S_3)){:}2$ | No | | | |

(ii) In TABLE 5 the first column represents the codewords of weight $l$ and the second column gives the parameters of the designs $\mathcal{D}_{w_l}$ which were constructed in Section 5.1. In the third column we list the number of blocks of $\mathcal{D}_{w_l}$. We test the primitivity for the action of HS:2 on $\mathcal{D}_{w_l}$ in the final column.

TABLE 5
1-designs $\mathcal{D}_{w_l}$ from HS:2

| $l$ | $\mathcal{D}_{w_l}$ | No. of blocks | Primitivity |
|---|---|---|---|
| 420 | 1-(1100, 420, 420) | 1100 | Yes |
| 480 | 1-(1100, 480, 6720) | 15400 | Yes |
| 484 | 1-(1100, 484, 44) | 100 | Yes |
| 500 | 1-(1100, 500, 10080) | 22176 | Yes |
| $(512)_1$ | 1-(1100, 512, 1792) | 3850 | Yes |
| $(512)_2$ | 1-(1100, 512, 1920) | 4125 | Yes |
| $(532)_1$ | 1-(1100, 532, 29792) | 61600 | No |
| $(532)_2$ | 1-(1100, 532, 63840) | 132000 | No |
| $(544)_1$ | 1-(1100, 544, 38080) | 77000 | No |
| $(544)_2$ | 1-(1100, 544, 171360) | 346500 | No |
| $(544)_3$ | 1-(1100, 544, 182784) | 369600 | No |
| $(548)_1$ | 1-(1100, 548, 19180) | 38500 | No |
| $(548)_2$ | 1-(1100, 548, 230160) | 462000 | No |
| 564 | 1-(1100, 564, 157920) | 308000 | No |
| 576 | 1-(1100, 576, 120960) | 231000 | No |
| 612 | 1-(1100, 612, 12852) | 23100 | No |
| 672 | 1-(1100, 672, 672) | 1100 | Yes |

## 6 Binary codes from the complementary design

It is often of interest to know whether a given code contains the all-one vector. We showed in Proposition 5 that $\jmath \in C^{\perp}$. Since $\jmath \notin C$ we know that $C \neq \tilde{C}$, where $\tilde{C}$ is the code of the complementary 1-$(1100, 680, 680)$ design $\tilde{\mathcal{D}}$. In Proposition 9 below we collect the properties of $\tilde{C}$. Observe by the weight distribution that $C$ and $\tilde{C}$ are complementary codes.

**Proposition 9.** *Let $\tilde{C}$ be the binary code defined by the incidence matrix of the design $\tilde{\mathcal{D}}$. Then $\tilde{C}$ is a self-orthogonal doubly-even $[1100, 21, 480]_2$ code. Its dual code $\tilde{C}^{\perp}$ is a $[1100, 1079, 4]_2$ with words of weight 4. Furthermore,* $\mathrm{Aut}(\tilde{C}) \cong$ HS:2.

*Proof.* The proof follows similar arguments to those used in the proof of Proposition 5. So we omit the details. ☐

*Remark* 10. The weight distribution of $\tilde{C}$ is listed in TABLE 6.

TABLE 6: Weight distribution of $\tilde{C}$

| $l$ | $A_l$ | $l$ | $A_l$ |
|---|---|---|---|
| 0 | 1 | 568 | 193600 |
| 480 | 15400 | 576 | 231000 |
| 488 | 23100 | 600 | 22176 |
| 512 | 7975 | 616 | 100 |
| 536 | 308000 | 672 | 1100 |
| 544 | 793100 | 680 | 1100 |
| 552 | 500500 | | |

A closer examination of TABLE 3 and 6 shows that the codewords of $C$ and $\tilde{C}$ appear in complementary pairs. Hence, an analysis of the structures of the stabilizers, their maximality and the primitivity of the corresponding designs can be dealt with in a manner similar to that in the previous results.

## 7 Concluding remarks

The codes $C$ and $\tilde{C}$ meet in their doubly-even self-orthogonal code $C_0$. It turns out that $C_0$ is isomorphic to the code constructed in [15]. $C_0$ consists just of the code vectors of $C$ whose weights are divisible by 32. Let $J = \langle \boldsymbol{\jmath} \rangle$ denote the repetition code generated by the all 1-vector $\boldsymbol{\jmath}$. Then $C_1 = C_0 + J$ is a self-orthogonal doubly-even $[1100, 21, 428]_2$ code which is isomorphic to the code of the complementary 1-$(1100, 428, 428)$ design discussed in [15]. We note that $C$, $\tilde{C}$ and $C_1$ are HS-invariant subcodes of $C_2 = C + J$ containing $C_0$ with codimension 1.

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